Visual Numerics
A Rogue Wave Software Company

IMSL®

C# Numerical Library

User Guide

VERSION 6.5

Visual Numerics
A Rogue Wave Software Company

IMSL® C# Numerical Library V.6.5

User's Guide

Trusted for Almost Four Decades

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Solving Systems of Linear Equations

A square system of linear equations has the form $Ax = b$, where A is a user-specified n x n matrix, b is a given right-hand side n vector, and x is the solution n vector. Each entry of A and b must be specified by the user. The entire vector x is returned as output.

When A is invertible, a unique solution to $Ax = b$ exists. The most commonly used direct method for solving $Ax = b$ factors the matrix A into a product of triangular matrices and solves the resulting triangular systems of linear equations. Functions that use direct methods for solving systems of linear equations all compute the solution to $Ax = b$.

Matrix Factorizations

In some applications, it is desirable to just factor the n x n matrix A into a product of two triangular matrices. This can be done by a constructor of a class for solving the system of linear equations $Ax = b$. The constructor of class `LU` computes the LU factorization of A.

Besides the basic matrix factorizations, such as LU and LL^T, additional matrix factorizations also are provided. For a real matrix A, its QR factorization can be computed using the class `QR`. The class for computing the singular value decomposition (SVD) of a matrix is discussed in a later section.

Matrix Inversions

The inverse of an n x n nonsingular matrix can be obtained by using the method `Inverse` in the classes for solving systems of linear equations. The inverse of a matrix need not be computed if the purpose is to *solve* one or more systems of linear equations. Even with multiple right-hand sides, solving a system of linear equations by computing the inverse and performing matrix multiplication is usually more expensive than the method discussed in the next section.

Multiple Right-Hand Sides

Consider the case where a system of linear equations has more than one right-hand side vector. It is most economical to find the solution vectors by first factoring the coefficient matrix A into products of triangular matrices. Then, the resulting triangular systems of linear equations are solved for each right-hand side. When A is a real general matrix, access to the LU factorization of A is computed by a constructor of LU. The solution x_k for the k-th right-hand side vector, b_k is then found by two triangular solves, $Ly_k = b_k$ and $Ux_k = y_k$. The method `Solve` in class `LU` is used to solve each right-hand side. These arguments are found in other functions for solving systems of linear equations.

Least-Squares Solutions and QR Factorizations

Least-squares solutions are usually computed for an over-determined system of linear equations $A_{m\times n}x = b$, where $m > n$. A least-squares solution x minimizes the Euclidean length of the residual vector $r = Ax - b$. The class QR computes a unique least-squares solution for x when A has full column rank. If A is rank-deficient, then the *base* solution for some variables is computed. These variables consist of the resulting columns after the interchanges. The *QR* decomposition, with column interchanges or pivoting, is computed such that $AP = QR$. Here, Q is orthogonal, R is upper-trapezoidal with its diagonal elements nonincreasing in magnitude, and P is the permutation matrix determined by the pivoting. The base solution x_B is obtained by solving $R(P^T)x = Q^T b$ for the base variables. For details, see class `QR`. The QR factorization of a matrix A such that $AP = QR$ with P specified by the user can be computed using keywords.

Singular Value Decompositions and Generalized Inverses

The SVD of an m x n matrix A is a matrix decomposition $A = USV^T$. With $q = \min(m,n)$, the factors $U_{m\times q}$ and $V_{n\times q}$ are orthogonal matrices, and $S_{q\times q}$ is a nonnegative diagonal matrix with nonincreasing diagonal terms. The class `SVD` computes the singular values of A by default. Part or all of the U and V matrices, an estimate of the rank of A, and the generalized inverse of A, also can be obtained.

Ill-Conditioning and Singularity

An m x n matrix A, is mathematically singular if there is an $x \neq 0$ such that $Ax = 0$. In this case, the system of linear equations $Ax = b$ does not have a unique solution. On the other hand, a matrix A is *numerically* singular if it is "close" to a mathematically singular matrix. Such problems are called *ill-conditioned*. If the numerical results with an ill-conditioned problem are unacceptable, users can obtain an *approximate* solution to the system using the SVD of A: If $q = \min(m,n)$ and

$$A = \sum_{i=1}^{q} s_{i,i} u_i v_i^T$$

then the approximate solution is given by the following:

$$x_k = \sum_{i=1}^{k} t_{i,i} \left(b^T u_i\right) v_i$$

The scalars $t_{i,i}$ are defined below.

$$t_{i,i} = \begin{cases} s_{i,i}^{-1} & \text{if } s_{i,i} \geq \text{tol} > 0 \\ 0 & \text{otherwise} \end{cases}$$

The user specifies the value of *tol*. This value determines how "close" the given matrix is to a singular matrix. Further restrictions may apply to the number of terms in the sum, $k \leq q$. For example, there may be a value of $k \leq q$ such that the scalars $\left|b^T u_i\right|, i > k$ are smaller than the average uncertainty in the right-hand side b. This means that these scalars can be replaced by zero; and hence, b is replaced by a vector that is within the stated uncertainty of the problem.

Sparse Matrix Storage Modes

All classes that work with sparse matrices (e.g. classes `SparseCholesky`, `SuperLU`, `ComplexSparseCholesky` and `ComplexSuperLU`) require a matrix input format that only stores information about the nonzero entries of these matrices. IMSL C# Numerical Library supports two such formats: The sparse coordinate storage (SCS) format and the Sparse Array format. The SCS format stores the value of each matrix entry together with that entry's row and column index. The Sparse Array

format stores the sparse matrix in two arrays of arrays. One array contains the references to the nonzero value arrays for each row of the matrix, the other contains the references to the associated column index arrays. As an example, consider the following sparse 6×6 matrix:

$$A = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & -3 & -1 & 0 & 0 \\ 0 & 0 & 15 & 0 & 0 & 0 \\ -2 & 0 & 0 & 10 & -1 & 0 \\ -1 & 0 & 0 & -5 & 1 & -3 \\ -1 & -2 & 0 & 0 & 0 & 6 \end{pmatrix}$$

This matrix has 15 nonzero elements, and the sparse coordinate representation would be

```
row   0   1   1   1   2   3   3   3   4   4  4   4   5   5  5
col   0   1   2   3   2   0   3   4   0   3  4   5   0   1  5
val  10  10  -3  -1  15  -2  10  -1  -1  -5  1  -3  -1  -2  6
```

Since this representation does not rely on the order of the matrix elements, an equivalent form would be

```
row   5   4   3   0   5   1   2   1   4    3    1   4   3  5   4
col   0   0   0   0   1   1   2   2   3    3    3   4   4  5   5
val  -1  -1  -2  10  -2  10  15  -3  -5   10   -1   1  -1  6  -3
```

In Sparse Array format, matrix *A* can be represented by the following two arrays, `index` and `values`:

```
index:
row 0: 0
row 1: 1, 2, 3
row 2: 2
row 3: 0, 3, 4
row 4: 0, 3, 4, 5
row 5: 0, 1, 5


values:
row 0: 10
row 1: 10, -3, -1
row 2: 15
row 3: -2, 10, -1
row 4: -1, -5, 1, -3
row 5: -1, -2, 6
```

In contrast to the SCS format, the row order is fixed here, but the entries within each row do not rely on any order.

In IMSL C# Numerical Library, sparse matrices are instances of classes `SparseMatrix/ComplexSparseMatrix`. Use of the SCS and Sparse Array input format is reflected by different constructors. The following code fragment shows how a user can construct a `SparseMatrix` for the matrix *A* above in SCS format:

```
// Generate an empty 6 by 6 SparseMatrix object
int nRows = nColumns = 6;
```

```
SparseMatrix a = new SparseMatrix(nRows, nColumns);
// Fill object a with the entries of matrix A,
// A given in SCS format
a.Set(0, 0, 10.0); a.Set(1, 1, 10.0); a.Set(1, 2, -3.0);
a.Set(1, 3, -1.0); a.Set(2, 2, 15.0); a.Set(3, 0, -2.0);
a.Set(3, 3, 10.0); a.Set(3, 4, -1.0); a.Set(4, 0, -1.0);
a.Set(4, 3, -5.0); a.Set(4, 4, 1.0); a.Set(4, 5, -3.0);
a.Set(5, 0, -1.0); a.Set(5, 1, -2.0); a.Set(5, 5, 6.0);
```

Similarly, the following code fragment shows how to construct a `SparseMatrix` in Sparse Array format:

```
// Generate matrix A in Sparse Array format
int nRows = nColumns = 6;
int[][] aIndex =   { {0},
                     {1, 2, 3},
                     {2},
                     {0, 3, 4},
                     {0, 3, 4, 5},
                     {0, 1, 5}};

int[][] aValues =  { {10.0},
                     {10.0, -3.0, -1.0},
                     {15.0},
                     {-2.0, 10.0, -1.0},
                     {-1.0, -5.0, 1.0, -3.0},
                     {-1.0, -2.0, 6.0}};

// Create SparseMatrix object based on A in Sparse Array format
SparseMatrix a = new SparseMatrix(nRows, nColumns, aIndex, aValues);
```

All IMSL C# Numerical Library algorithms working on sparse matrices use `SparseMatrix` or `ComplexSparseMatrix` as an input argument type. A square $n \times n$ matrix is called *symmetric* if $A^T = A$, where A^T denotes the tranpose of A. It is called *Hermitian* if $A^H = A$. Here, $A^H \equiv \bar{A}^T$ denotes the conjugate transpose of A. For these types of matrices, all matrix information is stored in the lower triangular part of the matrix. Therefore, for all IMSL C# Numerical Library sparse matrix classes working on symmetric or Hermitian matrices it is sufficient to store only the lower triangle of the matrix in the `SparseMatrix` or `ComplexSparseMatrix` input object.

The complex 4×4 matrix

$$H = \begin{pmatrix} 4 & 1-i & 0 & 0 \\ 1+i & 4 & 1-i & 0 \\ 0 & 1+i & 4 & 1-i \\ 0 & 0 & 1+i & 4 \end{pmatrix}$$

is Hermitian positive definite. Typically, matrices of this type are input to methods of class `ComplexSparseCholesky`.

The following code fragment shows how the lower triangular part of H can be constructed as a `ComplexSparseMatrix` type in SCS input format:

```
// Generate an empty 4 by 4 ComplexSparseMatrix object
int nRows = nColumns = 4;
ComplexSparseMatrix h = new ComplexSparseMatrix(nRows, nColumns);
// Store lower triangular part of H in object h
```

```
h.Set(0, 0, new Complex(4.0,0.0));
h.Set(1, 0, new Complex(1.0, 1.0)); h.Set(1, 1, new Complex(4.0,0.0));
h.Set(2, 1, new Complex(1.0,1.0)); h.Set(2, 2, new Complex(4.0, 0.0));
h.Set(3, 2, new Complex(1.0, 1.0)); h.Set(3, 3, new Complex(4.0, 0.0));
```

Matrix Class

`public class Imsl.Math.Matrix`

Matrix manipulation functions.

Methods

Add

`static public double[,] Add(double[,] a, double[,] b)`

Description

Add two rectangular matrixs, a + b.

Parameters

`a` – A `double` rectangular matrix.

`b` – A `double` rectangular matrix.

Returns

A `double` rectangular matrix representing the matrix sum of the two arguments.

Exception

`System.ArgumentException` is thrown when the matrices are not the same size

CheckSquareMatrix

`static public void CheckSquareMatrix(double[,] a)`

Description

Check that the matrix is square.

Parameter

`a` – A `double` matrix.

Exception

`System.ArgumentException` is thrown when the matrix is not square

FrobeniusNorm

`static public double FrobeniusNorm(double[,] a)`

Description

Return the frobenius norm of a matrix.

Parameter

`a` – A `double` rectangular matrix.

Returns

A `double` scalar value equal to the frobenius norm of the matrix.

InfinityNorm

```
static public double InfinityNorm(double[,] a)
```

Description

Return the infinity norm of a matrix.

Parameter

`a` – A `double` rectangular matrix.

Returns

A `double` scalar value equal to the maximum of the row sums of the absolute values of the matrix elements.

InverseLowerTriangular

```
static public double[,] InverseLowerTriangular(double[,] a)
```

Description

Returns the inverse of the lower triangular matrix `a`.

Parameter

`a` – A `double` square lower triangular matrix.

Returns

A `double` matrix containing the inverse of `a`.

InverseUpperTriangular

```
static public double[,] InverseUpperTriangular(double[,] a)
```

Description

Returns the inverse of the upper triangular matrix `a`.

Parameter

`a` – A `double` square upper triangular matrix.

Returns

A `double` matrix containing the inverse of `a`.

Multiply

```
static public double[] Multiply(double[] x, double[,] a)
```

Description

Return the product of the row matrix x and the rectangular matrix a.

Parameters

`x` – A `double` row matrix.

`a` – A `double` rectangular matrix.

Returns

A `double` vector representing the product of the arguments, `x*a`.

Exception

`System.ArgumentException` is thrown when the number of elements in the input row matrix is not equal to the number of rows of the matrix

Multiply

```
static public double[] Multiply(double[] x, double[,] a, int processors)
```

Description

Return the product of the row matrix x and the rectangular matrix a.

Parameters

`x` – A `double` row matrix.

`a` – A `double` rectangular matrix.

`processors` – An `int` which specifies the number of processors to use. If `processors` is less than 1, then `processors` = 1 is used.

Returns

A `double` vector representing the product of the arguments, `x*a`.

Exception

`System.ArgumentException` is thrown when the number of elements in the input row matrix is not equal to the number of rows of the matrix

Multiply

```
static public double[] Multiply(double[,] a, double[] x)
```

Description

Multiply the rectangular matrix a and the column matrix x.

Parameters

`a` – A `double` rectangular matrix.

`x` – A `double` column matrix.

Returns

A `double` vector representing the product of the arguments, `a * x`.

Exception

`System.ArgumentException` is thrown when the number of columns in the input matrix is not equal to the number of elements in the input column vector

Multiply

```
static public double[,] Multiply(double[,] a, double[,] b, int processors)
```

Description

Multiply two rectangular arrays, `a` * `b`, using multiple processors.

Parameters

`a` – A `double` rectangular array.

`b` – A `double` rectangular array.

`processors` – An `int` which specifies the number of processors to use. If `processors` is less than 1, then `processors` = 1 is used.

Returns

The `double` matrix product of `a` times `b`

Multiply

```
static public double[,] Multiply(double[,] a, double[,] b)
```

Description

Multiply two rectangular matrices, a * b.

Parameters

`a` – A `double` rectangular matrix.

`b` – A `double` rectangular matrix.

Returns

The `double` matrix product of `a * b`.

Exception

`System.ArgumentException` is thrown when the number of columns in `a` is not equal to the number of rows in `b`

OneNorm

```
static public double OneNorm(double[,] a)
```

Description

Return the matrix one norm.

Parameter

`a` – A `double` rectangular matrix.

Returns

A `double` value equal to the maximum of the column sums of the absolute values of the matrix elements.

Subtract

```
static public double[,] Subtract(double[,] a, double[,] b)
```

Description

Subtract two rectangular matrixs, a - b.

Parameters

`a` – A `double` rectangular matrix.

`b` – A `double` rectangular matrix.

Returns

A `double` rectangular matrix representing the matrix difference of the two arguments.

Exception

`System.ArgumentException` is thrown when the matrices are not the same size

Transpose

```
static public double[,] Transpose(double[,] a)
```

Description

Return the transpose of a matrix.

Parameter

`a` – A `double` matrix.

Returns

A `double` matrix which is the transpose of the argument.

Example: Matrix and PrintMatrix

The 1 norm of a matrix is found using a method from the Matrix class. The matrix is printed using the PrintMatrix class.

```
using System;
using Imsl.Math;

public class MatrixEx1
{
    public static void  Main(String[] args)
    {
        double nrm1;
        double[,] a = {
            {0.0, 1.0, 2.0, 3.0},
            {4.0, 5.0, 6.0, 7.0},
            {8.0, 9.0, 8.0, 1.0},
            {6.0, 3.0, 4.0, 3.0}
```

```
        };

        //    Get the 1 norm of matrix a
        nrm1 = Matrix.OneNorm(a);

        //    Construct a PrintMatrix object with a title
        PrintMatrix p = new PrintMatrix("A Simple Matrix");

        //    Print the matrix and its 1 norm
        p.Print(a);
        Console.Out.WriteLine("The 1 norm of the matrix is " + nrm1);
    }
}
```

Output

```
A Simple Matrix
   0  1  2  3
0  0  1  2  3
1  4  5  6  7
2  8  9  8  1
3  6  3  4  3

The 1 norm of the matrix is 20
```

ComplexMatrix Class

`public class Imsl.Math.ComplexMatrix`

Complex matrix manipulation functions.

Methods

Add

```
static public Imsl.Math.Complex[,] Add(Imsl.Math.Complex[,] a,
Imsl.Math.Complex[,] b)
```

Description

Add two rectangular `Complex` arrays, a + b.

Parameters

`a` – A `Complex` rectangular array.

`b` – A `Complex` rectangular array.

Returns

The Complex matrix sum of the two arguments.

Exception

System.ArgumentException is thrown when the matrices are not the same size.

CheckSquareMatrix

```
static public void CheckSquareMatrix(Imsl.Math.Complex[,] a)
```

Description

Check that the Complex matrix is square.

Parameter

a – A Complex matrix.

Exception

System.ArgumentException is thrown when the matrix is not square.

FrobeniusNorm

```
static public double FrobeniusNorm(Imsl.Math.Complex[,] a)
```

Description

Return the frobenius norm of a Complex matrix.

Parameter

a – A Complex rectangular matrix.

Returns

A double value equal to the frobenius norm of the matrix.

InfinityNorm

```
static public double InfinityNorm(Imsl.Math.Complex[,] a)
```

Description

Return the infinity norm of a Complex matrix.

Parameter

a – A Complex rectangular matrix.

Returns

A double value equal to the maximum of the row sums of the absolute values of the array elements.

Multiply

```
static public Imsl.Math.Complex[] Multiply(Imsl.Math.Complex[] x,
Imsl.Math.Complex[,] a)
```

Description

Returns the product of the row vector x and the rectangular array a, both Complex.

Parameters

`x` – A `Complex` row vector.

`a` – A `Complex` rectangular matrix.

Returns

A `Complex` vector containing the product of the arguments, `x * a`.

Exception

`System.ArgumentException` is thrown when the number of elements in the input vector is not equal to the number of rows of the matrix.

Multiply

```
static public Imsl.Math.Complex[] Multiply(Imsl.Math.Complex[] x,
Imsl.Math.Complex[,] a, int processors)
```

Description

Returns the product of the row vector x and the rectangular array a, both `Complex`.

Parameters

`x` – A `Complex` row vector.

`a` – A `Complex` rectangular matrix.

`processors` – An `int` which specifies the number of processors to use. If `processors` is less than 1, then `processors` = 1 is used.

Returns

A `Complex` vector containing the product of the arguments, `x * a`.

Exception

`System.ArgumentException` is thrown when the number of elements in the input vector is not equal to the number of rows of the matrix.

Multiply

```
static public Imsl.Math.Complex[] Multiply(Imsl.Math.Complex[,] a,
Imsl.Math.Complex[] x)
```

Description

Multiply the rectangular array a and the column vector x, both `Complex`.

Parameters

`a` – A `Complex` rectangular matrix.

`x` – A `Complex` vector.

Returns

A `Complex` vector containing the product of the arguments, `a * x`.

Exception

`System.ArgumentException` is thrown when the number of columns in the input matrix is not equal to the number of elements in the input vector.

Multiply

```
static public Imsl.Math.Complex[,] Multiply(Imsl.Math.Complex[,] a,
Imsl.Math.Complex[,] b)
```

Description

Multiply two `Complex` rectangular arrays, a * b.

Parameters

`a` – A `Complex` rectangular array.

`b` – A `Complex` rectangular array.

Returns

The `Complex` matrix product of `a` times `b`.

Exception

`System.ArgumentException` is thrown when the number of columns in `a` is not equal to the number of rows in `b`.

Multiply

```
static public Imsl.Math.Complex[,] Multiply(Imsl.Math.Complex[,] a,
Imsl.Math.Complex[,] b, int processors)
```

Description

Multiply two `Complex` rectangular arrays, `a` * `b`, using multiple processors.

Parameters

`a` – A `Complex` rectangular array.

`b` – A `Complex` rectangular array.

`processors` – An `int` which specifies the number of processors to use. If `processors` is less than 1, then `processors` = 1 is used.

Returns

The `Complex` matrix product of `a` times `b`

OneNorm

```
static public double OneNorm(Imsl.Math.Complex[,] a)
```

Description

Return the `Complex` matrix one norm.

Parameter

`a` – A `Complex` rectangular array.

Returns

A `double` value equal to the maximum of the column sums of the absolute values of the array elements.

Subtract

```
static public Imsl.Math.Complex[,] Subtract(Imsl.Math.Complex[,] a,
Imsl.Math.Complex[,] b)
```

Description

Subtract two `Complex` rectangular arrays, a - b.

Parameters

`a` – A `Complex` rectangular array.

`b` – A `Complex` rectangular array.

Returns

The `Complex` matrix difference of the two arguments.

Exception

`System.ArgumentException` is thrown when the matrices are not the same size.

Transpose

```
static public Imsl.Math.Complex[,] Transpose(Imsl.Math.Complex[,] a)
```

Description

Return the transpose of a `Complex` matrix.

Parameter

`a` – A `Complex` matrix.

Returns

The `Complex` matrix transpose of the argument.

Example: Print a Complex Matrix

A Complex matrix and its transpose is printed.

```
using System;
using Imsl.Math;

public class ComplexMatrixEx1
{
    public static void  Main(String[] args)
    {
        Complex[,] a = {
            {new Complex(1, 3), new Complex(3, 5), new Complex(7, 9)},
            {new Complex(8, 7), new Complex(9, 5), new Complex(1, 9)},
            {new Complex(2, 9), new Complex(6, 9), new Complex(7, 3)},
            {new Complex(5, 4), new Complex(8, 4), new Complex(5, 9)}
        };
```

```
        //    Print the matrix
        new PrintMatrix("Matrix A").Print(a);

        //    Print the transpose of the matrix
        new PrintMatrix("Transpose(A)").Print(ComplexMatrix.Transpose(a));
    }
}
```

Output

```
      Matrix A
    0     1     2
0  1+3i  3+5i  7+9i
1  8+7i  9+5i  1+9i
2  2+9i  6+9i  7+3i
3  5+4i  8+4i  5+9i

       Transpose(A)
    0     1     2     3
0  1+3i  8+7i  2+9i  5+4i
1  3+5i  9+5i  6+9i  8+4i
2  7+9i  1+9i  7+3i  5+9i
```

SparseMatrix Class

`public class Imsl.Math.SparseMatrix`

A general real sparse matrix intended to be efficiently and easily updated.

A `SparseMatrix` can be constructed from a set of arrays, or it can be abstractly created as an empty array and then incrementally built into final form. It is usually easier to create an empty `SparseMatrix` of a set size and then use the `Set` method to set the elements of the array. When setting the elements of the sparse array, their positions should be thought of as their positions in the dense array. Elements can be set in any order, but only the elements actually set are stored.

This class includes methods to update the sparse matrix. There are also methods to multiply a sparse matrix and a vector or to multiply two sparse matrices. To solve a sparse linear system use `SparseCholesky` or `SuperLU`.

See Also

Imsl.Math.SparseCholesky (p. 81), Imsl.Math.SuperLU (p. 49)

Properties

NumberOfColumns

```
public int NumberOfColumns {get; }
```

Description

The number of columns in the matrix.

Property Value

An `int` containing the number of columns in the matrix.

NumberOfNonZeros

```
public long NumberOfNonZeros {get; }
```

Description

The number of nonzeros in the matrix.

Property Value

A `long` containing the number of nonzeros in the matrix.

NumberOfRows

```
public int NumberOfRows {get; }
```

Description

The number of rows in the matrix.

Property Value

An `int` containing the number of rows in the matrix.

Constructors

SparseMatrix

```
public SparseMatrix(int nRows, int nColumns)
```

Description

Creates a new instance of `SparseMatrix`. Initially this is the zero matrix.

Parameters

`nRows` – An `int` specifying the number of rows in the sparse matrix.

`nColumns` – An `int` specifying the number of columns in the sparse matrix.

SparseMatrix

```
public SparseMatrix(Imsl.Math.SparseMatrix A)
```

Description

Creates a new instance of `SparseMatrix` which is a copy of another `SparseMatrix` object.

Parameter

`A` – A `SparseMatrix` object containing the matrix to be copied.

SparseMatrix

```
public SparseMatrix(Imsl.Math.SparseMatrix.SparseArray sparseArray)
```

Description

Constructs a sparse matrix from a `SparseArray`.

Parameter

`sparseArray` – A `SparseArray` used to initialize the sparse matrix. The field `NumberOfNonZeros` in `SparseArray` (p. 27) is not used for initialization, so it does not have to be set.

SparseMatrix

```
public SparseMatrix(int nRows, int nColumns, int[][] index, double[][] values)
```

Description

Constructs a sparse matrix from `SparseArray` data.

Parameters

`nRows` – An `int` specifying the number of rows in the sparse matrix.

`nColumns` – An `int` specifying the number of columns in the sparse matrix.

`index` – An `int` jagged array containing the column indices of all nonzero elements corresponding to the compressed representation of the sparse matrix in `values`. The size of `index` must be identical to the size of `values`. The *i*-th row contains the column indices of all nonzero elements of row *i* of the sparse matrix. The *j*-th element of row *i* is the column index of the value located at the same position in `values`.

`values` – A `double` jagged array containing the compressed representation of a real sparse matrix of size `nRows` by `nColumns`. The number of rows in `values` must be `nRows`. The *i*-th row contains all nonzero elements of row *i* of the full sparse matrix.

Methods

Add

```
static public Imsl.Math.SparseMatrix Add(double alpha, double beta,
Imsl.Math.SparseMatrix A, Imsl.Math.SparseMatrix B)
```

Description

Performs element-wise addition of two real sparse matrices `A`, `B` of type `SparseMatrix`, $C \leftarrow \alpha A + \beta B$.

Parameters

`alpha` – A `double` scalar value applied to `SparseMatrix A`.

`beta` – A `double` scalar value applied to `SparseMatrix B`.

`A` – A `SparseMatrix` matrix.

`B` – A `SparseMatrix` matrix.

Returns

A `SparseMatrix` matrix representing the computed sum.

Exception

`System.ArgumentException` is thrown when the matrices are not of the same size.

CheckSquareMatrix

```
public void CheckSquareMatrix()
```

Description

Check that the matrix is square.

Exception

`System.ArgumentException` is thrown if the matrix is not square.

FrobeniusNorm

```
public double FrobeniusNorm()
```

Description

Returns the Frobenius norm of the matrix.

Returns

A `double` scalar equal to the Frobenius norm of the matrix.

Get

```
public double Get(int iRow, int jColumn)
```

Description

Returns the value of an element in the matrix.

Parameters

`iRow` – An `int` specifying the row index of the element.

`jColumn` – An `int` specifying the column index of the element.

Returns

A `double` containing the value of the `iRow`-th and `jColumn`-th element. If the element was never set, its value is zero.

InfinityNorm

```
public double InfinityNorm()
```

Description

Returns the infinity norm of the matrix.

Returns

A `double` scalar equal to the maximum of the row sums of the absolute values of the array elements of the sparse matrix.

Multiply

```
public double[] Multiply(double[] x)
```

Description

Multiply the matrix by a vector.

Parameter

`x` – A `double` column array.

Returns

A `double` vector representing the product of the constructed matrix and *x*, *Ax*.

Exception

`System.ArgumentException` is thrown when the number of columns of the matrix represented by this object is not equal to the number of elements in the input column vector.

Multiply

```
static public double[] Multiply(Imsl.Math.SparseMatrix A, double[] x)
```

Description

Multiply sparse matrix `A` and column array `x`, *Ax*.

Parameters

`A` – A `SparseMatrix` matrix.

`x` – A `double` column array.

Returns

A `double` vector representing the product of the arguments, *Ax*.

Exception

`System.ArgumentException` is thrown when the number of columns in the input matrix is not equal to the number of elements in the input column vector.

Multiply

```
static public double[] Multiply(double[] x, Imsl.Math.SparseMatrix A)
```

Description

Multiply row array `x` and sparse matrix `A`, $x^T A$.

Parameters

`x` – A `double` row array.

`A` – A `SparseMatrix` matrix.

Returns

A `double` vector representing the product of the arguments, $x^T A$.

Exception

`System.ArgumentException` is thrown when the number of elements in the input vector is not equal to the number of rows of the matrix.

Multiply

```
static public Imsl.Math.SparseMatrix Multiply(Imsl.Math.SparseMatrix A,
Imsl.Math.SparseMatrix B)
```

Description

Multiply two sparse matrices, $C \leftarrow AB$.

Parameters

`A` – A `SparseMatrix` sparse matrix.

`B` – A `SparseMatrix` sparse matrix.

Returns

The `SparseMatrix` product AB of `A` and `B`.

Exception

`System.ArgumentException` is thrown when the number of columns of matrix `A` is not equal to the number of rows of matrix `B`.

MultiplySymmetric

```
static public double[] MultiplySymmetric(Imsl.Math.SparseMatrix A, double[] x)
```

Description

Multiply sparse symmetric matrix `A` and column vector `x`.

Parameters

`A` – A `SparseMatrix` sparse symmetric matrix, where only the lower triangular part of the matrix is to be used.

`x` – A `double` vector.

Returns

A `double` vector representing the product of the arguments, Ax.

Exception

`System.ArgumentException` is thrown when the input matrix is not square or the number of columns in the input matrix is not equal to the number of elements in the input column vector.

OneNorm

```
public double OneNorm()
```

Description

Returns the matrix one norm of the sparse matrix.

Returns

A `double` scalar containing the maximum of the column sums of the absolute values of the array elements.

PlusEquals

```
public double PlusEquals(int iRow, int jColumn, double x)
```

Description

Adds a value to an element in the matrix.

Parameters

`iRow` – An `int` specifying the row index of the element.

`jColumn` – An `int` specifying the column index of the element.

`x` – A `double` specifying the value to be added to the `iRow`-th and `jColumn`-th element.

Returns

A `double` containing the updated value of the `iRow`-th and `jColumn`-th element, which equals its old value plus *x*.

Set

```
public void Set(int iRow, int jColumn, double x)
```

Description

Sets the value of an element in the matrix.

Parameters

`iRow` – An `int` specifying the row index of the element.

`jColumn` – An `int` specifying the column index of the element.

`x` – A `double` specifying the value of the `iRow`-th and `jColumn`-th element.

ToDenseMatrix

```
public double[,] ToDenseMatrix()
```

Description

Returns the sparse matrix as a dense matrix.

Returns

A `double` rectangular array containing this matrix with all of the zeros explicitly present. The number of rows and columns in the returned matrix is the same as in the sparse matrix.

ToSparseArray

```
public Imsl.Math.SparseMatrix.SparseArray ToSparseArray()
```

Description

Returns the sparse matrix in the `SparseArray` form.

Returns

A `SparseArray` object representing the sparse matrix.

See Also: Imsl.Math.SparseMatrix.SparseArray (p. 27)

Transpose

```
public Imsl.Math.SparseMatrix Transpose()
```

Description

Returns the transpose of the matrix.

Returns

A `SparseMatrix` matrix containing the transpose of the constructed `SparseMatrix` object.

Example 1: SparseMatrix

The matrix product of two matrices is computed using a method from the `SparseMatrix` class. The one norm of the result is also computed. The matrix and its norm are printed.

```
using System;
using Imsl.Math;

public class SparseMatrixEx1
{
    public static void  Main(String[] args)
    {
        SparseMatrix b = new SparseMatrix(6, 6);
        b.Set(0, 0, 10.0);
        b.Set(1, 1, 10.0);
        b.Set(1, 2, -3.0);
        b.Set(1, 3, -1.0);
        b.Set(2, 2, 15.0);
        b.Set(3, 0, -2.0);
        b.Set(3, 3, 10.0);
        b.Set(3, 4, -1.0);
        b.Set(4, 0, -1.0);
        b.Set(4, 3, -5.0);
        b.Set(4, 4, 1.0);
        b.Set(4, 5, -3.0);
        b.Set(5, 0, -1.0);
        b.Set(5, 1, -2.0);
        b.Set(5, 5, 6.0);

        SparseMatrix c = new SparseMatrix(6, 3);
        c.Set(0, 0, 5.0);
        c.Set(1, 2, -3.0);
        c.Set(2, 0, 1.0);
        c.Set(2, 2, 7.0);
        c.Set(3, 0, 2.0);
        c.Set(4, 1, -5.0);
```

```
        c.Set(4, 2, 2.0);
        c.Set(5, 2, 4.0);

        SparseMatrix A = SparseMatrix.Multiply(b, c);

        // Get the one norm of matrix A
        Console.Out.WriteLine("The 1-norm of the matrix is "
            + A.OneNorm());
        SparseMatrix.SparseArray sa = A.ToSparseArray();

        // Print the matrix and its one norm
        Console.Out.WriteLine("row  column  value");
        for (int i = 0; i < sa.NumberOfRows; i++)
        {
            for (int j = 0; j < sa.Index[i].Length; j++)
            {
                int jj = sa.Index[i][j];
                Console.Out.WriteLine(" " + i + "     " + jj + "     "
                    + sa.Values[i][j]);
            }
        }
    }
}
```

Output

```
The 1-norm of the matrix is 198
row  column  value
 0     0     50
 1     0     -5
 1     2     -51
 2     0     15
 2     2     105
 3     0     10
 3     1     5
 3     2     -2
 4     0     -15
 4     1     -5
 4     2     -10
 5     0     -5
 5     2     30
```

Example 2: SparseMatrix Using Matrix Market Format

The matrix market exchange format is an ASCII file format that represents sparse matrices in coordinate format. It consists of three sections: The header section is the first line in the file and contains general informations about the matrix, e.g. data type and symmetry properties. This line is followed by the comments section which consists of zero or more lines of comments. The remainder of the file is the data section. Its first line contains the row number, column number and number of nonzeros of the matrix. The following lines contain location and value of all nonzero entries of the matrix, usually one per line. A file in Matrix Market format is read and converted to a `SparseMatrix`. Matrix information and its one norm are printed.

```
using System;
using Imsl.Math;

public class SparseMatrixEx2
{
    public class MTXReader
    {
        public String TypeCode
        {
            get
            {
                return this.typecode;
            }

        }
        public SparseMatrix Matrix
        {
            get
            {
                return this.matrix;
            }

        }
        private String typecode;
        private SparseMatrix matrix;

        public virtual void  read(String filename)
        {
            System.IO.FileStream aFile = System.IO.File.OpenRead(filename);
            System.IO.StreamReader br = new System.IO.StreamReader(aFile);

            // read type code initial line
            String line = br.ReadLine();
            typecode = line.Replace("%%", "");

            // read comment lines if any
            bool comment = true;
            while (comment)
            {
                line = br.ReadLine();
                comment = line.StartsWith("%");
            }

            // line now contains the size information which needs to be parsed
            String[] str = line.Split(new Char[] {' '});
            int nRows = (Int32.Parse(str[0].Trim()));
            int nColumns = (Int32.Parse(str[1].Trim()));
            int nNonZeros = (Int32.Parse(str[2].Trim()));

            // now we're into the data section
            matrix = new SparseMatrix(nRows, nColumns);
            while (true)
            {
                line = br.ReadLine();
                if (line == null)
                    break;
```

```
                str = line.Split(new Char[] {' '});
                int i = (Int32.Parse(str[0].Trim()));
                int j = (Int32.Parse(str[1].Trim()));

                double x=0.0;
                if (str[2].Trim() == "")
                {
                    x = Convert.ToDouble(str[3].Trim());
                }
                else
                {
                    x = Convert.ToDouble(str[2].Trim());
                }
                matrix.Set(i - 1, j - 1, x);
            }
            br.Close();
        }
    }


    public static void  Main(String[] args)
    {
        MTXReader mr = new MTXReader();
        mr.read("bcsstk01.mtx");
        SparseMatrix A = mr.Matrix;

        // Print the matrix type
        Console.Out.WriteLine("The matrix type is " + mr.TypeCode);

        // Print the matrix information and its one norm
        Console.Out.WriteLine("The number of rows is " + A.NumberOfRows);
        int ncols = A.NumberOfColumns;
        Console.Out.WriteLine("The number of columns is " + ncols);
        long nnz = A.NumberOfNonZeros;
        Console.Out.WriteLine("The number of nonzero elements is " + nnz);
        Console.Out.WriteLine();
        Console.Out.WriteLine("The 1 norm of the matrix is " + A.OneNorm());
    }
}
```

Output

```
The matrix type is MatrixMarket matrix coordinate real symmetric
The number of rows is 48
The number of columns is 48
The number of nonzero elements is 224

The 1 norm of the matrix is 3009444444.44744
```

SparseMatrix.SparseArray Class

`public class Imsl.Math.SparseMatrix.SparseArray`

The `SparseArray` class uses public fields to hold the data for a sparse matrix in the Sparse Array format.

This format came about as a means for storing sparse matrices. In this format, a sparse matrix is represented by two arrays of arrays. One array contains the references to the nonzero value arrays for each row of the sparse matrix. The other contains the references to the associated column index arrays.

As an example, consider the following real sparse matrix:

$$A = \begin{pmatrix} 10.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 10.0 & -3.0 & -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 15.0 & 0.0 & 0.0 & 0.0 \\ -2.0 & 0.0 & 0.0 & 10.0 & -1.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & -5.0 & 1.0 & -3.0 \\ -1.0 & -2.0 & 0.0 & 0.0 & 0.0 & 6.0 \end{pmatrix}$$

In `SparseArray`, this matrix can be represented by the two jagged arrays, `values` and `index`, where `values` refers to the nonzero entries in *A* and `index` to the column indices:

```
double values[][] = {{10.0},
                     {10.0, -3.0, -1.0},
                     {15.0},
                     {-2.0, 10.0, -1.0},
                     {-1.0, -5.0, 1.0, -3.0},
                     {-1.0, -2.0, 6.0}
                    };

int index[][] = {{0},
                 {1, 2, 3},
                 {2},
                 {0, 3, 4},
                 {0, 3, 4, 5},
                 {0, 1, 5}
                };
```

Fields

Index

`public int[][] Index`

Description

Jagged array containing column indices. The length of this array equals `numberOfRows`. The length of each row equals the number of nonzeros in that row of the sparse matrix.

NumberOfColumns

```
public int NumberOfColumns
```

Description

Number of columns in the matrix.

NumberOfNonZeros

```
public long NumberOfNonZeros
```

Description

Number of nonzeros in the matrix.

NumberOfRows

```
public int NumberOfRows
```

Description

Number of rows in the matrix.

Values

```
public double[][] Values
```

Description

Jagged array containing sparse array values. This array must have the same shape as `Index`.

Constructor

SparseArray

```
public SparseArray()
```

Description

Initializes a new instance of the Imsl.Math.SparseMatrix.SparseArray (p. 27) class.

ComplexSparseMatrix Class

```
public class Imsl.Math.ComplexSparseMatrix
```

A general complex sparse matrix which is intended to be efficiently and easily updated.

A `ComplexSparseMatrix` can be constructed from a set of arrays, or it can be abstractly created as an empty array and then incrementally built into final form. It is usually easier to create an empty `ComplexSparseMatrix` of set size and then use the `Set` method to set the elements of the array. When setting the elements of the sparse array, their positions should be thought of as their positions in the dense array. Elements can be set in any order, but only the elements set are stored.

This class includes methods to update the sparse matrix. There are also methods to multiply a sparse matrix and a vector or to multiply two sparse matrices. To solve a sparse linear system use `ComplexSparseCholesky` or `ComplexSuperLU`.

See Also

ComplexSparseCholesky (p. 88), ComplexSuperLU (p. 63)

Properties

NumberOfColumns

```
public int NumberOfColumns {get; }
```

Description

The number of columns in the matrix.

Property Value

An `int` containing the number of columns in the matrix.

NumberOfNonZeros

```
public long NumberOfNonZeros {get; }
```

Description

The number of nonzeros in the matrix.

Property Value

A `long` containing the number of nonzeros in the matrix.

NumberOfRows

```
public int NumberOfRows {get; }
```

Description

The number of rows in the matrix.

Property Value

An `int` containing the number of rows in the matrix.

Constructors

ComplexSparseMatrix

```
public ComplexSparseMatrix(int nRows, int nColumns)
```

Description

Creates a new instance of `ComplexSparseMatrix`. Initially this is the zero matrix.

Parameters

`nRows` – An `int` containing the number of rows in the sparse matrix.

`nColumns` – An `int` containing the number of columns in the sparse matrix.

ComplexSparseMatrix

```
public ComplexSparseMatrix(Imsl.Math.ComplexSparseMatrix A)
```

Description

Creates a new instance of `ComplexSparseMatrix` which is a copy of another `ComplexSparseMatrix` object.

Parameter

`A` – A `ComplexSparseMatrix` object containing the complex sparse matrix to be copied.

ComplexSparseMatrix

```
public ComplexSparseMatrix(Imsl.Math.ComplexSparseMatrix.SparseArray
sparseArray)
```

Description

Constructs a complex sparse matrix from a `SparseArray` object.

Parameter

`sparseArray` – A `ComplexSparseMatrix.SparseArray` used to initialize the sparse matrix. The field `NumberOfNonZeros` in `SparseArray` (p. 37) is not used for initialization, therefore does not have to be set.

ComplexSparseMatrix

```
public ComplexSparseMatrix(int nRows, int nColumns, int[][] index,
Imsl.Math.Complex[][] values)
```

Description

Constructs a sparse matrix from `SparseArray` data.

Parameters

`nRows` – An `int` containing the number of rows in the sparse matrix.

`nColumns` – An `int` containing the number of columns in the sparse matrix.

`index` – An `int` jagged array containing the column indices of all nonzero elements corresponding to the compressed representation of the sparse matrix in `values`. The size of `index` must be identical to the size of `values`. The *i*-th row contains the column indices of all nonzero elements of row *i* of the sparse matrix. The *j*-th element of row *i* is the column index of the value located at the same position in `values`.

`values` – A `Complex` jagged array containing the compressed representation of a complex sparse matrix of size `nRows` by `nColumns`. The number of rows in `values` must be `nRows`. The *i*-th row contains all nonzero elements of row *i* of the full sparse matrix.

Methods

Add

```
static public Imsl.Math.ComplexSparseMatrix Add(Imsl.Math.Complex alpha,
Imsl.Math.Complex beta, Imsl.Math.ComplexSparseMatrix A,
Imsl.Math.ComplexSparseMatrix B)
```

Description

Performs element-wise addition of two complex sparse matrices `A`, `B` of type `ComplexSparseMatrix`, $C \leftarrow \alpha A + \beta B$.

Parameters

`alpha` – A `Complex` scalar value applied to `ComplexSparseMatrix A`.

`beta` – A `Complex` scalar value applied to `ComplexSparseMatrix B`.

`A` – A `ComplexSparseMatrix` matrix.

`B` – A `ComplexSparseMatrix` matrix.

Returns

A `ComplexSparseMatrix` matrix representing the computed sum.

Exception

`System.ArgumentException` is thrown when the matrices are not of the same size.

CheckSquareMatrix

```
public void CheckSquareMatrix()
```

Description

Check that the matrix is square.

Exception

`System.ArgumentException` is thrown if the matrix is not square.

ConjugateTranspose

```
public Imsl.Math.ComplexSparseMatrix ConjugateTranspose()
```

Description

Returns the conjugate transpose of the matrix.

Returns

A `ComplexSparseMatrix` object which is the conjugate transpose of the constructed `ComplexSparseMatrix`.

FrobeniusNorm

```
public double FrobeniusNorm()
```

Description

Returns the Frobenius norm of the matrix.

Returns

A `double` containing the Frobenius norm of the matrix.

Get

```
public Imsl.Math.Complex Get(int iRow, int jColumn)
```

Description

Returns the value of an element in the matrix.

Parameters

`iRow` – An `int` containing the row index of the element.

`jColumn` – An `int` containing the column index of the element.

Returns

A `Complex` containing the value of the `iRow`-th and `jColumn`-th element. If the element was never set, its value is zero.

InfinityNorm

```
public double InfinityNorm()
```

Description

Returns the infinity norm of the matrix.

Returns

A `double` containing the maximum of the row sums of the absolute values of the array elements of the sparse matrix.

Multiply

```
public Imsl.Math.Complex[] Multiply(Imsl.Math.Complex[] x)
```

Description

Multiply the matrix by a vector.

Parameter

`x` – A `Complex` column array.

Returns

A `Complex` vector representing the product of the constructed matrix and *x*, *Ax*.

Exception

`System.ArgumentException` is thrown when the number of columns of the matrix represented by this object is not equal to the number of elements in the input column vector.

Multiply

```
static public Imsl.Math.Complex[] Multiply(Imsl.Math.ComplexSparseMatrix A,
Imsl.Math.Complex[] x)
```

Description

Multiply sparse matrix `A` and column array `x`, *Ax*.

Parameters

`A` – A `ComplexSparseMatrix` matrix.

`x` – A `Complex` column array.

Returns

A `Complex` vector representing the product of the arguments, *Ax*.

Exception

`System.ArgumentException` is thrown when the number of columns in the input matrix is not equal to the number of elements in the input column vector.

Multiply

```
static public Imsl.Math.Complex[] Multiply(Imsl.Math.Complex[] x,
Imsl.Math.ComplexSparseMatrix A)
```

Description

Multiply row array `x` and sparse matrix `A`, $x^T A$.

Parameters

`x` – A `Complex` row array.

`A` – A `ComplexSparseMatrix` matrix.

Returns

A `Complex` vector representing the product of the arguments, $x^T A$.

Exception

`System.ArgumentException` is thrown when the number of elements in the input vector is not equal to the number of rows of the matrix.

Multiply

```
static public Imsl.Math.ComplexSparseMatrix
Multiply(Imsl.Math.ComplexSparseMatrix A, Imsl.Math.ComplexSparseMatrix B)
```

Description

Multiply two sparse matrices, $C \leftarrow AB$.

Parameters

`A` – A `ComplexSparseMatrix` sparse matrix.

`B` – A `ComplexSparseMatrix` sparse matrix.

Returns

The `ComplexSparseMatrix` product AB of `A` and `B`.

Exception

`System.ArgumentException` is thrown when the column number of matrix A is not equal to the row number of matrix B.

MultiplyHermitian

```
static public Imsl.Math.Complex[]
MultiplyHermitian(Imsl.Math.ComplexSparseMatrix A, Imsl.Math.Complex[] x)
```

Description

Multiply sparse Hermitian matrix `A` and column vector `x`.

Parameters

`A` – A Hermitian `ComplexSparseMatrix`, where only the lower triangular part of the matrix is to be used.

`x` – A `Complex` vector.

Returns

A `Complex` vector representing the product of the arguments, Ax.

Exception

`System.ArgumentException` is thrown when the input matrix is not square or the number of columns in the input matrix is not equal to the number of elements in the input column vector.

OneNorm

```
public double OneNorm()
```

Description

Returns the matrix one norm of the sparse matrix.

Returns

A `double` containing the maximum of the column sums of the absolute values of the array elements.

PlusEquals

```
public Imsl.Math.Complex PlusEquals(int iRow, int jColumn, Imsl.Math.Complex x)
```

Description

Adds a value to an element in the matrix.

Parameters

`iRow` – An `int` containing the row index of the element.

`jColumn` – An `int` containing the column index of the element.

`x` – A `Complex` containing the value to be added to the `iRow`-th and `jColumn`-th element.

Returns

A `Complex` containing the updated value of the `iRow`-th and `jColumn`-th element, which equals its old value plus *x*.

Set

```
public void Set(int iRow, int jColumn, Imsl.Math.Complex x)
```

Description

Sets the value of an element in the matrix.

Parameters

`iRow` – An `int` containing the row index of the element.

`jColumn` – An `int` containing the column index of the element.

`x` – A `Complex` containing the value of the `iRow`-th and `jColumn`-th element.

ToDenseMatrix

```
public Imsl.Math.Complex[,] ToDenseMatrix()
```

Description

Returns the sparse matrix as a dense matrix.

Returns

A `Complex` rectangular matrix containing this matrix with all of the zeros explicitly present. The number of rows and columns in the returned matrix is the same as in the sparse matrix.

ToSparseArray

```
public Imsl.Math.ComplexSparseMatrix.SparseArray ToSparseArray()
```

Description

Returns the sparse matrix in the `SparseArray` form.

Returns

A `ComplexSparseMatrix.SparseArray` containing the complex sparse matrix in sparse array form.

See Also: Imsl.Math.ComplexSparseMatrix.SparseArray (p. 37)

Example: ComplexSparseMatrix

The matrix product of two complex sparse matrices is computed using a method from the `ComplexSparseMatrix` class. The one-norm of the result is also computed. The matrix and its norm are printed.

```
using System;
using Imsl.Math;

public class ComplexSparseMatrixEx1
{
    public static void  Main(String[] args)
    {
        ComplexSparseMatrix b = new ComplexSparseMatrix(6, 6);
        b.Set(0, 0, new Complex(10.0, -3.0));
        b.Set(1, 1, new Complex(10.0, 0.0));
        b.Set(1, 2, new Complex(-3.0, 2.0));
        b.Set(1, 3, new Complex(-1.0, -1.0));
        b.Set(2, 2, new Complex(15.0, 5.0));
        b.Set(3, 0, new Complex(-2.0, 0.0));
        b.Set(3, 3, new Complex(10.0, 1.0));
        b.Set(3, 4, new Complex(-1.0, -2.0));
        b.Set(4, 0, new Complex(-1.0, 0.0));
        b.Set(4, 3, new Complex(-5.0, 7.0));
        b.Set(4, 4, new Complex(1.0, -3.0));
        b.Set(4, 5, new Complex(-3.0, 0.0));
        b.Set(5, 0, new Complex(-1.0, 4.0));
        b.Set(5, 1, new Complex(-2.0, 1.0));
        b.Set(5, 5, new Complex(6.0, -5.0));


        ComplexSparseMatrix c = new ComplexSparseMatrix(6, 3);
        c.Set(0, 0, new Complex(5.0, 0.0));
        c.Set(1, 2, new Complex(-3.0, -4.0));
        c.Set(2, 0, new Complex(1.0, 2.0));
        c.Set(2, 2, new Complex(7.0, 1.0));
        c.Set(3, 0, new Complex(2.0, -7.0));
        c.Set(4, 1, new Complex(-5.0, 2.0));
        c.Set(4, 2, new Complex(2.0, 1.0));
        c.Set(5, 2, new Complex(4.0, 0.0));

        // A = b * c
        ComplexSparseMatrix A = ComplexSparseMatrix.Multiply(b, c);

        // Get the one norm of matrix A
        Console.Out.WriteLine("The 1-norm of the matrix is "
            + A.OneNorm());

        ComplexSparseMatrix.SparseArray sa = A.ToSparseArray();

        Console.Out.WriteLine("row  column  value");
        for (int i = 0; i < sa.NumberOfRows; i++)
        {
            for (int j = 0; j < sa.Index[i].Length; j++)
            {
                Console.Out.WriteLine(" " + i + "     "
                    + sa.Index[i][j] + "     " + "("
                    + Complex.Real(sa.Values[i][j])
                    + "," + Complex.Imag(sa.Values[i][j]) + ")");
            }
        }
    }
```

```
}
```

Output

```
The 1-norm of the matrix is 253.937005114344
row  column  value
 0     0     (50,-15)
 1     0     (-16,1)
 1     2     (-53,-29)
 2     0     (5,35)
 2     2     (100,50)
 3     0     (17,-68)
 3     1     (9,8)
 3     2     (0,-5)
 4     0     (34,49)
 4     1     (1,17)
 4     2     (-7,-5)
 5     0     (-5,20)
 5     2     (34,-15)
```

ComplexSparseMatrix.SparseArray Class

`public class Imsl.Math.ComplexSparseMatrix.SparseArray`

The `SparseArray` class uses public fields to hold the data for a sparse matrix in the Sparse Array format.

This format came about as a means for storing sparse matrices. In this format, a sparse matrix is represented by two arrays of arrays. One array contains the references to the nonzero value arrays for each row of the sparse matrix. The other contains the references to the associated column index arrays.

As an example, consider the following complex sparse matrix:

$$A = \begin{pmatrix} 10.0-3.0i & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 10.0 & -3.0+2.0i & -1.0-1.0i & 0.0 & 0.0 \\ 0.0 & 0.0 & 15.0+5.0i & 0.0 & 0.0 & 0.0 \\ -2.0 & 0.0 & 0.0 & 10.0+1.0i & -1.0-2.0i & 0.0 \\ -1.0 & 0.0 & 0.0 & -5.0+7.0i & 1.0-3.0i & -3.0 \\ -1.0+4.0i & -2.0+1.0i & 0.0 & 0.0 & 0.0 & 6.0-5.0i \end{pmatrix}$$

In `SparseArray`, this matrix can be represented by the two jagged arrays, `values` and `index`, where `values` refers to the nonzero entries in A and `index` to the column indices: `Complex values[][] = { {new Complex(10.0, -3.0)}, {new Complex(10.0, 0.0), new Complex(-3.0, 2.0), new Complex(-1.0,-1.0)}, {new Complex(15.0, 5.0)}, {new Complex(-2.0, 0.0), new Complex(10.0, 1.0), new Complex(-1.0, -2.0)}, {new Complex(-1.0, 0.0), new Complex(-5.0, 7.0), new Complex(1.0, -3.0), new Complex(-3.0, 0.0)}, {new Complex(-1.0, 4.0), new Complex(-2.0, 1.0), new Complex(6.0, -5.0)} }; int index[][] = { {0}, {1, 2, 3}, {2}, {0, 3, 4}, {0, 3, 4, 5}, {0, 1, 5} };`

Fields

Index
```
public int[][] Index
```

Description

Jagged array containing column indices. The length of this array equals `numberOfRows`. The length of each row equals the number of nonzeros in that row of the sparse matrix.

NumberOfColumns
```
public int NumberOfColumns
```

Description

Number of columns in the matrix.

NumberOfNonZeros
```
public long NumberOfNonZeros
```

Description

Number of nonzeros in the matrix.

NumberOfRows
```
public int NumberOfRows
```

Description

Number of rows in the matrix.

Values
```
public Imsl.Math.Complex[][] Values
```

Description

Jagged array containing sparse array values. This array must have the same shape as `Index`.

Constructor

SparseArray
```
public SparseArray()
```

Description

Initializes a new instance of the Imsl.Math.ComplexSparseMatrix.SparseArray (p. 37) class.

LU Class

`public class Imsl.Math.LU`

LU factorization of a matrix of type `double`.

`LU` performs an *LU* factorization of a real general coefficient matrix. The `Condition` method estimates the reciprocal of the L_1 condition number of the matrix. The LU factorization is done using scaled partial pivoting. Scaled partial pivoting differs from partial pivoting in that the pivoting strategy is the same as if each row were scaled to have the same infinity norm.

The L_1 condition number of the matrix A is defined to be $\kappa(A) = ||A||_1||A^{-1}||_1$. Since it is expensive to compute $||A^{-1}||_1$, the condition number is only estimated. The estimation algorithm is the same as used by LINPACK and is described in a paper by Cline et al. (1979).

Note that `A` is not retained for use by other methods of this class, only the factorization of `A` is retained. Thus, `A` is a required parameter to the `condition` method.

An estimated condition number greater than $1/\varepsilon$ (where ε is machine precision) indicates that very small changes in A can cause very large changes in the solution x. Iterative refinement can sometimes find the solution to such a system. If there is conern about the input matrix being ill-conditioned, the user of this class should check the condition number of the input matrix using the `condition` method before using one of the other class methods.

`LU` fails if U, the upper triangular part of the factorization, has a zero diagonal element. This can occur only if A either is singular or is very close to a singular matrix.

Use the `Solve` method to solve systems of equations. The `Determinant` method can be called to compute the determinant of the coefficient matrix.

`LU` is based on the LINPACK routine `SGECO`; see Dongarra et al. (1979). `SGECO` uses unscaled partial pivoting.

Property

NumberOfProcessors

`public int NumberOfProcessors {get; set; }`

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Constructor

LU

```
public LU(double[,] a)
```

Description

Creates the LU factorization of a square matrix of type `double`.

Parameter

`a` – The `double` square matrix to be factored.

Exception

`Imsl.Math.SingularMatrixException` is thrown when the input matrix is singular

Methods

Condition

```
public double Condition(double[,] a)
```

Description

Return an estimate of the reciprocal of the L1 condition number of a matrix.

Parameter

`a` – The `double` square matrix for which the reciprocal of the L1 condition number is desired.

Returns

A `double` value representing an estimate of the reciprocal of the L1 condition number of the matrix.

Determinant

```
public double Determinant()
```

Description

Return the determinant of the matrix used to construct this instance.

Returns

A `double` scalar containing the determinant of the matrix used to construct this instance.

GetL

```
public double[,] GetL()
```

Description

Returns the lower triangular portion of the *LU* factorization of `a`.

Returns

A `double` matrix containing *L*, the lower triangular portion of the *LU* factorization of `a`.

Remarks

Scaled partial pivoting is used to achieve the *LU* factorization. The resulting factorization is such that $AP = LU$, where *A* is the input matrix `a`, P is the permutation matrix returned by `GetPermutationMatrix`, *L* is the lower triangular matrix returned by `GetL`, and *U* is the unit upper triangular matrix returned by `GetU`.

GetPermutationMatrix

```
public double[,] GetPermutationMatrix()
```

Description

Returns the permutation matrix which results from the *LU* factorization of `a`.

Returns

A `double` matrix containing the permuted identity matrix as a result of the *LU* factorization of `a`.

Remarks

Scaled partial pivoting is used to achieve the *LU* factorization. The resulting factorization is such that $AP = LU$, where *A* is the input matrix `a`, *P* is the permutation matrix returned by `GetPermutationMatrix`, *L* is the lower triangular matrix returned by `GetL`, and *U* is the unit upper triangular matrix returned by `GetU`.

GetU

```
public double[,] GetU()
```

Description

Returns the unit upper triangular portion of the *LU* factorization of `a`.

Returns

A `double` matrix containing *U*, the unit upper triangular portion of the *LU* factorization of `a`.

Remarks

Scaled partial pivoting is used to achieve the *LU* factorization. The resulting factorization is such that $AP = LU$, where *A* is the input matrix `a`, *P* is the permutation matrix returned by `GetPermutationMatrix`, *L* is the lower triangular matrix returned by `GetL`, and *U* is the unit upper triangular matrix returned by `GetU`.

Inverse

```
public double[,] Inverse()
```

Description

Returns the inverse of the matrix used to construct this instance.

Returns

A `double` matrix representing the inverse of the matrix used to construct this instance.

Solve

```
public double[] Solve(double[] b)
```

Description

Solve ax=b for x using the LU factorization of `a`.

Parameter

`b` – A `double` array containing the right-hand side of the linear system.

Returns

A `double` array containing the solution to the linear system of equations.

Solve

```
static public double[] Solve(double[,] a, double[] b)
```

Description

Solve ax=b for x using the LU factorization of `a`.

Parameters

`a` – A `double` square matrix.

`b` – A `double` column vector.

Returns

A `double` column vector containing the solution to the linear system of equations.

Exceptions

`System.ArgumentException` is thrown when the number of rows in the input matrix is not equal to the number of elements in x

`Imsl.Math.SingularMatrixException` is thrown when the matrix is singular

SolveTranspose

```
public double[] SolveTranspose(double[] b)
```

Description

Return the solution x of the linear system transpose(A)x = b.

Parameter

`b` – A `double` array containing the right-hand side of the linear system.

Returns

A `double` array containing the solution to the linear system of equations.

Example: LU Factorization of a Matrix

The LU Factorization of a Matrix is performed. A linear system is then solved using the factorization. The inverse, determinant, and condition number of the input matrix are also computed.

```
using System;
using Imsl.Math;

public class LUEx1
{
   public static void Main(String[] args)
   {
      double[,] a = {
            {1, 3, 3},
            {1, 3, 4},
            {1, 4, 3}
        };
      double[] b = new double[] { 12, 13, 14 };

      // Compute the LU factorization of A
      LU lu = new LU(a);

      // Solve Ax = b
      double[] x = lu.Solve(b);
      new PrintMatrix("x").Print(x);

      // Find the inverse of A.
      double[,] ainv = lu.Inverse();
      new PrintMatrix("ainv").Print(ainv);

      // Find the condition number of A.
      double condition = lu.Condition(a);
      Console.Out.WriteLine("condition number = " + condition);
      Console.Out.WriteLine();

      // Find the determinant of A.
      double determinant = lu.Determinant();
      Console.Out.WriteLine("determinant = " + determinant);
   }
}
```

Output

```
  x
   0
0  3
1  2
2  1

               ainv
   0             1            2
0   7  -3                     -3
1  -1   2.22044604925031E-16   1
2  -1   1                      0
```

```
condition number = 0.0151202749140893

determinant = -1
```

ComplexLU Class

```
public class Imsl.Math.ComplexLU
```

LU factorization of a matrix of type `Complex`.

`ComplexLU` performs an *LU* factorization of a complex general coefficient matrix. `ComplexLU`'s method `Condition` estimates the condition number of the matrix. The *LU* factorization is done using scaled partial pivoting. Scaled partial pivoting differs from partial pivoting in that the pivoting strategy is the same as if each row were scaled to have the same infinity norm.

The L_1 condition number of the matrix A is defined to be $\kappa(A) = \|A\|_1 \left\|A^{-1}\right\|_1$. Since it is expensive to compute $\left\|A^{-1}\right\|_1$, the condition number is only estimated. The estimation algorithm is the same as used by LINPACK and is described by Cline et al. (1979).

Note that `A` is not retained for use by other methods of this class, only the factorization of `A` is retained. Thus, `A` is a required parameter to the `condition` method.

An estimated condition number greater than $1/\varepsilon$ (where ε is machine precision) indicates that very small changes in A can cause very large changes in the solution x. Iterative refinement can sometimes find the solution to such a system.

`ComplexLU` fails if U, the upper triangular part of the factorization, has a zero diagonal element. This can occur only if A either is singular or is very close to a singular matrix.

The `Solve` method can be used to solve systems of equations. The method `Determinant` can be called to compute the determinant of the coefficient matrix.

ComplexLU is based on the LINPACK routine CGECO; see Dongarra et al. (1979). `CGECO` uses unscaled partial pivoting.

Property

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Constructor

ComplexLU

```
public ComplexLU(Imsl.Math.Complex[,] a)
```

Description

Creates the LU factorization of a square matrix of type `Complex`.

Parameter

`a` – The `Complex` square matrix to be factored.

Exception

`Imsl.Math.SingularMatrixException` is thrown when the input matrix is singular

Methods

Condition

```
public double Condition(Imsl.Math.Complex[,] a)
```

Description

Return an estimate of the reciprocal of the L1 condition number.

Parameter

`a` – A `Complex` matrix.

Returns

A `double` scalar value representing the estimate of the reciprocal of the L1 condition number of the matrix `a`.

Determinant

```
public Imsl.Math.Complex Determinant()
```

Description

Return the determinant of the matrix used to construct this instance.

Returns

A `Complex` scalar containing the determinant of the matrix used to construct this instance.

GetL

```
public Imsl.Math.Complex[,] GetL()
```

Description

Returns the lower triangular portion of the *LU* factorization of `a`.

Returns

A `Complex` matrix containing *L*, the lower triangular portion of the *LU* factorization of `a`.

Remarks

Scaled partial pivoting is used to achieve the *LU* factorization. The resulting factorization is such that $AP = LU$, where *A* is the input matrix `a`, P is the permutation matrix returned by `GetPermutationMatrix`, *L* is the lower triangular matrix returned by `GetL`, and *U* is the unit upper triangular matrix returned by `GetU`.

GetPermutationMatrix

```
public Imsl.Math.Complex[,] GetPermutationMatrix()
```

Description

Returns the permutation matrix which results from the *LU* factorization of `a`.

Returns

A `double` matrix containing the permuted identity matrix as a result of the *LU* factorization of `a`.

Remarks

Scaled partial pivoting is used to achieve the *LU* factorization. The resulting factorization is such that $AP = LU$, where *A* is the input matrix `a`, *P* is the permutation matrix returned by `GetPermutationMatrix`, *L* is the lower triangular matrix returned by `GetL`, and *U* is the unit upper triangular matrix returned by `GetU`.

GetU

```
public Imsl.Math.Complex[,] GetU()
```

Description

Returns the unit upper triangular portion of the *LU* factorization of `a`.

Returns

A `Complex` matrix containing *U*, the unit upper triangular portion of the *LU* factorization of `a`.

Remarks

Scaled partial pivoting is used to achieve the *LU* factorization. The resulting factorization is such that $AP = LU$, where *A* is the input matrix `a`, *P* is the permutation matrix returned by `GetPermutationMatrix`, *L* is the lower triangular matrix returned by `GetL`, and *U* is the unit upper triangular matrix returned by `GetU`.

Inverse

```
public Imsl.Math.Complex[,] Inverse()
```

Description

Compute the inverse of a matrix of type `Complex`.

Returns

A `Complex` matrix containing the inverse of the matrix used to construct this object.

Solve

```
public Imsl.Math.Complex[] Solve(Imsl.Math.Complex[] b)
```

Description

Return the solution x of the linear system ax = b using the LU factorization of a.

Parameter

`b` – A `Complex` array containing the right-hand side of the linear system.

Returns

A `Complex` array containing the solution to the linear system of equations.

Solve

```
static public Imsl.Math.Complex[] Solve(Imsl.Math.Complex[,] a,
Imsl.Math.Complex[] b)
```

Description

Return the solution x of the linear system ax = b using the LU factorization of a.

Parameters

`a` – A `Complex` square matrix.

`b` – A `Complex` column vector.

Returns

A `Complex` column vector containing the solution to the linear system of equations.

Exceptions

`System.ArgumentException` is thrown when the number of rows in `a` is not equal to the length of `b`

`Imsl.Math.SingularMatrixException` is thrown when the matrix is singular

SolveTranspose

```
public Imsl.Math.Complex[] SolveTranspose(Imsl.Math.Complex[] b)
```

Description

Return the solution x of the linear system transpose(A)x = b.

Parameter

`b` – A `Complex` array containing the right-hand side of the linear system.

Returns

A `Complex` array containing the solution to the linear system of equations.

Example: LU Decomposition of a Complex Matrix

The Complex structure is used to convert a real matrix to a Complex matrix. An LU decomposition of the matrix is performed. The reciprocal of the condition number of the Matrix is then computed and checked against machine precision to determine whether or not to issue a Warning about the results. A linear system is then solved using the factorization. The determinant of the input matrix is also computed.

```
using System;
using Imsl.Math;

public class ComplexLUEx1
{
   public static void  Main(String[] args)
   {
      double[,] ar = {{1, 3, 3},
                      {1, 3, 4},
                      {1, 4, 3}};
      double[] br = {12, 13, 14};

      Complex[,] a = new Complex[3,3];
      Complex[] b = new Complex[3];

      for (int i = 0; i < 3; i++)
      {
         b[i] = new Complex(br[i]);
         for (int j = 0; j < 3; j++)
         {
            a[i,j] = new Complex(ar[i,j]);
         }
      }

      // Compute the LU factorization of A
      ComplexLU clu = new ComplexLU(a);

      // Check the reciprocal of the condition number of A
      //   against machine precision
      double   condition = clu.Condition(a);

      if(condition <= 2.220446049250313e-16){
         Console.Out.WriteLine("WARNING. The matrix is too ill-conditioned.");
         Console.Out.WriteLine("An estimate of the reciprocal of its L1 " +
                               "condition number is "+condition+".");
         Console.Out.WriteLine("Results based on this factorization may " +
                               "not be accurate.");
      }

      // Solve Ax = b
      Complex[] x = clu.Solve(b);
      Console.Out.WriteLine("The solution is:");
      Console.Out.WriteLine(" ");
      new PrintMatrix("x").Print(x);

      // Print the condition number of A.
      Console.Out.WriteLine("The condition number = " + condition);
      Console.Out.WriteLine();
```

```
      // Find the determinant of A.
      Complex determinant = clu.Determinant();
      Console.Out.WriteLine("The determinant = " + determinant);
   }
}
```

Output

```
The solution is:

  x
   0
0  3
1  2
2  1

The condition number = 0.0148867313915858

The determinant = -0.99999999999999978
```

SuperLU Class

`public class Imsl.Math.SuperLU`

Computes the LU factorization of a general sparse matrix of type `SparseMatrix` by a column method and solves the real sparse linear system of equations $Ax = b$.

Consider the sparse linear system of equations

$$Ax = b.$$

Here, A is a general square, nonsingular, n by n sparse matrix, and x and b are vectors of length n. All entries in A, x and b are of type `double`.

Gaussian elimination, applied to the system above, can be shortly described as follows:

1. Compute a triangular factorization $P_r D_r A D_c P_c = LU$. Here, D_r and D_c are positive definite diagonal matrices to equilibrate the system and P_r and P_c are permutation matrices to ensure numerical stability and preserve sparsity. L is a unit lower triangular matrix and U is an upper triangular matrix.

2. Solve $Ax = b$ by evaluating

$$x = A^{-1}b = D_c(P_c(U^{-1}(L^{-1}(P_r(D_r b))))).$$

This is done efficiently by multiplying from right to left in the last expression: Scale the rows of b by D_r. Multiplying $P_r(D_r b)$ means permuting the rows of $D_r b$. Multiplying $L^{-1}(P_r D_r b)$ means solving the triangular system of equations with matrix L by substitution. Similarly, multiplying $U^{-1}(L^{-1}(P_r D_r b))$ means solving the triangular system with U.

Class `SuperLU` handles step 1 above in the `Solve` method if it has not been computed prior to step 2. More precisely, before $Ax = b$ is solved the following steps are performed:

1. Equilibrate matrix A, i.e. compute diagonal matrices D_r and D_c so that $\hat{A} = D_r A D_c$ is "better conditioned" than A, i.e. $\hat{A}^{-1}$ is less sensitive to perturbations in $\hat{A}$ than A^{-1} is to perturbations in A.
2. Order the columns of $\hat{A}$ to increase the sparsity of the computed L and U factors, i.e. replace $\hat{A}$ by $\hat{A}P_c$ where P_c is a column permutation matrix.
3. Compute the LU factorization of $\hat{A}P_c$. For numerical stability, the rows of $\hat{A}P_c$ are eventually permuted through the factorization process by scaled partial pivoting, leading to the decomposition $\tilde{A} := P_r \hat{A} P_c = LU$. The LU factorization is done by a left looking supernode-panel algorithm with 2-D blocking. See Demmel, Eisenstat, Gilbert et al. (1999) for further information on this technique.
4. Compute the reciprocal pivot growth factor

$$\max_{1 \le j \le n} \frac{\|\tilde{A}_j\|_\infty}{\|U_j\|_\infty}$$

   where $\tilde{A}_j$ and U_j denote the j-th column of matrices $\tilde{A}$ and U, respectively.
5. Estimate the reciprocal of the condition number of matrix $\tilde{A}$.

Method `Solve` uses this information to perform the following steps:

1. Solve the system $Ax = b$ using the computed triangular factors.
2. Iteratively refine the solution, again using the computed triangular factors. This is equivalent to Newton's method.
3. Compute forward and backward error bounds for the solution vector x.

Some of the steps mentioned above are optional. Their settings can be controlled by the `Set` methods and properties of class `SuperLU`.

Class `SuperLU` is based on the SuperLU code written by Demmel, Gilbert, Li et al. For more detailed explanations of the factorization and solve steps, see the SuperLU Users' Guide (1999).

Properties

ColumnOrderingMethod

`public Imsl.Math.SuperLU.ColumnOrdering ColumnOrderingMethod {get; set; }`

Description

The method used to permute the columns of the input matrix.

Property Value

A `ColumnOrdering` scalar specifying how the columns of the input matrix are to be permuted for sparsity preservation.

value	*method*
`Natural`	natural ordering, that is $P_c = I$, I the identity matrix.
`MinimumDegreeAtPlusA`	minimum degree ordering on the structure of $A^T + A$
`MinimumDegreeAtA`	minimum degree ordering on the structure of $A^T A$
`ColumnApproximateMinimumDegree`	column approximate minimum degree ordering

Remarks

By default, `value = SuperLU.ColumnOrderingMethod.ColumnApproximateMinimumDegree`.

Exception

`System.ArgumentException` is thrown if `value` is not one of the above values.

ConditionNumber

```
public double ConditionNumber {get; }
```

Description

The estimate of the reciprocal condition number of the matrix *A*.

Property Value

A `double` scalar containing the reciprocal condition number of the matrix *A* after equilibration and permutation of rows/columns (if done). If the reciprocal condition number is less than machine precision, in particular if it is equal to 0, the matrix is singular to working precision.

DiagonalPivotThreshold

```
public double DiagonalPivotThreshold {get; set; }
```

Description

The threshold used for a diagonal entry to be an acceptable pivot.

Property Value

A `double` scalar specifying the threshold used for a diagonal entry to be an acceptable pivot.

Remarks

By default, `DiagonalPivotThreshold=1.0`, i.e. classical partial pivoting.

Exception

`System.ArgumentException` is thrown if `DiagonalPivotThreshold` is not in the interval $[0.0, 1.0]$.

Equilibrate

```
public bool Equilibrate {get; set; }
```

Description

Specifies if input matrix *A* is equilibrated before factorization.

Property Value

A `bool` specifying whether or not matrix `A` is equilibrated before factorization. If `Equilibrate` is `true` the system is equilibrated, if `Equilibrate` is `false`, no equilibration is performed.

Remarks

By default, `Equilibrate` = `true`.

EquilibrationMethod

```
public Imsl.Math.SuperLU.Scaling EquilibrationMethod {get; }
```

Description

The type of equilibration used before matrix factorization.

Property Value

A `Scaling` value specifying the equilibration option used.

Remarks

EquilibrationMethod	*Option description*
`None`	No equilibration is performed.
`Row`	Equilibration is performed with row scaling.
`Column`	Equilibration is performed with column scaling.
`RowAndColumn`	Equilibration is performed with row and column scaling.

ForwardErrorBound

```
public double ForwardErrorBound {get; }
```

Description

The estimated forward error bound for the solution vector.

Property Value

A `double` containing the estimated forward error bound for the solution vector. The estimate is as reliable as the estimate for the reciprocal condition number, and is almost always a slight overestimate of the true error. If iterative refinement is not used, the return value is set to `1.0`.

IterativeRefinement

```
public bool IterativeRefinement {get; set; }
```

Description

The iterative refinement option.

Property Value

A `bool` specifying whether to use iterative refinement.

Remarks

For iterative refinement, set `IterativeRefinement = true`, otherwise set `IterativeRefinement = false`.

By default, `IterativeRefinement = false`.

PivotGrowth

```
public bool PivotGrowth {get; set; }
```

Description

Specifies whether to compute the reciprocal pivot growth factor.

Property Value

A `bool` specifying whether to compute the reciprocal pivot growth factor. `true` indicates the reciprocal pivot factor will be computed, `false` indicates it will not be computed.

Remarks

By default, `PivotGrowth = false`.

ReciprocalPivotGrowthFactor

```
public double ReciprocalPivotGrowthFactor {get; }
```

Description

The reciprocal pivot growth factor.

Property Value

A `double` scalar representing the reciprocal growth factor

$$\max_{1 \le j \le n} \frac{\|\tilde{A}_j\|_\infty}{\|U_j\|_\infty}.$$

If the value is much less than 1, the stability of the LU factorization could be poor.

RelativeBackwardError

```
public double RelativeBackwardError {get; }
```

Description

The componentwise relative backward error of the solution vector.

Property Value

A `double` containing the componentwise relative backward error of the solution vector `x`. If `IterativeRefinement` is not set to `true`, then `RelativeBackwardError` returns `1.0`.

SymmetricMode

```
public bool SymmetricMode {get; set; }
```

Description

The symmetric mode option.

Property Value

A `bool` indicating if symmetric mode is to be used.

Remarks

The symmetric mode option should be applied if the input matrix A is diagonally dominant or nearly so. The user should then define a small diagonal pivot threshold (e.g. 0.0 or 0.01) by property `DiagonalPivotThreshold` and choose an $(A^T + A)$-based column permutation algorithm (e.g. column permutation method `SuperLU.ColumnOrdering.MinimumDegreeAtPlusA`). `SymmetricMode=true` implies symmetric mode is used.

By default, `SymmetricMode=false`.

Constructor

SuperLU

```
public SuperLU(Imsl.Math.SparseMatrix A)
```

Description

Constructor for `SuperLU`.

Parameter

`A` – A `SparseMatrix` containing the sparse square input matrix.

Methods

GetPerformanceTuningParameters

```
public int
GetPerformanceTuningParameters(Imsl.Math.SuperLU.PerformanceParameters
parameter)
```

Description

Returns a performance tuning parameter value.

Parameter

`parameter` – A `PerformanceParameters` scalar that specifies the parameter for which the value is to be returned.

`parameter`	*Description*
`PanelSize`	The panel size.
`RelaxationParameter`	The relaxation parameter to control supernode amalgamation.
`MaximumSupernodeSize`	The maximum allowable size for a supernode.
`MinimumRowDimension`	The minimum row dimension to be used for 2D blocking.
`MinimumColumnDimension`	The minimum column dimension to be used for 2D blocking.
`FillFactor`	The estimated fill factor for L and U, compared with A.

Returns

An `int` containing the current value used for the specified tuning parameter.

SetPerformanceTuningParameters

```
public void
SetPerformanceTuningParameters(Imsl.Math.SuperLU.PerformanceParameters
parameter, int tunedValue)
```

Description

Sets performance tuning parameters.

Parameters

`parameter` – A `PerformanceParameters` scalar that specifies the parameter to be tuned.

`tunedValue` – An `int` scalar that specifies the value to be used for the specified tuning parameter.

`parameter`	*Description*	*Default*
`PanelSize`	The panel size.	10
`RelaxationParameter`	The relaxation parameter to control supernode amalgamation.	5
`MaximumSupernodeSize`	The maximum allowable size for a supernode.	100
`MinimumRowDimension`	The minimum row dimension to be used for 2D blocking.	200
`MinimumColumnDimension`	The minimum column dimension to be used for 2D blocking.	40
`FillFactor`	The estimated fill factor for L and U, compared with A.	20

Exception

`System.ArgumentException` is thrown if `tunedValue` is not greater than zero.

Solve

```
public double[] Solve(double[] b)
```

Description

Computation of the solution vector for the system $Ax = b$.

Parameter

`b` – A `double` vector of length `n`, `n` the order of input matrix `A`, containing the right hand side.

Returns

A `double` vector containing the solution to the system $Ax = b$. Optionally, the solution is improved by iterative refinement, if `IterativeRefinement = true`. Method `Solve` internally first factorizes matrix A (step 1 of the introduction) if the factorization has not been done before.

SolveTranspose

```
public double[] SolveTranspose(double[] b)
```

Description

Computation of the solution vector for the system $A^T x = b$.

Parameter

`b` – A `double` vector of length `n` containing the right hand side, `n` is the order of input matrix `A`.

Returns

A `double` vector containing the solution to the system $A^T x = b$. Optionally, the solution is improved by iterative refinement, if `IterativeRefinement = true`. Method `SolveTranspose` internally first factorizes matrix A (step 1 of the introduction) if the factorization has not been done before.

Example: LU Factorization of a Sparse Matrix

The LU Factorization of the sparse 6×6 matrix

$$A = \begin{pmatrix} 10.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 10.0 & -3.0 & -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 15.0 & 0.0 & 0.0 & 0.0 \\ -2.0 & 0.0 & 0.0 & 10.0 & -1.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & -5.0 & 1.0 & -3.0 \\ -1.0 & -2.0 & 0.0 & 0.0 & 0.0 & 6.0 \end{pmatrix}$$

is computed. The sparse coordinate form for A is given by a series of row, column, value triplets:

row	column	value
0	0	10.0
1	1	10.0
1	2	-3.0
1	3	-1.0
2	2	15.0
3	0	-2.0
3	3	10.0
3	4	-1.0
4	0	-1.0
4	3	-5.0
4	4	1.0
4	5	-3.0
5	0	-1.0
5	1	-2.0
5	5	6.0

Let

$$y^T = (1.0, 2.0, 3.0, 4.0, 5.0, 6.0)$$

so that

$$b_1 := Ay = (10.0, 7.0, 45.0, 33.0, -34.0, 31.0)^T$$

and

$$b_2 := A^T y = (-9.0, 8.0, 39.0, 13.0, 1.0, 21.0)^T .$$

The LU factorization of A is used to solve the sparse linear systems $Ax = b_1$ and $A^T x = b_2$ with iterative refinement. The reciprocal pivot growth factor and the reciprocal condition number are also computed.

```
using System;
using Imsl.Math;

public class SuperLUEx1
{
    public static void  Main(String[] args)
    {
        int m;
```

```
SuperLU slu;
double conditionNumber, recip_pivot_growth;
double[] sol = null;
double Ferr, Berr;

double[] b1 = {10.0, 7.0, 45.0, 33.0, -34.0, 31.0};
double[] b2 = {-9.0, 8.0, 39.0, 13.0, 1.0, 21.0};

// Initialize matrix A.
m = 6;
SparseMatrix a = new SparseMatrix(m, m);

a.Set(0, 0, 10.0);
a.Set(1, 1, 10.0);
a.Set(1, 2, -3.0);
a.Set(1, 3, -1.0);
a.Set(2, 2, 15.0);
a.Set(3, 0, -2.0);
a.Set(3, 3, 10.0);
a.Set(3, 4, -1.0);
a.Set(4, 0, -1.0);
a.Set(4, 3, -5.0);
a.Set(4, 4, 1.0);
a.Set(4, 5, -3.0);
a.Set(5, 0, -1.0);
a.Set(5, 1, -2.0);
a.Set(5, 5, 6.0);

// Compute the sparse LU factorization of a

slu = new SuperLU(a);

slu.Equilibrate = false;
slu.ColumnOrderingMethod = SuperLU.ColumnOrdering.Natural;
slu.PivotGrowth = true;

// Set option of iterative refinement
slu.IterativeRefinement = true;

// Solve sparse system A*x = b1
Console.Out.WriteLine();
Console.Out.WriteLine("Solve sparse System Ax=b1");
Console.Out.WriteLine("========================");
Console.Out.WriteLine();

sol = slu.Solve(b1);
new PrintMatrix("Solution").Print(sol);

Ferr = slu.ForwardErrorBound;
Berr = slu.RelativeBackwardError;

Console.Out.WriteLine();
Console.Out.WriteLine("Forward error bound: " + Ferr);
Console.Out.WriteLine("Relative backward error: " + Berr);
Console.Out.WriteLine();
Console.Out.WriteLine();
```

```
        // Solve sparse system (A^T)*x = b2
        Console.Out.WriteLine();
        Console.Out.WriteLine("Solve sparse System (A^T)*x=b2");
        Console.Out.WriteLine("==============================");
        Console.Out.WriteLine();

        sol = slu.SolveTranspose(b2);
        new PrintMatrix("Solution").Print(sol);

        Ferr = slu.ForwardErrorBound;
        Berr = slu.RelativeBackwardError;

        System.Console.Out.WriteLine();
        System.Console.Out.WriteLine("Forward error bound: " + Ferr);
        System.Console.Out.WriteLine("Relative backward error: " + Berr);
        System.Console.Out.WriteLine();
        System.Console.Out.WriteLine();

        // Compute reciprocal pivot growth factor and condition number

        recip_pivot_growth = slu.ReciprocalPivotGrowthFactor;
        conditionNumber = slu.ConditionNumber;

        Console.Out.WriteLine("Pivot growth factor and condition number");
        Console.Out.WriteLine("========================================");
        Console.Out.WriteLine();

        Console.Out.WriteLine("Reciprocal pivot growth factor: " +
         recip_pivot_growth);
        Console.Out.WriteLine("Reciprocal condition number: " + conditionNumber);
        Console.Out.WriteLine();
    }
}
```

Output

```
Solve sparse System Ax=b1
=========================

Solution
   0
0  1
1  2
2  3
3  4
4  5
5  6


Forward error bound: 2.74489425897801E-14
Relative backward error: 0
```

```
Solve sparse System (A^T)*x=b2
==============================

Solution
   0
0  1
1  2
2  3
3  4
4  5
5  6


Forward error bound: 2.41379101136191E-15
Relative backward error: 0


Pivot growth factor and condition number
========================================

Reciprocal pivot growth factor: 1
Reciprocal condition number: 0.0244510978043912
```

SuperLU.ColumnOrdering Enumeration

public enumeration Imsl.Math.SuperLU.ColumnOrdering

The column permutation method to be used.

Fields

ColumnApproximateMinimumDegree

public Imsl.Math.SuperLU.ColumnOrdering ColumnApproximateMinimumDegree

Description

For column ordering, use column approximate minimum degree ordering.

MinimumDegreeAtA

public Imsl.Math.SuperLU.ColumnOrdering MinimumDegreeAtA

Description

For column ordering, use minimum degree ordering on the structure of $A^T A$.

MinimumDegreeAtPlusA

`public Imsl.Math.SuperLU.ColumnOrdering MinimumDegreeAtPlusA`

Description

For column ordering, use minimum degree ordering on the structure of $A^T + A$.

Natural

`public Imsl.Math.SuperLU.ColumnOrdering Natural`

Description

For column ordering, use the natural ordering.

SuperLU.PerformanceParameters Enumeration

`public enumeration Imsl.Math.SuperLU.PerformanceParameters`

Performance tuning parameters which can be adjusted via method `SetPerformanceTuningParameters`.

Fields

FillFactor

`public Imsl.Math.SuperLU.PerformanceParameters FillFactor`

Description

The estimated fill factor for L and U, compared with A.

MaximumSupernodeSize

`public Imsl.Math.SuperLU.PerformanceParameters MaximumSupernodeSize`

Description

The maximum allowable size for a supernode.

MinimumColumnDimension

`public Imsl.Math.SuperLU.PerformanceParameters MinimumColumnDimension`

Description

The minimum column dimension to be used for 2D blocking.

MinimumRowDimension

`public Imsl.Math.SuperLU.PerformanceParameters MinimumRowDimension`

Description

The minimum row dimension to be used for 2D blocking.

PanelSize

```
public Imsl.Math.SuperLU.PerformanceParameters PanelSize
```

Description

The panel size, that is, the number of consecutive columns of *A* being factorized at a time.

RelaxationParameter

```
public Imsl.Math.SuperLU.PerformanceParameters RelaxationParameter
```

Description

The relaxation parameter to control supernode amalgamation.

SuperLU.Scaling Enumeration

```
public enumeration Imsl.Math.SuperLU.Scaling
```

One of several possible equilibration option return values for property `EquilibrationMethod`.

Fields

Column

```
public Imsl.Math.SuperLU.Scaling Column
```

Description

Indicates that input matrix *A* was column scaled before factorization. This is a return value for `EquilibrationMethod`.

None

```
public Imsl.Math.SuperLU.Scaling None
```

Description

Indicates that input matrix *A* was not equilibrated before factorization. This is a return value for `EquilibrationMethod`.

Row

```
public Imsl.Math.SuperLU.Scaling Row
```

Description

Indicates that input matrix A was row scaled before factorization. This is a return value for `EquilibrationMethod`.

RowAndColumn

```
public Imsl.Math.SuperLU.Scaling RowAndColumn
```

Description

Indicates that input matrix A was row and column scaled before factorization. This is a return value for `EquilibrationMethod`.

ComplexSuperLU Class

```
public class Imsl.Math.ComplexSuperLU
```

Computes the LU factorization of a general sparse matrix of type `ComplexSparseMatrix` by a column method and solves a sparse linear system of equations $Ax = b$.

Consider the sparse linear system of equations

$$Ax = b.$$

Here, A is a general square, nonsingular, n by n sparse matrix, and x and b are vectors of length n. All entries in A, x and b are of type `Complex`.

Gaussian elimination, applied to the system above, can be shortly described as follows:

1. Compute a triangular factorization $P_r D_r A D_c P_c = LU$. Here, D_r and D_c are positive definite diagonal matrices to equilibrate the system and P_r and P_c are permutation matrices to ensure numerical stability and preserve sparsity. L is a unit lower triangular matrix and U is an upper triangular matrix.
2. Solve $Ax = b$ by evaluating

   $$x = A^{-1}b = D_c(P_c(U^{-1}(L^{-1}(P_r(D_r b))))).$$

   This is done efficiently by multiplying from right to left in the last expression: Scale the rows of b by D_r. Multiplying $P_r(D_r b)$ means permuting the rows of $D_r b$. Multiplying $L^{-1}(P_r D_r b)$ means solving the triangular system of equations with matrix L by substitution. Similarly, multiplying $U^{-1}(L^{-1}(P_r D_r b))$ means solving the triangular system with U.

Class `ComplexSuperLU` handles step 1 above in the `Solve` method if it is has not been computed prior to step 2. More precisely, before $Ax = b$ is solved the following steps are performed:

1. Equilibrate matrix A, i.e. compute diagonal matrices D_r and D_c so that $\hat{A} = D_r A D_c$ is "better conditioned" than A, i.e. $\hat{A}^{-1}$ is less sensitive to perturbations in $\hat{A}$ than A^{-1} is to perturbations in A.

2. Order the columns of $\hat{A}$ to increase the sparsity of the computed L and U factors, i.e. replace $\hat{A}$ by $\hat{A}P_c$ where P_c is a column permutation matrix.

3. Compute the LU factorization of $\hat{A}P_c$. For numerical stability, the rows of $\hat{A}P_c$ are eventually permuted through the factorization process by scaled partial pivoting, leading to the decomposition $\tilde{A} := P_r\hat{A}P_c = LU$. The LU factorization is done by a left looking supernode-panel algorithm with 2-D blocking. See Demmel, Eisenstat, Gilbert et al. (1999) for further information on this technique.

4. Compute the reciprocal pivot growth factor

$$\max_{1\leq j\leq n} \frac{\|\tilde{A}_j\|_\infty}{\|U_j\|_\infty},$$

   where $\tilde{A}_j$ and U_j denote the j-th column of matrices $\tilde{A}$ and U, respectively.

5. Estimate the reciprocal of the condition number of matrix $\tilde{A}$.

Method `Solve` uses this information to perform the following steps:

1. Solve the system $Ax = b$ using the computed triangular factors.

2. Iteratively refine the solution, again using the computed triangular factors. This is equivalent to Newton's method.

3. Compute forward and backward error bounds for the solution vector x.

Some of the steps mentioned above are optional. Their settings can be controlled by the `Set` methods and properties of class `ComplexSuperLU`.

Class `ComplexSuperLU` is based on the SuperLU code written by Demmel, Gilbert, Li et al. For more detailed explanations of the factorization and solve steps, see the SuperLU Users' Guide (1999).

Properties

ColumnOrderingMethod

```
public Imsl.Math.ComplexSuperLU.ColumnOrdering ColumnOrderingMethod {get; set;
}
```

Description

The method used to permute the columns of the input matrix.

Property Value

A `ColumnOrdering` scalar specifying how the columns of the input matrix are to be permuted for sparsity preservation.

value	*method*
`Natural`	natural ordering, that is $P_c = I$, I the identity matrix
`MinimumDegreeAtPlusA`	minimum degree ordering on the structure of $A^T + A$
`MinimumDegreeAtA`	minimum degree ordering on the structure of $A^T A$
`ColumnApproximateMinimumDegree`	column approximate minimum degree ordering

Remarks

By default, `ColumnOrderingMethod` = `ComplexSuperLU.ColumnOrdering.ColumnApproximateMinimumDegree`.

Exception

`System.ArgumentException` is thrown if `value` is not one of the above values.

ConditionNumber

```
public double ConditionNumber {get; }
```

Description

The estimate of the reciprocal condition number of the matrix A.

Property Value

A `double` scalar containing the reciprocal condition number of the matrix *A* after equilibration and permutation of rows/columns (if done). If the reciprocal condition number is less than machine precision, in particular if it is equal to 0, the matrix is singular to working precision.

DiagonalPivotThreshold

```
public double DiagonalPivotThreshold {get; set; }
```

Description

The threshold used for a diagonal entry to be an acceptable pivot.

Property Value

A `double` scalar specifying the threshold used for a diagonal entry to be an acceptable pivot.

Remarks

By default, `DiagonalPivotThreshold=1.0`, i.e. classical partial pivoting.

Exception

`System.ArgumentException` is thrown if `DiagonalPivotThreshold` is not in the interval $[0.0, 1.0]$.

Equilibrate

```
public bool Equilibrate {get; set; }
```

Description

Specifies if input matrix *A* is equilibrated before factorization.

Property Value

A `bool` specifying whether or not matrix `A` is equilibrated before factorization. If `Equilibrate` is `true` the system is equilibrated, if `Equilibrate` is `false`, no equilibration is performed.

Remarks

By default, `Equilibrate` = `true`.

EquilibrationMethod

```
public Imsl.Math.ComplexSuperLU.Scaling EquilibrationMethod {get; }
```

Description

The type of equilibration used before matrix factorization.

Property Value

A `Scaling` value specifying the equilibration option used.

Remarks

EquilibrationMethod	*Option description*
`None`	No equilibration is performed.
`Row`	Equilibration is performed with row scaling.
`Column`	Equilibration is performed with column scaling.
`RowAndColumn`	Equilibration is performed with row and column scaling.

ForwardErrorBound

```
public double ForwardErrorBound {get; }
```

Description

The estimated forward error bound for the solution vector.

Property Value

A `double` containing the estimated forward error bound for the solution vector. The estimate is as reliable as the estimate for the reciprocal condition number, and is almost always a slight overestimate of the true error. If iterative refinement is not used, the return value is set to `1.0`.

IterativeRefinement

```
public bool IterativeRefinement {get; set; }
```

Description

Specifies whether to perform iterative refinement.

Property Value

A `bool` specifying whether to use iterative refinement.

Remarks

For iterative refinement, set `IterativeRefinement = true`, otherwise set `IterativeRefinement = false`.

By default, `IterativeRefinement = false`.

PivotGrowth

```
public bool PivotGrowth {get; set; }
```

Description

Specifies whether to compute the reciprocal pivot growth factor.

Property Value

A `bool` specifying whether to compute the reciprocal pivot growth factor. `true` indicates the reciprocal pivot factor will be computed, `false` indicates it will not be computed.

Remarks

By default, `PivotGrowth = false`.

ReciprocalPivotGrowthFactor

```
public double ReciprocalPivotGrowthFactor {get; }
```

Description

The reciprocal pivot growth factor.

Property Value

A `double` scalar representing the reciprocal growth factor

$$\max_{1 \le j \le n} \frac{\|\tilde{A}_j\|_\infty}{\|U_j\|_\infty}.$$

If the value is much less than 1, the stability of the *LU* factorization could be poor.

RelativeBackwardError

```
public double RelativeBackwardError {get; }
```

Description

The componentwise relative backward error of the solution vector.

Property Value

A `double` containing the componentwise relative backward error of the solution vector `x`. If `IterativeRefinement` is not set to `true`, then `RelativeBackwardError` returns `1.0`.

SymmetricMode

```
public bool SymmetricMode {get; set; }
```

Description

Specifies whether to use the symmetric mode.

Property Value

A `bool` indicating if symmetric mode is to be used.

Remarks

The symmetric mode option should be applied if the input matrix A is diagonally dominant or nearly so. The user should then define a small diagonal pivot threshold (e.g. 0.0 or 0.01) by property `DiagonalPivotThreshold` and choose an $(A^T + A)$-based column permutation algorithm (e.g. column permutation method `SuperLU.ColumnOrdering.MinimumDegreeAtPlusA`). `SymmetricMode=true` implies symmetric mode is used.

By default, `SymmetricMode=false`.

Constructor

ComplexSuperLU

```
public ComplexSuperLU(Imsl.Math.ComplexSparseMatrix A)
```

Description

Constructor for `ComplexSuperLU`.

Parameter

`A` – A `ComplexSparseMatrix` containing the sparse square input matrix.

Methods

GetPerformanceTuningParameters

```
public int
GetPerformanceTuningParameters(Imsl.Math.ComplexSuperLU.PerformanceParameters
parameter)
```

Description

Returns a performance tuning parameter value.

Parameter

`parameter` – a `PerformanceParameters` scalar that specifies the parameter for which the value is to be returned.

`parameter`	*Description*
`PanelSize`	The panel size.
`RelaxationParameter`	The relaxation parameter to control supernode amalgamation.
`MaximumSupernodeSize`	The maximum allowable size for a supernode.
`MinimumRowDimension`	The minimum row dimension to be used for 2D blocking.
`MinimumColumnDimension`	The minimum column dimension to be used for 2D blocking.
`FillFactor`	The estimated fill factor for L and U, compared with A.

Returns

An `int` containing the current value used for the specified tuning parameter.

SetPerformanceTuningParameters

```
public void
SetPerformanceTuningParameters(Imsl.Math.ComplexSuperLU.PerformanceParameters
parameter, int tunedValue)
```

Description

Sets performance tuning parameters.

Parameters

`parameter` – A `PerformanceParameters` that specifies the parameter to be tuned.

`tunedValue` – An `int` scalar that specifies the value to be used for the specified tuning parameter.

parameter	*Description*	*Default*
PanelSize	The panel size.	10
RelaxationParameter	The relaxation parameter to control supernode amalgamation.	5
MaximumSupernodeSize	The maximum allowable size for a supernode.	100
MinimumRowDimension	The minimum row dimension to be used for 2D blocking.	200
MinimumColumnDimension	The minimum column dimension to be used for 2D blocking.	40
FillFactor	The estimated fill factor for L and U, compared with A.	20

Exception

`System.ArgumentException` is thrown if `tunedValue` is not greater than zero.

Solve

```
public Imsl.Math.Complex[] Solve(Imsl.Math.Complex[] b)
```

Description

Computation of the solution vector for the system $Ax = b$.

Parameter

`b` – A `Complex` vector of length `n`, `n` the order of input matrix `A`, containing the right hand side.

Returns

A `Complex` vector containing the solution to the system $Ax = b$. Optionally, the solution is improved by iterative refinement, if `IterativeRefinement` is set to `true`. Method `Solve` internally first factorizes matrix A (step 1 of the introduction) if the factorization has not been done before.

SolveConjugateTranspose

```
public Imsl.Math.Complex[] SolveConjugateTranspose(Imsl.Math.Complex[] b)
```

Description

Computation of the solution vector for the system $A^H x = b$.

Parameter

`b` – A `Complex` vector of length `n`, `n` the order of input matrix `A`, containing the right hand side.

Returns

A `Complex` vector containing the solution to the system $A^H x = b$. Optionally, the solution is improved by iterative refinement, if `IterativeRefinement` is set to `true`. Method `SolveConjugateTranspose` internally first factorizes matrix A (step 1 of the introduction) if the factorization has not been done before.

SolveTranspose

```
public Imsl.Math.Complex[] SolveTranspose(Imsl.Math.Complex[] b)
```

Description

Computation of the solution vector for the system $A^T x = b$.

Parameter

`b` – A `Complex` vector of length `n` containing the right hand side, `n` is the order of input matrix `A`.

Returns

A `Complex` vector containing the solution to the system $A^T x = b$. Optionally, the solution is improved by iterative refinement, if `IterativeRefinement` is set to `true`. Method `SolveTranspose` internally first factorizes matrix A (step 1 of the introduction) if the factorization has not been done before.

Example: LU Factorization of a Complex Sparse Matrix

The LU Factorization of the sparse complex 6×6 matrix

$$A = \begin{pmatrix} 10+7i & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3+2i & -3+0i & -1+2i & 0.0 & 0.0 \\ 0.0 & 0.0 & 4+2i & 0.0 & 0.0 & 0.0 \\ -2-4i & 0.0 & 0.0 & 1+6i & -1+3i & 0.0 \\ -5+4i & 0.0 & 0.0 & -5+0i & 12+2i & -7+7i \\ -1+12i & -2+8i & 0.0 & 0.0 & 0.0 & 3+7i \end{pmatrix}$$

is computed. The sparse coordinate form for A is given by row, column, value triplets:

row	column	value
0	0	10+7i
1	1	3+2i
1	2	-3+0i
1	3	-1+2i
2	2	4+2i
3	0	-2-4i
3	3	1+6i
3	4	-1+3i
4	0	-5+4i
4	3	-5+0i
4	4	12+2i
4	5	-7+7i
5	0	-1+12i
5	1	-2+8i
5	5	3+7i

Let

$$x^T = (1+i, 2+2i, 3+3i, 4+4i, 5+5i, 6+6i)$$

so that

$$b_1 := Ax = (3+17i, -19+5i, 6+18i, -38+32i, -63+49i, -57+83i)^T$$

and

$$b_2 := A^H x = (54-112i, 46-58i, 12, 5-51i, 78+34i, 60-94i)^T .$$

The LU factorization of A is used to solve the complex sparse linear systems $Ax = b_1$ and $A^H x = b_2$ with iterative refinement. The reciprocal pivot growth factor and the reciprocal condition number are also computed.

```
using System;
using Imsl.Math;

public class ComplexSuperLUEx1
{
    public static void Main(String[] args)
    {
        int m;
        ComplexSuperLU cslu;
        double conditionNumber, recip_pivot_growth;
        Complex[] sol = null;
        double Ferr, Berr;
        Complex[] b1 = { new Complex(3.0, 17.0), new Complex(-19.0, 5.0),
                         new Complex(6.0, 18.0), new Complex(-38.0, 32.0),
                         new Complex(-63.0, 49.0), new Complex(-57.0, 83.0)
                       };
        Complex[] b2 = { new Complex(54.0, -112.0), new Complex(46.0, -58.0),
                         new Complex(12.0, 0.0), new Complex(5.0, -51.0),
                         new Complex(78.0, 34.0), new Complex(60.0, -94.0)
                       };
        // Initialize input matrix A.
        m = 6;
        ComplexSparseMatrix a = new ComplexSparseMatrix(m, m);
        a.Set(0, 0, new Complex(10.0, 7.0));
        a.Set(1, 1, new Complex(3.0, 2.0));
        a.Set(1, 2, new Complex(-3.0, 0.0));
        a.Set(1, 3, new Complex(-1.0, 2.0));
        a.Set(2, 2, new Complex(4.0, 2.0));
        a.Set(3, 0, new Complex(- 2.0, -4.0));
        a.Set(3, 3, new Complex(1.0, 6.0));
        a.Set(3, 4, new Complex(-1.0, 3.0));
        a.Set(4, 0, new Complex(-5.0, 4.0));
        a.Set(4, 3, new Complex(-5.0, 0.0));
        a.Set(4, 4, new Complex(12.0, 2.0));
        a.Set(4, 5, new Complex(-7.0, 7.0));
        a.Set(5, 0, new Complex(-1.0, 12.0));
        a.Set(5, 1, new Complex(-2.0, 8.0));
        a.Set(5, 5, new Complex(3.0, 7.0));

        // Compute the sparse LU factorization of a
        cslu = new ComplexSuperLU(a);
        cslu.Equilibrate = false;
        cslu.ColumnOrderingMethod =
                ComplexSuperLU.ColumnOrdering.Natural;
        cslu.PivotGrowth =true;

        // Set option of iterative refinement
        cslu.IterativeRefinement=true;

        // Solve sparse system A*x = b1
        Console.Out.WriteLine();
```

```
        Console.Out.WriteLine("Solve sparse System A*x=b1");
        Console.Out.WriteLine("==========================");
        Console.Out.WriteLine();
        sol = cslu.Solve(b1);
        new PrintMatrix("Solution").Print(sol);

        // Determine error bounds
        Ferr = cslu.ForwardErrorBound;
        Berr = cslu.RelativeBackwardError;
        Console.Out.WriteLine();
        Console.Out.WriteLine("Forward error bound: " + Ferr);
        Console.Out.WriteLine("Relative backward error: " + Berr);
        Console.Out.WriteLine();
        Console.Out.WriteLine();

        // Solve sparse system (A^H)*x = b2
        Console.Out.WriteLine();
        Console.Out.WriteLine("Solve sparse System (A^H)*x=b2");
        Console.Out.WriteLine("==============================");
        Console.Out.WriteLine();
        sol = cslu.SolveConjugateTranspose(b2);
        new PrintMatrix("Solution").Print(sol);

        // Determine error bounds
        Ferr = cslu.ForwardErrorBound;
        Berr = cslu.RelativeBackwardError;
        Console.Out.WriteLine();
        Console.Out.WriteLine("Forward error bound: " + Ferr);
        Console.Out.WriteLine("Relative backward error: " + Berr);
        Console.Out.WriteLine();
        Console.Out.WriteLine();

        // Compute reciprocal pivot growth factor and condition number
        recip_pivot_growth = cslu.ReciprocalPivotGrowthFactor;
        conditionNumber = cslu.ConditionNumber;
        Console.Out.WriteLine("Pivot growth factor and condition number");
        Console.Out.WriteLine("========================================");
        Console.Out.WriteLine();
        Console.Out.WriteLine("Reciprocal pivot growth factor: " + recip_pivot_growth);
        Console.Out.WriteLine("Reciprocal condition number: " + conditionNumber);
        Console.Out.WriteLine();
    }
}
```

Output

```
Solve sparse System A*x=b1
==========================

Solution
    0
0  1+1i
1  2+2i
2  3+3i
3  4+4i
```

```
4  5+5i
5  6+6i


Forward error bound: 2.83933305928053E-15
Relative backward error: 1.70803542250024E-16



Solve sparse System (A^H)*x=b2
==============================

        Solution
            0
0  1+1i
1  2+2.000000000000001i
2  3+3i
3  4+4i
4  5+5i
5  6+6i


Forward error bound: 8.54834098797111E-15
Relative backward error: 1.02977208081174E-16


Pivot growth factor and condition number
========================================

Reciprocal pivot growth factor: 0.799382716049383
Reciprocal condition number: 0.0700654479096751
```

ComplexSuperLU.ColumnOrdering Enumeration

`public enumeration Imsl.Math.ComplexSuperLU.ColumnOrdering`

The column permutation method to be used.

Fields

ColumnApproximateMinimumDegree

`public Imsl.Math.ComplexSuperLU.ColumnOrdering ColumnApproximateMinimumDegree`

Description

Indicates column approximate minimum degree ordering.

MinimumDegreeAtA

```
public Imsl.Math.ComplexSuperLU.ColumnOrdering MinimumDegreeAtA
```

Description

Indicates minimum degree ordering on the structure of A^TA.

MinimumDegreeAtPlusA

```
public Imsl.Math.ComplexSuperLU.ColumnOrdering MinimumDegreeAtPlusA
```

Description

Indicates minimum degree ordering on the structure of $A^T + A$.

Natural

```
public Imsl.Math.ComplexSuperLU.ColumnOrdering Natural
```

Description

Indicates natural ordering.

ComplexSuperLU.PerformanceParameters Enumeration

```
public enumeration Imsl.Math.ComplexSuperLU.PerformanceParameters
```

Performance tuning parameters which can be adjusted via method `SetPerformanceTuningParameters`.

Fields

FillFactor

```
public Imsl.Math.ComplexSuperLU.PerformanceParameters FillFactor
```

Description

The estimated fill factor for L and U, compared with A.

MaximumSupernodeSize

```
public Imsl.Math.ComplexSuperLU.PerformanceParameters MaximumSupernodeSize
```

Description

The maximum allowable size for a supernode.

MinimumColumnDimension

```
public Imsl.Math.ComplexSuperLU.PerformanceParameters MinimumColumnDimension
```

Description

The minimum column dimension to be used for 2D blocking.

MinimumRowDimension

```
public Imsl.Math.ComplexSuperLU.PerformanceParameters MinimumRowDimension
```

Description

The minimum row dimension to be used for 2D blocking.

PanelSize

```
public Imsl.Math.ComplexSuperLU.PerformanceParameters PanelSize
```

Description

The panel size.

RelaxationParameter

```
public Imsl.Math.ComplexSuperLU.PerformanceParameters RelaxationParameter
```

Description

The relaxation parameter to control supernode amalgamation.

ComplexSuperLU.Scaling Enumeration

```
public enumeration Imsl.Math.ComplexSuperLU.Scaling
```

Equilibration method before factorization, this setting is returned from `GetEquilibrationMethod`.

Fields

Column

```
public Imsl.Math.ComplexSuperLU.Scaling Column
```

Description

Indicates that input matrix *A* was column scaled before factorization.

None

`public Imsl.Math.ComplexSuperLU.Scaling None`

Description

Indicates that input matrix *A* was not equilibrated before factorization.

Row

`public Imsl.Math.ComplexSuperLU.Scaling Row`

Description

Indicates that input matrix *A* was row scaled before factorization.

RowAndColumn

`public Imsl.Math.ComplexSuperLU.Scaling RowAndColumn`

Description

Indicates that input matrix *A* was row and column scaled before factorization.

Cholesky Class

`public class Imsl.Math.Cholesky`

Cholesky factorization of a matrix of type `double`.

Class `Cholesky` is based on the LINPACK routine `SCHDC`; see Dongarra et al. (1979).

Before the decomposition is computed, initial elements are moved to the leading part of *A* and final elements to the trailing part of *A*. During the decomposition only rows and columns corresponding to the free elements are moved. The result of the decomposition is an lower triangular matrix *R* and a permutation matrix *P* that satisfy $P^T AP = RR^T$, where *P* is represented by `ipvt`.

The method `Update` is based on the LINPACK routine `SCHUD`; see Dongarra et al. (1979).

The Cholesky factorization of a matrix is $A = RR^T$, where *R* is an lower triangular matrix. Given this factorization, `Downdate` computes the factorization

$$A - xx^T = \tilde{R}^T \tilde{R}$$

`Downdate` determines an orthogonal matrix *U* as the product $G_N \ldots G_1$ of Givens rotations, such that

$$U \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{R} \\ x^T \end{bmatrix}$$

By multiplying this equation by its transpose and noting that $U^T U = I$, the desired result

$$R^T R - xx^T = \tilde{R}^T \tilde{R}$$

is obtained.

Let a be the solution of the linear system $R^T a = x$ and let

$$\alpha = \sqrt{1 - \|a\|_2^2}$$

The Givens rotations, G_i, are chosen such that

$$G_1 \cdots G_N \begin{bmatrix} a \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The G_i, are *(N + 1) * (N + 1)* matrices of the form

$$G_i = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & c_i & 0 & -s_i \\ 0 & 0 & I_{N-i} & 0 \\ 0 & s_i & 0 & c_i \end{bmatrix}$$

where I_k is the identity matrix of order k; and $c_i = \cos\theta_i, s_i = \sin\theta_i$ for some θ_i.

The Givens rotations are then used to form

$$\tilde{R},\ G_1 \cdots G_N \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{R} \\ \tilde{x}^T \end{bmatrix}$$

The matrix

$$\tilde{R}$$

is lower triangular and

$$\tilde{x} = x$$

because

$$x = \left(R^T 0\right) \begin{bmatrix} a \\ \alpha \end{bmatrix} = \left(R^T 0\right) U^T U \begin{bmatrix} a \\ \alpha \end{bmatrix} = \left(\tilde{R}^T \tilde{x}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \tilde{x}$$

.

Constructor

Cholesky

```
public Cholesky(double[,] a)
```

Description

Create the Cholesky factorization of a symmetric positive definite matrix of type `double`.

Parameter

`a` – A `double` square matrix to be factored.

Exceptions

`Imsl.Math.SingularMatrixException` is thrown when the input matrix `a` is singular

`Imsl.Math.NotSPDException` is thrown when the input matrix is not symmetric, positive definite.

Methods

Downdate

```
public void Downdate(double[] x)
```

Description

Downdates the factorization by subtracting a rank-1 matrix.

Parameter

`x` – A `double` array which specifies the rank-1 matrix. `x` is not modified by this function.

Remarks

The object will contain the Cholesky factorization of a - x * transpose(x), where a is the previously factor matrix.

Exception

`Imsl.Math.NotSPDException` is thrown if a - x * transpose(x) is not symmetric positive-definite.

GetR

```
public double[,] GetR()
```

Description

The R matrix that results from the Cholesky factorization.

Returns

A `double` matrix which contains the lower triangular R matrix that results from the Cholesky factorization, $A = RR^T$.

Inverse

```
public double[,] Inverse()
```

Description

Returns the inverse of this matrix.

Returns

A `double` matrix containing the inverse.

Solve

```
public double[] Solve(double[] b)
```

Description

Solve Ax = b where A is a positive definite matrix with elements of type `double`.

Parameter

`b` – A `double` array containing the right-hand side of the linear system.

Returns

A `double` array containing the solution to the system of linear equations.

Update

```
public void Update(double[] x)
```

Description

Updates the factorization by adding a rank-1 matrix.

Parameter

`x` – A `double` array which specifies the rank-1 matrix. `x` is not modified by this function.

Remarks

The object will contain the Cholesky factorization of a + x * transpose(x), where a is the previously factored matrix.

Example: Cholesky Factorization

The Cholesky Factorization of a matrix is performed as well as its inverse.

```
using System;
using Imsl.Math;

public class CholeskyEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {
            {1, - 3, 2},
            {- 3, 10, - 5},
            {2, - 5, 6}
        };
        double[] b = new double[]{27, - 78, 64};

        // Compute the Cholesky factorization of A
        Cholesky cholesky = new Cholesky(a);

        // Solve Ax = b
```

```
        double[] x = cholesky.Solve(b);
        new PrintMatrix("x").Print(x);

        // Find the inverse of A.
        double[,] ainv = cholesky.Inverse();
        new PrintMatrix("ainv").Print(ainv);
    }
}
```

Output

```
   x
   0
0   1
1  -4
2   7

     ainv
   0   1   2
0  35   8  -5
1   8   2  -1
2  -5  -1   1
```

SparseCholesky Class

`public class Imsl.Math.SparseCholesky`

Sparse Cholesky factorization of a matrix of type `SparseMatrix`.

Class `SparseCholesky` computes the Cholesky factorization of a sparse symmetric positive definite matrix A. This factorization can then be used to compute the solution of the linear system $Ax = b$.

Typically, the solution of a large sparse positive definite system $Ax = b$ is done in 4 steps:

1. In step one, an ordering algorithm is used to preserve sparsity in the Cholesky factor L of matrix A during the numerical factorization process. The new order can be described by a permutation matrix P.
2. Step two consists of setting up the data structure for the Cholesky factor L, where $PAP^T = LL^T$. This step is called the symbolic factorization phase of the computation. During symbolic factorization, only the sparsity pattern of sparse matrix A, i.e., the locations of the nonzero entries of matrix A are needed but not any of the elements themselves.
3. In step 3, the numerical factorization phase, the Cholesky factorization is done numerically.
4. Step 4 is the solution phase. Here, the numerical solution, x, to the original system is obtained by solving the two triangular systems $Ly_1 = Pb$, $L^T y_2 = y_1$ and the permutation $x = P^T y_2$.

Class SparseCholesky realizes all four steps by algorithms described in George and Liu (1981). Especially, step one, is a realization of a minimum degree ordering algorithm. The numerical factorization in its standard form is based on a sparse compressed storage scheme. Alternatively, a multifrontal method can be used. The multifrontal method requires more storage but will be faster than the standard method in certain cases. The multifrontal method is based on the routines in Liu (1987). For a detailed description of this method, see Liu (1990), also Duff and Reid(1983, 1984), Ashcraft (1987) et al. (1987), and Liu (1986, 1989, 1992).The numerical factorization method can be specified by using the NumericFactorizationMethod (p. 82) property.

The Solve (p. 85) method will compute the symbolic and numeric factorizations if they have not already been computed or supplied by the user through FactorSymbolically (p. 84), FactorNumerically (p. 83), SetNumericFactor (p. 85), or SetSymbolicFactor (p. 85). These factorizations are retained for later use by the Solve method when different right-hand sides are to be solved.

There is a special situation where computations can be simplified. If an application generates different sparse symmetric positive definite coefficient matrices that all have the same sparsity pattern, then by using SetSymbolicFactor (p. 85) the symbolic factorization need only be computed once.

Properties

LargestDiagonalElement

```
public double LargestDiagonalElement {get; }
```

Description

The largest diagonal element of the Cholesky factor.

Property Value

A double value specifying the largest diagonal element of the Cholesky factor. Use of this method is only sensible if a numeric factorization of the input matrix was done beforehand.

NumberOfNonzeros

```
public long NumberOfNonzeros {get; }
```

Description

The number of nonzeros in the Cholesky factor.

Property Value

A long containing the number of nonzeros (including the diagonal) of the Cholesky factor.

NumericFactorizationMethod

```
public Imsl.Math.SparseCholesky.NumericFactorization NumericFactorizationMethod
{get; set; }
```

Description

The method used in the numerical factorization of the permuted input matrix.

Property Value

A `NumericFactorization` value equal to `StandardMethod` (p. 87) or `MultiFrontalMethod` (p. 87) representing the method used in the numeric factorization of the permuted input matrix.

Remarks

Method Name	*Description*
`StandardMethod`	standard method as described by George/Liu (1981). This is the default.
`MultiFrontalMethod`	multifrontal method

SmallestDiagonalElement

```
public double SmallestDiagonalElement {get; }
```

Description

The smallest diagonal element of the Cholesky factor.

Property Value

A `double` value specifying the smallest diagonal element of the Cholesky factor. Use of this method is only sensible if a numeric factorization of the input matrix was done beforehand.

Constructor

SparseCholesky

```
public SparseCholesky(Imsl.Math.SparseMatrix A)
```

Description

Constructs the matrix structure for the Cholesky factorization of a sparse symmetric positive definite matrix of type `SparseMatrix`.

Parameter

`A` – The `SparseMatrix` symmetric positive definite matrix to be factored. Only the lower triangular part of the input matrix is used.

Methods

FactorNumerically

```
public void FactorNumerically()
```

Description

Computes the numeric factorization of a sparse real symmetric positive definite matrix.

Remarks

This method numerically factors the instance of the constructed matrix *A*, where *A* is of type `SparseMatrix` and is symmetric positive definite. The factorization is obtained in several steps:

1. First, matrix *A* is permuted to reduce fill-in, leading to a sparse symmetric positive definite matrix PAP^T.
2. Then, matrix PAP^T is symbolically and numerically factored.

Note that the symbolic factorization is not done if the symbolic factor has been supplied by the user through the `SetSymbolicFactor` method.

Exception

`Imsl.Math.NotSPDException` is thrown when the input matrix is not symmetric, positive definite.

FactorSymbolically

```
public void FactorSymbolically()
```

Description

Computes the symbolic factorization of a sparse real symmetric positive definite matrix.

Remarks

This method symbolically factors the instance of the constructed matrix *A*, where *A* is of type `SparseMatrix` and is symmetric positive definite. The factorization is obtained in several steps:

1. First, matrix *A* is permuted to reduce fill-in, leading to a sparse symmetric positive definite matrix PAP^T.
2. Then, matrix PAP^T is symbolically factored.

Exception

`Imsl.Math.NotSPDException` is thrown when the input matrix is not symmetric, positive definite.

GetNumericFactor

```
public Imsl.Math.SparseCholesky.NumericFactor GetNumericFactor()
```

Description

Returns the numeric Cholesky factor.

Returns

A `NumericFactor` containing the numeric Cholesky factor.

GetSymbolicFactor

```
public Imsl.Math.SparseCholesky.SymbolicFactor GetSymbolicFactor()
```

Description

Returns the symbolic Cholesky factor.

Returns

A `SymbolicFactor` containing the symbolic Cholesky factor.

SetNumericFactor

```
public void SetNumericFactor(Imsl.Math.SparseCholesky.NumericFactor
numericFactor)
```

Description

Sets the numeric Cholesky factor to use in solving of a sparse positive definite system of linear equations $Ax = b$.

Parameter

`numericFactor` – A `NumericFactor` containing the numeric Cholesky factor. By default the numeric factorization is computed.

SetSymbolicFactor

```
public void SetSymbolicFactor(Imsl.Math.SparseCholesky.SymbolicFactor
symbolicFactor)
```

Description

Sets the symbolic Cholesky factor to use in solving a sparse positive definite system of linear equations $Ax = b$.

Parameter

`symbolicFactor` – A `SymbolicFactor` containing the symbolic Cholesky factor. By default the symbolic factorization is computed.

Solve

```
public double[] Solve(double[] b)
```

Description

Computes the solution of a sparse real symmetric positive definite system of linear equations $Ax = b$.

Parameter

`b` – A `double` vector of length equal to the order of matrix `A` representing the right-hand side of the linear system.

Returns

A `double` vector of length equal to the order of matrix `A` representing the solution to the system of linear equations $Ax = b$.

Remarks

This method solves the linear system $Ax = b$, where A is symmetric positive definite. The solution is obtained in several steps:

1. First, matrix A is permuted to reduce fill-in, leading to a sparse symmetric positive definite system $PAP^T(Px) = Pb$.
2. Then, matrix PAP^T is symbolically and numerically factored.

3. The final solution is obtained by solving the systems $Ly_1 = Pb, L^T y_2 = y_1$ and $x = P^T y_2$.

By default this method implements all of the above steps. The factorizations are retained for later use by subsequent solves. By choosing appropriate methods within this class, the computation can be reduced to the solution of the system $Ax = b$ for a given or precomputed symbolic or numeric factor.

Exception

`Imsl.Math.NotSPDException` is thrown when the input matrix is not symmetric, positive definite.

Example: Sparse Cholesky Factorization

The Cholesky Factorization of a sparse symmetric positive definite matrix is computed. Some additional information about the Cholesky factorization is also computed.

```
using System;
using Imsl.Math;

public class SparseCholeskyEx1
{
    public static void Main(String[] args)
    {
        SparseMatrix A = new SparseMatrix(5, 5);
        A.Set(0, 0, 10.0);
        A.Set(1, 1, 20.0);
        A.Set(2, 0, 1.0);
        A.Set(2, 2, 30.0);
        A.Set(3, 2, 4.0);
        A.Set(3, 3, 40.0);
        A.Set(4, 0, 2.0);
        A.Set(4, 1, 3.0);
        A.Set(4, 3, 5.0);
        A.Set(4, 4, 50.0);

        double[] b = { 55.0, 83.0, 103.0, 97.0, 82.0};

        SparseCholesky cholesky = new SparseCholesky(A);

        // Choose Multifrontal method as numeric factorization method
        cholesky.NumericFactorizationMethod =
         SparseCholesky.NumericFactorization.MultiFrontalMethod;

        // Compute solution
        double[] solution = cholesky.Solve(b);
        new PrintMatrix("Computed Solution").Print(solution);

        // Print additional information about the factorization
        Console.Out.Write("Smallest diagonal element of Cholesky factor: ");
        Console.Out.WriteLine(cholesky.SmallestDiagonalElement);
        Console.Out.Write("Largest diagonal element of Cholesky factor: ");
        Console.Out.WriteLine(cholesky.LargestDiagonalElement);
        Console.Out.Write("Number of nonzeros in Cholesky factor: ");
        Console.Out.WriteLine(cholesky.NumberOfNonzeros);
    }
```

```
}
```

Output

```
Computed Solution
   0
0  5
1  4
2  3
3  2
4  1

Smallest diagonal element of Cholesky factor: 3.16227766016838
Largest diagonal element of Cholesky factor: 7.01070609853244
Number of nonzeros in Cholesky factor: 11
```

SparseCholesky.NumericFactorization Enumeration

`public enumeration Imsl.Math.SparseCholesky.NumericFactorization`

Numeric factorization methods.

Fields

MultiFrontalMethod

`public Imsl.Math.SparseCholesky.NumericFactorization MultiFrontalMethod`

Description

Indicates the multifrontal method will be used for numeric factorization.

StandardMethod

`public Imsl.Math.SparseCholesky.NumericFactorization StandardMethod`

Description

Indicates the method of George/Liu (1981) will be used for numeric factorization.

SparseCholesky.NumericFactor Class

`public class Imsl.Math.SparseCholesky.NumericFactor`

The numeric Cholesky factorization of a matrix.

Used by `GetNumericFactor` and `SetNumericFactor` to hold the numeric Cholesky factorization of a matrix.

SparseCholesky.SymbolicFactor Class

`public class Imsl.Math.SparseCholesky.SymbolicFactor`

The symbolic Cholesky factorization of a matrix.

Used by `GetSymbolicFactor` and `SetSymbolicFactor` to hold the symbolic Cholesky factorization of a matrix.

ComplexSparseCholesky Class

`public class Imsl.Math.ComplexSparseCholesky`

Sparse Cholesky factorization of a matrix of type `ComplexSparseMatrix`.

Class `ComplexSparseCholesky` computes the Cholesky factorization of a sparse Hermitian positive definite matrix A. This factorization can then be used to compute the solution of the linear system $Ax = b$.

Typically, the solution of a large sparse positive definite system $Ax = b$ is done in four steps.

1. In step one, an ordering algorithm is used to preserve sparsity in the Cholesky factor L of matrix A during the numerical factorization process. The new order can be described by a permutation matrix P.
2. Step two consists of setting up the data structure for the Cholesky factor L, where $PAP^T = LL^T$. This step is called the symbolic factorization phase of the computation. During symbolic factorization, only the sparsity pattern of sparse matrix A, i.e., the locations of the nonzero entries of matrix A are needed but not any of the elements themselves.
3. In step 3, the numerical factorization phase, the Cholesky factorization is done numerically.

4. Step 4 is the solution phase. Here, the numerical solution, x, to the original system is obtained by solving the two triangular systems $Ly_1 = Pb$, $L^T y_2 = y_1$ and the permutation $x = P^T y_2$.

Class `ComplexSparseCholesky` realizes all four steps by algorithms described in George and Liu (1981). Especially, step one, is a realization of a minimum degree ordering algorithm. The numerical factorization in its standard form is based on a sparse compressed storage scheme. Alternatively, a multifrontal method can be used. The multifrontal method requires more storage but will be faster than the standard method in certain cases. The multifrontal method is based on the routines in Liu (1987). For a detailed description of this method, see Liu (1990), also Duff and Reid(1983, 1984), Ashcraft (1987) et al. (1987), and Liu (1986, 1989, 1992). The numerical factorization method can be specified by using the `NumericFactorizationMethod` (p. 89) property.

The `Solve` (p. 92) method will compute the symbolic and numeric factorizations if they have not already been computed or supplied by the user through the `FactorSymbolically` (p. 91), `FactorNumerically` (p. 91), `SetNumericFactor` (p. 92), or `SetSymbolicFactor` (p. 92) methods. These factorizations are retained for later use by the `Solve` method when different right-hand sides are to be solved.

There is a special situation where computations can be simplified.If an application generates different sparse Hermitian positive definite coefficient matrices that all have the same sparsity pattern, then by using methods `GetSymbolicFactor` (p. 92) and `SetSymbolicFactor` (p. 92) the symbolic factorization needs only be computed once.

Properties

LargestDiagonalElement

```
public double LargestDiagonalElement {get; }
```

Description

The largest diagonal element of the Cholesky factor.

Property Value

A `double` containing the largest diagonal element of the Cholesky factor. Use of this method is only sensible if a numeric factorization of the input matrix was done beforehand.

NumberOfNonzeros

```
public long NumberOfNonzeros {get; }
```

Description

The number of nonzeros in the Cholesky factor.

Property Value

A `long` containing the number of nonzeros (including the diagonal) of the Cholesky factor.

NumericFactorizationMethod

```
public Imsl.Math.ComplexSparseCholesky.NumericFactorization
NumericFactorizationMethod {get; set; }
```

Description

The method used in the numerical factorization of the permuted input matrix.

Property Value

An `int` value equal to `StandardMethod` (p. 87) or `MultiFrontalMethod` (p. 87) representing the method used during the numeric factorization phase:

Method Name	*Description*
`StandardMethod` (p. 94)	standard method as described by George/Liu (1981). This is the default.
`MultiFrontalMethod` (p. 94)	multifrontal method

Exception

`System.ArgumentException` is thrown when the value for `NumericFactorizationMethod` is not `NumericFactorization.StandardMethod` or `NumericFactorization.MultiFrontalMethod`.

SmallestDiagonalElement

```
public double SmallestDiagonalElement {get; }
```

Description

The smallest diagonal element of the Cholesky factor.

Property Value

A `double` containing the smallest diagonal element of the Cholesky factor. Use of this method is only sensible if a numeric factorization of the input matrix was done beforehand.

Constructor

ComplexSparseCholesky

```
public ComplexSparseCholesky(Imsl.Math.ComplexSparseMatrix A)
```

Description

Constructs the matrix structure for the Cholesky factorization of a sparse Hermitian positive definite matrix of type `ComplexSparseMatrix`.

Parameter

`A` – The `ComplexSparseMatrix` Hermitian positive definite matrix to be factored. Only the lower triangular part of the input matrix is used.

Methods

FactorNumerically

```
public void FactorNumerically()
```

Description

Computes the numeric factorization of a sparse Hermitian positive definite matrix.

Remarks

This method numerically factors the instance of the constructed matrix *A*, where *A* is of type `ComplexSparseMatrix` and is Hermitian positive definite. The factorization is obtained in several steps:

1. First, matrix *A* is permuted to reduce fill-in, leading to a sparse Hermitian positive definite matrix PAP^T.
2. Then, matrix PAP^T is symbolically and numerically factored.

Note that the symbolic factorization is not done if the symbolic factor has been supplied by the user through the `SetSymbolicFactor` method.

Exception

`Imsl.Math.NotSPDException` is thrown if the input matrix is not Hermitian, positive definite.

FactorSymbolically

```
public void FactorSymbolically()
```

Description

Computes the symbolic factorization of a sparse Hermitian positive definite matrix.

Remarks

This method symbolically factors the instance of the constructed matrix *A*, where *A* is of type `ComplexSparseMatrix` and is Hermitian positive definite. The factorization is obtained in several steps:

1. First, matrix *A* is permuted to reduce fill-in, leading to a sparse Hermitian positive definite matrix PAP^T.
2. Then, matrix PAP^T is symbolically factored.

Exception

`Imsl.Math.NotSPDException` is thrown if the input matrix is not Hermitian, positive definite.

GetNumericFactor

```
public Imsl.Math.ComplexSparseCholesky.NumericFactor GetNumericFactor()
```

Description

Returns the numeric Cholesky factor.

Returns

A `NumericFactor` containing the numeric Cholesky factor.

GetSymbolicFactor

```
public Imsl.Math.ComplexSparseCholesky.SymbolicFactor GetSymbolicFactor()
```

Description

Returns the symbolic Cholesky factor.

Returns

A `SymbolicFactor` containing the symbolic Cholesky factor.

SetNumericFactor

```
public void SetNumericFactor(Imsl.Math.ComplexSparseCholesky.NumericFactor
numericFactor)
```

Description

Sets the numeric Cholesky factor to use in solving a sparse complex Hermitian positive definite system of linear equations $Ax = b$.

Parameter

`numericFactor` – A `NumericFactor` containing the numeric Cholesky factor. By default the numeric factorization is computed.

SetSymbolicFactor

```
public void SetSymbolicFactor(Imsl.Math.ComplexSparseCholesky.SymbolicFactor
symbolicFactor)
```

Description

Sets the symbolic Cholesky factor to use in solving a sparse complex Hermitian positive definite system of linear equations $Ax = b$.

Parameter

`symbolicFactor` – a `SymbolicFactor` containing the symbolic Cholesky factor. By default the symbolic factorization is computed.

Solve

```
public Imsl.Math.Complex[] Solve(Imsl.Math.Complex[] b)
```

Description

Computes the solution of a sparse Hermitian positive definite system of linear equations $Ax = b$.

Parameter

`b` – A `Complex` vector of length equal to the order of matrix A containing the right-hand side.

Returns

A `Complex` vector of length equal to the order of matrix A containing the solution of the system $Ax = b$.

Remarks

This method solves the linear system $Ax = b$, where A is Hermitian positive definite. The solution is obtained in several steps:

1. First, matrix A is permuted to reduce fill-in, leading to a sparse Hermitian positive definite system $PAP^T(Px) = Pb$.
2. Then, matrix PAP^T is symbolically and numerically factored.
3. The final solution is obtained by solving the systems $Ly_1 = Pb, L^T y_2 = y_1$ and $x = P^T y_2$.

By default this method implements all of the above steps. The factorizations are retained for later use by subsequent solves. By choosing appropriate methods within this class, the computation can be reduced to the solution of the system $Ax = b$ for a given or precomputed symbolic or numeric factor.

Exception

`Imsl.Math.NotSPDException` is thrown when the input matrix is not Hermitian, positive definite.

Example: Complex Sparse Cholesky Factorization

The Cholesky Factorization of a sparse Hermitian positive definite matrix is computed. Some additional information about the Cholesky factorization is also computed.

```
using System;
using Imsl.Math;

public class ComplexSparseCholeskyEx1
{
    public static void  Main(String[] args)
    {
        ComplexSparseMatrix A = new ComplexSparseMatrix(3, 3);

        A.Set(0, 0, new Complex(2.0, 0.0));
        A.Set(1, 0, new Complex(-1.0, -1.0));
        A.Set(1, 1, new Complex(4.0, 0.0));
        A.Set(2, 1, new Complex(1.0, -2.0));
        A.Set(2, 2, new Complex(10.0, 0.0));

        Complex[] b = { new Complex(-2.0, 2.0), new Complex(5.0, 15.0),
                        new Complex(36.0, 28.0)};

        ComplexSparseCholesky cholesky = new ComplexSparseCholesky(A);

        // Choose Multifrontal method as numeric factorization method
        cholesky.NumericFactorizationMethod =
            ComplexSparseCholesky.NumericFactorization.MultiFrontalMethod;

        // Compute solution
        Complex[] solution = cholesky.Solve(b);

        PrintMatrix p = new PrintMatrix("Computed solution");
        p.Print(solution);
```

```
        // Compute additional information about the factorization

        Console.Out.Write("Smallest diagonal element of Cholesky factor: ");
        Console.Out.WriteLine(cholesky.SmallestDiagonalElement);
        Console.Out.Write("Largest diagonal element of Cholesky factor: ");
        Console.Out.WriteLine(cholesky.LargestDiagonalElement);
        Console.Out.Write("Number of nonzeros in Cholesky factor: ");
        Console.Out.WriteLine(cholesky.NumberOfNonzeros);
    }
}
```

Output

```
Computed solution
    0
0  1+1i
1  2+2i
2  3+3i

Smallest diagonal element of Cholesky factor: 1.4142135623731
Largest diagonal element of Cholesky factor: 2.88675134594813
Number of nonzeros in Cholesky factor: 5
```

ComplexSparseCholesky.NumericFactorization Enumeration

```
public enumeration Imsl.Math.ComplexSparseCholesky.NumericFactorization
```

Numeric Factorization methods.

Fields

MultiFrontalMethod

```
public Imsl.Math.ComplexSparseCholesky.NumericFactorization MultiFrontalMethod
```

Description

Indicates the multifrontal method will be used for numeric factorization.

StandardMethod

```
public Imsl.Math.ComplexSparseCholesky.NumericFactorization StandardMethod
```

Description

Indicates that the method of George/Liu (1981) is used for numeric factorization.

ComplexSparseCholesky.NumericFactor Class

`public class Imsl.Math.ComplexSparseCholesky.NumericFactor`

Data structures and functions for the numeric Cholesky factor.

Used by `GetNumericFactor` and `SetNumericFactor` to hold the numeric Cholesky factorization of a matrix.

ComplexSparseCholesky.SymbolicFactor Class

`public class Imsl.Math.ComplexSparseCholesky.SymbolicFactor`

Data structures and functions for the symbolic Cholesky factor.

Used by `GetSymbolicFactor` and `SetSymbolicFactor` to hold the symbolic Cholesky factorization of a matrix.

QR Class

`public class Imsl.Math.QR`

QR Decomposition of a matrix.

Class `QR` computes the *QR* decomposition of a matrix using Householder transformations. It is based on the LINPACK routine `SQRDC`; see Dongarra et al. (1979).

`QR` determines an orthogonal matrix *Q*, a permutation matrix *P*, and an upper trapezoidal matrix *R* with diagonal elements of nonincreasing magnitude, such that *AP* = *QR*. The Householder transformation for column *k* is of the form

$$I - \frac{u_k u_k^T}{P_k}$$

for $k = 1, 2, \ldots,$ min(number of rows of *A*, number of columns of *A*), where u has zeros in the first *k - 1* positions. The matrix *Q* is not produced directly by `QR`. Instead the information needed to reconstruct the Householder transformations is saved. If the matrix *Q* is needed explicitly, use the `Q` property. This method accumulates *Q* from its factored form.

Before the decomposition is computed, initial columns are moved to the beginning of the array A and the final columns to the end. Both initial and final columns are frozen in place during the computation. Only free columns are pivoted. Pivoting is done on the free columns of largest reduced norm.

Property

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Constructor

QR

```
public QR(double[,] a)
```

Description

Constructs the QR decomposition of a matrix with elements of type `double`.

Parameter

`a` – A `double` matrix to be factored.

Methods

GetPermute

```
public int[] GetPermute()
```

Description

Returns an `int` array containing information about the permutation of the elements of the matrix during pivoting.

Returns

The *k*-th element contains the index of the column of the matrix that has been interchanged into the *k*-th column.

GetQ

```
public double[,] GetQ()
```

Description

The orthogonal or unitary matrix Q.

Returns

A `double` matrix containing the accumulated orthogonal matrix Q from the QR decomposition.

GetR

```
public double[,] GetR()
```

Description

The upper trapezoidal matrix R.

Returns

The upper trapezoidal `double` matrix R of the QR decomposition.

GetRank

```
public int GetRank()
```

Description

Returns the rank of the matrix used to construct this instance.

Returns

An `int` specifying the rank of the matrix used to construct this instance.

GetRank

```
public int GetRank(double tolerance)
```

Description

Returns the rank of the matrix given an input tolerance.

Parameter

`tolerance` – A `double` scalar value used in determining the rank of the matrix.

Returns

An `int` specifying the rank of the matrix.

Solve

```
public double[] Solve(double[] b)
```

Description

Returns the solution to the least-squares problem Ax = b.

Parameter

`b` – A `double` array to be manipulated.

Returns

A `double` array containing the solution vector to Ax = b with components corresponding to the unused columns set to zero.

Exception

`Imsl.Math.SingularMatrixException` is thrown when the upper triangular matrix R resulting from the QR factorization is singular

Solve

```
public double[] Solve(double[] b, double tol)
```

Description

Returns the solution to the least-squares problem Ax = b using an input tolerance.

Parameters

`b` – A `double` array to be manipulated.

`tol` – A `double` scalar value used in determining the rank of A.

Returns

A `double` array containing the solution vector to Ax = b with components corresponding to the unused columns set to zero.

Exception

`Imsl.Math.SingularMatrixException` is thrown when the upper triangular matrix R resulting from the QR factorization is singular

Example: QR Factorization of a Matrix

The QR Factorization of a Matrix is performed. A linear system is then solved using the factorization. The rank of the input matrix is also computed.

```
using System;
using Imsl.Math;

public class QREx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {
            {1, 2, 4},
            {1, 4, 16},
            {1, 6, 36},
            {1, 8, 64}
        };
        double[] b = new double[]{16.99, 57.01, 120.99, 209.01};
```

```
        // Compute the QR factorization of A
        QR qr = new QR(a);

        // Solve Ax = b
        double[] x = qr.Solve(b);
        new PrintMatrix("x").Print(x);

        // Print Q and R.
        new PrintMatrix("Q").Print(qr.GetQ());
        new PrintMatrix("R").Print(qr.GetR());

        // Find the rank of A.
        int rank = qr.GetRank();
        Console.Out.WriteLine("rank = " + rank);
    }
}
```

Output

```
          x
           0
0  0.990000000000019
1  2.00199999999999
2  3

                                        Q
            0                   1                   2                   3
0  -0.0531494003452735  -0.54217094609664    0.808223859120487  -0.22360679774998
1  -0.212597601381094   -0.657435635424271  -0.269407953040162   0.670820393249937
2  -0.478344603107461   -0.345794067982896  -0.449013255066938  -0.670820393249936
3  -0.850390405524374    0.392753756227487   0.269407953040163   0.223606797749979

                             R
           0                 1                   2
0  -75.2595508889071  -10.6298800690547   -1.5944820103582
1    0                 -2.64681879196785  -1.15264689327632
2    0                  0                  0.359210604053549
3    0                  0                  0

rank = 3
```

SVD Class

`public class Imsl.Math.SVD`

Singular Value Decomposition (SVD) of a rectangular matrix of type `double`.

`SVD` is based on the LINPACK routine `SSVDC`; see Dongarra et al. (1979).

Let *n* be the number of rows in *A* and let *p* be the number of columns in *A*. For any

n x p matrix A, there exists an n x n orthogonal matrix U and a p x p orthogonal matrix V such that

$$U^T AV = \begin{cases} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} & \text{if } n \geq p \\ [\Sigma \; 0] & \text{if } n \leq p \end{cases}$$

where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_m)$, and $m = \min(n, p)$. The scalars $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$ are called the *singular values* of A. The columns of U are called the *left singular vectors* of A. The columns of V are called the *right singular vectors* of A.

The estimated rank of A is the number of σ_k that is larger than a tolerance η. If τ is the parameter `tol` in the program, then

$$\eta = \begin{cases} \tau & \text{if } \tau > 0 \\ |\tau| \, \|A\|_\infty & \text{if } \tau < 0 \end{cases}$$

The Moore-Penrose generalized inverse of the matrix is computed by partitioning the matrices U, V and Σ as $U = (U_1, U_2)$, $V = (V_1, V_2)$ and $\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_k)$ where the "1" matrices are k by k. The Moore-Penrose generalized inverse is $V_1 \Sigma_1^{-1} U_1^T$.

Properties

Info

```
public int Info {get; }
```

Description

Returns the index of the first singular value for which the algorithm converged.

Property Value

Convergence was obtained for the `Info`, `Info`+1, ..., Min(nrows,ncols) singular values and their corresponding vectors. Here, nrows and ncols represent the number of rows and columns of the input matrix respectively.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Rank

```
public int Rank {get; }
```

Description

Returns the rank of the matrix used to construct this instance.

Property Value

An `int` scalar containing the rank of the matrix used to construct this instance. The estimated rank of the input matrix is the number of singular values which are larger than a tolerance.

Constructors

SVD

```
public SVD(double[,] a, double tol)
```

Description

Construct the singular value decomposition of a rectangular matrix with a given tolerance.

Parameters

`a` – A `double` matrix for which the singular value decomposition is to be computed.

`tol` – A `double` scalar containing the tolerance used to determine when a singular value is negligible.

Remarks

If `tol` is positive, then a singular value is considered negligible if the singular value is less than or equal to `tol`. If `tol` is negative, then a singular value is considered negligible if the singular value is less than or equal to the absolute value of the product of `tol` and the infinity norm of the input matrix. In the latter case, the absolute value of `tol` generally contains an estimate of the level of the relative error in the data.

Exception

`Imsl.Math.DidNotConvergeException` is thrown when the rank cannot be determined because convergence was not obtained for all singular values

SVD

```
public SVD(double[,] a)
```

Description

Construct the singular value decomposition of a rectangular matrix with default tolerance.

Parameter

a – A `double` matrix for which the singular value decomposition is to be computed.

Remarks

The tolerance used is 2.2204460492503e-14. This tolerance is used to determine rank. A singular value is considered negligible if the singular value is less than or equal to this tolerance.

Methods

GetS

```
public double[] GetS()
```

Description

Returns the singular values.

Returns

A `double` array containing the singular values of the matrix.

GetU

```
public double[,] GetU()
```

Description

Returns the left singular vectors.

Returns

A `double` matrix containing the left singular vectors.

GetV

```
public double[,] GetV()
```

Description

Returns the right singular vectors.

Returns

A `double` matrix containing the right singular vectors

Inverse

```
public double[,] Inverse()
```

Description

Compute the Moore-Penrose generalized inverse of a real matrix.

Returns

A `double` matrix containing the generalized inverse of the matrix used to construct this instance.

Example: Singular Value Decomposition of a Matrix

The singular value decomposition of a matrix is performed. The rank of the matrix is also computed.

```
using System;
using Imsl.Math;

public class SVDEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {
            {1, 2, 1, 4},
            {3, 2, 1, 3},
            {4, 3, 1, 4},
            {2, 1, 3, 1},
            {1, 5, 2, 2},
            {1, 2, 2, 3}
        };

        // Compute the SVD factorization of A
        SVD svd = new SVD(a);

        // Print U, S and V.
        new PrintMatrix("U").SetPageWidth(80).Print(svd.GetU());
        new PrintMatrix("S").SetPageWidth(80).Print(svd.GetS());
        new PrintMatrix("V").SetPageWidth(80).Print(svd.GetV());

        // Find the rank of A.
        int rank = svd.Rank;
        Console.Out.WriteLine("rank = " + rank);
    }
}
```

Output

```
                              U
           0                  1                  2
0  -0.380475586320569   0.119670992640587   0.439082824383239
1  -0.403753713172442   0.345110837105607  -0.0565761852901658
2  -0.545120486248343   0.429264893493195   0.0513926928086694
3  -0.264784294004146  -0.0683195253271513  -0.883860867430429
4  -0.446310112301556  -0.816827623278282   0.141899675060401
5  -0.354628656614145  -0.102147399162125  -0.00431844397986985

           3                  4                  5
0  -0.565399585908374   0.0243115161463761  -0.57258686109915
1   0.214775576522681   0.80890058872827     0.11929741721493
2   0.432144162809737  -0.572327648171096    0.0403309248707933
3  -0.215253698182974  -0.0625209225900579  -0.30621669907105
4   0.321269584269887   0.0621337820958098  -0.0799352679998222
5  -0.545800221853259  -0.0987946265624981   0.745739576113111

          S
           0
```

```
0  11.4850179115597
1   3.2697512144125
2   2.65335616200783
3   2.08872967244092

                              V
          0                   1                   2
0  -0.444294128842354   0.555531257799947  -0.435378966673942
1  -0.558067238190387  -0.654298740112323   0.277456900458814
2  -0.32438610320628   -0.351360645592513  -0.732099533429598
3  -0.621238553843379   0.37393031038343    0.444401954223745

          3
0   0.55175438744187
1   0.428336065179864
2  -0.485128463324533
3  -0.526066236587424

rank = 4
```

GenMinRes Class

```
public class Imsl.Math.GenMinRes
```

Linear system solver using the restarted Generalized Minimum Residual (GMRES) method.

`GenMinRes` implements restarted GMRES to generate an approximate solution to $Ax = b$. It is based on GMRES by Homer Walker (1988).

The GMRES method begins with an approximate solution x_0 and an initial residual $r_0 = b - Ax_0$. At iteration m, a correction z_m is determined in the Krylov subspace

$$\kappa_m(v) = \text{span}(v, \text{A}v, \ldots, \text{A}^{\text{m}-1}v)$$

$v = r_0$ which solves the least squares problem

$$\min_{z \in \kappa_m(r_0)} \|b - A(x_0 + z)\|_2$$

Then at iteration m, $x_m = x_0 + z_m$.

There are four distinct GMRES implementations, selectable through property `Method`. The first Gram-Schmidt implementation is essentially the original algorithm by Saad and Schultz (1986). The second Gram-Schmidt implementation, developed by Homer Walker and Lou Zhou, is simpler than the first implementation. The least squares problem is constructed in upper-triangular form and the residual vector updating at the end of a GMRES cycle is cheaper. The first Householder implementation is algorithm 2.2 of Walker (1988), but with more efficient correction accumulation at the end of each GMRES cycle. The second Householder implementation is algorithm 3.1 of Walker (1988). The products of Householder transformations are expanded as sums, allowing most work to be formulated as large scale matrix-vector operations.

The Gram-Schmidt implementations are less expensive than the Householder, the latter requiring about twice as many computations beyond the coefficient matrix/vector products. However, the Householder implementations may be more reliable near the limits of residual reduction. See Walker (1988) for details. Issues such as the cost of coefficient matrix/vector products, availability of effective preconditioners, and features of particular computing environments may serve to mitigate the extra expense of the Householder implementations.

Properties

Iterations

```
public int Iterations {get; }
```

Description

The actual number of GMRES iterations used.

Property Value

An `int` scalar representing the number of iterations used.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` specifying the maximum number of iterations allowed.

Remarks

By default, `MaxIterations` = 1000.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is less than or equal to 0.

MaxKrylovDim

```
public int MaxKrylovDim {get; set; }
```

Description

The maximum Krylov subspace dimension.

Property Value

An `int` scalar representing the maximum Krylov subspace dimension, that is, the maximum allowable number of GMRES iterations allowed before restarting.

Remarks

By default, `MaxKrylovDim` is set to the minimum of `n` and 20 where `n` is the order of the system to be solved.

Exception

`System.ArgumentException` is thrown if `MaxKrylovDim` is less than 1 or greater than `n`.

Method

```
public Imsl.Math.GenMinRes.ImplementationMethod Method {get; set; }
```

Description

The implementation method to be used.

Property Value

An `ImplementationMethod` value specifying the implementation method to be used.

Remarks

`ImplementationMethod`	*Method Used*
`FirstGramSchmidt`	Use the first Gram-Schmidt implementation. This is the default value used.
`SecondGramSchmidt`	Use the second Gram-Schmidt implementation.
`FirstHouseholder`	Use the first Householder implementation.
`SecondHouseholder`	Use the second Householder implementation.

By default, `Method` = `GenMinRes.ImplementationMethod.FirstGramSchmidt`.

PreconditionerSolves

```
public int PreconditionerSolves {get; }
```

Description

The total number of GMRES right preconditioner solves.

Property Value

An `int` representing the number of GMRES right preconditioner solves.

Products

```
public int Products {get; }
```

Description

The total number of GMRES matrix-vector products used.

Property Value

An `int` representing the number of GMRES matrix-vector products used.

RelativeError

```
public double RelativeError {get; set; }
```

Description

The stopping tolerance.

Property Value

A `double` scalar value specifying the stopping tolerance.

Remarks

The algorithm attempts to generate x such that $\|b - Ax\|_2 \leq t\|b\|_2$, where t = `RelativeError`. By default, `RelativeError` is set to 1.4901161193847656e-08.

Exception

`System.ArgumentException` is thrown if `RelativeError` is less than or equal to 0.0.

ResidualNorm

```
public double ResidualNorm {get; }
```

Description

The final residual norm, $\|b - Ax\|_2$.

Property Value

A `double` scalar value specifying the final residual norm.

ResidualUpdating

```
public Imsl.Math.GenMinRes.ResidualMethod ResidualUpdating {get; set; }
```

Description

The residual updating method to be used.

Property Value

A `ResidualMethod` value specifying the residual updating method to be used.

Remarks

`ResidualMethod`	*Updating Method Used*
`LinearAtRestartOnly`	Update by linear combination upon restarting only. This is the default value used.
`LinearAtRestartAndTermination`	Update by linear combination upon restarting and at termination.
`DirectAtRestartOnly`	Update by direct evaluation upon restarting only.
`DirectAtRestartAndTermination`	Update by direct evaluation upon restarting and at termination.

By default, `ResidualUpdating` is set to `LinearAtRestartOnly`.

Constructor

GenMinRes

```
public GenMinRes(int n, Imsl.Math.GenMinRes.IFunction argF)
```

Description

GMRES linear system solver constructor.

Parameters

`n` – An `int` scalar value which defines the order of the system to be solved.

`argF` – An `IFunction` that defines the user-supplied function which computes $z = Ap$. If `argF` implements `IPreconditioner` then right preconditioning is performed using this user supplied function. Otherwise, no preconditioning is performed. Note that `argF` can be used to act upon the coefficients of matrix A stored in different storage modes. See the examples.

Methods

GetGuess

```
public double[] GetGuess()
```

Description

Returns the initial guess of the solution.

Returns

A `double` array of length `n` containing the initial guess of the solution.

GetVectorProducts

```
public Imsl.Math.GenMinRes.IVectorProducts GetVectorProducts()
```

Description

Returns the user-supplied functions for the inner product and, optionally, the norm used in the Gram-Schmidt implementations.

Returns

An `IVectorProducts` that defines the user-supplied functions for the inner product and, optionally, the norm used in the Gram-Schmidt implementations.

SetGuess

```
public void SetGuess(double[] xGuess)
```

Description

Sets the initial guess of the solution.

Parameter

`xGuess` – A `double` array of length `n` containing the initial guess of the solution.

Remarks

By default the elements of this array are set to 0.0.

SetVectorProducts

```
public void SetVectorProducts(Imsl.Math.GenMinRes.IVectorProducts argP)
```

Description

Sets the user-supplied functions for the inner product and, optionally, the norm to be used in the Gram-Schmidt implementations.

Parameter

`argP` – An `IVectorProducts` specifying the user-defined function for the inner product and, optionally, the norm in the Gram-Schmidt implementations. If this member function is not called, the dot product will be used for the inner product and the L_2 norm will be used for the norm.

Solve

```
public double[] Solve(double[] b)
```

Description

Generate an approximate solution to $Ax = b$ using the Generalized Residual Method.

Parameter

`b` – A `double` array which defines the right-hand side of the linear system.

Returns

A `double` array containing the solution of the linear system.

Exception

`System.ArgumentException` is thrown if the length of `b` is not consistent with `n`.

Example 1: Solve a Small Linear System

A solution to a small linear system is found. The coefficient matrix is stored as a full matrix and no preconditioning is used. Typically, preconditioning is required to achieve convergence in a reasonable number of iterations.

```
using System;
using Imsl.Math;
using IMSLException = Imsl.IMSLException;

public class GenMinResEx1 : GenMinRes.IFunction
{
    private static double[,] a = {
                    {33.0, 16.0, 72.0},
                    {-24.0, -10.0, -57.0},
                    {18.0, -11.0, 7.0}
                };
    private static double[] b = {129.0, -96.0, 8.5};
    // If A were to be read in from some outside source the   //
    // code to read the matrix could reside in a constructor. //

    public void  Amultp(double[] p, double[] z)
    {
        double[] result;
        result = Matrix.Multiply(a, p);
        Array.Copy(result, 0, z, 0, z.Length);
    }
```

```
    public static void  Main(String[] args)
    {
        int n = 3;

        GenMinResEx1 atp = new GenMinResEx1();

        // Construct a GenMinRes object
        GenMinRes gnmnrs = new GenMinRes(n, atp);

        // Solve Ax = b
        new PrintMatrix("x").Print(gnmnrs.Solve(b));
    }
}
```

Output

```
          x
           0
0  0.999999999999999
1  1.5
2  1
```

Example 2: Solve a Small Linear System with User Supplied Inner Product

A solution to a small linear system is found. The coefficient matrix is stored as a full matrix and no preconditioning is used. Typically, preconditioning is required to achieve convergence in a reasonable number of iterations. The user supplies a function to compute the inner product and norm within the Gram-Schmidt implementation.

```
using System;
using Imsl.Math;
using IMSLException = Imsl.IMSLException;

public class GenMinResEx2 : GenMinRes.IFunction, GenMinRes.IVectorProducts
{
    private static double[,] a = {
                    {33.0, 16.0, 72.0},
                    {-24.0, -10.0, -57.0},
                    {18.0, -11.0, 7.0}
                                  };
    private static double[] b = {129.0, -96.0, 8.5};
    // If A were to be read in from some outside source the   //
    // code to read the matrix could reside in a constructor. //

    public void  Amultp(double[] p, double[] z)
    {
        double[] result;
        result = Matrix.Multiply(a, p);
        Array.Copy(result, 0, z, 0, z.Length);
    }
```

```
    public double Innerproduct(double[] x, double[] y)
    {
        int n = x.Length;
        double tmp = 0.0;
        for (int i = 0; i < n; i++)
        {
            tmp += x[i] * y[i];
        }
        return tmp;
    }

    public double Norm(double[] x)
    {
        int n = x.Length;
        double tmp = 0.0;
        for (int i = 0; i < n; i++)
        {
            tmp += x[i] * x[i];
        }
        return System.Math.Sqrt(tmp);
    }

    public static void  Main(String[] args)
    {
        int n = 3;

        GenMinResEx2 atp = new GenMinResEx2();

        // Construct a GenMinRes object
        GenMinRes gnmnrs = new GenMinRes(n, atp);
        gnmnrs.SetVectorProducts(atp);

        // Solve Ax = b
        new PrintMatrix("x").Print(gnmnrs.Solve(b));
    }
}
```

Output

```
          x
           0
0  0.999999999999999
1  1.5
2  1
```

Example 3: Solve a Small Linear System Stored in Sparse Form

A solution to a small linear system in which the coefficient matrix has been stored in `SparseMatrix` form is found. An initial guess of ones is set before solving the system.

```
using System;
```

```
using Imsl.Math;
public class GenMinResEx3 : GenMinRes.IFunction
{
    private static SparseMatrix A;
    private static double[] a = {6.0, 10.0, 15.0, -3.0, 10.0, -1.0,
                                 -1.0, -3.0, -5.0, 1.0, 10.0, -1.0,
                                 -2.0, -1.0, -2.0};
    private static int[] irow = {5, 1, 2, 1, 3, 3, 4, 4, 4, 4, 0, 5, 5, 1, 3};
    private static int[] jcol = {5, 1, 2, 2, 3, 4, 0, 5, 3, 4, 0, 0, 1, 3, 0};
    private static double[] b = {10.0, 7.0, 45.0, 33.0, -34.0, 31.0};
    private static double[] xguess = {1.0, 1.0, 1.0, 1.0, 1.0, 1.0};

    public void  Amultp(double[] p, double[] z)
    {
        double[] result;
        result = Imsl.Math.SparseMatrix.Multiply(A, p);
        Array.Copy(result, 0, z, 0, z.Length);
    }

    public static void  Main(String[] args)
    {
        int n = 6;

        A = new SparseMatrix(n, n);
        for (int i = 0; i < a.Length; i++)
        {
            A.Set(irow[i], jcol[i], a[i]);
        }

        GenMinResEx3 atp = new GenMinResEx3();

        // Construct a GenMinRes object
        GenMinRes gnmnrs = new GenMinRes(n, atp);
        gnmnrs.SetGuess(xguess);
        // Solve Ax = b
        new PrintMatrix("x").Print(gnmnrs.Solve(b));
    }
}
```

Output

```
  x
   0
0  1
1  2
2  3
3  4
4  5
5  6
```

Example 4: Solve a Small Linear System Stored in Sparse Form With Preconditioning

A solution to a small linear system in which the coefficient matrix has been stored in `SparseMatrix` form is found. An initial guess of ones is set before solving the system and preconditioning is used.

```
using System;
using Imsl.Math;
public class GenMinResEx4 : GenMinRes.IPreconditioner
{
    private static SparseMatrix A;
    private static double[] a = {6.0, 10.0, 15.0, -3.0, 10.0, -1.0,
                                 -1.0, -3.0, -5.0, 1.0, 10.0, -1.0,
                                 -2.0, -1.0, -2.0};
    private static double[] b = {10.0, 7.0, 45.0, 33.0, -34.0, 31.0};
    private static double[] xguess = {1.0, 1.0, 1.0, 1.0, 1.0, 1.0};
    private static double[] diagin = {0.1, 0.1, 0.066666666666667, 0.1,
                                      1.0, 0.1666666666666667};
    private static int[] irow = {5, 1, 2, 1, 3, 3, 4, 4, 4, 4, 0, 5, 5, 1, 3};
    private static int[] jcol = {5, 1, 2, 2, 3, 4, 0, 5, 3, 4, 0, 0, 1, 3, 0};

    public virtual void  Amultp(double[] p, double[] z)
    {
        double[] result;
        result = Imsl.Math.SparseMatrix.Multiply(A, p);
        Array.Copy(result, 0, z, 0, z.Length);
    }

    public void  Preconditioner(double[] r, double[] z)
    {
        int n = z.Length;
        for (int i = 0; i < n; i++)
        {
            z[i] = diagin[i] * r[i];
        }
    }

    public static void  Main(String[] args)
    {
        int n = 6;
        A = new SparseMatrix(n, n);
        for (int i = 0; i < a.Length; i++)
        {
            A.Set(irow[i], jcol[i], a[i]);
        }
        GenMinResEx4 atp = new GenMinResEx4();

        // Construct a GenMinRes object
        GenMinRes gnmnrs = new GenMinRes(n, atp);
        gnmnrs.SetGuess(xguess);
        // Solve Ax = b
        new PrintMatrix("x").Print(gnmnrs.Solve(b));
    }
}
```

Output

```
          x
          0
0  1
1  2
2  3
3  4
4  5.00000000000001
5  6
```

Example 5: The Second Householder Implementation

The coefficient matrix in this example corresponds to the five-point discretization of the 2-d Poisson equation with the Dirichlet boundary condition. Assuming the natural ordering of the unknowns, and moving all boundary terms to the right hand side, we obtain a block tridiagonal matrix. (Consider the tridiagonal matrix T which has the value 4.0 down the main diagonal and -1.0 along the upper and lower co-diagonals. Then the coefficient matrix is the block tridiagonal matrix consisting of T's down the main diagonal and -I along the upper and lower codiagonals where I is the identity matrix.) Discretizing on a 20 x 20 grid implies that the coefficient matrix is 400 x 400. In the solution, the second Householder implementation is selected and we choose to update the residual vector by direct evaluation.

```
using System;
using Imsl.Math;
public class GenMinResEx5 : GenMinRes.IFunction
{
    //Creates a new instance of GenMinResEx5
    public GenMinResEx5()
    {
    }
    public void  Amultp(double[] p, double[] z)
    {
        int n = z.Length;
        int k = (int) Math.Sqrt(n);
        // Multiply by diagonal blocks
        for (int i = 0; i < n; i++)
        {
            z[i] = 4.0 * p[i];
        }
        for (int i = 0; i < n - 2; i++)
        {
            z[i] = (- 1.0) * p[i + 1] + z[i];
        }
        for (int i = 0; i < n - 2; i++)
        {
            z[i + 1] = (- 1.0) * p[i] + z[i + 1];
        }
        // Correct for terms not properly in block diagonal
        for (int i = k - 1; i < n - k; i = i + k)
        {
            z[i] += p[i + 1];
            z[i + 1] += p[i];
        }
```

```
        // Do the super and subdiagonal blocks, the -I's
        for (int i = 0; i < n - k; i++)
        {
            z[i] = (- 1.0) * p[i + k] + z[i];
        }
        for (int i = 0; i < n - k; i++)
        {
            z[i + k] = (- 1.0) * p[i] + z[i + k];
        }
    }

    public static void  Main(String[] args)
    {
        int n = 400;
        double[] b = new double[n];
        double[] xguess = new double[n];

        GenMinResEx5 atp = new GenMinResEx5();

        // Construct a GenMinRes object
        GenMinRes gnmnrs = new GenMinRes(n, atp);
        // Set right hand side and initial guess to ones    */
        for (int i = 0; i < n; i++)
        {
            b[i] = 1.0;
            xguess[i] = 1.0;
        }
        gnmnrs.SetGuess(xguess);
        gnmnrs.Method =
         Imsl.Math.GenMinRes.ImplementationMethod.SecondHouseholder;
        gnmnrs.ResidualUpdating =
         Imsl.Math.GenMinRes.ResidualMethod.DirectAtRestartOnly;
        // Solve Ax = b
        gnmnrs.Solve(b);
        int iterations = gnmnrs.Iterations;
        Console.Out.WriteLine("The number of iterations used = " + iterations);
        double resnorm = gnmnrs.ResidualNorm;
        Console.Out.WriteLine("The final residual norm is " + resnorm);
    }
}
```

Output

```
The number of iterations used = 92
The final residual norm is 2.52648529541037E-07
```

Example 6: The Second Householder Implementation With Preconditioning

The coefficient matrix in this example corresponds to the five-point discretization of the 2-d Poisson equation with the Dirichlet boundary condition. Assuming the natural ordering of the unknowns, and moving all boundary terms to the right hand side, we obtain a block tridiagonal matrix. (Consider the tridiagonal matrix T which has the value 4.0 down the main diagonal and -1.0 along the upper and lower

co-diagonals. Then the coefficient matrix is the block tridiagonal matrix consisting of T's down the main diagonal and -I along the upper and lower codiagonals where I is the identity matrix.) Discretizing on a 20 x 20 grid implies that the coefficient matrix is 400 x 400. In the solution, the second Householder implementation is selected and we choose to update the residual vector by direct evaluation. Preconditioning is used with the preconditioning matrix being a diagonal matrix with 4.0 down the main diagonal and -1.0 along the upper and lower co-diagonals. This `Preconditioner` method solves this tridiagonal matrix.

```
using System;
using Imsl.Math;
public class GenMinResEx6 : GenMinRes.IPreconditioner
{
    private double[] precondA, precondB, precondC;

    // Creates a new instance of GenMinResEx6
    public GenMinResEx6(int n)
    {
        precondA = new double[n];
        precondB = new double[n];
        precondC = new double[n];
        // Define the preconditioning matrix
        for (int i = 0; i < n; i++)
        {
            precondA[i] = 4.0;
            precondB[i] = - 1.0;
            precondC[i] = - 1.0;
        }
    }

    public void  Amultp(double[] p, double[] z)
    {
        int m = z.Length;
        int n = (int) Math.Sqrt(m);
        // Multiply by diagonal blocks
        for (int i = 0; i < m; i++)
        {
            z[i] = 4.0 * p[i];
        }
        for (int i = 0; i < m - 2; i++)
        {
            z[i] -= p[i + 1];
        }
        for (int i = 0; i < m - 2; i++)
        {
            z[i + 1] -= p[i];
        }
        // Correct for terms not properly in block diagonal
        for (int i = n - 1; i < m - n; i = i + n)
        {
            z[i] += p[i + 1];
            z[i + 1] += p[i];
        }
        // Do the super and subdiagonal blocks, the -I's
        for (int i = 0; i < m - n; i++)
```

```
    {
        z[i] -= p[i + n];
    }
    for (int i = 0; i < m - n; i++)
    {
        z[i + n] -= p[i];
    }
}

/// Solve the tridiagonal preconditioning matrix problem for z.
public void  Preconditioner(double[] r, double[] z)
{
    int n = z.Length;
    double[] w = new double[n];
    double[] v = new double[n];
    double[] u = new double[n];
    w[0] = precondA[0];
    v[0] = precondC[0] / w[0];
    u[0] = r[0] / w[0];
    for (int i = 1; i < n; i++)
    {
        w[i] = precondA[i] - precondB[i] * v[i - 1];
        v[i] = precondC[i] / w[i];
        u[i] = (r[i] - precondB[i] * u[i - 1]) / w[i];
    }
    z[n - 1] = u[n - 1];
    for (int j = n - 2; j >= 0; j--)
    {
        z[j] = u[j] - v[j] * z[j + 1];
    }
}

public static void  Main(String[] args)
{
    int n = 400;
    double[] b = new double[n];
    double[] xguess = new double[n];

    GenMinResEx6 atp = new GenMinResEx6(n);

    // Construct a GenMinRes object
    GenMinRes gnmnrs = new GenMinRes(n, atp);
    // Set right hand side and initial guess to ones
    for (int i = 0; i < n; i++)
    {
        b[i] = 1.0;
        xguess[i] = 1.0;
    }
    gnmnrs.SetGuess(xguess);
    gnmnrs.Method = Imsl.Math.GenMinRes.ImplementationMethod.SecondHouseholder;
    gnmnrs.ResidualUpdating = Imsl.Math.GenMinRes.ResidualMethod.
        DirectAtRestartOnly;
    // Solve Ax = b
    gnmnrs.Solve(b);
    int iterations = gnmnrs.Iterations;
    Console.Out.WriteLine("The number of iterations used = " + iterations);
```

```
        double resnorm = gnmnrs.ResidualNorm;
        Console.Out.WriteLine("The final residual norm is " + resnorm);
    }
}
```

Output

```
The number of iterations used = 60
The final residual norm is 2.84141491724154E-07
```

GenMinRes.ImplementationMethod Enumeration

`public enumeration Imsl.Math.GenMinRes.ImplementationMethod`

Implementation methods.

Fields

FirstGramSchmidt

`public Imsl.Math.GenMinRes.ImplementationMethod FirstGramSchmidt`

Description

Indicates the first Gram-Schmidt implementation method is to be used.

FirstHouseholder

`public Imsl.Math.GenMinRes.ImplementationMethod FirstHouseholder`

Description

Indicates the first Householder implementation method is to be used.

SecondGramSchmidt

`public Imsl.Math.GenMinRes.ImplementationMethod SecondGramSchmidt`

Description

Indicates the second Gram-Schmidt implementation method is to be used.

SecondHouseholder

`public Imsl.Math.GenMinRes.ImplementationMethod SecondHouseholder`

Description

Indicates the second Householder implementation method is to be used.

GenMinRes.ResidualMethod Enumeration

```
public enumeration Imsl.Math.GenMinRes.ResidualMethod
```

Residual updating methods.

Fields

DirectAtRestartAndTermination

```
public Imsl.Math.GenMinRes.ResidualMethod DirectAtRestartAndTermination
```

Description

Indicates residual updating is to be done by direct evaluation upon restarting and at termination.

DirectAtRestartOnly

```
public Imsl.Math.GenMinRes.ResidualMethod DirectAtRestartOnly
```

Description

Indicates residual updating is to be done by direct evaluation upon restarting only.

LinearAtRestartAndTermination

```
public Imsl.Math.GenMinRes.ResidualMethod LinearAtRestartAndTermination
```

Description

Indicates residual updating is to be done by linear combination upon restarting and at termination.

LinearAtRestartOnly

```
public Imsl.Math.GenMinRes.ResidualMethod LinearAtRestartOnly
```

Description

Indicates residual updating is to be done by linear combination upon restarting only.

GenMinRes.IFunction Interface

```
public interface Imsl.Math.GenMinRes.IFunction
```

Public interface for the user supplied function to `GenMinRes`.

Method

Amultp

```
abstract public void Amultp(double[] p, double[] z)
```

Description

Used to compute $z = Ap$ where `A` is the matrix of coefficients to solve and `p` and `z` are arrays of length `n`, the order of matrix `A`.

Parameters

`p` – An input `double` array of length `n` generated during the implementation of the `Solve` method.

`z` – An output `double` array of length `n`.

GenMinRes.IPreconditioner Interface

```
public interface Imsl.Math.GenMinRes.IPreconditioner :
Imsl.Math.GenMinRes.IFunction
```

Public interface for the user supplied function to `GenMinRes` used for preconditioning.

Method

Preconditioner

```
abstract public void Preconditioner(double[] r, double[] z)
```

Description

Used to compute $z = M^{-1}r$ where M is the preconditioning matrix and r and z are arrays of length `n`, the order of matrix M.

Parameters

`r` – An input `double` array of length `n` generated during the implementation of the `Solve` method.

`z` – An output `double` array of length `n`.

GenMinRes.IVectorProducts Interface

```
public interface Imsl.Math.GenMinRes.IVectorProducts
```

Public interface for the user supplied function to the `GenMinRes` object used for the inner product when the Gram-Schmidt implementation is used.

Method

Innerproduct

```
abstract public double Innerproduct(double[] x, double[] y)
```

Description

Used to compute the inner product of two vectors for the Gram-Schmidt implementation.

Parameters

`x` – The first input `double` vector which is to take part in the inner product.

`y` – The second input `double` vector which is to take part in the inner product.

Returns

A `double`, the value of the inner product of `x` and `y`.

Remarks

If this function is not implemented, the dot product is used for the inner product.

GenMinRes.INorm Interface

```
public interface Imsl.Math.GenMinRes.INorm :
Imsl.Math.GenMinRes.IVectorProducts
```

Public interface for the user supplied function to the `GenMinRes` object used for the norm $||X||$ when the Gram-Schmidt implementation is used.

Method

Norm

```
abstract public double Norm(double[] x)
```

Description

Used to compute the norm $||X||$ in the Gram-Schmidt implementation.

Parameter

`x` – An input `double` vector for which the norm will be computed.

Returns

A `double`, the value of the norm of `x`.

Remarks

If this function is not implemented, the L_2 norm is used.

ConjugateGradient Class

`public class Imsl.Math.ConjugateGradient`

Solves a real symmetric definite linear system using the conjugate gradient method with optional preconditioning.

Class `ConjugateGradient` solves the symmetric positive or negative definite linear system $Ax = b$ using the conjugate gradient method with optional preconditioning. This method is described in detail by Golub and Van Loan (1983, Chapter 10), and in Hageman and Young (1981, Chapter 7).

The preconditioning matrix M is a matrix that approximates A, and for which the linear system $Mz=r$ is easy to solve. These two properties are in conflict; balancing them is a topic of current research. If no preconditioning matrix is specified, M is set to the identity, i.e. $M = I$.

The number of iterations needed depends on the matrix and the error tolerance. As a rough guide,

$$\text{itmax} = \sqrt{n} \text{ for } n \gg 1,$$

where n is the order of matrix A. See the references for details.

Let M be the preconditioning matrix, let b,p,r,x and z be vectors and let τ be the desired relative error. Then the algorithm used is as follows:

- $\lambda = -1$
- $p_0 = x_0$
- $r_1 = b - Ap_0$
- for $k = 1, \ldots,$ `itmax`
  - $z_k = M^{-1} r_k$
  - if $k = 1$ then
    - $\beta_k = 1$
    - $p_k = z_k$
  - else
    - $\beta_k = (z_k^T r_k)/(z_{k-1}^T r_{k-1})$
    - $p_k = z_k + \beta_k p_{k-1}$

  - endif
  - $\alpha_k = (r_k^T z_k)/(p_k^T A p_k)$
  - $x_k = x_{k-1} + \alpha_k p_k$
  - $r_{k+1} = r_k - \alpha_k A p_k$
  - if $(\|Ap_k\|_2 \leq \tau(1-\lambda)\|x_k\|_2)$ then
    * recompute λ
    * if $(\|Ap_k\|_2 \leq \tau(1-\lambda)\|x_k\|_2)$ exit
  - endif
- `endfor`

Here, λ is an estimate of $\lambda_{\max}(\Gamma)$, the largest eigenvalue of the iteration matrix $\Gamma = I - M^{-1}A$. The stopping criterion is based on the result (Hageman and Young 1981, pp. 148-151)

$$\frac{\|x_k - x\|_M}{\|x\|_M} \leq \left(\frac{1}{1-\lambda_{\max}(\Gamma)}\right)\left(\frac{\|z_k\|_M}{\|x_k\|_M}\right),$$

where

$$\|x\|_M^2 = x^T M x.$$

It is also known that

$$\lambda_{\max}(T_1) \leq \lambda_{\max}(T_2) \leq \ldots \leq \lambda_{\max}(\Gamma) < 1,$$

where the T_l are the symmetric, tridiagonal matrices

$$T_l = \begin{bmatrix} \mu_1 & \omega_2 & & & \\ \omega_2 & \mu_2 & \omega_3 & & \\ & \omega_3 & \mu_3 & \ddots & \\ & & \ddots & \ddots & \omega_l \\ & & & \omega_l & \mu_l \end{bmatrix}$$

with $\mu_1 = 1 - 1/\alpha_1$ and, for $k = 2, \ldots, l$,

$$\mu_k = 1 - \beta_k/\alpha_{k-1} - 1/\alpha_k \quad \text{and} \quad \omega_k = \sqrt{\beta_k}/\alpha_{k-1}.$$

Usually, the eigenvalue computation is needed for only a few of the iterations.

Properties

Iterations

`public int Iterations {get; }`

Description

The number of iterations needed by the conjugate gradient algorithm.

Property Value

An `int` value indicating the number of iterations needed.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` value specifying the maximum number of iterations allowed.

Remarks

By default, `MaxIterations` = $\max(1000, \sqrt{n})$.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is less than or equal to 0.

RelativeError

```
public double RelativeError {get; set; }
```

Description

The relative error used for stopping the algorithm.

Property Value

A `double` specifying the relative error.

Remarks

By default, `RelativeError` = 1.49e-08, the square root of the precision.

Exception

`System.ArgumentException` is thrown if `RelativeError` is less than 0.

Constructor

ConjugateGradient

```
public ConjugateGradient(int n, Imsl.Math.ConjugateGradient.IFunction argF)
```

Description

Conjugate gradient constructor.

Parameters

`n` – An `int` scalar value defining the order of the matrix.

`argF` – An `IFunction` that defines the user-supplied function which computes $z = Ap$. If `argF` implements `IPreconditioner` then right preconditioning is performed using this user supplied function. Otherwise, no preconditioning is performed. Note that `argF` can be used to act upon the coefficients of matrix A stored in different storage modes.

Methods

GetJacobi

```
public double[] GetJacobi()
```

Description

Returns the Jacobi preconditioning matrix.

Returns

a `double` vector `diagonal` containing the diagonal of the Jacobi preconditioner M, that is, `diagonal[i]`$=A_{i,i}$, A the input matrix.

SetJacobi

```
public void SetJacobi(double[] diagonal)
```

Description

Defines a Jacobi preconditioner as the preconditioning matrix, that is, M is the diagonal of A.

Parameter

`diagonal` – A `double` vector containing the diagonal of the Jacobi preconditioner M, that is, `diagonal[i]`$=A_{i,i}$, A the input matrix.

Exception

`System.ArgumentException` is thrown if the length of vector `diagonal` is not equal to the order `n` of input matrix A.

Solve

```
public double[] Solve(double[] b)
```

Description

Solves a real symmetric positive or negative definite system $Ax = b$ using a conjugate gradient method with or without preconditioning.

Parameter

`b` – A `double` vector of length `n` containing the right-hand side.

Returns

A `double` vector of length `n` containing the approximate solution to the linear system.

Exceptions

`System.ArgumentException` is thrown if the length of `b` is not consistent with the order `n` of `A`.

`Imsl.Math.SingularPreconditionMatrixException` is thrown if the preconditioning matrix is singular.

`Imsl.Math.NotDefinitePreconditionMatrixException` is thrown if the preconditioning matrix is not definite.

`Imsl.Math.SingularMatrixException` is thrown if input matrix *A* is singular.

`Imsl.Math.NotDefiniteAMatrixException` is thrown if matrix *A* is not definite.

`Imsl.Math.NoConvergenceException` is thrown if the algorithm is not convergent within `MaxIterations` iterations.

`Imsl.Math.NotDefiniteJacobiPreconditionerException` is thrown if the Jacobi preconditioner is not definite.

Conjugate Gradient Example 1:

The solution to a positive definite linear system is found. The coefficient matrix is stored as a full matrix.

```
using System;
using Imsl.Math;

public class ConjugateGradientEx1 : ConjugateGradient.IFunction
{
    private double[,] a = {
                {1.0, - 3.0, 2.0},
                {- 3.0, 10.0, - 5.0},
                {2.0, - 5.0, 6.0} };

    public void  Amultp(double[] p, double[] z)
    {

        double[] w = Matrix.Multiply(a, p);
        Array.Copy(w, 0, z, 0, z.Length);
    }

    public static void  Main(String[] args)
    {
        int n = 3;
        double[] b = {27.0, -78.0, 64.0};
        double[] solution = null;

        ConjugateGradientEx1 atp = new ConjugateGradientEx1();

        // Construct Cg object
        ConjugateGradient cg = new ConjugateGradient(n, atp);

        // Solve Ax=b
        solution = cg.Solve(b);
        new PrintMatrix("Solution").Print(solution);
    }
}
```

Output

```
       Solution
           0
0   1.00000000000116
1  -3.99999999999972
2   6.99999999999984
```

Conjugate Gradient Example 2:

In this example, two different preconditioners are used to find the solution of a sparse positive definite linear system which occurs in a finite difference solution of Laplace's equation on a regular $c \times c$ grid, $c = 50$. The matrix is $A = E(c^2, c)$. For the first solution, Jacobi preconditioning with preconditioner $M = \text{diag}(A)$ is used. The required iteration number and maximum absolute error are printed. Next, a more complicated preconditioning matrix, consisting of the symmetric tridiagonal part of A, is used. Again, the iteration number and the maximum absolute error are printed. The iteration number is substantially reduced.

```
using System;
using Imsl.Math;

public class ConjugateGradientEx2 : ConjugateGradient.IPreconditioner
{
    private SparseMatrix A;
    private SparseCholesky M;

    public ConjugateGradientEx2(int n, int c)
    {

        // Create matrix E(n,c), n>1, 1<c<n-1
        // See Osterby and Zlatev(1982), pp. 7-8
        A = new SparseMatrix(n, n);
        for (int j = 0; j < n; j++)
        {
            if (j - c >= 0)
                A.Set(j, j - c, - 1.0);
            if (j - 1 >= 0)
                A.Set(j, j - 1, - 1.0);
                A.Set(j, j, 4.0);
            if (j + 1 < n)
                A.Set(j, j + 1, - 1.0);
            if (j + c < n)
                A.Set(j, j + c, - 1.0);
        }

        // Create and factor preconditioning matrix
        SparseMatrix C = new SparseMatrix(n, n);
        for (int j = 0; j < n; j++)
        {
            if (j - 1 >= 0)
                C.Set(j, j - 1, - 1.0);
            C.Set(j, j, 4.0);
            if (j + 1 < n)
```

```
            C.Set(j, j + 1, - 1.0);
        }
        M = new SparseCholesky(C);
        M.FactorSymbolically();
        M.FactorNumerically();
    }

    public void Amultp(double[] p, double[] z)
    {
        double[] w = A.Multiply(p);
        Array.Copy(w, 0, z, 0, w.Length);
    }

    public void Preconditioner(double[] r, double[] z)
    {
            double[] w = M.Solve(r);
            Array.Copy(w, 0, z, 0, w.Length);
    }

    public static void  Main(String[] args)
    {
        int n = 2500;
        int c = 50;

        ConjugateGradientEx2 atp = new ConjugateGradientEx2(n, c);

        // Set a predetermined answer and diagonal
        double[] expected = new double[n];
        double[] diagonal = new double[n];
        for (int i = 0; i < n; i++)
        {
            expected[i] = (double) (i % 5);
            diagonal[i] = 4.0;
        }

        // Get right-hand side
        double[] b = new double[n];
        atp.Amultp(expected, b);

        // Solve system with Jacobi preconditioning
        ConjugateGradient cgJacobi = new ConjugateGradient(n, atp);
        cgJacobi.SetJacobi(diagonal);
        double[] solution = cgJacobi.Solve(b);

        // Compute inf-norm of computed solution - exact solution, print results
        double norm = 0.0;
        for (int i = 0; i < n; i++)
        {
            norm = Math.Max(norm, Math.Abs(solution[i] - expected[i]));
        }
        Console.Out.WriteLine("Jacobi preconditioning");
        Console.Out.WriteLine("Iterations= " + cgJacobi.Iterations + ", norm= "
         + norm);
        Console.Out.WriteLine();
```

```
        /*
        * Solve the same system, with Cholesky preconditioner
        */
        ConjugateGradient cgCholesky = new ConjugateGradient(n, atp);
        solution = cgCholesky.Solve(b);

        norm = 0.0;
        for (int i = 0; i < n; i++)
        {
            norm = Math.Max(norm, Math.Abs(solution[i] - expected[i]));
        }
        Console.Out.WriteLine("More general preconditioning");
        Console.Out.WriteLine("Iterations= " + cgCholesky.Iterations + ", norm= "
         + norm);
    }
}
```

Output

```
Jacobi preconditioning
Iterations= 187, norm= 4.46344072813076E-10

More general preconditioning
Iterations= 127, norm= 5.13725062489812E-10
```

ConjugateGradient.IFunction Interface

`public interface Imsl.Math.ConjugateGradient.IFunction`

Public interface for the user supplied function to `ConjugateGradient`.

Method

Amultp

`abstract public void Amultp(double[] p, double[] z)`

Description

A user-supplied function which computes $z=Ap$.

Parameters

`p` – An input `double` vector of length dimension of A.

`z` – An output `double` vector containing the matrix-vector product Ap.

ConjugateGradient.IPreconditioner Interface

```
public interface Imsl.Math.ConjugateGradient.IPreconditioner :
Imsl.Math.ConjugateGradient.IFunction
```

Public interface for the user supplied function to `ConjugateGradient` used for preconditioning.

Method

Preconditioner

```
abstract public void Preconditioner(double[] r, double[] z)
```

Description

Used to compute $z = M^{-1}r$ where M is the preconditioning matrix and r and z are arrays of length `n`, the order of matrix M.

Parameters

`r` – An input `double` array of length `n` generated during the implementation of the `Solve` method.

`z` – An output `double` array of length `n`.

Chapter 2: Eigensystem Analysis

Types

Usage Notes

An ordinary linear eigensystem problem is represented by the equation $Ax = \lambda x$ where A denotes an n x n matrix. The value λ is an *eigenvalue* and $x \neq 0$ is the corresponding *eigenvector*. The eigenvector is determined up to a scalar factor. In all functions, we have chosen this factor so that x has Euclidean length one, and the component of x of largest magnitude is positive. If x is a complex vector, this component of largest magnitude is scaled to be real and positive. The entry where this component occurs can be arbitrary for eigenvectors having nonunique maximum magnitude values.

Error Analysis and Accuracy

Except in special cases, functions will not return the exact eigenvalue-eigenvector pair for the ordinary eigenvalue problem $Ax = \lambda x$. Typically, the computed pair

$$\tilde{x},\ \tilde{\lambda}$$

is an exact eigenvector-eigenvalue pair for a "nearby" matrix A + E. Information about E is known only in terms of bounds of the form $\|E\|_2 \leq f(n)\|A\|_2 \varepsilon$. The value of *f(n)* depends on the algorithm, but is typically a small fractional power of n. The parameter ε is the machine precision. By a theorem due to Bauer and Fike (see Golub and Van Loan 1989, p. 342),

$$\min\left|\tilde{\lambda} - \lambda\right| \leq \kappa(X)\|E\|_2 \quad \text{for all}\, \lambda\, \text{in}\, \sigma(A)$$

where $\sigma(A)$ is the set of all eigenvalues of A (called the *spectrum* of A), X is the matrix of eigenvectors, $\|\cdot\|_2$ is Euclidean length, and $\kappa(X)$ is the condition number of X defined as $\kappa(X) = \|X\|_2 \left\|X^{-1}\right\|_2$. If A is a real symmetric or complex Hermitian matrix, then its eigenvector matrix X is respectively orthogonal or unitary. For these matrices, $\kappa(X) = 1$.

The accuracy of the computed eigenvalues

$$\tilde{\lambda}_j$$

and eigenvectors

$$\tilde{x}_j$$

can be checked by computing their performance index τ. The performance index is defined to be

$$\tau = \max_{1 \le j \le n} \frac{\left\| A\tilde{x}_j - \tilde{\lambda}_j \tilde{x}_j \right\|_2}{n\varepsilon \|A\|_2 \left\| \tilde{x}_j \right\|_2}$$

where ε is again the machine precision.

The performance index τ is related to the error analysis because

$$\left\| E\tilde{x}_j \right\|_2 = \left\| A\tilde{x}_j - \tilde{\lambda}_j \tilde{x}_j \right\|_2$$

where E is the "nearby" matrix discussed above.

While the exact value of τ is precision and data dependent, the performance of an eigensystem analysis function is defined as excellent if $\tau < 1$, good if $1 \le \tau \le 100$, and poor if $\tau > 100$. This is an arbitrary definition, but large values of τ can serve as a warning that there is a significant error in the calculation.

If the condition number $\kappa(X)$ of the eigenvector matrix X is large, there can be large errors in the eigenvalues even if τ is small. In particular, it is often difficult to recognize near multiple eigenvalues or unstable mathematical problems from numerical results. This facet of the eigenvalue problem is often difficult for users to understand. Suppose the accuracy of an individual eigenvalue is desired. This can be answered approximately by computing the *condition number of an individual eigenvalue*(see Golub and Van Loan 1989, pp. 344-345). For matrices A, such that the computed array of normalized eigenvectors X is invertible, the condition number of λ_i is

$$\kappa_j = \left\| e_j^T X^{-1} \right\|,$$

the Euclidean length of the j-th row of X^{-1}. Users can choose to compute this matrix using the class LU in "Linear Systems." An approximate bound for the accuracy of a computed eigenvalue is then given by $\kappa_j \varepsilon \|A\|$. To compute an approximate bound for the relative accuracy of an eigenvalue, divide this bound by $\left|\lambda_j\right|$.

Eigen Class

```
public class Imsl.Math.Eigen
```

Collection of Eigen System functions.

`Eigen` computes the eigenvalues and eigenvectors of a real matrix. The matrix is first balanced. Orthogonal similarity transformations are used to reduce the balanced matrix to a real upper Hessenberg matrix. The implicit double-shifted QR algorithm is used to compute the eigenvalues and eigenvectors of this Hessenberg matrix. The eigenvectors are normalized such that each has Euclidean length of value one. The largest component is real and positive.

The balancing routine is based on the EISPACK routine `BALANC`. The reduction routine is based on the EISPACK routines `ORTHES` and `ORTRAN`. The QR algorithm routine is based on the EISPACK routine `HQR2`. See Smith et al. (1976) for the EISPACK routines. Further details, some timing data, and credits are given in Hanson et al. (1990).

While the exact value of the performance index, τ, is highly machine dependent, the performance of `Eigen` is considered excellent if $\tau < 1$, good if $1 \leq \tau \leq 100$, and poor if $\tau > 100$.

The performance index was first developed by the EISPACK project at Argonne National Laboratory; see Smith et al. (1976, pages 124-125).

Properties

MaxIterations

```
virtual public int MaxIterations {get; set; }
```

Description

The maximum number of iterations.

Property Value

An `int` containing the maximum number of iterations.

Default: `MaxIterations` = 50.

Remarks

The maximum number of iterations must be greater than 0.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`,

`Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Constructor

Eigen

```
public Eigen()
```

Description

Constructor for `Eigen`.

Methods

GetValues

```
public Imsl.Math.Complex[] GetValues()
```

Description

Returns the eigenvalues of a matrix of type `double`.

Returns

A `Complex` array containing the eigenvalues of this matrix in descending order.

GetVectors

```
public Imsl.Math.Complex[,] GetVectors()
```

Description

Returns the eigenvectors.

Returns

A `Complex` matrix containing the eigenvectors. The eigenvector corresponding to the j-th eigenvalue is stored in the j-th column. Each vector is normalized to have Euclidean length one.

PerformanceIndex

```
public double PerformanceIndex(double[,] a)
```

Description

Returns the performance index of a real eigensystem.

Parameter

`a` – A `double` matrix.

Returns

A `double` scalar value indicating how well the algorithms which have computed the eigenvalue and eigenvector pairs have performed.

Remarks

A performance index less than 1 is considered excellent, 1 to 100 is good, while greater than 100 is considered poor.

Solve

```
virtual public void Solve(double[,] a, bool computeVectors)
```

Description

Solves for the eigenvalues and (optionally) the eigenvectors of a real square matrix.

Parameters

`a` – A `double` square matrix for which the eigenvalues and (optionally) the eigenvectors are to be found.

`computeVectors` – A `bool` value of `true` if the eigenvectors are to be computed.

Exception

`Imsl.Math.DidNotConvergeException` is thrown when the algorithm fails to converge on the eigenvalues of the matrix

Example: Eigensystem Analysis

The eigenvalues and eigenvectors of a matrix are computed.

```
using System;
using Imsl.Math;

public class EigenEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {
            {8, - 1, - 5},
            {- 4, 4, - 2},
            {18, - 5, - 7}
        };
        Eigen eigen = new Eigen();
        eigen.Solve(a, true);
        new PrintMatrix("Eigenvalues").SetPageWidth(80).Print(eigen.GetValues());
        new PrintMatrix("Eigenvectors").SetPageWidth(80).Print(eigen.GetVectors());
    }
}
```

Output

```
       Eigenvalues
            0
0                   2+4i
1                   2-4i
2  0.999999999999997
```

```
                Eigenvectors
                        0
0     0.316227766016838-0.316227766016838i
1     0.632455532033676
2  1.66533453693773E-16-0.632455532033676i

                        1
0     0.316227766016838+0.316227766016838i
1     0.632455532033676
2  1.66533453693773E-16+0.632455532033676i

           2
0  0.408248290463863
1  0.816496580927725
2  0.408248290463864
```

SymEigen Class

`public class Imsl.Math.SymEigen`

Computes the eigenvalues and eigenvectors of a real symmetric matrix.

Orthogonal similarity transformations are used to reduce the matrix to an equivalent symmetric tridiagonal matrix. These transformations are accumulated. An implicit rational QR algorithm is used to compute the eigenvalues of this tridiagonal matrix. The eigenvectors are computed using the eigenvalues as perfect shifts, Parlett (1980, pages 169, 172). The reduction routine is based on the EISPACK routine `TRED2`. See Smith et al. (1976) for the EISPACK routines. Further details, some timing data, and credits are given in Hanson et al. (1990).

Let M = the number of eigenvalues, λ = the array of eigenvalues, and x_j is the associated eigenvector with jth eigenvalue.

Also, let ε be the machine precision. The performance index, τ, is defined to be

$$\tau = \max_{1 \le j \le M} \frac{\left\| Ax_j - \lambda_j x_j \right\|_1}{10N\varepsilon \left\| A \right\|_1 \left\| x_j \right\|_1}$$

While the exact value of τ is highly machine dependent, the performance of `SymEigen` is considered excellent if $\tau < 1$, good if $1 \le 100$, and poor if $\tau > 100$. The performance index was first developed by the EISPACK project at Argonne National Laboratory; see Smith et al. (1976, pages 124-125).

Constructors

SymEigen

```
public SymEigen(double[,] a)
```

Description

Constructs the eigenvalues and the eigenvectors for a real symmetric matrix.

Parameter

`a` – The symmetric matrix whose eigensystem is to be constructed.

SymEigen

```
public SymEigen(double[,] a, bool computeVectors)
```

Description

Constructs the eigenvalues and (optionally) the eigenvectors for a real symmetric matrix.

Parameters

`a` – A `double` symmetric matrix whose eigensystem is to be constructed.

`computeVectors` – A `bool`, `true` if the eigenvectors are to be computed.

Methods

GetValues

```
public double[] GetValues()
```

Description

Returns the eigenvalues.

Returns

A `double` array containing the eigenvalues in descending order.

Remarks

If the algorithm fails to converge on an eigenvalue, that eigenvalue is set to `NaN`.

GetVectors

```
public double[,] GetVectors()
```

Description

Return the eigenvectors of a symmetric matrix of type `double`.

Returns

A `double` array containing the eigenvectors.

Remarks

The j-th column of the eigenvector matrix corresponds to the j-th eigenvalue. The eigenvectors are normalized to have Euclidean length one. If the eigenvectors were not computed by the constructor, then null is returned.

PerformanceIndex

```
public double PerformanceIndex(double[,] a)
```

Description

Returns the performance index of a real symmetric eigensystem.

Parameter

`a` – A `double` symmetric matrix.

Returns

A `double` scalar value indicating how well the algorithms which have computed the eigenvalue and eigenvector pairs have performed.

Remarks

A performance index less than 1 is considered excellent, 1 to 100 is good, while greater than 100 is considered poor.

Example: Eigenvalues and Eigenvectors of a Symmetric Matrix

The eigenvalues and eigenvectors of a symmetric matrix are computed.

```
using System;
using Imsl.Math;

public class SymEigenEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {
            {1, 1, 1},
            {1, 1, 1},
            {1, 1, 1}
        };

        SymEigen eigen = new SymEigen(a);
        new PrintMatrix("Eigenvalues").Print(eigen.GetValues());
        new PrintMatrix("Eigenvectors").Print(eigen.GetVectors());
    }
}
```

Output

```
       Eigenvalues
             0
```

```
0   3
1  -3.53439879482695E-16
2  -2.22044604925031E-16

                        Eigenvectors
           0                  1                  2
0  0.577350269189626   0.816496580927726   0
1  0.577350269189626  -0.408248290463863  -0.707106781186547
2  0.577350269189626  -0.408248290463863   0.707106781186548
```

Chapter 3: Interpolation and Approximation

Types

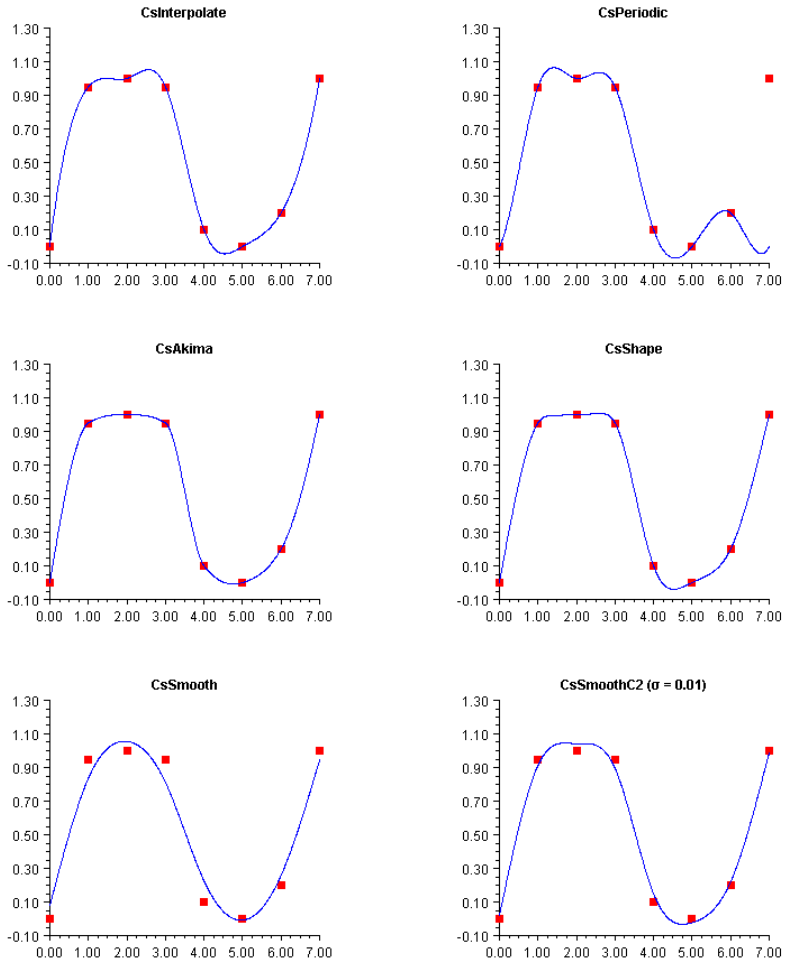
Usage Notes

This chapter contains classes to interpolate and approximate data with cubic splines. Interpolation means that the fitted curve passes through all of the specified data points. An approximation spline does not have to pass through any of the data points. An appoximating curve can therefore be smoother than an interpolating curve.

Cubic splines are smooth C^1 or C^2 fourth-order piecewise-polynomial (pp) functions. For historical and

other reasons, cubic splines are the most heavily used pp functions.

This chapter contains four cubic spline interpolation classes and two approximation classes. These classes are dervived from the base class `Spline`, which provides basic services, such as spline evaluation and integration.



The chart shows how the six cubic splines in this chapter fit a single data set.

Class `CsInterpolate` allows the user to specify various endpoint conditions (such as the value of the first and second derviatives at the right and left endpoints).

Class `CsPeriodic` is used to fit periodic (repeating) data. The sample data set used is not periodic and so the curve does not pass through the final data point.

Class `CsAkima` keeps the shape of the data while minimizing oscillations.

Class `CsShape` keeps the shape of the data by preserving its convexity.

Class `CsSmooth` constructs a smooth spline from noisy data.

Class `CsSmoothC2` constructs a smooth spline from noisy data using cross-validation and a user-supplied smoothing parameter.

Class `BSpline` is the abstract base class for univariate B-splines. B-splines provide a particularly convenient and suitable basis for a given class of smooth piecewise polynomial (ppoly) functions. Such a class is specified by giving its breakpoint sequence, its order *k*, and the required smoothness across each of the interior breakpoints. The corresponding B-spline basis is specified by giving its knot sequence $\mathbf{t} \in \mathbf{R}$. The specification rule is as follows: if the class is to have all derivatives up to and including the *j*-th derivative continuous across the interior breakpoint ξ_i, then the number ξ_i should occur *k-j-1* times in the knot sequence. Assuming that ξ_0 and ξ_{n-1} are the endpoints of the interval of interest, choose the first *k* knots equal to ξ_0 and the last *k* knots equal to ξ_{n-1}. This can be done because the B-splines are defined to be right continuous near ξ_0 and left continuous near ξ_{n-1}.

When the above construction is completed, a knot sequence **t** of length *M* is generated, and there are *m* = *M-k* B-splines of order *k*, for example $B_0, \ldots, B_{m-1}$, spanning the ppoly functions on the interval with the indicated smoothness. That is, each ppoly function in this class has a unique representation $p = a_0 B_0 + a_1 B_1 + \cdots + a_{m-1} B_{m-1}$ as a linear combination of B-splines. A B-spline is a particularly compact ppoly function. B_i is a nonnegative function that is nonzero only on the interval $[\mathbf{t}_i, \mathbf{t}_{i+k}]$ More precisely, the support of the *i*-th B-spline is $[\mathbf{t}_i, \mathbf{t}_{i+k}]$. No ppoly function in the same class (other than the zero function) has smaller support (i.e., vanishes on more intervals) than a B-spline. This makes B-splines particularly attractive basis functions since the influence of any particular B-spline coefficient extends only over a few intervals. When it is necessary to emphasize the dependence of the B-spline on its parameters, we will use the notation $B_{i,j,\mathbf{t}}$ to denote the *i*-th B-spline of order *k* for the knot sequence **t**.

Class `BsInterpolate` extends `BSpline` and creates a B-spline by interpolating data points.

Class `BsLeastSquares` extends `BSpline` and creates a B-spline by computing a least squares spline approximation to data points.

Class `Spline2D` is the abstract base class for the two-dimensional, tensor-product splines.

Class `Spline2DInterpolate` computes a `Spline2D` using interpolation from two-dimensional, tensor-product data.

Class `RadialBasis` computes an approximation to scattered data in $\mathbf{R}^M$ using radial-basis functions.

Spline Class

```
public class Imsl.Math.Spline
```

`Spline` represents and evaluates univariate piecewise polynomial splines.

A univariate piecewise polynomial (function) *p(x)* is specified by giving its breakpoint sequence `breakPoint[]`= $\xi \in \mathbf{R}^n$, the order *k* (degree *k*-1) of its polynomial pieces,and the $k \times (n-1)$ matrix `coef`=*c* of its local polynomial coefficients. In terms of this information, the piecewise polynomial

(ppoly) function is given by

$$p(x) = \sum_{j=1}^{k} c_{ji} \frac{(x - \xi_i)^{j-1}}{(j-1)!} \quad \text{for } \xi_i \leq x \leq \xi_{i+1}$$

The breakpoint sequence ξ is assumed to be strictly increasing, and we extend the ppoly function to the entire real axis by extrapolation from the first and last intervals.

Constructor

Spline

`Spline()`

Description

Initializes a new instance of the Imsl.Math.Spline (p. 143) class.

Methods

Derivative

```
virtual public double Derivative(double x)
```

Description

Returns the value of the first derivative of the spline at a point.

Parameter

`x` – A `double`, the point at which the derivative is to be evaluated.

Returns

A `double` containing the value of the first derivative of the spline at the point x.

Derivative

```
virtual public double Derivative(double x, int ideriv)
```

Description

Returns the value of the derivative of the spline at a point.

Parameters

`x` – A `double`, the point at which the derivative is to be evaluated.

`ideriv` – An `int` specifying the derivative to be computed. If zero, the function value is returned. If one, the first derivative is returned, etc.

Returns

A `double` containing the value of the derivative of the spline at the point x.

Derivative

```
virtual public double[] Derivative(double[] x, int ideriv)
```

Description

Returns the value of the derivative of the spline at each point of an array.

Parameters

`x` – A `double` array of points at which the derivative is to be evaluated.

`ideriv` – An `int` specifying the derivative to be computed. If zero, the function value is returned. If one, the first derivative is returned, etc.

Returns

A `double` array containing the value of the derivative the spline at each point of the array x.

Eval

```
virtual public double Eval(double x)
```

Description

Returns the value of the spline at a point.

Parameter

`x` – A `double`, the point at which the spline is to be evaluated.

Returns

A `double` giving the value of the spline at the point x.

Eval

```
virtual public double[] Eval(double[] x)
```

Description

Returns the value of the spline at each point of an array.

Parameter

`x` – A `double` array of points at which the spline is to be evaluated.

Returns

A `double` array containing the value of the spline at each point of the array x.

GetBreakpoints

```
public double[] GetBreakpoints()
```

Description

Returns a copy of the breakpoints.

Returns

A `double` array containing a copy of the breakpoints.

Integral

```
virtual public double Integral(double a, double b)
```

Description

Returns the value of an integral of the spline.

Parameters

`a` – A `double` specifying the lower limit of integration.

`b` – A `double` specifying the upper limit of integration.

Returns

A `double`, the integral of the spline from a to b.

CsAkima Class

```
public class Imsl.Math.CsAkima :  Spline
```

Extension of the Spline class to handle the Akima cubic spline.

Class `CsAkima` computes a C^1 cubic spline interpolant to a set of data points (x_i, f_i) for $i = 0, \ldots, n-1$. The breakpoints of the spline are the abscissas. Endpoint conditions are automatically determined by the program; see Akima (1970) or de Boor (1978).

If the data points arise from the values of a smooth, say C^4, function f, i.e. $f_i = f(x_i)$, then the error will behave in a predictable fashion. Let ξ be the breakpoint vector for the above spline interpolant. Then, the maximum absolute error satisfies

$$\|f - s\|_{[\xi_0, \xi_{n-1}]} \leq C \left\| f^{(2)} \right\|_{[\xi_0, \xi_{n-1}} |\xi|^2$$

where

$$|\xi| := \max_{i=1,\ldots,n-1} |\xi_i - \xi_{i-1}|$$

`CsAkima` is based on a method by Akima (1970) to combat wiggles in the interpolant. The method is nonlinear; and although the interpolant is a piecewise cubic, cubic polynomials are not reproduced. (However, linear polynomials are reproduced.)

Constructor

CsAkima

```
public CsAkima(double[] xData, double[] yData)
```

Description

Constructs the Akima cubic spline interpolant to the given data points.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data.

Exception

`System.ArgumentException` is thrown if the arrays `xData` and `yData` do not have the same length

Example: The Akima cubic spline interpolant

A cubic spline interpolant to a function is computed. The value of the spline at point 0.25 is printed.

```
using System;
using Imsl.Math;

public class CsAkimaEx1
{
    public static void  Main(String[] args)
    {
        int n = 11;
        double[] x = new double[n];
        double[] y = new double[n];

        for (int k = 0; k < n; k++)
        {
            x[k] = (double) k / (double) (n - 1);
            y[k] = Math.Sin(15.0 * x[k]);
        }

        CsAkima cs = new CsAkima(x, y);
        double csv = cs.Eval(0.25);
        Console.Out.WriteLine("The computed cubic spline value at " +
                              "point .25 is " + csv);
    }
}
```

Output

```
The computed cubic spline value at point .25 is -0.478185519991867
```

CsTCB Class

```
public class Imsl.Math.CsTCB : Spline
```

Extension of the Spline class to handle a tension-continuity-bias (TCB) cubic spline, also known as a Kochanek-Bartels spline and is a generalization of the Catmull-Rom spline.

Let $x =$ `xData`, $y =$ `yData`, and $n =$ the length of `xData` and `yData`. Class `CsTCB` computes the Kochanek-Bartels spline, a piecewise cubic Hermite spline interpolant to the set of data points $\{x_i, y_i\}$ for $i = 0, \ldots, n-1$. The breakpoints of the spline are the abscissas. As with all of the univariate interpolation functions, the abscissas need not be sorted.

The $\{x_i\}$ values are the knots, so the i-th interval is $[x_i, x_{i+1}]$. (To simplify the explanation, it is assumed that the data points are given in increasing order.) The cubic Hermite in the i-th segment has a starting value of y_i and an ending value of y_{i+1}. Its incoming tangent is

$$DS_i = \frac{1}{2}(1-t_i)(1-c_i)(1+b_i)\frac{y_i - y_{i-1}}{x_{i+1} - x_i} + \frac{1}{2}(1-t_i)(1+c_i)(1-b_i)\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

where t_i is the i-th tension value, c_i is the i-th continuity value, and b_i is the i-th bias value. Its outgoing tangent is

$$DD_i = \frac{1}{2}(1-t_i)(1+c_i)(1+b_i)\frac{y_i - y_{i-1}}{x_{i+1} - x_i} + \frac{1}{2}(1-t_i)(1-c_i)(1-b_i)\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

The value of the tangent at the left endpoint is given as:

$$\frac{y_0 - y_{-1}}{x_1 - x_0}$$

The value of the tangent at the right endpoint is given as:

$$\frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

By default the values of the tangents at the leftmost and rightmost endpoints are zero. These values can be reset via the `LeftEndTangent` and `RightEndTangent` properties.

The spline has a continuous first derivative (C^{-1}) if at each data point the left and right tangents are equal. This is true if the continuity parameters, c_i, are all zero. For any values of the parameters the spline is continuous (C^0).

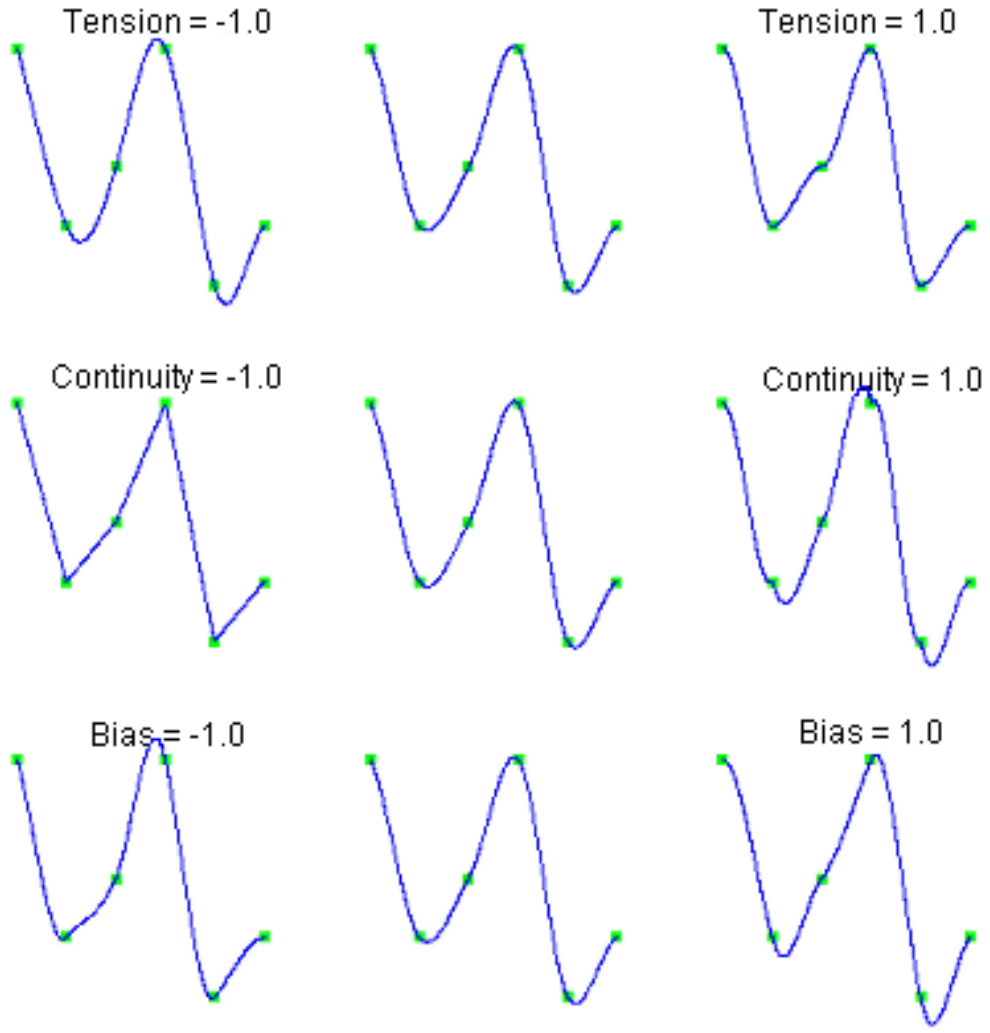
If $t_i = c_i = b_i = 0$ for all i, then the curve is the Catmull-Rom spline.

The following chart shows the same data points interpolated with different parameter values. All of the tension, continuity, and bias parameters are zero except for the labeled parameter, which has the indicated value at all data points.

Tension controls how sharply the spline bends at the data points. The tension values can be set via the `SetTension` method. If `tension` values are near +1, the curve tightens. If the `tension` values are near -1, the curve slackens.

The continuity parameter controls the continuity of the first derivative. The continuity values can be set via the `SetContinuity` method. If the `continuity` value is zero, the spline's first derivative is continuous, so the spline is C^{-1}.

The bias parameter controls the weighting of the left and right tangents. If zero, the tangents are equally weighted. If the bias parameter is near +1, the left tangent dominates. If the bias parameter is near -1, the right tangent dominates. The bias values can be set via the `SetBias` method.



Properties

LeftEndTangent

```
virtual public double LeftEndTangent {get; set; }
```

Description

The value of the tangent at the leftmost endpoint.

Property Value

A `double` containing the value of the tangent at the leftmost endpoint.

Default: `LeftEndTangent` is zero.

RightEndTangent

```
virtual public double RightEndTangent {get; set; }
```

Description

The value of the tangent at the rightmost endpoint.

Property Value

A `double` containing the value of the tangent at the rightmost endpoint.

Default: `RightEndTangent` is zero.

Constructor

CsTCB

```
public CsTCB(double[] xData, double[] yData)
```

Description

Constructs the tension-continuity-bias (TCB) cubic spline interpolant to the given data points.

Parameters

`xData` – A `double` array containing the x-coordinates of the data.

`yData` – A `double` array containing the y-coordinates of the data.

Remarks

Values must be distinct. `xData` and `yData` must be of the same length.

Methods

Compute

```
virtual public void Compute()
```

Description

Computes the tension-continuity-bias (TCB) cubic spline interpolant.

SetBias

```
virtual public void SetBias(double[] bias)
```

Description

Sets the bias values at the data points.

Parameter

`bias` – A `double` array of length `xData.Length` which contains bias values in the interval [-1,1]. For each point, if the bias value is zero, the left and right side tangents are equally weighted. If the value is near +1, the left-side tangent dominates. If the value is near -1, the right side tangent dominates.

Default: All values of `bias` are zero.

SetContinuity

```
virtual public void SetContinuity(double[] continuity)
```

Description

Sets the continuity values at the data points.

Parameter

`continuity` – A `double` array of length `xData.Length` which contains continuity values in the interval [-1,1]. For each point, if the continuity value is zero, the curve is C^1 at that point. Otherwise, the curve has a corner at that point, but is still continuous (C^0).

Default: All values of `continuity` are zero.

SetTension

```
virtual public void SetTension(double[] tension)
```

Description

Sets the tension values at the data points.

Parameter

`tension` – A `double` array of length `xData.Length` which contains tension values in the interval [-1,1]. For each point, if the tension value is near +1, the curve is tightened at that point. If it is near -1, the curve is slack.

Default: All values of `tension` are zero.

Example: The Kochanek-Bartels cubic spline interpolant

This example interpolates to a set of points. At x = 3, the continuity and tension parameters are -1. At all other points, they are zero. Interpolated values are then printed.

```
using System;
using Imsl.Math;
```

```
public class CsTCBEx1
{
    public static void  Main(String[] args)
    {
        double[] xdata = {0.0, 1.0, 2.0, 3.0, 4.0, 5.0};
        double[] ydata = {5.0, 2.0, 3.0, 5.0, 1.0, 2.0};
        double[] continuity = {0.0, 0.0, 0.0, - 1.0, 0.0, 0.0};
        double[] tension = {0.0, 0.0, 0.0, - 1.0, 0.0, 0.0};

        CsTCB cs = new CsTCB(xdata, ydata);
        cs.SetContinuity(continuity);
        cs.SetTension(tension);
        cs.Compute();
        Console.Out.WriteLine("    x    " + "  value   ");

        for (int k = 0; k < 11; k++)
        {
            double x = (double) k / 2.0;
            double y = cs.Eval(x);
            Console.Out.WriteLine(" {0,5:0.0000}    {1,5:0.0000}",x, y);
        }
    }
}
```

Output

```
    x      value
 0.0000    5.0000
 0.5000    3.4375
 1.0000    2.0000
 1.5000    2.1875
 2.0000    3.0000
 2.5000    3.6875
 3.0000    5.0000
 3.5000    2.1875
 4.0000    1.0000
 4.5000    1.2500
 5.0000    2.0000
```

CsInterpolate Class

```
public class Imsl.Math.CsInterpolate :  Spline
```

Extension of the Spline class to interpolate data points.

`CsInterpolate` computes a C^2 cubic spline interpolant to a set of data points (x_i, f_i) for $i = 0, \ldots, n-1$. The breakpoints of the spline are the abscissas. Endpoint conditions are automatically determined by the program. These conditions correspond to the "not-a-knot" condition (see de Boor 1978), which requires that the third derivative of the spline be continuous at the second and next-to-last breakpoint. If n is 2 or

3, then the linear or quadratic interpolating polynomial is computed, respectively.

If the data points arise from the values of a smooth, say, C^4 function f, i.e. $f_i = f(x_i)$, then the error will behave in a predictable fashion. Let ξ be the breakpoint vector for the above spline interpolant. Then, the maximum absolute error satisfies

$$|f - s|_{[\xi_0,\xi_n]} \leq C \left\| f^{(4)} \right\|_{[\xi_0,\xi_n]} |\xi|^4$$

where

$$|\xi| := \max_{i=0,\ldots,n-1} |\xi_{i+1} - \xi_i|$$

For more details, see de Boor (1978, pages 55-56).

Constructors

CsInterpolate

```
public CsInterpolate(double[] xData, double[] yData)
```

Description

Constructs a cubic spline that interpolates the given data points.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

CsInterpolate

```
public CsInterpolate(double[] xData, double[] yData,
Imsl.Math.CsInterpolate.Condition typeLeft, double valueLeft,
Imsl.Math.CsInterpolate.Condition typeRight, double valueRight)
```

Description

Constructs a cubic spline that interpolates the given data points with specified derivative endpoint conditions.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`typeLeft` – A `CsInterpolate.Condition` denoting the type of condition at the left endpoint. This can be `NotAKnot`, `FirstDerivative` or `SecondDerivative`.

`valueLeft` – A `double` value at the left endpoint. If `typeLeft` is `NotAKnot` this is ignored, Otherwise, it is the value of the specified derivative.

`typeRight` – A `CsInterpolate.Condition` denoting the type of condition at the right endpoint. This can be `NotAKnot`, `FirstDerivative` or `SecondDerivative`.

`valueRight` – A `double` value at the right endpoint.

Example: The cubic spline interpolant

A cubic spline interpolant to a function is computed. The value of the spline at point 0.25 is printed.

```
using System;
using Imsl.Math;

public class CsInterpolateEx1
{
    public static void  Main(String[] args)
    {
        int n = 11;
        double[] x = new double[n];
        double[] y = new double[n];

        for (int k = 0; k < n; k++)
        {
            x[k] = (double) k / (double) (n - 1);
            y[k] = System.Math.Sin(15.0 * x[k]);
        }

        CsInterpolate cs = new CsInterpolate(x, y);
        double csv = cs.Eval(0.25);
        Console.Out.WriteLine("The computed cubic spline value at " +
                              "point .25 is " + csv);
    }
}
```

Output

```
The computed cubic spline value at point .25 is -0.548772503812158
```

CsInterpolate.Condition Enumeration

`public enumeration Imsl.Math.CsInterpolate.Condition`

Denotes the type of condition at an endpoint.

Fields

FirstDerivative

```
public Imsl.Math.CsInterpolate.Condition FirstDerivative
```

Description

Satisfies the endpoint condition of the first derivative at the right and left points.

NotAKnot

```
public Imsl.Math.CsInterpolate.Condition NotAKnot
```

Description

Satisfies the “not-a-knot” condition.

SecondDerivative

```
public Imsl.Math.CsInterpolate.Condition SecondDerivative
```

Description

Satisfies the endpoint condition of the second derivative at the right and left points.

CsPeriodic Class

```
public class Imsl.Math.CsPeriodic :  Spline
```

Extension of the Spline class to interpolate data points with periodic boundary conditions.

Class `CsPeriodic` computes a C^2 cubic spline interpolant to a set of data points (x_i, f_i) for $i = 0, \ldots n-1$. The breakpoints of the spline are the abscissas. The program enforces periodic endpoint conditions. This means that the spline *s* satisfies *s(a) = s(b)*, $s'(a) = s'(b)$, and $s''(a) = s''(b)$, where *a* is the leftmost abscissa and *b* is the rightmost abscissa. If the ordinate values corresponding to *a* and *b* are not equal, then a warning message is issued. The ordinate value at *b* is set equal to the ordinate value at *a* and the interpolant is computed.

If the data points arise from the values of a smooth (say C^4) periodic function *f*, i.e. $f_i = f(x_i)$, then the error will behave in a predictable fashion. Let ξ be the breakpoint vector for the above spline interpolant. Then, the maximum absolute error satisfies

$$|f - s|_{[\xi_0, \xi_{n-1}]} \leq C |f^{(4)}|_{[\xi_0, \xi_{n-1}]} |\xi|^4$$

where

$$|\xi| := \max_{i=1,\ldots,n-1} |\xi_i - \xi_{i-1}|$$

For more details, see de Boor (1978, pages 320-322).

Constructor

CsPeriodic

```
public CsPeriodic(double[] xData, double[] yData)
```

Description

Constructs a cubic spline that interpolates the given data points with periodic boundary conditions.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. There must be at least 4 data points and values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

Example: The cubic spline interpolant with periodic boundary conditions

A cubic spline interpolant to a function is computed. The value of the spline at point 0.23 is printed.

```
using System;
using Imsl.Math;

public class CsPeriodicEx1
{
    public static void  Main(String[] args)
    {
        int n = 11;
        double[] x = new double[n];
        double[] y = new double[n];

        double h = 2.0 * System.Math.PI / 15.0 / 10.0;
        for (int k = 0; k < n; k++)
        {
            x[k] = h * (double) (k);
            y[k] = System.Math.Sin(15.0 * x[k]);
        }

        CsPeriodic cs = new CsPeriodic(x, y);
        double csv = cs.Eval(0.23);
        Console.Out.WriteLine("The computed cubic spline value at " +
                              "point .23 is " + csv);
    }
}
```

Output

```
The computed cubic spline value at point .23 is -0.303401472606451
```

CsShape Class

```
public class Imsl.Math.CsShape :  Spline
```

Extension of the Spline class to interpolate data points consistent with the concavity of the data.

Class `CsShape` computes a cubic spline interpolant to n data points x_i, f_i for $i = 0, \ldots, n-1$. For ease of explanation, we will assume that $x_i < x_{i+1}$, although it is not necessary for the user to sort these data values. If the data are strictly convex, then the computed spline is convex, C^2, and minimizes the expression

$$\int_{x_1}^{x_n} \left(g''\right)^2$$

over all convex C^1 functions that interpolate the data. In the general case when the data have both convex and concave regions, the convexity of the spline is consistent with the data and the above integral is minimized under the appropriate constraints. For more information on this interpolation scheme, we refer the reader to Micchelli et al. (1985) and Irvine et al. (1986).

One important feature of the splines produced by this class is that it is not possible, a priori, to predict the number of breakpoints of the resulting interpolant. In most cases, there will be breakpoints at places other than data locations. The method is nonlinear; and although the interpolant is a piecewise cubic, cubic polynomials are not reproduced. However, linear polynomials are reproduced.) This routine should be used when it is important to preserve the convex and concave regions implied by the data.

Constructor

CsShape

```
public CsShape(double[] xData, double[] yData)
```

Description

Construct a cubic spline interpolant which is consistent with the concavity of the data.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

Exceptions

`Imsl.Math.TooManyIterationsException` is thrown if the iteration did not converge.

`Imsl.Math.SingularMatrixException` is thrown if matrix is singular.

Example: The shape preserving cubic spline interpolant

A cubic spline interpolant to a function is computed consistent with the concavity of the data. The spline value at 0.05 is printed.

```
using System;
using Imsl.Math;

public class CsShapeEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{0.00, 0.10, 0.20, 0.30, 0.40,
                                  0.50, 0.60, 0.80, 1.00};
        double[] y = new double[]{0.00, 0.90, 0.95, 0.90, 0.10,
                                  0.05, 0.05, 0.20, 1.00};

        CsShape cs = new CsShape(x, y);
        double csv = cs.Eval(0.05);
        Console.Out.WriteLine("The computed cubic spline value at " +
                              "point .05 is " + csv);
    }
}
```

Output

```
The computed cubic spline value at point .05 is 0.55823122286482
```

CsSmooth Class

```
public class Imsl.Math.CsSmooth :  Spline
```

Extension of the Spline class to construct a smooth cubic spline from noisy data points.

Class `CsSmooth` is designed to produce a C^2 cubic spline approximation to a data set in which the function values are noisy. This spline is called a smoothing spline. It is a natural cubic spline with knots at all the data abscissas x = `xData`, but it does not interpolate the data (x_i, f_i). The smoothing spline S is the unique C^2 function that minimizes

$$\int_a^b S''(x)^2 \, dx$$

subject to the constraint

$$\sum_{i=0}^{n-1} |(S(x_i) - f_i)w_i|^2 \leq \sigma$$

where σ is the smoothing parameter. The reader should consult Reinsch (1967) for more information concerning smoothing splines. `CsSmooth` solves the above problem when the user provides the smoothing parameter σ. `CsSmoothC2` attempts to find the "optimal" smoothing parameter using the statistical technique known as cross-validation. This means that (in a very rough sense) one chooses the value of σ so that the smoothing spline (S_σ) best approximates the value of the data at x_I, if it is computed using all the data except the *i*-th; this is true for all $i = 0, \ldots, n-1$. For more information on this topic, we refer the reader to Craven and Wahba (1979).

Constructors

CsSmooth

```
public CsSmooth(double[] xData, double[] yData)
```

Description

Constructs a smooth cubic spline from noisy data using cross-validation to estimate the smoothing parameter. All of the points have equal weights.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

CsSmooth

```
public CsSmooth(double[] xData, double[] yData, double[] weight)
```

Description

Constructs a smooth cubic spline from noisy data using cross-validation to estimate the smoothing parameter. Weights are supplied by the user.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`weight` – A `double` array containing the relative weights. This array must havethe same length as `xData`.

Example: The cubic spline interpolant to noisy data

A cubic spline interpolant to noisy data is computed using cross-validation to estimate the smoothing parameter. The value of the spline at point 0.3010 is printed.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class CsSmoothEx1
{
    public static void  Main(String[] args)
    {
        int n = 300;
        double[] x = new double[n];
        double[] y = new double[n];
        for (int k = 0; k < n; k++)
        {
            x[k] = (3.0 * k) / (n - 1);
            y[k] = 1.0 / (0.1 + System.Math.Pow(3.0 * (x[k] - 1.0), 4));
        }

        //    Seed the random number generator
        Imsl.Stat.Random rn = new Imsl.Stat.Random(1234579);
        rn.Multiplier = 16807;

        //    Contaminate the data
        for (int i = 0; i < n; i++)
        {
            y[i] += 2.0 * (float) rn.NextDouble() - 1.0;
        }

        //    Smooth the data
        CsSmooth cs = new CsSmooth(x, y);
        double csv = cs.Eval(0.3010);
        Console.Out.WriteLine("The computed cubic spline value at " +
                              "point .3010 is " + csv);
    }
}
```

Output

```
The computed cubic spline value at point .3010 is 0.0101201298963992
```

CsSmoothC2 Class

```
public class Imsl.Math.CsSmoothC2 :  Spline
```

Extension of the Spline class used to construct a spline for noisy data points using an alternate method.

Class `CsSmoothC2` is designed to produce a C^2 cubic spline approximation to a data set in which the function values are noisy. This spline is called a smoothing spline. It is a natural cubic spline with knots at all the data abscissas x, but it does not interpolate the data (x_i, f_i). The smoothing spline S_σ is the

unique C^2 function that minimizes

$$\int_a^b s''_\sigma(x)^2\,dx$$

subject to the constraint

$$\sum_{i=0}^{n-1} |s_\sigma(x_i) - f_i|^2 \le \sigma$$

.

Recommended values for σ depend on the weights, w. If an estimate for the standard deviation of the error in the y-values is availiable, then w_i should be set to this value and the smoothing parameter should be choosen in the confidence interval corresponding to the left side of the above inequality. That is,

$$n - \sqrt{2n} \le \sigma \le n + \sqrt{2n}$$

`CsSmoothC2` is based on an algorithm of Reinsch (1967). This algorithm is also discussed in de Boor (1978, pages 235-243).

Constructors

CsSmoothC2

```
public CsSmoothC2(double[] xData, double[] yData, double sigma)
```

Description

Constructs a smooth cubic spline from noisy data using an algorithm based on Reinsch (1967). All of the points have equal weights.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`sigma` – A `double` value specifying the smoothing parameter. `sigma` must not be negative.

CsSmoothC2

```
public CsSmoothC2(double[] xData, double[] yData, double[] weight, double
sigma)
```

Description

Constructs a smooth cubic spline from noisy data using an algorithm based on Reinsch (1967) with weights supplied by the user.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`weight` – A `double` array containing the weights. The arrays `xData` and `weight` must have the same length.

`sigma` – A `double` value specifying the smoothing parameter. `sigma` must not be negative.

Example: The cubic spline interpolant to noisy data with supplied weights

A cubic spline interpolant to noisy data is computed using supplied weights and smoothing parameter. The value of the spline at point 0.3010 is printed.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class CsSmoothC2Ex1
{
    public static void  Main(String[] args)
    {
        //    Set up a grid
        int n = 300;
        double[] x = new double[n];
        double[] y = new double[n];
        for (int k = 0; k < n; k++)
        {
            x[k] = 3.0 * ((double) (k) / (double) (n - 1));
            y[k] = 1.0 / (.1 + System.Math.Pow(3.0 * (x[k] - 1.0), 4));
        }

        //    Seed the random number generator
        Imsl.Stat.Random rn = new Imsl.Stat.Random(1234579);
        rn.Multiplier = 16807;

        //    Contaminate the data
        for (int i = 0; i < n; i++)
        {
            y[i] = y[i] + 2.0 * (float) rn.NextDouble() - 1.0;
        }

        //    Set the weights
        double sdev = 1.0 / System.Math.Sqrt(3.0);
        double[] weights = new double[n];
        for (int i = 0; i < n; i++)
        {
            weights[i] = sdev;
        }

        //    Set the smoothing parameter
```

```
        double smpar = (double) n;

        //    Smooth the data
        CsSmoothC2 cs = new CsSmoothC2(x, y, weights, smpar);
        double csv = cs.Eval(0.3010);
        Console.Out.WriteLine("The computed cubic spline value at " +
                              "point .3010 is " + csv);
    }
}
```

Output

```
The computed cubic spline value at point .3010 is 0.0335028881575695
```

CsTCB Class

```
public class Imsl.Math.CsTCB : Spline
```

Extension of the Spline class to handle a tension-continuity-bias (TCB) cubic spline, also known as a Kochanek-Bartels spline and is a generalization of the Catmull-Rom spline.

Let $x =$ `xData`, $y =$ `yData`, and $n =$ the length of `xData` and `yData`. Class `CsTCB` computes the Kochanek-Bartels spline, a piecewise cubic Hermite spline interpolant to the set of data points $\{x_i, y_i\}$ for $i = 0, \ldots, n-1$. The breakpoints of the spline are the abscissas. As with all of the univariate interpolation functions, the abscissas need not be sorted.

The $\{x_i\}$ values are the knots, so the i-th interval is $[x_i, x_{i+1}]$. (To simplify the explanation, it is assumed that the data points are given in increasing order.) The cubic Hermite in the i-th segment has a starting value of y_i and an ending value of y_{i+1}. Its incoming tangent is

$$DS_i = \frac{1}{2}(1-t_i)(1-c_i)(1+b_i)\frac{y_i - y_{i-1}}{x_{i+1} - x_i} + \frac{1}{2}(1-t_i)(1+c_i)(1-b_i)\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

where t_i is the i-th tension value, c_i is the i-th continuity value, and b_i is the i-th bias value. Its outgoing tangent is

$$DD_i = \frac{1}{2}(1-t_i)(1+c_i)(1+b_i)\frac{y_i - y_{i-1}}{x_{i+1} - x_i} + \frac{1}{2}(1-t_i)(1-c_i)(1-b_i)\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

The value of the tangent at the left endpoint is given as:

$$\frac{y_0 - y_{-1}}{x_1 - x_0}$$

The value of the tangent at the right endpoint is given as:

$$\frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

By default the values of the tangents at the leftmost and rightmost endpoints are zero. These values can be reset via the `LeftEndTangent` and `RightEndTangent` properties.

The spline has a continuous first derivative (C^{-1}) if at each data point the left and right tangents are equal. This is true if the continuity parameters, c_i, are all zero. For any values of the parameters the spline is continuous (C^0).

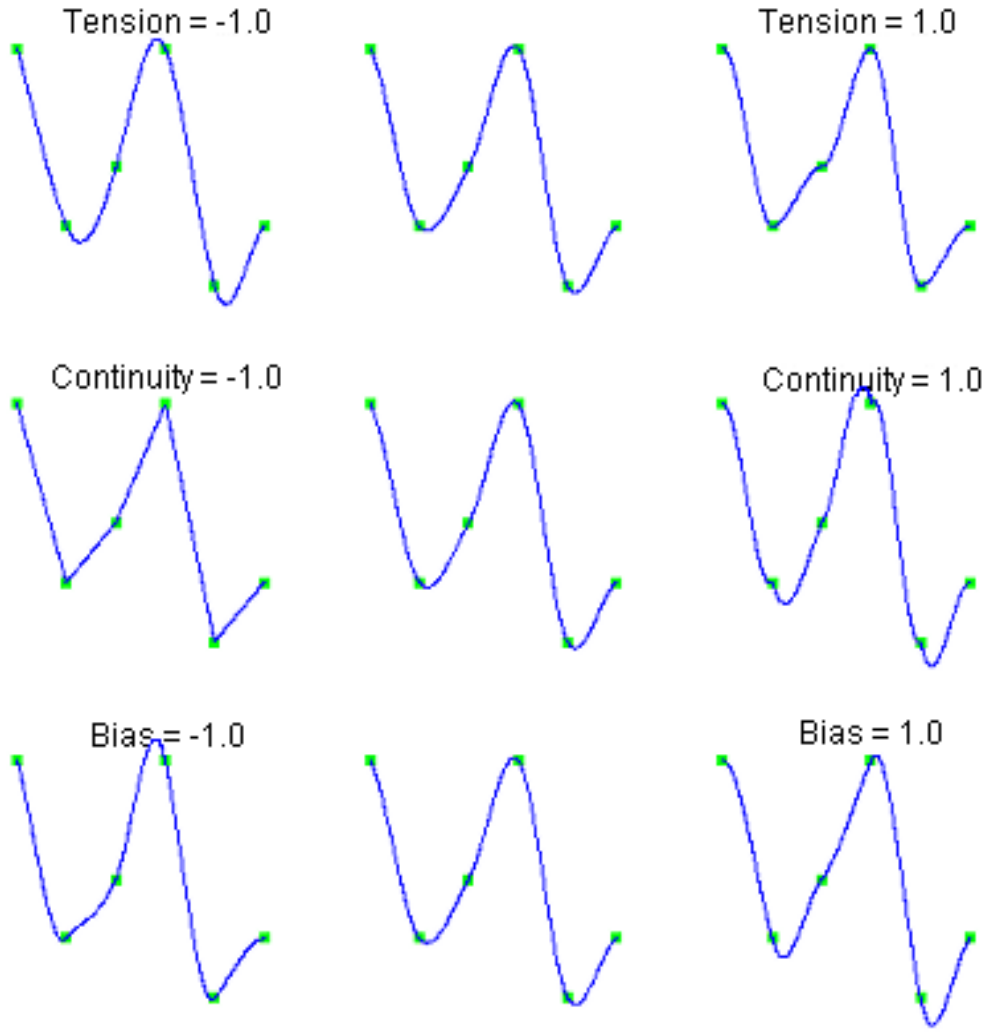
If $t_i = c_i = b_i = 0$ for all i, then the curve is the Catmull-Rom spline.

The following chart shows the same data points interpolated with different parameter values. All of the tension, continuity, and bias parameters are zero except for the labeled parameter, which has the indicated value at all data points.

Tension controls how sharply the spline bends at the data points. The tension values can be set via the `SetTension` method. If `tension` values are near +1, the curve tightens. If the `tension` values are near -1, the curve slackens.

The continuity parameter controls the continuity of the first derivative. The continuity values can be set via the `SetContinuity` method. If the `continuity` value is zero, the spline's first derivative is continuous, so the spline is C^{-1}.

The bias parameter controls the weighting of the left and right tangents. If zero, the tangents are equally weighted. If the bias parameter is near +1, the left tangent dominates. If the bias parameter is near -1, the right tangent dominates. The bias values can be set via the `SetBias` method.



Properties

LeftEndTangent

```
virtual public double LeftEndTangent {get; set; }
```

Description

The value of the tangent at the leftmost endpoint.

Property Value

A `double` containing the value of the tangent at the leftmost endpoint.

Default: `LeftEndTangent` is zero.

RightEndTangent

```
virtual public double RightEndTangent {get; set; }
```

Description

The value of the tangent at the rightmost endpoint.

Property Value

A `double` containing the value of the tangent at the rightmost endpoint.

Default: `RightEndTangent` is zero.

Constructor

CsTCB

```
public CsTCB(double[] xData, double[] yData)
```

Description

Constructs the tension-continuity-bias (TCB) cubic spline interpolant to the given data points.

Parameters

`xData` – A `double` array containing the x-coordinates of the data.

`yData` – A `double` array containing the y-coordinates of the data.

Remarks

Values must be distinct. `xData` and `yData` must be of the same length.

Methods

Compute

```
virtual public void Compute()
```

Description

Computes the tension-continuity-bias (TCB) cubic spline interpolant.

SetBias

```
virtual public void SetBias(double[] bias)
```

Description

Sets the bias values at the data points.

Parameter

`bias` – A `double` array of length `xData.Length` which contains bias values in the interval [-1,1]. For each point, if the bias value is zero, the left and right side tangents are equally weighted. If the value is near +1, the left-side tangent dominates. If the value is near -1, the right side tangent dominates.

Default: All values of `bias` are zero.

SetContinuity

```
virtual public void SetContinuity(double[] continuity)
```

Description

Sets the continuity values at the data points.

Parameter

`continuity` – A `double` array of length `xData.Length` which contains continuity values in the interval [-1,1]. For each point, if the continuity value is zero, the curve is C^1 at that point. Otherwise, the curve has a corner at that point, but is still continuous (C^0).

Default: All values of `continuity` are zero.

SetTension

```
virtual public void SetTension(double[] tension)
```

Description

Sets the tension values at the data points.

Parameter

`tension` – A `double` array of length `xData.Length` which contains tension values in the interval [-1,1]. For each point, if the tension value is near +1, the curve is tightened at that point. If it is near -1, the curve is slack.

Default: All values of `tension` are zero.

Example: The Kochanek-Bartels cubic spline interpolant

This example interpolates to a set of points. At x = 3, the continuity and tension parameters are -1. At all other points, they are zero. Interpolated values are then printed.

```
using System;
using Imsl.Math;

public class CsTCBEx1
{
    public static void  Main(String[] args)
    {
        double[] xdata = {0.0, 1.0, 2.0, 3.0, 4.0, 5.0};
        double[] ydata = {5.0, 2.0, 3.0, 5.0, 1.0, 2.0};
        double[] continuity = {0.0, 0.0, 0.0, - 1.0, 0.0, 0.0};
        double[] tension = {0.0, 0.0, 0.0, - 1.0, 0.0, 0.0};

        CsTCB cs = new CsTCB(xdata, ydata);
```

```
        cs.SetContinuity(continuity);
        cs.SetTension(tension);
        cs.Compute();
        Console.Out.WriteLine("    x    " + "  value   ");

        for (int k = 0; k < 11; k++)
        {
            double x = (double) k / 2.0;
            double y = cs.Eval(x);
            Console.Out.WriteLine(" {0,5:0.0000}    {1,5:0.0000}",x, y);
        }
    }
}
```

Output

```
    x      value
 0.0000    5.0000
 0.5000    3.4375
 1.0000    2.0000
 1.5000    2.1875
 2.0000    3.0000
 2.5000    3.6875
 3.0000    5.0000
 3.5000    2.1875
 4.0000    1.0000
 4.5000    1.2500
 5.0000    2.0000
```

BSpline Class

`public class Imsl.Math.BSpline`

Spline represents and evaluates univariate B-splines.

B-splines provide a particularly convenient and suitable basis for a given class of smooth ppoly functions. Such a class is specified by giving its breakpoint sequence, its order *k*, and the required smoothness across each of the interior breakpoints. The corresponding B-spline basis is specified by giving its knot sequence $\mathbf{t} \in \mathbf{R}^M$. The specification rule is as follows: If the class is to have all derivatives up to and including the j-th derivative continuous across the interior breakpoint ξ_i, then the number ξ_i should occur *k* - *j* - 1 times in the knot sequence. Assuming that ξ_1 and ξ_n are the endpoints of the interval of interest, choose the first *k* knots equal to ξ_1 and the last *k* knots equal to ξ_n. This can be done because the B-splines are defined to be right continuous near ξ_1 and left continuous near ξ_n.

When the above construction is completed, a knot sequence **t** of length *M* is generated, and there are *m*: = *M*-*k* B-splines of order *k*, for example $B_0, \ldots, B_{m-1}$, spanning the ppoly functions on the interval with the indicated smoothness. That is, each ppoly function in this class has a unique representation $p = a_0 B_0 + a_1 B_1 + \ldots + a_{m-1} B_{m-1}$ as a linear combination of B-splines. A B-spline is a particularly

compact ppoly function. B_i is a nonnegative function that is nonzero only on the interval $[\mathbf{t}_i, \mathbf{t}_{i+k}]$. More precisely, the support of the i-th B-spline is $[t_i, t_{i+k}]$. No ppoly function in the same class (other than the zero function) has smaller support (i.e., vanishes on more intervals) than a B-spline. This makes B-splines particularly attractive basis functions since the influence of any particular B-spline coefficient extends only over a few intervals.

Constructor

BSpline

```
BSpline()
```

Description

Initializes a new instance of the Imsl.Math.BSpline (p. 168) class.

Methods

Derivative

```
public double Derivative(double x)
```

Description

Returns the value of the first derivative of the B-spline at a point.

Parameter

`x` – A `double` which specifies the point at which the derivative is to be evaluated.

Returns

A `double` containing the value of the first derivative of the B-spline at the point `x`.

Derivative

```
public double Derivative(double x, int ideriv)
```

Description

Returns the value of the derivative of the B-spline at a point.

Parameters

`x` – A `double` which specifies the point at which the derivative is to be evaluated.

`ideriv` – A `int` specifying the derivative to be computed.

Returns

A `double` containing the value of the derivative of the B-spline at the point x.

Remarks

If `ideriv` is zero, the function value is returned. If one, the first derivative is returned, etc.

Derivative

```
public double[] Derivative(double[] x, int ideriv)
```

Description

Returns the value of the derivative of the B-spline at each point of an array.

Parameters

`x` – A `double` array of points at which the derivative is to be evaluated.

`ideriv` – A `int` specifying the derivative to be computed.

Returns

A `double` array containing the value of the derivative the B-spline at each point of the array `x`.

Remarks

If `ideriv` is zero, the function value is returned. If one, the first derivative is returned, etc.

Eval

```
public double Eval(double x)
```

Description

Returns the value of the B-spline at a point.

Parameter

`x` – A `double` which specifies the point at which the B-spline is to be evaluated.

Returns

A `double` giving the value of the B-spline at the point `x`.

Eval

```
public double[] Eval(double[] x)
```

Description

Returns the value of the B-spline at each point of an array.

Parameter

`x` – A `double` array of points at which the B-spline is to be evaluated.

Returns

A `double` array containing the value of the B-spline at each point of the array `x`.

GetKnots

```
public double[] GetKnots()
```

Description

Returns a copy of the knot sequence.

Returns

A `double` array containing a copy of the knot sequence.

GetSpline

```
public Imsl.Math.Spline GetSpline()
```

Description

Returns a `Spline` representation of the B-spline.

Returns

A `Spline` representation of the B-spline.

Integral

```
public double Integral(double a, double b)
```

Description

Returns the value of an integral of the B-spline.

Parameters

`a` – A `double` specifying the lower limit of integration.

`b` – A `double` specifying the upper limit of integration.

Returns

A `double` which specifies the integral of the B-spline from `a` to `b`.

Example: The B-spline interpolant

A B-Spline interpolant to data is computed. The value of the spline at point .23 is printed.

```
using System;
using Imsl.Math;

public class BsInterpolateEx1
{
    public static void  Main(String[] args)
    {
        int n = 11;
        double[] x = new double[n];
        double[] y = new double[n];

        double h = 2.0 * System.Math.PI / 15.0 / 10.0;
        for (int k = 0; k < n; k++)
        {
            x[k] = h * (double) (k);
            y[k] = System.Math.Sin(15.0 * x[k]);
        }

        BsInterpolate bs = new BsInterpolate(x, y);
        double bsv = bs.Eval(0.23);
        Console.Out.WriteLine("The computed B-spline value at point "
```

```
                                + ".23 is " + bsv);
    }
}
```

Output

```
The computed B-spline value at point .23 is -0.303418399276769
```

Example: The B-spline least squares fit

A B-Spline least squares fit to data is computed. The value of the spline at point 4.5 is printed.

```
using System;
using Imsl.Math;

public class BsLeastSquaresEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{0, 1, 2, 3, 4, 5, 8, 9, 10};
        double[] y = new double[]{1.0, 0.8, 2.4, 3.1, 4.5,
                                    5.8, 6.2, 4.9, 3.7};

        BsLeastSquares bs = new BsLeastSquares(x, y, 5);
        double bsv = bs.Eval(4.5);
        Console.Out.WriteLine("The computed B-spline value at point " +
                              "4.5 is " + bsv);
    }
}
```

Output

```
The computed B-spline value at point 4.5 is 5.22855432359694
```

BsInterpolate Class

```
public class Imsl.Math.BsInterpolate :  BSpline
```

Extension of the BSpline class to interpolate data points.

Given the data points x = xData, f = yData, and n the number of elements in xData and yData, the default action of BsInterpolate computes a cubic (order = 4) spline interpolant s to the data using a default “not-a-knot” knot sequence. Constructors are also provided that allow the order and knot sequence to be specified. This algorithm is based on the routine SPLINT by de Boor (1978, p. 204).

First, the xData vector is sorted and the result is stored in x. The elements of yData are permuted appropriately and stored in f, yielding the equivalent data (x_i, f_i) for i = 0 to n-1. The following preliminary checks are performed on the data, with k = order. We verify that

$x_i < x_{i+1}$ for $i = 0, \ldots, n-2$

$\mathbf{t}_i < \mathbf{t}_{i+k}$ for $i = 0, \ldots, n-1$

$\mathbf{t}_i < \mathbf{t}_{i+1}$ for $i = 0, \ldots, n+k-2$

The first test checks to see that the abscissas are distinct. The second and third inequalities verify that a valid knot sequence has been specified.

In order for the interpolation matrix to be nonsingular, we also check $\mathbf{t}_{k-1} \leq x_i \leq \mathbf{t}_n$ for *i* = 0 to *n-1*. This first inequality in the last check is necessary since the method used to generate the entries of the interpolation matrix requires that the *k* possibly nonzero B-splines at x_i, $B_{j-k+1}, \ldots, B_j$ where *j* satisfies $\mathbf{t}_j \leq x_i < \mathbf{t}_{j+1}$ be well-defined (that is, $j - k + 1 \geq 0$).

Constructors

BsInterpolate

```
public BsInterpolate(double[] xData, double[] yData)
```

Description

Constructs a B-spline that interpolates the given data points. The computed B-spline will be order 4 (cubic) and have a default “not-a-knot” spline knot sequence.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

BsInterpolate

```
public BsInterpolate(double[] xData, double[] yData, int order)
```

Description

Constructs a B-spline that interpolates the given data points and order, using a default “not-a-knot” spline knot sequence.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`order` – An `int` denoting the order of the B-spline.

BsInterpolate

```
public BsInterpolate(double[] xData, double[] yData, int order, double[] knot)
```

Description

Constructs a B-spline that interpolates the given data points, using the specified order and knots.

Parameters

`xData` – A `double` array containing the x-coordinates of the data. Values must be distinct.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`order` – A `int` denoting the order of the spline.

`knot` – A `double` array containing the knot sequence for the B-spline.

Example: The B-spline interpolant

A B-Spline interpolant to data is computed. The value of the spline at point .23 is printed.

```
using System;
using Imsl.Math;

public class BsInterpolateEx1
{
    public static void  Main(String[] args)
    {
        int n = 11;
        double[] x = new double[n];
        double[] y = new double[n];

        double h = 2.0 * System.Math.PI / 15.0 / 10.0;
        for (int k = 0; k < n; k++)
        {
            x[k] = h * (double) (k);
            y[k] = System.Math.Sin(15.0 * x[k]);
        }

        BsInterpolate bs = new BsInterpolate(x, y);
        double bsv = bs.Eval(0.23);
        Console.Out.WriteLine("The computed B-spline value at point "
                              + ".23 is " + bsv);
    }
}
```

Output

```
The computed B-spline value at point .23 is -0.303418399276769
```

BsLeastSquares Class

```
public class Imsl.Math.BsLeastSquares :  BSpline
```

Extension of the BSpline class to compute a least squares spline approximation to data points.

Let's make the identifications

n = xData.Length

x = xData

f = yData

m = nCoef

k = order

For convenience, we assume that the sequence x is increasing, although the class does not require this.

By default, $k = 4$, and the knot sequence we select equally distributes the knots through the distinct $x_i's$. In particular, the $m + k$ knots will be generated in $[x_1, x_n]$ with k knots stacked at each of the extreme values. The interior knots will be equally spaced in the interval.

Once knots **t** and weights w are determined, then the spline least-squares fit to the data is computed by minimizing over the linear coefficients a_j

$$\sum_{i=0}^{n-1} w_i \left[f_i - \sum_{j=1}^{m} a_j B_j(x_i) \right]^2$$

where the $B_j, j = 1, \ldots, m$ are a (B-spline) basis for the spline subspace.

This algorithm is based on the routine L2APPR by deBoor (1978, p. 255).

Constructors

BsLeastSquares

```
public BsLeastSquares(double[] xData, double[] yData, int nCoef)
```

Description

Constructs a least squares B-spline approximation to the given data points.

Parameters

`xData` – A `double` array containing the x-coordinates of the data.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`nCoef` – A `int` denoting the linear dimension of the spline subspace. It should be smaller than the number of data points and greater than or equal to the order of the spline (whose default value is 4).

BsLeastSquares

```
public BsLeastSquares(double[] xData, double[] yData, int nCoef, int order)
```

Description

Constructs a least squares B-spline approximation to the given data points.

Parameters

`xData` – A `double` array containing the x-coordinates of the data.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`nCoef` – A `int` denoting the linear dimension of the spline subspace. It should be smaller than the number of data points and greater than or equal to the order of the spline.

`order` – A `int` denoting the order of the spline.

BsLeastSquares

```
public BsLeastSquares(double[] xData, double[] yData, int nCoef, int order,
double[] weight, double[] knot)
```

Description

Constructs a least squares B-spline approximation to the given data points.

Parameters

`xData` – A `double` array containing the x-coordinates of the data.

`yData` – A `double` array containing the y-coordinates of the data. The arrays `xData` and `yData` must have the same length.

`nCoef` – A `int` denoting the linear dimension of the spline subspace. It should be smaller than the number of data points and greater than or equal to the order of the spline.

`order` – A `int` denoting the order of the spline.

`weight` – A `double` array containing the weights for the data. The arrays `xData`, `yDataa` and `weight` must have the same length.

`knot` – A `double` array containing the knot sequence for the spline.

Example: The B-spline least squares fit

A B-Spline least squares fit to data is computed. The value of the spline at point 4.5 is printed.

```
using System;
using Imsl.Math;

public class BsLeastSquaresEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{0, 1, 2, 3, 4, 5, 8, 9, 10};
        double[] y = new double[]{1.0, 0.8, 2.4, 3.1, 4.5,
                                  5.8, 6.2, 4.9, 3.7};

        BsLeastSquares bs = new BsLeastSquares(x, y, 5);
        double bsv = bs.Eval(4.5);
        Console.Out.WriteLine("The computed B-spline value at point " +
                              "4.5 is " + bsv);
    }
}
```

Output

```
The computed B-spline value at point 4.5 is 5.22855432359694
```

Spline2D Class

```
public class Imsl.Math.Spline2D
```

Represents and evaluates tensor-product splines.

The simplest method of obtaining multivariate interpolation and approximation functions is to take univariate methods and form a multivariate method via tensor products. In the case of two-dimensional spline interpolation, the derivation proceeds as follows: Let t_x be a knot sequence for splines of order k_x, and t_y be a knot sequence for splines of order k_y. Let $N_x + k_x$ be the length of t_x, and $N_y + k_x$ be the length of t_y. Then, the tensor-product spline has the following form:

$$\sum_{m=0}^{N_y-1}\sum_{n=0}^{N_x-1} c_{nm} B_{n,k_x,t_x}(x)\, B_{m,k_y,t_y}(y)$$

Given two sets of points

$$\{x_i\}_{i=1}^{N_x}$$

and

$$\{y_j\}_{j=1}^{N_y}$$

for which the corresponding univariate interpolation problem can be solved, the tensor-product interpolation problem finds the coefficients c_{nm} so that

$$\sum_{m=0}^{N_y-1}\sum_{n=0}^{N_x-1} c_{nm} B_{n,k_x,t_x}(x_i)\, B_{m,k_y,t_y}(y_j) = f_{ij}$$

This problem can be solved efficiently by repeatedly solving univariate interpolation problems as described in de Boor (1978, p. 347).

Constructor

Spline2D

```
Spline2D()
```

Description

Initializes a new instance of the Imsl.Math.Spline2D (p. 177) class.

Methods

Derivative

```
virtual public double Derivative(double x, double y, int xPartial, int
yPartial)
```

Description

Returns the value of the partial derivative of the tensor-product spline at the point *(x, y)*.

Parameters

`x` – A `double` scalar specifying the *x*-coordinate of the evaluation point for the tensor-product spline.

`y` – A `double` scalar specifying the *y*-coordinate of the evaluation point for the tensor-product spline.

`xPartial` – An `int` scalar specifying the *x*-partial derivative.

`yPartial` – An `int` scalar specifying the *y*-partial derivative.

Returns

A `double` scalar containing the value of the partial derivative

$$\frac{\partial^{i+j} s}{\partial^i x \, \partial^j y}$$

where i = `xPartial` and j = `yPartial`, at *(x, y)*.

Derivative

```
virtual public double[,] Derivative(double[] xVec, double[] yVec, int xPartial,
int yPartial)
```

Description

Returns the values of the partial derivative of the tensor-product spline of an array of points.

Parameters

`xVec` – A `double` array specifying the *x*-coordinates at which the spline is to be evaluated.

`yVec` – A `double` array specifying the *y*-coordinates at which the spline is to be evaluated.

`xPartial` – An `int` scalar specifying the *x*-partial derivative.

`yPartial` – An `int` scalar specifying the *y*-partial derivative.

Returns

An `double` matrix containing the values of the partial derivatives

$$\frac{\partial^{i+j} s}{\partial^i x \, \partial^j y}$$

where i = `xPartial` and j = `yPartial`, at each *(x, y)*.

GetCoefficients

```
virtual public double[,] GetCoefficients()
```

Description

Returns the coefficients for the tensor-product spline.

Returns

A `double` matrix containing the coefficients.

GetXKnots

```
virtual public double[] GetXKnots()
```

Description

Returns the knot sequences in the *x*-direction.

Returns

A `double` array of containing the knot sequences of the spline in the *x*-direction.

GetYKnots

```
virtual public double[] GetYKnots()
```

Description

Returns the knot sequences in the *y*-direction.

Returns

A `double` array containing the knot sequences of the spline in the *y*-direction.

Integral

```
public double Integral(double a, double b, double c, double d)
```

Description

Returns the value of an integral of a tensor-product spline on a rectangular domain.

Parameters

`a` – A `double` specifying the lower limit for the first variable of the tensor-product spline.

`b` – A `double` specifying the upper limit for the first variable of the tensor-product spline.

`c` – A `double` specifying the lower limit for the second variable of the tensor-product spline.

`d` – A `double` specifying the upper limit for the second variable of the tensor-product spline.

Returns

A `double`, the integral of the tensor-product spline over the rectangle `[a, b]` by `[c, d]`.

Remarks

If *s* is the spline, then the `Integral` method returns

$$\int_a^b \int_c^d s(x,y)dydx$$

This method uses the (univariate integration) identity (22) in de Boor (1978, p. 151)

$$\int_{t_0}^{x} \sum_{i=0}^{n-1} \alpha_i B_{i,k}(\tau)\, d\tau = \sum_{i=0}^{r-1} \left[\sum_{j=0}^{i} \alpha_j \frac{t_{j+k} - t_j}{k} \right] B_{i,k+1}(x)$$

where $t_0 \leq x \leq t_r$.

It assumes (for all knot sequences) that the first and last k knots are stacked, that is, $t_0 = \ldots = t_{k-1}$ and $t_n = \ldots = t_{n+k-1}$, where k is the order of the spline in the x or y direction.

Value

```
virtual public double Value(double x, double y)
```

Description

Returns the value of the tensor-product spline at the point *(x, y)*.

Parameters

`x` – A `double` scalar specifying the *x*-coordinate of the evaluation point for the tensor-product spline.

`y` – A `double` scalar specifying the *y*-coordinate of the evaluation point for the tensor-product spline.

Returns

A `double` scalar containing the value of the tensor-product spline.

Value

```
virtual public double[,] Value(double[] xVec, double[] yVec)
```

Description

Returns the values of the tensor-product spline of an array of points.

Parameters

`xVec` – A `double` array specifying the *x*-coordinates at which the spline is to be evaluated.

`yVec` – A `double` array specifying the *y*-coordinates at which the spline is to be evaluated.

Returns

A `double` matrix containing the values evaluated.

Spline2DInterpolate Class

```
public class Imsl.Math.Spline2DInterpolate :  Spline2D
```

Computes a two-dimensional, tensor-product spline interpolant from two-dimensional, tensor-product data.

The class `Spline2DInterpolate` computes a tensor-product spline interpolant. The tensor-product spline interpolant to data $\{(x_i, y_j, f_{ij})\}$, where $0 \leq i \leq (n_x - 1)$ and $0 \leq j \leq (n_y - 1)$ has the form

$$\sum_{m=0}^{n_y-1} \sum_{n=0}^{n_x-1} c_{nm} B_{n,k_x,t_x}(x) B_{m,k_y,t_y}(y)$$

where k_x and k_y are the orders of the splines. These numbers are defaulted to be 4, but can be set to any positive integer using `xOrder` and `yOrder` in the constructor. Likewise, t_x and t_y are the corresponding knot sequences (`xKnots` and `yKnots`). These default values are selected by `Spline2DInterpolate`. The algorithm requires that

$$t_x(k_x - 1) \le x_i \le t_x(n_x) \qquad 0 \le i \le n_x - 1$$

$$t_y(k_y - 1) \le y_j \le t_y(n_y - 1) \qquad 0 \le j \le n_y - 1$$

Tensor-product spline interpolants in two dimensions can be computed quite efficiently by solving (repeatedly) two univariate interpolation problems.

The computation is motivated by the following observations. It is necessary to solve the system of equations

$$\sum_{m=0}^{n_y-1} \sum_{n=0}^{n_x-1} c_{nm} B_{n,k_x,t_x}(x_i) B_{m,k_y,t_y}(y_j) = f_{ij}$$

Setting

$$h_{mi} = \sum_{n=0}^{n_x-1} c_{nm} B_{n,k_x,t_x}(x_i)$$

note that for each fixed i from 0 to $n_x - 1$, we have n_y linear equations in the same number of unknowns as can be seen below:

$$\sum_{m=0}^{n_y-1} h_{mi} B_{n,k_y,t_y}(y_j) = f_{ij}$$

$$\sum_{m=0}^{n_y-1} \sum_{n=0}^{n_x-1} c_{nm} B_{n,k_x,t_x}(x_i) B_{m,k_y,t_y}(y_j) = f_{ij}$$

Setting

$$h_{mi} = \sum_{n=0}^{n_x-1} c_{nm} B_{n,k_x,t_x}(x_i)$$

note that for each fixed i from 0 to $n_x - 1$, we have $n_y - 1$ linear equations in the same number of unknowns as can be seen below:

$$\sum_{m=0}^{n_y-1} h_{mi} B_{m,k_y,t_y}(y_j) = f_{ij}$$

The same matrix appears in all of the equations above:

$$\left[B_{m,k_y,t_y}(y_j)\right] \qquad 0 \le m, j \le n_y - 1$$

Thus, only factor this matrix once and then apply this factorization to the n_x right-hand sides. Once this is done and h_{mi} is computed, then solve for the coefficients c_{nm} using the relation

$$\sum_{n=0}^{n_x-1} c_{nm} B_{n,k_x,t_x}(x_i) = h_{mi}$$

for n from 0 to $n_y - 1$, which again involves one factorization and n_y solutions to the different right-hand sides. The class `Spline2DInterpolate` is based on the routine SPLI2D by de Boor (1978, p. 347).

Constructors

Spline2DInterpolate

```
public Spline2DInterpolate(double[] xData, double[] yData, double[,] fData)
```

Description

Constructor for `Spline2DInterpolate`.

Parameters

`xData` – A `double` array containing the data points in the *x*-direction.

`yData` – A `double` array containing the data points in the *y*-direction.

`fData` – A `double` matrix of size `xData.Length` by `yData.Length` containing the values to be interpolated.

Spline2DInterpolate

```
public Spline2DInterpolate(double[] xData, double[] yData, double[,] fData, int
xOrder, int yOrder)
```

Description

Constructor for `Spline2DInterpolate`.

Parameters

`xData` – A `double` array containing the data points in the *x*-direction.

`yData` – A `double` array containing the data points in the *y*-direction.

`fData` – A `double` matrix of size `xData.Length` by `yData.Length` containing the values to be interpolated.

`xOrder` – An `int` scalar value specifying the order of the spline in the *x*-direction. `xOrder` must be at least 1. By default, `xOrder` = 4, tensor-product cubic spline.

`yOrder` – An `int` scalar value specifying the order of the spline in the *y*-direction. `yOrder` must be at least 1. By default, `yOrder` = 4, tensor-product cubic spline.

Spline2DInterpolate

```
public Spline2DInterpolate(double[] xData, double[] yData, double[,] fData, int
xOrder, int yOrder, double[] xKnots, double[] yKnots)
```

Description

Constructor for `Spline2DInterpolate`.

Parameters

`xData` – A `double` array containing the data points in the *x*-direction.

`yData` – A `double` array containing the data points in the *y*-direction.

`fData` – A `double` matrix of size `xData.Length` by `yData.Length` containing the values to be interpolated.

`xOrder` – An `int` scalar value specifying the order of the spline in the *x*-direction. `xOrder` must be at least 1. By default, `xOrder` = 4, tensor-product cubic spline.

`yOrder` – An `int` scalar value specifying the order of the spline in the *y*-direction. `yOrder` must be at least 1. By default, `yOrder` = 4, tensor-product cubic spline.

`xKnots` – A `double` array of size `xData.Length + xOrder` specifying the knot sequences of the spline in the *x*-direction. By default, knot sequences are selected by the class.

`yKnots` – A `double` array of size `yData.Length + yOrder` specifying the knot sequences of the spline in the *y*-direction. By default, knot sequences are selected by the class.

Example 1

A tensor-product spline interpolant to a function

$$f(x,y) = x^3 + y^2$$

is computed. The values of the interpolant and the error on a 4 x 4 grid are displayed.

```
using System;
using Imsl.Math;

public class Spline2DInterpolateEx1
{

    private static double F(double x, double y)
    {
        return (x * x * x + y * y);
    }

    public static void  Main(String[] args)
    {
        int nData = 11;
        int outData = 2;
        double[,] fdata = new double[nData,nData];
        double[] xData = new double[nData];
        double[] yData = new double[nData];
        double x, y, z;

        // Set up grid
        for (int i = 0; i < nData; i++)
        {
            xData[i] = yData[i] = (double) i / ((double) (nData - 1));
        }

        for (int i = 0; i < nData; i++)
        {
            for (int j = 0; j < nData; j++)
            {
                fdata[i,j] = F(xData[i], yData[j]);
            }
        }

        // Compute tensor-product interpolant
```

```
        Spline2DInterpolate spline = new Spline2DInterpolate(xData,
            yData, fdata);

        // Print results
        Console.Out.WriteLine("   x        y     F(x, y)   "
            + "Interpolant   Error");
        for (int i = 0; i < outData; i++)
        {
            x = (double) i / (double) (outData);
            for (int j = 0; j < outData; j++)
            {
                y = (double) j / (double) (outData);
                z = spline.Value(x, y);
                Console.Out.WriteLine(x.ToString("0.0000")+"   "
                    +y.ToString("0.0000")+"   "+F(x,y).ToString("0.0000")
                    +"       "+z.ToString("0.0000")+"     "
                    +Math.Abs(F(x, y) - z).ToString("0.0000"));
            }
        }
    }
}
```

Output

```
   x        y     F(x, y)   Interpolant   Error
0.0000   0.0000   0.0000       0.0000     0.0000
0.0000   0.5000   0.2500       0.2500     0.0000
0.5000   0.0000   0.1250       0.1250     0.0000
0.5000   0.5000   0.3750       0.3750     0.0000
```

Example 2

A tensor-product spline interpolant to a function

$$f(x,y) = x^3 + y^2$$

is computed. The values of the interpolant and the error on a 4 x 4 grid are displayed. Notice that the first interpolant with order = 3 does not reproduce the cubic data, while the second interpolant with order = 6 does reproduce the data.

```
using System;
using Imsl.Math;

public class Spline2DInterpolateEx2
{

    private static double F(double x, double y)
    {
        return (x * x * x + y * y);
    }

    public static void  Main(String[] args)
    {
```

```
        int nData = 7;
        int outData = 4;
        double[,] fData = new double[nData,nData];
        double[] xData = new double[nData];
        double[] yData = new double[nData];
        double x, y, z;

        // Set up grid
        for (int i = 0; i < nData; i++)
        {
            xData[i] = yData[i] = (double) i / ((double) (nData - 1));
        }

        for (int i = 0; i < nData; i++)
        {
            for (int j = 0; j < nData; j++)
            {
                fData[i,j] = F(xData[i], yData[j]);
            }
        }


        for (int order = 3; order < 7; order += 3)
        {
            // Compute tensor-product interpolant
            Spline2DInterpolate spline = new Spline2DInterpolate(xData, yData,
            fData, order, order);

            // Print results
            Console.Out.WriteLine("\nThe order of the spline is " + order);
            Console.Out.WriteLine("   x        y     F(x, y)   "
            + "Interpolant   Error");

            for (int i = 0; i < outData; i++)
            {
                x = (double) i / (double) (outData);

                for (int j = 0; j < outData; j++)
                {
                    y = (double) j / (double) (outData);
                    z = spline.Value(x, y);

                    Console.Out.WriteLine(x.ToString("0.0000")+"   "
                        +y.ToString("0.0000")+"   "+F(x,y).ToString("0.0000")
                        +"       "+z.ToString("0.0000")+"     "
                        +Math.Abs(F(x, y) - z).ToString("0.0000"));
                }
            }
        }
    }
}
```

Output

```
The order of the spline is 3
```

```
   x        y     F(x, y)   Interpolant   Error
0.0000   0.0000   0.0000       0.0000     0.0000
0.0000   0.2500   0.0625       0.0625     0.0000
0.0000   0.5000   0.2500       0.2500     0.0000
0.0000   0.7500   0.5625       0.5625     0.0000
0.2500   0.0000   0.0156       0.0158     0.0002
0.2500   0.2500   0.0781       0.0783     0.0002
0.2500   0.5000   0.2656       0.2658     0.0002
0.2500   0.7500   0.5781       0.5783     0.0002
0.5000   0.0000   0.1250       0.1250     0.0000
0.5000   0.2500   0.1875       0.1875     0.0000
0.5000   0.5000   0.3750       0.3750     0.0000
0.5000   0.7500   0.6875       0.6875     0.0000
0.7500   0.0000   0.4219       0.4217     0.0002
0.7500   0.2500   0.4844       0.4842     0.0002
0.7500   0.5000   0.6719       0.6717     0.0002
0.7500   0.7500   0.9844       0.9842     0.0002

The order of the spline is 6
   x        y     F(x, y)   Interpolant   Error
0.0000   0.0000   0.0000       0.0000     0.0000
0.0000   0.2500   0.0625       0.0625     0.0000
0.0000   0.5000   0.2500       0.2500     0.0000
0.0000   0.7500   0.5625       0.5625     0.0000
0.2500   0.0000   0.0156       0.0156     0.0000
0.2500   0.2500   0.0781       0.0781     0.0000
0.2500   0.5000   0.2656       0.2656     0.0000
0.2500   0.7500   0.5781       0.5781     0.0000
0.5000   0.0000   0.1250       0.1250     0.0000
0.5000   0.2500   0.1875       0.1875     0.0000
0.5000   0.5000   0.3750       0.3750     0.0000
0.5000   0.7500   0.6875       0.6875     0.0000
0.7500   0.0000   0.4219       0.4219     0.0000
0.7500   0.2500   0.4844       0.4844     0.0000
0.7500   0.5000   0.6719       0.6719     0.0000
0.7500   0.7500   0.9844       0.9844     0.0000
```

Example 3

A spline interpolant *s* to a function

$$f(x,y) = x^3y^2$$

is constructed. Then, the values of the partial derivative

$$\frac{\partial^3 s(x,y)}{\partial x^2 2y}$$

and the error are computed on a 4 x 4 grid.

```
using System;
using Imsl.Math;

public class Spline2DInterpolateEx3
{
```

```
    private static double F(double x, double y)
    {
        return (x * x * x * y * y);
    }

    private static double F21(double x, double y)
    {
        return (6.0 * x * 2.0 * y);
    }

    public static void  Main(String[] args)
    {
        int nData = 11;
        int outData = 2;
        double[,] fData = new double[nData,nData];
        double[] xData = new double[nData];
        double[] yData = new double[nData];
        double x, y, z;

        // Set up grid
        for (int i = 0; i < nData; i++)
        {
            xData[i] = yData[i] = (double) i / ((double) (nData - 1));
        }

        for (int i = 0; i < nData; i++)
        {
            for (int j = 0; j < nData; j++)
            {
                fData[i,j] = F(xData[i], yData[j]);
            }
        }

        // Compute tensor-product interpolant
        Spline2DInterpolate spline = new Spline2DInterpolate(xData, yData, fData);

        // Print results
        Console.Out.WriteLine("   x        y     F21(x, y)   "
        + "21InterpDeriv   Error");
        for (int i = 0; i < outData; i++)
        {
            x = (double) (1 + i) / (double) (outData + 1);

            for (int j = 0; j < outData; j++)
            {
                y = (double) (1 + j) / (double) (outData + 1);
                z = spline.Derivative(x, y, 2, 1);
                Console.Out.WriteLine(x.ToString("0.0000")+"   "
                    +y.ToString("0.0000")+"   "+F21(x,y).ToString("0.0000")
                    +"       "+z.ToString("0.0000")+"     "
                    +Math.Abs(F21(x, y) - z).ToString("0.0000"));
            }
        }
    }
}
```

Output

```
   x        y     F21(x, y)   21InterpDeriv   Error
0.3333   0.3333   1.3333       1.3333     0.0000
0.3333   0.6667   2.6667       2.6667     0.0000
0.6667   0.3333   2.6667       2.6667     0.0000
0.6667   0.6667   5.3333       5.3333     0.0000
```

Example 4

This example integrates a two-dimensional, tensor-product spline over the rectangle [0, x] by [0, y].

```
using System;
using Imsl.Math;

public class Spline2DInterpolateEx4
{

    // Define function
    private static double F(double x, double y)
    {
        return (x*x*x+y*y);
    }

    // The integral of F from 0 to x and 0 to y
    private static double FI(double x, double y)
    {
        return (y*x*x*x*x/4.0 + x*y*y*y/3.0);
    }

    public static void  Main(String[] args)
    {
        int nData = 11, outData = 2;
        double[,] fData = new double[nData, nData];
        double[] xData = new double[nData];
        double[] yData = new double[nData];
        double x, y, z;

        // Set up grid
        for (int i = 0; i < nData; i++)
        {
            xData[i] = yData[i] = (double) i / ((double)(nData-1));
        }

        for (int i = 0; i < nData; i++)
        {
            for (int j = 0; j < nData; j++)
            {
                fData[i, j] = F(xData[i], yData[j]);
            }
        }

        // Compute tensor-product interpolant
        Spline2DInterpolate spline =
            new Spline2DInterpolate(xData, yData, fData);
```

```
        // Print results
        Console.Out.WriteLine("   x       y     FI(x, y)  Integral   Error");
        for (int i = 0; i < outData; i++)
        {
            x = (double) (1+i) / (double) (outData+1);
            for (int j = 0; j < outData; j++)
            {
                y = (double) (1+j) / (double) (outData+1);
                z = spline.Integral(0.0, x, 0.0, y);
                Console.Out.WriteLine(x.ToString("0.0000") + "  " +
                    y.ToString("0.0000") + "    " +
                    FI(x, y).ToString("0.0000") + "    " +
                    z.ToString("0.0000") + "    " +
                    Math.Abs(FI(x,y)-z).ToString("0.0000"));
            }
        }
    }
}
```

Output

```
   x       y     FI(x, y)  Integral   Error
0.3333  0.3333    0.0051    0.0051    0.0000
0.3333  0.6667    0.0350    0.0350    0.0000
0.6667  0.3333    0.0247    0.0247    0.0000
0.6667  0.6667    0.0988    0.0988    0.0000
```

Spline2DLeastSquares Class

`public class Imsl.Math.Spline2DLeastSquares :  Spline2D`

Computes a two-dimensional, tensor-product spline approximant using least squares.

The Spline2DLeastSquares class computes a tensor-product spline least-squares approximation to weighted tensor-product data. The input consists of data vectors to specify the tensor-product grid for the data, two vectors with the weights, the values of the surface on the grid, and the specification for the tensor-product spline. The grid is specified by the two vectors x = `xData` and y = `yData` of length n = `xData.Length` and m = `yData.Length`, respectively. A two-dimensional array f = `fData` contains the data values which are to be fit. The two vectors w_x = `xWeights` and w_y = `yWeights` contain the weights for the weighted least-squares problem. The information for the approximating tensor-product spline can be provided using the `SetXOrder`, `SetYOrder`, `SetXKnots` and `SetYKnots` methods. This information is contained in k_x = `xOrder`, t_x = `xKnots`, and N = `xSplineSpaceDim` for the spline in the first variable, and in k_y = `yOrder`, t_y = `yKnots` and M = `ySplineSpaceDim` for the spline in the second variable. This class computes coefficients for the tensor-product spline by solving the normal equations in tensor-product form as discussed in de Boor (1978, Chapter 17). The interested reader might also want to study the paper by Grosse (1980).

As the computation proceeds, we obtain coefficients c minimizing

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_x(i)\,w_y(j)\left[\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} c_{kl}B_{kl}(x_i,y_i) - f_{ij}\right]^2$$

where the function B_{kl} is the tensor-product of two B-splines of order k_x and k_y. Specifically, we have

$$B_{kl}(x,y) = B_{k,k_x,t_x}(x)\,B_{l,k_y,t_y}(y)$$

The spline

$$\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} c_{kl}B_{kl}$$

and its partial derivatives can be evaluated using the `Value` method.

Constructor

Spline2DLeastSquares

```
public Spline2DLeastSquares(double[] xData, double[] yData, double[,] fData,
int xSplineSpaceDim, int ySplineSpaceDim)
```

Description

Constructor for `Spline2DLeastSquares`.

Parameters

`xData` – A `double` array containing the data points in the *x*-direction.

`yData` – A `double` array containing the data points in the *y*-direction.

`fData` – A `double` matrix of size `xData.Length` by `yData.Length` containing the values to be approximated.

`xSplineSpaceDim` – An `int` scalar value specifying the linear dimension of the spline subspace for the *x* variable. It should be smaller than `xData.Length` and greater than or equal to `xOrder` (whose default value is 4).

`ySplineSpaceDim` – An `int` scalar value specifying the linear dimension of the spline subspace for the *y* variable. It should be smaller than `yData.Length` and greater than or equal to `yOrder` (whose default value is 4).

Methods

Compute

```
public void Compute()
```

Description

Computes a two-dimensional, tensor-product spline approximant using least squares.

GetErrorSumOfSquares

```
public double GetErrorSumOfSquares()
```

Description

Returns the weighted error sum of squares.

Returns

A `double` scalar containing the weighted error sum of squares.

GetXOrder

```
virtual public int GetXOrder()
```

Description

Returns the order of the spline in the *x*-direction.

Returns

An `int` scalar containing the order of the spline in the *x*-direction.

GetXWeights

```
virtual public double[] GetXWeights()
```

Description

Returns the weights for the least-squares fit in the *x*-direction.

Returns

A `double` array containing the weights for the least-squares fit in the *x*-direction.

GetYOrder

```
virtual public int GetYOrder()
```

Description

Returns the order of the spline in the *y*-direction.

Returns

An `int` scalar containing the order of the spline in the *y*-direction.

GetYWeights

```
virtual public double[] GetYWeights()
```

Description

Returns the weights for the least-squares fit in the *y*-direction.

Returns

A `double` array containing the weights for the least-squares fit in the *y*-direction.

SetXKnots

```
virtual public void SetXKnots(double[] xKnots)
```

Description

Sets the knot sequences of the spline in the *x*-direction.

Parameter

`xKnots` – A `double` array of size `xSplineSpaceDim` + `xOrder` specifying the knot sequences of the spline in the *x*-direction.

Default: Knot sequences are selected by the class.

SetXOrder

```
virtual public void SetXOrder(int xOrder)
```

Description

Sets the order of the spline in the *x*-direction.

Parameter

`xOrder` – An `int` scalar value specifying the order of the spline in the *x*-direction. `xOrder` must be greater than or equal to 1.

Default: `xOrder` = 4, implying a tensor-product cubic spline.

SetXWeights

```
virtual public void SetXWeights(double[] xWeights)
```

Description

Sets the weights for the least-squares fit in the *x*-direction.

Parameter

`xWeights` – A `double` array of size `xData.Length` specifying the weights for the least-squares fit in the *x*-direction.

Default: All weights are equal to 1.

SetYKnots

```
virtual public void SetYKnots(double[] yKnots)
```

Description

Sets the knot sequences of the spline in the *y*-direction.

Parameter

`yKnots` – A `double` array of size `ySplineSpaceDim` + `yOrder` specifying the knot sequences of the spline in the *y*-direction.

Default: Knot sequences are selected by the class.

SetYOrder

```
virtual public void SetYOrder(int yOrder)
```

Description

Sets the order of the spline in the *y*-direction.

Parameter

`yOrder` – An `int` scalar value specifying the order of the spline in the *y*-direction. `yOrder` must be greater than or equal to 1.

Default: `yOrder` = 4, implying a tensor-product cubic spline.

SetYWeights

```
virtual public void SetYWeights(double[] yWeights)
```

Description

Sets the weights for the least-squares fit in the *y*-direction.

Parameter

`yWeights` – A `double` array of size `yData.Length` specifying the weights for the least-squares fit in the *y*-direction.

Default: All weights are equal to 1.

Example

The data for this example comes from the function $e^x sin(x+y)$ on the rectangle [0, 3] x [0, 5]. This function is first sampled on a 50 x 25 grid. Next, an attempt to recover it by using tensor-product cubic splines is performed. The values of the function $e^x sin(x+y)$ are printed on a 2 x 2 grid and compared with the values of the tensor-product spline least-squares fit.

```
using System;
using Imsl.Math;

public class Spline2DLeastSquaresEx1 {

    private static double F(double x, double y) {
        return (Math.Exp(x) * Math.Sin(x + y));
    }

    public static void  Main(String[] args)
    {
        int nxData = 50, nyData = 25, outData = 2;
        double[] xData = new double[nxData];
        double[] yData = new double[nyData];
        double[,] fData = new double[nxData, nyData];
        double x, y, z;

        // Set up grid
        for (int i = 0; i < nxData; i++) {
            xData[i] = 3.0*(double) i / ((double) (nxData - 1));
        }
        for (int i = 0; i < nyData; i++) {
            yData[i] = 5.0*(double) i / ((double) (nyData - 1));
        }

        // Compute function values on grid
        for (int i = 0; i < nxData; i++) {
            for (int j = 0; j < nyData; j++) {
```

```
                fData[i, j] = F(xData[i], yData[j]);
            }
        }

        // Compute tensor-product approximant
        Spline2DLeastSquares spline =
                new Spline2DLeastSquares(xData, yData, fData, 5, 7);

        spline.Compute();
        x = spline.GetErrorSumOfSquares();

        // Print results
        Console.Out.WriteLine("The error sum of squares is " +
                x.ToString("0.0000") + "\n");

        double[,] output = new double[outData*outData, 5];
        for (int i = 0, idx = 0; i < outData; i++) {
            x = (double) i / (double) (outData);
            for (int j = 0; j < outData; j++) {
                y = (double) j / (double) (outData);
                z = spline.Value(x, y);
                output[idx, 0] = x;
                output[idx, 1] = y;
                output[idx, 2] = F(x,y);
                output[idx, 3] = z;
                output[idx, 4] = Math.Abs(F(x,y)-z);
                idx++;
            }
        }

        String[] labels = {"x", "y", "F(x, y)", "Fitted Values", "Error"};
        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.0000";
        pmf.SetColumnLabels(labels);
        new PrintMatrix().Print(pmf, output);
    }
}
```

Output

```
The error sum of squares is 3.7532

     x       y     F(x, y)  Fitted Values  Error
0  0.0000  0.0000  0.0000      -0.0204     0.0204
1  0.0000  0.5000  0.4794       0.5002     0.0208
2  0.5000  0.0000  0.7904       0.8158     0.0253
3  0.5000  0.5000  1.3874       1.3842     0.0031
```

RadialBasis Class

```
public class Imsl.Math.RadialBasis
```

`RadialBasis` computes a least-squares fit to scattered data in $\mathbf{R}^d$, where d is the dimension. More precisely, we are given data points

$$x_0, \ldots, x_{n-1} \in \mathbf{R}^d$$

and function values

$$f_0, \ldots, f_{n-1} \in \mathbf{R}^1$$

The radial basis fit to the data is a function F which approximates the above data in the sense that it minimizes the sum-of-squares error

$$\sum_{i=0}^{n-1} w_i \left(F(x_i) - f_i\right)^2$$

where w are the weights. Of course, we must restrict the functional form of F. Here we assume it is a linear combination of radial functions:

$$F(x) \equiv \sum_{j=0}^{m-1} \alpha_j \phi(\|x - c_j\|)$$

The c_j are the *centers*.

A radial function, $\phi(r)$, maps $[0, \infty)$ into $\mathbf{R}^1$. The default radial function is the Hardy multiquadric,

$$\phi(r) \equiv \sqrt{r^2 + \delta^2}$$

with $\delta = 1$. An alternate radial function is the Gaussian, e^{-ax^2}.

By default, the centers are points in a Faure sequence, scaled to cover the box containing the data.

Two `Update` methods allow the user to specify weights for each data point in the approximation scheme. In this way the user can influence the fit of the radial basis function. For example, if weights are in the range [0,1] then 0-weighted points are effectively removed from computations and 1-weighted points will have more influence than any others. When the number of centers equals the number of data points, the RBF fit will be "exact", otherwise it will be an approximation (useful for large or noisy data sets). Provided the ratios of the weights are not too extreme, weights will not appreciably change the accuracy of the fit to the data, but they will affect the shape of the approximating function away from the data: Greater weights result in greater influence at greater distances.

Properties

ANOVA

```
public Imsl.Stat.ANOVA ANOVA {get; }
```

Description

Returns the ANOVA statistics from the linear regression.

Property Value

An ANOVA table and related statistics.

See Also: Imsl.Stat.LinearRegression (p. 634), Imsl.Stat.ANOVA (p. 708)

RadialFunction

```
public Imsl.Math.RadialBasis.IFunction RadialFunction {get; set; }
```

Description

The radial function.

Property Value

A RadialBasis.IFunction which is the current radial function.

Constructor

RadialBasis

```
public RadialBasis(int nDim, int nCenters)
```

Description

Creates a new instance of RadialBasis.

Parameters

nDim – An int specifying the number of dimensions.

nCenters – An int specifying the number of centers.

Methods

Eval

```
public double Eval(double[] x)
```

Description

Returns the value of the radial basis approximation at a point.

Parameter

x – A double array containing the locations of the data point at which the approximation is to be computed.

Returns

A `double` containing the value of the radial basis approximation at `x`.

Eval

```
public double[] Eval(double[,] x)
```

Description

Returns the value of the radial basis at a point.

Parameter

`x` – A `double` matrix of size *n* by *nDim* containing the points at which the radial basis is to be evaluated.

Returns

A `double` array giving the value of the radial basis at the point `x`

Gradient

```
public double[] Gradient(double[] x)
```

Description

Returns the gradient of the radial basis approximation at a point.

Parameter

`x` – A `double` array containing the locations of the data point at which the approximation's gradient is to be computed.

Returns

A `double` array, of length `nDim` containing the value of the gradient of the radial basis approximation at `x`.

Update

```
public void Update(double[] x, double f)
```

Description

Adds a data point with weight = 1.

Parameters

`x` – A `double` array containing the locations of the data point.

`f` – A `double` containing the function value at the data point.

Update

```
public void Update(double[] x, double f, double w)
```

Description

Adds a data point with a specified weight.

Parameters

`x` – A `double` array containing the locations of the data point.

`f` – A `double` containing the function value at the data point.

`w` – A `double` containing the weight of this data point.

Update

```
public void Update(double[,] x, double[] f)
```

Description

Adds a set of data points, all with weight = 1.

Parameters

`x` – A `double` matrix of size *n* by *nDim* containing the locations of the data points for each dimension.

`f` – A `double` array containing the function values at the data points.

Update

```
public void Update(double[,] x, double[] f, double[] w)
```

Description

Adds a set of data points with user-specified weights.

Parameters

`x` – A `double` matrix of size *n* by *nDim* containing the locations of the data points for each dimension.

`f` – A `double` array containing the function values at the data points.

`w` – A `double` array containing the weights associated with the data points.

Example: Radial Basis Function Approximation

The function

$$e^{-\|\vec{x}\|^2/d}$$

where *d* is the dimension, is evaluated at a set of randomly chosen points. Random noise is added to the values and a radial basis approximated to the noisy data is computed. The radial basis fit is then compared to the original function at another set of randomly chosen points. Both the average error and the maximum error are computed and printed.

In this example, the dimension *d*=10. The function is sampled at 200 random points, in the $[-1,1]^d$ cube, to which what noise in the range [-0.2,0.2] is added. The error is computed at 1000 random points, also from the $[-1,1]^d$ cube. The compute errors are less than the added noise.

```
using System;
using Imsl.Math;

public class RadialBasisEx1
{
```

```
public static void  Main(String[] args)
{
    int nDim = 10;

    // Sample, with noise, the function at 100 randomly chosen points
    int nData = 200;
    double[,] xData = new double[nData,nDim];
    double[] fData = new double[nData];
    Imsl.Stat.Random rand = new Imsl.Stat.Random(234567);
    double[] tmp = new double[nDim];
    for (int k = 0; k < nData; k++)
    {
        for (int i = 0; i < nDim; i++)
        {
            tmp[i] = xData[k,i] = 2.0 * rand.NextDouble() - 1.0;
        }
        // noisy sample
        fData[k] =
            fcn(tmp) + 0.20 * (2.0 * rand.NextDouble() - 1.0);
    }

    // Compute the radial basis approximation using 25 centers
    int nCenters = 25;
    RadialBasis rb = new RadialBasis(nDim, nCenters);
    rb.Update(xData, fData);

    // Compute the error at a randomly selected set of points
    int nTest = 1000;
    double maxError = 0.0;
    double aveError = 0.0;
    double[] x = new double[nDim];
    for (int k = 0; k < nTest; k++)
    {
        for (int i = 0; i < nDim; i++)
        {
            x[i] = 2.0 * rand.NextDouble() - 1.0;
        }
        double error = Math.Abs(fcn(x) - rb.Eval(x));
        aveError += error;
        maxError = Math.Max(error, maxError);
        double f = fcn(x);
    }
    aveError /= nTest;

    Console.WriteLine("average error is " + aveError);
    Console.WriteLine("maximum error is " + maxError);
}


// The function to approximate
internal static double fcn(double[] x)
{
    double sum = 0.0;
    for (int k = 0; k < x.Length; k++)
    {
        sum += x[k] * x[k];
```

```
        }
        sum /= x.Length;
        return Math.Exp(-sum);
    }
}
```

Output

```
average error is 0.0419789795502543
maximum error is 0.171666811944547
```

Example: “Custom” Radial Basis Function Approximation

Data is generated from the function

$$e^{\frac{y}{2.0}} \sin x \cos \frac{y}{2.0}$$

where a number of (x,y) pairs make up a set of randomly chosen points. Random noise is added to the values, a “custom” Polyharmonic Spline radial basis function is specified

$$\varphi(r) = \begin{cases} r^k & k = 1,3,5,\ldots \\ r^k \ln r & k = 2,4,6,\ldots \end{cases}$$

and a radial basis approximation of the noisy data is computed. The radial basis fit is then compared to the original function at another set of randomly chosen points. Both the average normalized error and the maximum normalized error are computed and printed.

In this example, the order of the Polyharmonic Spline, k=2. The function is sampled at 200 random points and the error is computed at 10000 random points.

```
using System;
using Imsl.Math;

public class RadialBasisEx2
{

    // The function to approximate
    static double fcn(double[] x)
    {
        return Math.Exp((x[1]) / 2.0) * Math.Sin(x[0]) -
            Math.Cos((x[1]) / 2.0);
    }

    public class PolyHarmonicSpline : RadialBasis.IFunction
    {
        virtual public int Order
        {
            get
            {
                return order;
            }

        }
```

```
    virtual public bool EvenOrder
    {
        get
        {
            return isEven;
        }

    }

    private int order = 3;
    private bool isEven = false;

    public PolyHarmonicSpline(int order)
    {
        this.isEven = order % 2 == 0;
        this.order = order;
    }

    public virtual double F(double x)
    {
        if (this.isEven)
        {
            return Math.Pow(x, order) * Math.Log(x);
        }
        return Math.Pow(x, order);
    }

    public virtual double G(double x)
    {
        if (order == 1)
        {
            return 1;
        }
        if (this.isEven)
        {
            return order * Math.Pow(x, order - 1) * Math.Log(x) +
                Math.Pow(x, order - 1);
        }
        return order * Math.Pow(x, order - 1);
    }
}


public static void  Main(String[] args)
{
    int nDim = 2;

    // Sample, with noise, the function at 100 randomly chosen points
    int nData = 200;
    double[,] xData = new double[nData,nDim];
    double[] fData = new double[nData];
    double[] row = new double[nDim];
    Imsl.Stat.Random rand = new Imsl.Stat.Random(123457);
    rand.Multiplier = 16807;
    double[] noise = new double[nData * nDim];
    for (int k = 0; k < nData; k++)
```

```
        {
            for (int i = 0; i < nDim; i++)
            {
                noise[k * 2 + i] = 1.0d - 2.0d * rand.NextDouble();
                xData[k,i] = 3 * noise[k * 2 + i];
            }
            // noisy sample
            for(int j = 0; j<nDim; j++)
                row[j]=xData[k,j];
            fData[k] = fcn(row) + noise[k * 2] / 10;
        }

        // Compute the radial basis approximation using 100 centers
        int nCenters = 100;
        RadialBasis rb = new RadialBasis(nDim, nCenters);
        rb.RadialFunction = new PolyHarmonicSpline(2);
        rb.Update(xData, fData);

        // Compute the error at a randomly selected set of points
        int nTest = 10000;
        double maxError = 0.0;
        double aveError = 0.0;
        double maxMagnitude = 0.0;
        double[][] x = new double[nTest][];
        for (int i2 = 0; i2 < nTest; i2++)
        {
            x[i2] = new double[nDim];
        }
        noise = new double[nTest * nDim];

        for (int i = 0; i < nTest; i++)
        {
            for (int j = 0; j < nDim; j++)
            {
                noise[i * 2 + j] = 1.0d - 2.0d * (double) rand.NextDouble();
                x[i][j] = 3 * noise[i * 2 + j];
            }
            double error = Math.Abs(fcn(x[i]) - rb.Eval(x[i]));
            maxMagnitude = Math.Max(Math.Abs(fcn(x[i])), maxMagnitude);
            aveError += error;
            maxError = Math.Max(error, maxError);
        }
        aveError /= nTest;

        Console.WriteLine("Average normalized error is " + aveError /
            maxMagnitude);
        Console.WriteLine("Maximum normalized error is " + maxError /
            maxMagnitude);
        Console.WriteLine("Using even order equation: " +
            ((PolyHarmonicSpline) rb.RadialFunction).EvenOrder);
    }
}
```

Output

```
Average normalized error is 0.0180558727060151
Maximum normalized error is 0.257565310407464
Using even order equation: True
```

Example: Multiquadric Radial Basis Function Approximation

Data is generated from the function

$$e^{\frac{y}{2.0}} \sin x \cos \frac{y}{2.0}$$

where a number of (*x*,*y*) pairs make up a set of randomly chosen points. Random noise is added to the values, a Hardy multiquadric radial basis function is specified $\sqrt{r^2 + \delta^2}$ and a radial basis approximation of the noisy data is computed. The radial basis fit is then compared to the original function at another set of randomly chosen points. Both the average normalized error and the maximum normalized error are computed and printed.

In this example, the parameter of the Hardy multiquadric radial basis function $\delta = 5.5$. The function is sampled at 100 random points and the error is computed at 10000 random points.

```
using System;
using Imsl.Math;

public class RadialBasisEx3
{

    public static void  Main(String[] args)
    {
        int nDim = 2;

        // Sample, with noise, the function at 100 randomly chosen points
        int nData = 100;
        double[,] xData = new double[nData, nDim];
        double[] fData = new double[nData];
        double[] row = new double[nDim];

        Imsl.Stat.Random rand = new Imsl.Stat.Random(123457);
        rand.Multiplier = 16807;
        double[] noise = new double[nData * nDim];
        for (int k = 0; k < nData; k++)
        {
            for (int i = 0; i < nDim; i++)
            {
                noise[k * 2 + i] = 1.0d - 2.0d * (double) rand.NextDouble();
                xData[k, i] = 3 * noise[k * 2 + i];
            }
            // noisy sample
            for(int j = 0; j<nDim; j++)
                row[j]=xData[k,j];
            fData[k] = fcn(row) + noise[k * 2] / 10;
        }

        // Compute the radial basis approximation using 100 centers
        int nCenters = 100;
```

```
        RadialBasis rb = new RadialBasis(nDim, nCenters);
        rb.RadialFunction = new RadialBasis.HardyMultiquadric(5.5);
        rb.Update(xData, fData);

        // Compute the error at a randomly selected set of points
        int nTest = 10000;
        double maxError = 0.0;
        double aveError = 0.0;
        double maxMagnitude = 0.0;
        Imsl.WarningObject w = Imsl.Warning.WarningObject;
        Imsl.Warning.WarningObject = null;
        double[][] x = new double[nTest][];
        for (int i2 = 0; i2 < nTest; i2++)
        {
            x[i2] = new double[nDim];
        }
        noise = new double[nTest * nDim];

        for (int i = 0; i < nTest; i++)
        {
            for (int j = 0; j < nDim; j++)
            {
                noise[i * 2 + j] = 1.0d - 2.0d * rand.NextDouble();
                x[i][j] = 3 * noise[i * 2 + j];
            }
            double error = Math.Abs(fcn(x[i]) - rb.Eval(x[i]));
            maxMagnitude = Math.Max(Math.Abs(fcn(x[i])),
                maxMagnitude);
            aveError += error;
            maxError = Math.Max(error, maxError);
        }
        aveError /= nTest;

        Imsl.Warning.WarningObject = w;
        Console.WriteLine("Average normalized error is " +
            aveError / maxMagnitude);
        Console.WriteLine("Maximum normalized error is " +
            maxError / maxMagnitude);
    }

    // The function to approximate
    internal static double fcn(double[] x)
    {
        return Math.Exp((x[1]) / 2.0) * Math.Sin(x[0]) -
            Math.Cos((x[1]) / 2.0);
    }
}
```

Output

```
Average normalized error is 0.0110861431448538
Maximum normalized error is 0.054260728660165
```

Example: Gaussian Radial Basis Function Approximation

Data is generated from the function

$$e^{\frac{y}{2.0}} \sin x \cos \frac{y}{2.0}$$

where a number of (*x*,*y*) pairs make up a set of randomly chosen points. Random noise is added to the values, a Gaussian radial basis function is specified e^{-ax^2} and a radial basis approximation of the noisy data is computed. The radial basis fit is then compared to the original function at another set of randomly chosen points. Both the average normalized error and the maximum normalized error are computed and printed.

In this example, the parameter of the Gaussian radial basis function $a = 0.1$. The function is sampled at 100 random points and the error is computed at 10000 random points.

```
using System;
using Imsl.Math;

public class RadialBasisEx4
{

    public static void  Main(String[] args)
    {
        int nDim = 2;

        // Sample, with noise, the function at 100 randomly chosen points
        int nData = 100;
        double[,] xData = new double[nData,nDim];
        double[] fData = new double[nData];
        double[] row = new double[nDim];

        Imsl.Stat.Random rand = new Imsl.Stat.Random(123457);
        rand.Multiplier = 16807;
        double[] noise = new double[nData * nDim];
        for (int k = 0; k < nData; k++)
        {
            for (int i = 0; i < nDim; i++)
            {
                noise[k * 2 + i] = 1.0d - 2.0d * (double) rand.NextDouble();
                xData[k,i] = 3 * noise[k * 2 + i];
            }
            // noisy sample
            for(int j = 0; j<nDim; j++)
                row[j]=xData[k,j];
            fData[k] = fcn(row) + noise[k * 2] / 10;
        }

        // Compute the radial basis approximation using 100 centers
        int nCenters = 100;
        RadialBasis rb = new RadialBasis(nDim, nCenters);
        rb.RadialFunction = new RadialBasis.Gaussian(.1);
        rb.Update(xData, fData);

        // Compute the error at a randomly selected set of points
        int nTest = 10000;
```

```
        double maxError = 0.0;
        double aveError = 0.0;
        double maxMagnitude = 0.0;
        Imsl.WarningObject w = Imsl.Warning.WarningObject;
        Imsl.Warning.WarningObject = null;
        double[][] x = new double[nTest][];
        for (int i2 = 0; i2 < nTest; i2++)
        {
            x[i2] = new double[nDim];
        }
        noise = new double[nTest * nDim];

        for (int i = 0; i < nTest; i++)
        {
            for (int j = 0; j < nDim; j++)
            {
                noise[i * 2 + j] = 1.0d - 2.0d * rand.NextDouble();
                x[i][j] = 3 * noise[i * 2 + j];
            }
            double error = Math.Abs(fcn(x[i]) - rb.Eval(x[i]));
            maxMagnitude = Math.Max(Math.Abs(fcn(x[i])),
                maxMagnitude);
            aveError += error;
            maxError = Math.Max(error, maxError);
        }
        aveError /= nTest;

        Imsl.Warning.WarningObject = w;
        Console.WriteLine("Average normalized error is " +
            aveError / maxMagnitude);
        Console.WriteLine("Maximum normalized error is " +
            maxError / maxMagnitude);
    }

    // The function to approximate
    internal static double fcn(double[] x)
    {
        return Math.Exp((x[1]) / 2.0) * Math.Sin(x[0]) -
            Math.Cos((x[1]) / 2.0);
    }
}
```

Output

```
Average normalized error is 0.0109549594616543
Maximum normalized error is 0.0230039785369829
```

RadialBasis.IFunction Interface

```
public interface Imsl.Math.RadialBasis.IFunction
```

Public interface for the user supplied function to the `RadialBasis` object.

Methods

F

```
abstract public double F(double x)
```

Description

A radial basis function.

Parameter

`x` – A `double`, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at `x`.

G

```
abstract public double G(double x)
```

Description

The derivative of the radial basis function used to calculate the gradient of the radial basis approximation.

Parameter

`x` – A `double`, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at `x`.

RadialBasis.Gaussian Class

```
public class Imsl.Math.RadialBasis.Gaussian :  Imsl.Math.RadialBasis.IFunction
```

The Gaussian basis function, e^{-ax^2}.

Constructor

Gaussian

```
public Gaussian(double a)
```

Description

Creates a Gaussian basis function e^{-ax^2}.

Parameter

`a` – A `double` specifying the value of the function parameter. Decreasing the Gaussian parameter decreases fitting-error but may increase computational effort.

Methods

F

```
Final public double F(double x)
```

Description

A Gaussian basis function.

Parameter

`x` – A `double`, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at `x`.

G

```
Final public double G(double x)
```

Description

The derivative of the Gaussian basis function used to calculate the `Gradient` of the radial basis approximation.

Parameter

`x` – A `double`, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at `x`.

RadialBasis.HardyMultiquadric Class

```
public class Imsl.Math.RadialBasis.HardyMultiquadric :
Imsl.Math.RadialBasis.IFunction
```

The Hardy multiquadric basis function, $\sqrt{r^2 + \delta^2}$.

Constructor

HardyMultiquadric

```
public HardyMultiquadric(double delta)
```

Description

Creates a Hardy multiquadric basis function $\sqrt{r^2 + \delta^2}$.

Parameter

`delta` – A `double` specifying the value of the function parameter. Increasing the multiquadric parameter decreases fitting-error but generally increases computational effort.

Default: delta = 1.0.

Methods

F

```
Final public double F(double x)
```

Description

A Hardy multiquadric basis function.

Parameter

`x` – A `double`, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at `x`.

G

```
Final public double G(double x)
```

Description

The derivative of the Hardy multiquadric basis function used to calculate the `Gradient` of the radial basis approximation.

Parameter

`x` – A `double`, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at `x`.

Chapter 4: Quadrature

Types

Usage Notes

Univariate Quadrature

Class Quadrature computes approximations to integrals of the form

$$\int_c^b f(x)dx$$

One situation that occasionally arises in univariate quadrature concerns the approximation of integrals when only tabular data are given. The functions described above do not directly address this question. However, the standard method for handling this problem is first to interpolate the data, and then to integrate the interpolant. This can be accomplished by using a IMSL C# Library spline interpolation class derived from `Imsl.Math.Spline` and the method `Imsl.Math.Spline.Integral` (a,b)

Multivariate Quadrature

The class HypercubeQuadrature computes an approximation to the integral of a function of *n* variables over a hyper-rectangle.

$$\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)dx_n \ldots dx_1$$

Return Values from User-Supplied Functions

All values returned by user-supplied functions must be valid real numbers. It is the user's responsibility to check that the values returned by a user-supplied function do not contain NaN, infinity, or negative infinity values.

Example: Minimum of a smooth function

The minimum of $e^x - 5x$ is found using function evaluations only.

```
using System;
using Imsl.Math;

public class OptimizationIntroEx1 : MinUncon.IFunction
{
   public double F(double x)
   {
      double y = Math.Exp(x) - 5.0 * x;
      if (!(Double.IsNaN(y)))
      {
         return y;
      }
      else
      {
         return 0.0;
      }
   }

   public static void Main(String[] args)
   {
      MinUncon zf = new MinUncon();
      zf.Guess = 0.0;
      zf.Accuracy = 0.001;
      MinUncon.IFunction fcn = new OptimizationIntroEx1();
      Console.Write("Minimum is " + zf.ComputeMin(fcn));
   }
}
```

Quadrature Class

`public class Imsl.Math.Quadrature`

`Quadrature` is a general-purpose integrator that uses a globally adaptive scheme in order to reduce the absolute error.

`Quadrature` subdivides the interval [*A*, *B*] and uses a $(2k+1)$-point Gauss-Kronrod rule to estimate the integral over each subinterval. The error for each subinterval is estimated by comparison with the *k*-point Gauss quadrature rule. The subinterval with the largest estimated error is then bisected and the same procedure is applied to both halves. The bisection process is continued until either the error criterion is satisfied, roundoff error is detected, the subintervals become too small, or the maximum number of subintervals allowed is reached. The Class `Quadrature` is based on the subroutine `QAG` by Piessens et al. (1983).

If the function to be integrated has endpoint singularities then extrapolation should be enabled. As described above, the integral's value is approximated by applying a quadrature rule to a series of

subdivisions of the interval. The sequence of approximate values converges to the integral's value. The ε-algorithm can be used to extrapolate from the initial terms of the sequence to its limit. Without extrapolation, the quadrature approximation sequence converges slowly if the function being integrated has endpoint singularities. The ε-algorithm accelerates convergence of the sequence in this case. The class EpsilonAlgorithm (p. 513) implements the ε-algorithm. With extrapolation, this class is similar to the subroutine `QAGS` by Piessens et al. (1983).

The desired absolute error, ε, can be set using `AbsoluteError` property. The desired relative error, ρ, can be set using `RelativeError` property. The method `Eval` computes the approximate integral value $R \approx \int_a^b f(x)dx$. It also computes an error estimate E, which can be retrieved using `ErrorEstimate` property. These are related by the following equation:

$$\left| \int_a^b f(x)dx - R \right| \leq E \leq \max \left\{ \varepsilon, \rho \left| \int_a^b f(x)dx \right| \right\}$$

Properties

AbsoluteError

```
public double AbsoluteError {get; set; }
```

Description

The absolute error tolerance.

Property Value

A `double` scalar value specifying the absolute error.

ErrorEstimate

```
public double ErrorEstimate {get; }
```

Description

Returns an estimate of the relative error in the computed result.

Property Value

A `double` specifying an estimate of the relative error in the computed result.

ErrorStatus

```
public int ErrorStatus {get; }
```

Description

Returns the non-fatal error status.

Property Value

An `int` specifying the non-fatal error status:

Status	Meaning
0	No error.
1	Maximum number of subdivisions allowed has been achieved. One can allow more subdivisions by setting the `MaxSubintervals`. If this yields no improvement it is advised to analyze the integrand in order to determine the integration difficulties. If the position of a local difficulty can be determined (e.g. singularity, discontinuity within the interval) one will probably gain from splitting up the interval at this point and calling the integrator on the subranges. If possible, an appropriate special-purpose integrator should be used, which is designed for handling the type of difficulty involved.
2	The occurrence of roundoff error is detected, which prevents the requested tolerance from being achieved. The error may be underestimated.
3	Extremely bad integrand behavior occurs at some points of the integration interval.
4	The algorithm does not converge. Roundoff error is detected in the extrapolation table. It is presumed that the requested tolerance cannot be obtained.
5	The algorithm does not converge. Roundoff error is detected in the extrapolation table. It is presumed that the requested tolerance cannot be achieved, and that the returned result is the best which can be obtained.
6	The integral is probably divergent, or slowly convergent. It must be noted that divergence can occur with any other status value.

Extrapolation

```
public bool Extrapolation {get; set; }
```

Description

If `true`, the epsilon-algorithm for extrapolation is enabled.

Property Value

A `bool`, `true` if the epsilon-algorithm for extrapolation is to be enabled, `false` otherwise.

Remarks

The default is `false` (extrapolation is not used).

MaxSubintervals

```
public int MaxSubintervals {get; set; }
```

Description

The maximum number of subintervals allowed.

Property Value

An `int` specifying the maximum number of subintervals to be allowed.

Remarks

The default value is 500.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Parallel

```
public bool Parallel {get; set; }
```

Description

Enable or disable performing `Quadrature.IFunction.F` in parallel.

Property Value

A `bool` indicating whether or not the `Quadrature.IFunction.F` calculations are to be performed in parallel.

Remarks

By default, `Parallel` = `true`. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

RelativeError

```
public double RelativeError {get; set; }
```

Description

The relative error tolerance.

Property Value

A `double` scalar value specifying the relative error.

Rule

```
public int Rule {get; set; }
```

Description

The Gauss-Kronrod rule.

Property Value

An `int` specifying the rule to be used.

Remarks

The default is 3.

Rule	Data points used
1	7 - 15
2	10 - 21
3	15 - 31
4	20 - 41
5	25 - 51
6	30 - 61

Constructor

Quadrature

```
public Quadrature()
```

Description

Constructs a `Quadrature` object.

Method

Eval

```
public double Eval(Imsl.Math.Quadrature.IFunction f, double a, double b)
```

Description

Returns the value of the integral from a to b.

Parameters

`f` – The function to be integrated.

`a` – A `double` specifying the lower limit of integration.

`b` – A `double` specifying the upper limit of integration, either or both of a and b can be `Double.POSITIVE_INFINITY` or `Double.NEGATIVE_INFINITY`.

Returns

A `double` specifying the integral value from a to b.

Example 1: Integral of exp(2x)

The integral $\int_1^3 e^{2x}\,dx$ is computed and compared to its expected value.

```
using System;
using Imsl.Math;

public class QuadratureEx1 : Quadrature.IFunction
{
    public double F(double x)
    {
        return Math.Exp(2.0 * x);
    }

    public static void  Main(String[] args)
    {
        Quadrature q = new Quadrature();
        Quadrature.IFunction fcn = new QuadratureEx1();
        double result = q.Eval(fcn, 1.0, 3.0);

        double expect =
            (System.Math.Exp(6) - System.Math.Exp(2)) / 2.0;
        Console.Out.WriteLine("result = " + result);
        Console.Out.WriteLine("expect = " + expect);
    }
}
```

Output

```
result = 198.019868696902
expect = 198.019868696902
```

Example 2: Integral of exp(-x) from 0 to infinity

The integral $\int_0^\infty e^{-x}\,dx$ is computed and compared to its expected value.

```
using System;
using Imsl.Math;

public class QuadratureEx2 : Quadrature.IFunction
{
    public double F(double x)
    {
        return Math.Exp(- x);
    }


    public static void  Main(String[] args)
    {
        Quadrature q = new Quadrature();
        Quadrature.IFunction fcn = new QuadratureEx2();
        double result = q.Eval(fcn, 0.0, Double.PositiveInfinity);

        double expect = 1.0;
        Console.Out.WriteLine("result = " + result);
        Console.Out.WriteLine("expect = " + expect);
    }
}
```

Output

```
result = 0.999999999999999
expect = 1
```

Example 3: Integral of the entire real line

The integral $\int_{-\infty}^{\infty} \frac{x}{4e^x+9e^{-x}}\,dx$ is computed and compared to its expected value. This integral is evaluated in Gradshteyn and Ryzhik (equation 3.417.1).

```
using System;
using Imsl.Math;

public class QuadratureEx3 : Quadrature.IFunction
{
    public double F(double x)
    {
        return x / (4.0 * Math.Exp(x) + 9.0 * Math.Exp(-x));
    }


    public static void  Main(String[] args)
    {
        Quadrature q = new Quadrature();
        Quadrature.IFunction fcn = new QuadratureEx3();
        double result = q.Eval(fcn, Double.NegativeInfinity,
            Double.PositiveInfinity);

        double expect = System.Math.PI * System.Math.Log(1.5) / 12.0;
        Console.Out.WriteLine("result = " + result);
        Console.Out.WriteLine("expect = " + expect);
    }
}
```

Output

```
result = 0.106150517076628
expect = 0.106150517076633
```

Reference

Gradshteyn, I. S. and I. M. Ryzhik (1965), *Table of Integrals, Series, and Products*, Academic Press, New York.

Example 4: Integral of an oscillatory function

The integral of cos(ax) for $a = 10^4$ is computed and compared to its expected value. Because the function is highly oscillatory, the quadrature rule is set to 6. The relative error tolerance is also set.

```
using System;
```

```
using Imsl.Math;

public class QuadratureEx4 : Quadrature.IFunction
{
    private double a;

    public QuadratureEx4(double a)
    {
        this.a = a;
    }

    public double F(double x)
    {
        return Math.Cos(a * x);
    }

    public static void  Main(String[] args)
    {
        double a = 1.0e4;
        Quadrature.IFunction fcn = new QuadratureEx4(a);

        Quadrature q = new Quadrature();
        q.Rule = 6;
        q.RelativeError = 1e-10;
        double result = q.Eval(fcn, 0.0, 1.0);

        double expect = Math.Sin(a) / a;
        Console.Out.WriteLine("result = " + result);
        Console.Out.WriteLine("expect = " + expect);
        Console.Out.WriteLine
            ("relative error = " + (expect - result) / expect);
        Console.Out.WriteLine
            ("relative error estimate = " + q.ErrorEstimate);
    }
}
```

Output

```
result = -3.05614388902539E-05
expect = -3.05614388888252E-05
relative error = -4.67475872674725E-11
relative error estimate = 1.04883755430041E-08
```

Quadrature.IFunction Interface

`public interface Imsl.Math.Quadrature.IFunction`

Interface defining function for the `Quadrature` class.

Method

F

```
abstract public double F(double x)
```

Description

Function to be integrated.

Parameter

`x` – A `double` specifying the point at which the function is to be evaluated.

Returns

A `double` specifying the value of the function at x.

HyperRectangleQuadrature Class

```
public class Imsl.Math.HyperRectangleQuadrature
```

HyperRectangleQuadrature integrates a function over a hypercube.

This class is used to evaluate integrals of the form:

$$\int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_0}^{b_0} f(x_0, \ldots, x_{n-1})\, dx_0 \ldots dx_{n-1}$$

Integration of functions over hypercubes by Monte Carlo, in which the integral is evaluated as the value of the function averaged over a sequence of randomly chosen points. Under mild assumptions on the function, this method will converge like $1/\sqrt{n}$, where n is the number of points at which the function is evaluated.

It is possible to improve on the performance of Monte Carlo by carefully choosing the points at which the function is to be evaluated. Randomly distributed points tend to be non-uniformly distributed. The alternative to a sequence of random points is a low-discrepancy sequence. A low-discrepancy sequence is one that is highly uniform.

This function is based on the low-discrepancy Faure sequence as computed by Imsl.Stat.FaureSequence (p. 1216).

Properties

AbsoluteError

```
public double AbsoluteError {get; set; }
```

Description

Sets the absolute error tolerance.

Property Value

A `double` scalar value specifying the absolute error tolerance.

ErrorEstimate

```
public double ErrorEstimate {get; }
```

Description

Returns an estimate of the relative error in the computed result.

Property Value

A `double` specifying an estimate of the relative error in the computed result.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Parallel

```
public bool Parallel {get; set; }
```

Description

Enable or disable performing `HyperRectangleQuadrature.IFunction.F` in parallel.

Property Value

A `bool` indicating whether or not the `HyperRectangleQuadrature.IFunction.F` calculations are to be performed in parallel.

Remarks

By default, `Parallel` = `true`. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

RelativeError

```
public double RelativeError {get; set; }
```

Description

Sets the relative error tolerance.

Property Value

A `double` scalar value specifying the relative error tolerance.

Constructors

HyperRectangleQuadrature

```
public HyperRectangleQuadrature(int dimension)
```

Description

Constructs a HyperRectangleQuadrature object.

Parameter

`dimension` – A `int` which specifies the dimension of the Faure sequence.

HyperRectangleQuadrature

```
public HyperRectangleQuadrature(Imsl.Stat.IRandomSequence sequence)
```

Description

Constructs a `HyperRectangleQuadrature` object.

Parameter

`sequence` – A `IRandomSequence` object containing the random number sequence.

Methods

Eval

```
public double Eval(Imsl.Math.HyperRectangleQuadrature.IFunction f)
```

Description

Returns the value of the integral over the unit cube.

Parameter

`f` – A `IFunction` containing the function to be integrated.

Returns

A `double` containing the value of the integral over the unit cube.

Eval

```
public double Eval(Imsl.Math.HyperRectangleQuadrature.IFunction f, double[] a,
double[] b)
```

Description

Returns the value of the integral over a cube.

Parameters

`f` – A `IFunction` containing the function to be integrated.

`a` – A `double` specifying the lower limit of integration. If null all of the lower limits default to 0.

`b` – A `double` specifying the upper limit of integration. If null all of the upper limits default to 1.

Returns

A `double` containing the value of the integral over the unit cube.

Example: HyperRectangle Quadrature

This example evaluates the following multidimensional integral, with n=10.

$$\int_{a_{n-1}}^{b_{n-1}} \cdots \int_{a_0}^{b_0} \left[\sum_{i=0}^{n} (-1)^i \prod_{j=0}^{i} x_j \right] dx_0 \ldots dx_{n-1} = \frac{1}{3} \left[1 - \left(-\frac{1}{2} \right)^n \right]$$

```
using System;
using Imsl.Math;

public class HyperRectangleQuadratureEx1 :
    HyperRectangleQuadrature.IFunction
{
    public double F(double[] x)
    {
        int sign = 1;
        double sum = 0.0;
        for (int i = 0; i < x.Length; i++)
        {
            double prod = 1.0;
            for (int j = 0; j <= i; j++)
            {
                prod *= x[j];
            }
            sum += sign * prod;
            sign = - sign;
        }
        return sum;
    }

    public static void  Main(String[] args)
    {
        HyperRectangleQuadrature q = new HyperRectangleQuadrature(10);
        double result = q.Eval(new HyperRectangleQuadratureEx1());
        Console.Out.WriteLine("result = " + result);
    }
}
```

Output

```
result = 0.333125383208954
```

HyperRectangleQuadrature.IFunction Interface

`public interface Imsl.Math.HyperRectangleQuadrature.IFunction`

Interface for the HyperRectangleQuadrature function.

Method

F

`abstract public double F(double[] x)`

Description

Returns the value of the function at the given point.

Parameter

`x` – A `double` array specifying the point at which the function is to be evaluated.

Returns

A `double` specifying the value of the function at `x`.

Chapter 5: Differential Equations

Types

Usage Notes

Ordinary Differential Equations

An *ordinary differential equation* is an equation involving one or more dependent variables called y_i, one independent variable, t, and derivatives of the y_i with respect to t.

In the *initial-value problem* (IVP), the initial or starting values of the dependent variables y_i at a known value $t = t_0$ are given. Values of $y_i(t)$ for $t > 0$ or $t < t_0$ are required.

The `OdeRungeKutta` class solves the IVP for ODEs of the form

$$\frac{dy_i}{dt} = y_i' = f_i(t, y_1, \ldots, y_N) \quad i = 1, \ldots, N$$

with $y_i = (t = t_0)$ specified. Here, f_i is a user-supplied function that must be evaluated at any set of

values $(t, y_1, \ldots, y_N), i = 1, \ldots, N$.

This problem statement is abbreviated by writing it as a system of first-order ODEs,

$$y(t)\,[y_1(t), \ldots, y_N(t)]^T, [f_1(t,y), \ldots, f_N(t,y)]^T$$

so that the problem becomes $y' = f(t,y)$ with initial values $y(t_0)$.

The system

$$\frac{dy}{dt} = y' = f(t,y)$$

is said to be *stiff* if some of the eigenvalues of the Jacobian matrix

$$\left\{\partial y_i'/\partial y_j\right\}$$

are large and negative. This is frequently the case for differential equations modeling the behavior of physical systems, such as chemical reactions proceeding to equilibrium where subspecies effectively complete their reactions in different epochs. An alternate model concerns discharging capacitors such that different parts of the system have widely varying decay rates (or *time constants*).

Users typically identify stiff systems by the fact that numerical differential equation solvers such as `OdeRungeKutta` are inefficient, or else completely fail. Special methods are often required. The most common inefficiency is that a large number of evaluations of *f(t, y)* (and hence an excessive amount of computer time) are required to satisfy the accuracy and stability requirements of the software.

Partial Differential Equations

The `FeynmanKac` class solves the Feynman-Kac equation, a single partial differential equation, on a finite interval $[x_{\min}, x_{\max}]$. This equation often arises in applications from financial engineering and that is the primary focus of the documentation examples. The equation, initial conditions and Feynman-Kac boundary values are given by

$$f_t + \mu(x,t) + \frac{\sigma^2(x,t)}{2} f_{xx} - \kappa(x,t) f = \phi(f,x,t),$$

$$f(x,T) = p(x), \{f_t = \frac{\partial f}{\partial t}, \text{etc.}\},$$

$$a(x,t) f + b(x,t) f_x + c(x,t) f_{xx} = d(x,t),\ x = x_{\min} \text{ or } x = x_{\max}.$$

The solution is approximated by a piece-wise series of Hermite quintic polynomials on a grid of the interval $[x_{\min}, x_{\max}]$ that yields a twice differentiable solution.
To assist in the evaluation of the approximate solution and its derivatives there is method `GetSplineValue`.

ODE Class

```
public class Imsl.Math.ODE
```

ODE represents and solves an initial-value problem for ordinary differential equations.

Properties

Floor

```
virtual public double Floor {get; set; }
```

Description

The value used in the norm computation.

Property Value

A `double` used in the norm computation.

Default: `Floor` = 1.0.

Remarks

`Floor` must be greater than zero.

InitialStepsize

```
virtual public double InitialStepsize {get; set; }
```

Description

The initial internal step size.

Property Value

A `double` specifying the initial internal step size.

Default: `InitialStepsize` = 0.0.

Remarks

`InitialStepsize` must be greater than or equal to zero.

MaxSteps

```
virtual public int MaxSteps {get; set; }
```

Description

The maximum number of internal steps allowed.

Property Value

An `int` specifying the maximum number of internal steps allowed.

Default: `MaxSteps` = 500.

Remarks

`MaxSteps` must be greater than zero.

MinimumStepsize

```
virtual public double MinimumStepsize {get; set; }
```

Description

The minimum internal step size.

Property Value

A `double` specifying the minimum internal step size.

Default: `MinimumStepsize` = 0.0.

Remarks

`MinimumStepsize` must be greater than or equal to zero.

NormMethod

```
virtual public Imsl.Math.ODE.ErrorNormOptions NormMethod {get; set; }
```

Description

The error norm.

Property Value

An `ODE.ErrorNormOptions` specifying the error norm.

Default: `NormMethod = ODE.ErrorNormOptions.MinAbsRel`.

Remarks

`NormMethod` must be one of the values specified in the table which follows. In the following table, e_i is the absolute value for an estimate of the error in $y_i(t)$.

value	Constraint
`MinAbsRel`	Minimum of the absolute error and the relative error, equals the maximum of $e_i/max(\|y_i(t)\|, 1)$
`Abs`	Absolute error, equals $max(e_i)$
`Max`	Maximum of $e_i/max(\|y_i(t)\|, floor)$
`Euclidean`	Scaled Euclidean norm defined as $$s = \sqrt{\sum_{i=1}^{neq} \frac{{e_i}^2}{{w_i}^2}}$$ where $w_i = e_i/max(\|y_i(t)\|, 1.0)$ and neq is the number of equations

Scale

```
virtual public double Scale {get; set; }
```

Description

The scaling factor.

Property Value

A `double` specifying the scaling factor.

Default: `Scale` = 1.0.

Remarks

`Scale` must be greater than zero.

Tolerance

```
virtual public double Tolerance {get; set; }
```

Description

The error tolerance.

Property Value

A `double` specifying the error tolerance.

Default: `Tolerance` = 1.0e-6.

Remarks

`Tolerance` must be greater than zero.

Constructor

ODE

```
ODE()
```

Description

Initializes a new instance of the Imsl.Math.ODE (p. 227) class.

Methods

ExamineStep

```
virtual void ExamineStep(Imsl.Math.ODE.ExamineStepOptions state, double t,
double[] y)
```

Description

Called before and after each internal step.

Parameters

`state` – An `ODE.ExamineStepOptions`, one of `BeforeStep`, `AfterSuccessfulStep` or `AfterUnsuccessfulStep`.

`t` – A `double` representing the independent variable.

`y` – A `double` array containing the dependent variables.

Remarks

This method can be over-ridden by the user to examine intermediate values of `t` and `y`.

GetMaximumStepsize

```
virtual public double GetMaximumStepsize()
```

Description

Returns the maximum internal step size.

Returns

A `double` specifying the maximum internal step size.

SetMaximumStepsize

```
virtual public void SetMaximumStepsize(double stepsize)
```

Description

Sets the maximum internal step size.

Parameter

`stepsize` – A `double` specifying the maximum internal step size. `stepsize` must be greater than zero.

Remarks

Default: See `SetMaximumStepsize` in the subclasses for the default values used.

Vnorm

```
virtual double Vnorm(double[] v, double[] y, double[] ymax)
```

Description

Returns the norm of a vector.

Parameters

`v` – A `double` array containing the vector whose norm is to be computed.

`y` – A `double` array containing the values of the dependent variable.

`ymax` – A `double` array containing the maximum `y` values computed thus far.

Returns

A `double` scalar value representing the norm of the vector `v`.

Remarks

This method can be over-ridden by the user to supply a different norm than those available through `ODE.ErrorNormOptions`, see `NormMethod` property.

ODE.ExamineStepOptions Enumeration

`public enumeration Imsl.Math.ODE.ExamineStepOptions`

`ExamineStep` options

Fields

AfterSuccessfulStep

`public Imsl.Math.ODE.ExamineStepOptions AfterSuccessfulStep`

Description

Indicates examining after a successful step.

AfterUnsuccessfulStep

`public Imsl.Math.ODE.ExamineStepOptions AfterUnsuccessfulStep`

Description

Indicates examining after an unsuccessful step.

BeforeStep

`public Imsl.Math.ODE.ExamineStepOptions BeforeStep`

Description

Indicates examining before the next step.

ODE.ErrorNormOptions Enumeration

`public enumeration Imsl.Math.ODE.ErrorNormOptions`

`ErrorNorm` options

Fields

Abs

```
public Imsl.Math.ODE.ErrorNormOptions Abs
```

Description

Indicates that the error norm to be used is to be the absolute error, equals $max(|e_i|)$.

Euclidean

```
public Imsl.Math.ODE.ErrorNormOptions Euclidean
```

Description

Indicates that the error norm to be used is to be the scaled Euclidean norm defined as

$$s = \sqrt{\sum_{i=1}^{neq} \frac{{e_i}^2}{{w_i}^2}}$$

where $w_i = e_i / max(|y_i(t)|, 1.0)$ and *neq* is the number of equations.

Max

```
public Imsl.Math.ODE.ErrorNormOptions Max
```

Description

Indicates that the error norm to be used is to be the maximum of $e_i / max(|y_i(t)|, floor)$ where *floor* is set via the `Floor` property.

MinAbsRel

```
public Imsl.Math.ODE.ErrorNormOptions MinAbsRel
```

Description

Indicates that the error norm to be used is to be the minimum of the absolute error and the relative error, equals the maximum of $e_i / max(|y_i(t)|, 1)$.

OdeRungeKutta Class

```
public class Imsl.Math.OdeRungeKutta :  ODE
```

Solves an initial-value problem for ordinary differential equations using the Runge-Kutta-Verner fifth-order and sixth-order method.

Class `OdeRungeKutta` finds an approximation to the solution of a system of first-order differential equations of the form $\frac{dy}{dt} = y' = f(t,y)$ with given initial data. The class attempts to keep the global error proportional to a user-specified tolerance. This class is efficient for nonstiff systems where the derivative evaluations are not expensive.

`OdeRungeKutta` is based on a code designed by Hull, Enright and Jackson (1976, 1977). It uses Runge-Kutta formulas of order five and six developed by J. H. Verner.

Constructor

OdeRungeKutta

```
public OdeRungeKutta(Imsl.Math.OdeRungeKutta.IFunction function)
```

Description

Constructs an ODE solver to solve the initial value problem *dy/dt = f(t,y)*.

Parameter

`function` – Implementation of interface `IFunction` that defines the right-hand side function *f(t,y)*

Methods

SetMaximumStepsize

```
override public void SetMaximumStepsize(double stepsize)
```

Description

Sets the maximum internal step size.

Parameter

`stepsize` – A `double` specifying the maximum internal step size. `stepsize` must be greater than zero.

Default: `stepsize` = 2.

Solve

```
virtual public void Solve(double t, double tEnd, double[] y)
```

Description

Integrates the ODE system from `t` to `tEnd`.

Parameters

`t` – A `double` specifying the independent variable.

`tEnd` – A `double` specifying the value of `t` at which the solution is desired.

`y` – On input, a `double` array containing the initial values. On output, a `double` array containing the approximate solution.

Remarks

On all but the first call to `Solve`, the value of `t` must equal the value of `tEnd` from the previous call.

Exceptions

`Imsl.Math.DidNotConvergeException` is thrown if the number of internal steps exceeds `MaxSteps` (default 500). This can be an indication that the ODE system is stiff. This exception can also be thrown if the error tolerance condition could not be met.

`Imsl.Math.ToleranceTooSmallException` is thrown if the computation does not converge on some step.

Example: Runge-Kutta-Verner ordinary differential equation solver

An ordinary differential equation problem is solved using a solver which implements the Runge-Kutta-Verner method. The solution at time t=10 is printed.

```
using System;
using Imsl.Math;

public class OdeRungeKuttaEx1 : OdeRungeKutta.IFunction
{
    public void F(double t, double[] y, double[] yprime)
    {
        yprime[0] = 2.0 * y[0] * (1 - y[1]);
        yprime[1] = - y[1] * (1 - y[0]);
    }


    public static void  Main(String[] args)
    {
        double[] y = new double[]{1, 3};
        OdeRungeKutta q = new OdeRungeKutta(new OdeRungeKuttaEx1());
        int nsteps = 10;
        for (int k = 0; k < nsteps; k++)
        {
            q.Solve(k, k + 1, y);
        }
        Console.Out.WriteLine("Result = {" + y[0] + "," + y[1] + "}");
    }
}
```

Output

```
Result = {3.14434167651608,0.3488265985197}
```

OdeRungeKutta.IFunction Interface

`public interface Imsl.Math.OdeRungeKutta.IFunction`

Public interface for user supplied function to `OdeRungeKutta` object.

Method

F

```
abstract public void F(double t, double[] y, double[] yprime)
```

Description

Returns the value of the function at the given point.

Parameters

`t` – A `double`, the point at which the function is to be evaluated.

`y` – A `double` array which contains the dependent variable values.

`yprime` – A `double` array which contains the value of the function at (`t`,`y`).

OdeAdamsGear Class

```
public class Imsl.Math.OdeAdamsGear :  ODE
```

Extension of the ODE class to solve a stiff initial-value problem for ordinary differential equations using the Adams-Gear methods. Class `OdeAdamsGear` finds an approximation to the solution of a system of first-order differential equations of the form

$$\frac{dy}{dt} = y' = f(t,y)$$

with given initial conditions for y at the starting value for t. The class attempts to keep the global error proportional to a user-specified tolerance. The proportionality depends on the differential equation and the range of integration.

The code is based on using backward difference formulas not exceeding order five as outlined in Gear (1971) and implemented by Hindmarsh (1974). There is an optional use of the code that employs implicit Adams formulas. This use is intended for nonstiff problems with expensive functions $y' = f(t,y)$.

Properties

IntegrationMethod

```
virtual public Imsl.Math.OdeAdamsGear.IntegrationType IntegrationMethod {get;
set; }
```

Description

The integration method.

Property Value

An `OdeAdamsGear.IntegrationType` specifying the integration method to be used.

Default: `IntegrationMethod` = `OdeAdamsGear.IntegrationType.BDF`.

Remarks

`IntegrationMethod` must be one of the values specified in the table which follows.

value	Description
`Adams`	Use the implicit Adams method.
`BDF`	Use backward differentiation formula (BDF) methods.

MaximumFunctionEvaluations

```
virtual public int MaximumFunctionEvaluations {get; set; }
```

Description

The maximum number of function evaluations of y' allowed.

Property Value

An `int` specifying the maximum number of function evaluations of y' allowed.

Default: No limit is enforced. .

Remarks

`MaximumFunctionEvaluations` must be greater than zero.

MaxOrder

```
virtual public int MaxOrder {get; set; }
```

Description

The highest order formula to use of implicit `Adams` type or `BDF` type.

Property Value

An `int` specifying the highest order formula to use of implicit `Adams` type or `BDF` type.

Default: `MaxOrder` = 12 for `Adams` and `MaxOrder` = 5 for `BDF`.

Remarks

`MaxOrder` must be greater than zero.

NumberOfFcnEvals

```
virtual public int NumberOfFcnEvals {get; }
```

Description

Returns the number of function evaluations of y' made.

Property Value

An `int` specifying the number of function evaluations of y' made.

NumberOfJacobianEvals

```
virtual public int NumberOfJacobianEvals {get; }
```

Description

Returns the number of Jacobian matrix evaluations used.

Property Value

An `int` specifying the number of Jacobian matrix evaluations used.

NumberOfSteps

```
virtual public int NumberOfSteps {get; }
```

Description

Returns the number of internal steps taken.

Property Value

An `int` specifying the number of internal steps taken.

SolveMethod

```
virtual public Imsl.Math.OdeAdamsGear.SolveOption SolveMethod {get; set; }
```

Description

The method for solving the formula equations.

Property Value

An `OdeAdamsGear.SolveOption` specifying the method to be used for solving the formula equations.

Default: `SolveMethod` = `OdeAdamsGear.SolveOption.ChordComputedJacobian`.

Remarks

Note that if the problem is stiff and a chord or modified Newton method is most efficient, use `ChordUserJacobian` or `ChordComputedJacobian`. `SolveMethod` must be one of the values specified in the table which follows.

value	Description
`FunctionIteration`	Use a function iteration or successive substitution method.
`ChordUserJacobian`	Use a chord or modified Newton method and a user-supplied Jacobian.
`ChordComputedJacobian`	Use a chord or modified Newton method and a divided differences Jacobian.
`ChordComputedDiagonal`	Use a chord method and a diagonal matrix based on a directional directive.

Constructor

OdeAdamsGear

```
public OdeAdamsGear(Imsl.Math.OdeAdamsGear.IFunction function)
```

Description

Constructs an ODE solver to solve the initial value problem *dy/dt = f(t,y)*.

Parameter

`function` – Implementation of interface IFunction that defines the right-hand side function. $f(t,y)$

Methods

SetMaximumStepsize

```
override public void SetMaximumStepsize(double stepsize)
```

Description

Sets the maximum internal step size.

Parameter

`stepsize` – A `double` specifying the maximum internal step size. `stepsize` must be greater than zero.

Remarks

Default: `stepsize` = 1.7976931348623157e+308 .

Solve

```
virtual public void Solve(double t, double tEnd, double[] y)
```

Description

Integrates the ODE system from `t` to `tEnd`.

Parameters

`t` – A `double` specifying the independent variable.

`tEnd` – A `double` specifying the value of `t` at which the solution is desired.

`y` – On input, a `double` array containing the initial values. On output, a `double` array containing the approximate solution.

Remarks

On all but the first call to `Solve`, the value of `t` must equal the value of `tEnd` from the previous call.

Exceptions

`Imsl.Math.DidNotConvergeException` is thrown if the number of internal steps exceeds `MaxSteps` (default 500). This can be an indication that the ODE system is stiff. This exception can also be thrown if the error tolerance condition could not be met.

`Imsl.Math.ToleranceTooSmallException` is thrown if the computation does not converge on some step.

`Imsl.Math.MaxFcnEvalsExceededException` is thrown if the maximum number of function evaluations allowed has been exceeded.

`Imsl.Math.SingularMatrixException` is thrown if the factorization function encounters a singular matrix during LU decomposition.

Example: Adams-Gear ordinary differential equation solver

A mildly stiff ordinary differential equation problem is solved using a solver which implements the Adams-Gear method. The solution at time t=240 is printed.

```
using System;
using Imsl.Math;

public class OdeAdamsGearEx1
{
    private class MyFunction : OdeAdamsGear.IFunction
    {
        public MyFunction(double k1, double k2, double k3)
        {
            this.k1 = k1;
            this.k2 = k2;
            this.k3 = k3;
        }

        private double k1;
        private double k2;
        private double k3;
        public virtual double[] F(double t, double[] y)
        {
            double[] yprime = new double[y.Length];
            yprime[0] = - y[0] - y[0] * y[1] + k1 * y[1];
            yprime[1] = (- k2) * y[1] + k3 * (1.0 - y[1]) * y[0];
            return yprime;
        }
    }

    public static void  Main(String[] args)
    {
        double k1 = 294.0;
        double k2 = 3.0;
        double k3 = 0.01020408;

        OdeAdamsGear.IFunction f = new MyFunction(k1, k2, k3);

        double t = 0.0;
        double tend = 240.0;
        double[] y = {1.0, 0.0};
        OdeAdamsGear q = new OdeAdamsGear(f);
        q.NormMethod = OdeAdamsGear.ErrorNormOptions.Abs;
        q.SolveMethod =
            OdeAdamsGear.SolveOption.ChordComputedJacobian;
        q.Tolerance = 1e-3;
        q.Solve(t, tend, y);

        // Print Results

        Console.Out.WriteLine("Result = {{{0,5:0.####}, {1,5:0.####}}}",
            y[0], y[1]);
    }
}
```

Output

```
Result = {0.3924, 0.0013}
```

OdeAdamsGear.IntegrationType Enumeration

```
public enumeration Imsl.Math.OdeAdamsGear.IntegrationType
```

Integration type.

Fields

Adams

```
public Imsl.Math.OdeAdamsGear.IntegrationType Adams
```

Description

Use the implicit Adams method.

BDF

```
public Imsl.Math.OdeAdamsGear.IntegrationType BDF
```

Description

Use backward differentiation formula (BDF) methods.

OdeAdamsGear.SolveOption Enumeration

```
public enumeration Imsl.Math.OdeAdamsGear.SolveOption
```

Solve option.

Fields

ChordComputedDiagonal

```
public Imsl.Math.OdeAdamsGear.SolveOption ChordComputedDiagonal
```

Description

Use a chord method and a diagonal matrix based on a directional directive.

ChordComputedJacobian

```
public Imsl.Math.OdeAdamsGear.SolveOption ChordComputedJacobian
```

Description

Use a chord or modified Newton method and a Jacobian approximated by divided differences.

ChordUserJacobian

```
public Imsl.Math.OdeAdamsGear.SolveOption ChordUserJacobian
```

Description

Use a chord or modified Newton method and a user-supplied Jacobian.

FunctionIteration

```
public Imsl.Math.OdeAdamsGear.SolveOption FunctionIteration
```

Description

Use a function iteration or successive substitution method.

OdeAdamsGear.IFunction Interface

```
public interface Imsl.Math.OdeAdamsGear.IFunction
```

Public interface for user supplied function to `OdeAdamsGear` object.

Method

F

```
abstract public double[] F(double t, double[] y)
```

Description

Computes the value of the function $y' = f(t,y)$ at the given point.

Parameters

`t` – A `double`, the point at which the function is to be evaluated.

`y` – A `double` array which contains the dependent variable values.

Returns

A `double` array of length `y.Length` which contains the value of the function $\frac{dy}{dt} = y' = f(t,y)$.

OdeAdamsGear.IJacobian Interface

```
public interface Imsl.Math.OdeAdamsGear.IJacobian :
Imsl.Math.OdeAdamsGear.IFunction
```

Public interface for the user supplied function to evaluate the Jacobian matrix.

Method

Jacobian

```
abstract public double[,] Jacobian(double t, double[] y, double[] yprime)
```

Description

Used to compute the Jacobian of the function at `t`.

Parameters

`t` – A `double`, the point at which the function is to be evaluated.

`y` – A `double` array which contains the dependent variable values.

`yprime` – A `double` array which contains the value of the function $\frac{dy}{dt} = y' = f(t,y)$.

Returns

A `double` `y.Length` by `y.Length` matrix containing the value of the Jacobian of the function at `t`.

FeynmanKac Class

```
public class Imsl.Math.FeynmanKac
```

Solves the generalized Feynman-Kac PDE.

Class `FeynmanKac` solves the generalized Feynman-Kac PDE on a rectangular grid with boundary conditions using a finite element Galerkin method. The generalized Feynman-Kac differential equation has the form

$$f_t + \mu(x,t) f_x + \frac{\sigma^2(x,t)}{2} f_{xx} - \kappa(x,t) f = \phi(f,x,t),$$

where the initial data satisfies $f(x,T) = p(x)$. The derivatives are $f_t = \frac{\partial f}{\partial t}$, etc. Method `ComputeCoefficients` uses a finite element Galerkin method over the rectangle

$$[x_{\min}, x_{\max}] \times [\overline{T}, T]$$

in (x,t) to compute the approximate solution. The interval $[x_{\min}, x_{\max}]$ is decomposed with a grid

$$(x_{\min} =) x_1 < x_2 < \ldots < x_m (= x_{\max}).$$

On each subinterval the solution is represented by

$$f(x,t) = f_i b_0(z) + f_{i+1} b_0(1-z) + h_i f_i' b_1(z) - h_i f_{i+1}' b_1(1-z) + h_i^2 f_i'' b_2(z) + h_i^2 f_{i+1}'' b_2(1-z).$$

The values

$$f_i, f_i', f_i'', f_{i+1}, f_{i+1}', f_{i+1}''$$

are time-dependent coefficients associated with each interval. The basis functions b_0, b_1, b_2 are given for

$$x \in [x_i, x_{i+1}], h_i = x_{i+1} - x_i, z = (x - x_i)/h_i, z \in [0,1],$$

by

$$b_0(z) = -6z^5 + 15z^4 - 10z^3 + 1 = (1-z)^3(6z^2 + 3z + 1),$$
$$b_1(z) = -3z^5 + 8z^4 - 6z^3 + z = (1-z)^3 z(3z+1),$$
$$b_2(z) = \frac{1}{2}(-z^5 + 3z^4 - 3z^3 + z^2) = \frac{1}{2}(1-z)^3 z^2.$$

By adding the piece-wise definitions the unknown solution function may be arranged as the series

$$f(x,t) = \sum_{i=1}^{3m} y_i \beta_i(x),\ x \in [x_{\min}, x_{\max}],$$

where the time-dependent coefficients are defined by re-labeling:

$$y_{3i-2} = f_i, y_{3i-1} = f_i', y_{3i} = f_i'', i = 1, \ldots, m.$$

The Galerkin principle is then applied. Using the provided initial and boundary conditions leads to an Index 1 differential-algebraic equation for the coefficients y_i, $i = 1, \ldots, 3m$.

This system is integrated using the variable order, variable step algorithm `DDASLX` noted in Hanson, R. and Krogh, F. (2008), *Solving Constrained Differential-Algebraic Systems Using Projections*. Solution values and their time derivatives at a grid preceding time T, expressed in units of time remaining, are given back by methods `GetSplineCoefficients` and `GetSplineCoefficientsPrime`, respectively. For further details of deriving and solving the system see Hanson, R. (2008), *Integrating Feynman-Kac Equations Using Hermite Quintic Finite Elements*. To evaluate f or its partials $f_x, f_{xx}, f_{xxx}, f_t, f_{tx}, f_{txx}, f_{txxx}$ at any time point in the grid, use method `GetSplineValue`.

One useful application of the `FeynmanKac` class is financial analytics. This is illustrated in Example 2, which solves a diffusion model for call options which, in the special case $\alpha = 2$ reduces to the Black-Scholes (BS) model. Another useful application for the `FeynmanKac` class is the calculation of the Greeks, i.e. various derivatives of Feynman-Kac solutions applicable to, e.g., the pricing of options

and related financial derivatives. In Example 5, the `FeynmanKac` class is used to calculate eleven of the Greeks for the same diffusion model introduced in Example 2 in the special BS case. These Greeks are also calculated using the BS closed form Greek equations (see http://en.wikipedia.org/wiki/The_Greeks). The Feynman-Kac and BS solutions are output and compared. Example 5 illustrates that the `FeynmanKac` class can be used to explore the Greeks for a much wider class of financial models than can BS.

Properties

GaussLegendreDegree

```
public int GaussLegendreDegree {get; set; }
```

Description

The number of quadrature points used in the Gauss-Legendre quadrature formula.

Property Value

An `int` scalar, the degree of the polynomial used in the Gauss-Legendre quadrature.

Default: `GaussLegendreDegree = 6`.

Remarks

It is required that the degree of the polynomial is greater than or equal to 6.

InitialStepsize

```
public double InitialStepsize {get; set; }
```

Description

The starting step size for the integration.

Property Value

A `double` scalar, the starting step size used in the integrator.

Default: `InitialStepsize = -1.1102230246252e-16`.

Remarks

The starting step size must be less than zero since the integration is internally done from `t=0` to `t=tGrid[tGrid.Length-1]` in a negative direction.

MaximumBDFOrder

```
public int MaximumBDFOrder {get; set; }
```

Description

The maximum order of the backward differentiation formulas (BDF).

Property Value

An `int`, the maximum order of the BDF used in the integrator.

Default: `MaximumBdfOrder = 5`.

Remarks

The maximum order of the BDF must be greater than zero and less than 6.

MaximumStepsize

```
public double MaximumStepsize {get; set; }
```

Description

The maximum internal step size used by the integrator.

Property Value

A positive scalar of type `double`, the maximum internal step size.

Default: `MaximumStepsize = Double.MaxValue`, the largest possible machine number.

MaxSteps

```
public int MaxSteps {get; set; }
```

Description

The maximum number of internal steps allowed.

Property Value

An `int` scalar specifying the maximum number of internal steps allowed between each output point of the integration.

Default: `MaxSteps = 500000`.

Method

```
public Imsl.Math.FeynmanKac.PDEStepControlMethod Method {get; set; }
```

Description

The step control method used in the integration of the Feynman-Kac PDE.

Property Value

A `PDEStepControlMethod` value specifying the step control method used in the integration.

Default: `Method = FeynmanKac.PDEStepControlMethod.MethodOfSoederlind`.

Remarks

`PDEStepControlMethod`	*Description*
`MethodOfSoederlind`	Use method of Soederlind.
`MethodOfPetzold`	Use method from the original Petzold code DASSL.

TimeBarrier

```
public double TimeBarrier {get; set; }
```

Description

Sets or returns the barrier used for integration in the time direction.

Property Value

A `double` scalar, the time point beyond which the integrator should not integrate.

Default: `TimeBarrier = tGrid[tGrid.Length-1]`.

Remarks

This time barrier controls whether the integrator should integrate in the time direction beyond a special point, `TimeBarrier`, and then interpolate to get the Hermite quintic spline coefficients and its derivatives at the points in `tGrid`. It is required that `TimeBarrier` be greater than or equal to `tGrid[tGrid.Length-1]`.

Constructor

FeynmanKac

```
public FeynmanKac(Imsl.Math.FeynmanKac.IPdeCoefficients pdeCoeffs)
```

Description

Constructs a PDE solver to solve the Feynman-Kac PDE.

Parameter

`pdeCoeffs` – Implementation of interface IPdeCoefficients that computes the values of the Feynman-Kac coefficients at a given point `(t,x)`.

Methods

ComputeCoefficients

```
public void ComputeCoefficients(int numLeftBounds, int numRightBounds,
Imsl.Math.FeynmanKac.IBoundaries pdeBounds, double[] xGrid, double[] tGrid)
```

Description

Determines the coefficients of the Hermite quintic splines that represent an approximate solution for the Feynman-Kac PDE.

Parameters

`numLeftBounds` – An `int` scalar, the number of left boundary conditions. It is required that $1 \leq$ `numLeftBounds` ≤ 3.

`numRightBounds` – An `int` scalar, the number of right boundary conditions. It is required that $1 \leq$ `numRightBounds` ≤ 3.

`pdeBounds` – Implementation of interface `IBoundaries` that computes the boundary coefficients and terminal condition for given `(t,x)`.

`xGrid` – A `double` array containing the breakpoints for the Hermite quintic splines used in the `x` discretization. The length of `xGrid` must be at least 2, `xGrid.Length` $\geq$ 2, and the elements in `xGrid` must be in strictly increasing order.

`tGrid` – A `double` array containing the set of time points (in time-remaining units) where an approximate solution is returned. The elements in array `tGrid` must be positive and in strictly increasing order.

Exceptions

`Imsl.Math.ToleranceTooSmallException` is thrown if the absolute or relative error tolerances used in the integrator are too small.

`Imsl.Math.TooManyStepsException` is thrown if the integrator needs too many iteration steps.

`Imsl.Math.ErrorTestException` is thrown if the error test used in the integrator failed repeatedly.

`Imsl.Math.CorrectorConvergenceException` is thrown if the corrector failed to converge repeatedly.

`Imsl.Math.IterationMatrixSingularException` is thrown if one of the iteration matrices used in the integrator is singular.

`Imsl.Math.TimeIntervalTooSmallException` is thrown if the distance between an intermediate starting and end point for the integration is too small.

`Imsl.Math.TcurrentTstopInconsistentException` is thrown if during the integration the current integration time and given stepsize is inconsistent with the endpoint of the integration.

`Imsl.Math.TEqualsToutException` is thrown if during the integration process the actual integration time and the end time of the integration are identical.

`Imsl.Math.InitialConstraintsException` is thrown if at the initial integration point some of the constraints are inconsistent.

`Imsl.Math.ConstraintsInconsistentException` is thrown if during the integration process the constraints for the actual time point and given stepsize are inconsistent.

`Imsl.Math.SingularMatrixException` is thrown if one of the matrices used outside the integrator is singular.

`Imsl.Math.BoundaryInconsistentException` is thrown if the boundary conditions are inconsistent.

GetAbsoluteErrorTolerances

```
public double[] GetAbsoluteErrorTolerances()
```

Description

Returns absolute error tolerances.

Returns

A `double` array of length `3*xGrid.Length` containing absolute error tolerances for the solutions.

GetRelativeErrorTolerances

```
public double[] GetRelativeErrorTolerances()
```

Description

Returns relative error tolerances.

Returns

A `double` array of length `3*xGrid.Length` containing relative error tolerances for the solutions.

GetSplineCoefficients

```
public double[,] GetSplineCoefficients()
```

Description

Returns the coefficients of the Hermite quintic splines that represent an approximate solution of the Feynman-Kac PDE.

Returns

A `double` matrix of dimension `(tGrid.Length+1)` by `(3*xGrid.Length)` containing the coefficients of the Hermite quintic spline representation of the approximate solution for the Feynman-Kac PDE at time points `0, tGrid[0],...,tGrid[tGrid.Length-1]`. Setting `ntGrid = tGrid.Length` and `nxGrid = xGrid.Length` the approximate solution is given by

$$f(x,t) = \sum_{j=0}^{3*\text{nxGrid}-1} y_{ij}\beta_j(x) \text{ for } t = \text{tGrid}[i-1], i = 1,\ldots,\text{ntGrid}.$$

The representation for the initial data at `t=0` is

$$p(x) = \sum_{j=0}^{3*\text{nxGrid}-1} y_{0j}\beta_j(x).$$

The `(ntGrid+1)` by `(3*nxGrid)` matrix

$$(y_{ij})_{i=0,\ldots,\text{ntGrid}}^{j=0,\ldots,3*\text{nxGrid}-1}$$

is stored row-wise in the returned array.

Remarks

The `ComputeCoefficients` method must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetSplineCoefficientsPrime

```
public double[,] GetSplineCoefficientsPrime()
```

Description

Returns the first derivatives of the Hermite quintic spline coefficients that represent an approximate solution of the Feynman-Kac PDE.

Returns

A `double` matrix of dimension `(tGrid.Length+1)` by `(3*xGrid.Length)` containing the first derivatives (in time) of the coefficients of the Hermite quintic spline representation of the approximate solution for the Feynman-Kac PDE at time points `0, tGrid[0],...,tGrid[tGrid.Length-1]`. The approximate solution itself is given by

$$f_t(x,\bar{t}) = \sum_{j=0}^{3*\text{nxGrid}-1} y'_{ij}\beta_j(x) \quad \text{for } \bar{t} = \text{tGrid}[i-1], i = 1,\ldots,\text{ntGrid},$$

and

$$f_t(x,\bar{t}) = \sum_{j=0}^{3*\text{nxGrid}-1} y'_{0j}\beta_j(x) \quad \text{for } \bar{t} = 0\,.$$

The `(ntGrid+1)` by `(3*nxGrid)` matrix

$$(y'_{ij})_{i=0,\ldots,\text{ntGrid}}^{j=0,\ldots,3*\text{nxGrid}-1}$$

is stored row-wise in the returned array.

Remarks

The `ComputeCoefficients` method must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetSplineValue

```
public double[] GetSplineValue(double[] evaluationPoints, double[]
coefficients, int ideriv)
```

Description

Evaluates for time value 0 or a time value in `tGrid` the derivative of the Hermite quintic spline interpolant at evaluation points within the range of `xGrid`.

Parameters

`evaluationPoints` – A `double` array containing the points in `x`-direction at which the Hermite quintic spline representing the approximate solution to the Feynman-Kac PDE or one of its derivatives is to be evaluated. It is required that all elements in array `evaluationPoints` are greater than or equal to `xGrid[0]` and less than or equal to `xGrid[xGrid.Length-1]`.

`coefficients` – A `double` array of length `3*xGrid.Length` containing the coefficients of the Hermite quintic spline representing the approximate solution f or f_t to the Feynman-Kac PDE. These coefficients are the rows of the arrays `splineCoeffs` and `splineCoeffsPrime` returned by methods `GetSplineCoefficients` and `GetSplineCoefficientsPrime`. If the user wants to compute approximate solutions f or f_x, f_{xx}, f_{xxx} to the Feynman-Kac PDE at time point `0`, one must assign row `0` of `splineCoeffs` to array `coefficients`. If the user wants to compute these approximate solutions for time points `t=tGrid[i], i=0,...,tGrid.Length-1`, one must assign row `i+1` of `splineCoeffs` to array `coefficients`. The same reasoning applies to the computation of approximate solutions f_t and f_{tx}, f_{txx}, f_{txxx} and assignment of rows of array `splineCoeffsPrime` to array `coefficients`.

`ideriv` – An `int` specifying the derivative to be computed. It must be 0, 1, 2 or 3.

Returns

A `double` array containing the derivative of order `ideriv` of the Hermite quintic spline representing the approximate solution f or f_t to the Feynman Kac PDE at `evaluationPoints`. If `ideriv=0`, then the spline values are returned. If `ideriv=1`, then the first derivative is returned, etc.

SetAbsoluteErrorTolerances

```
public void SetAbsoluteErrorTolerances(double[] atol)
```

Description

Sets the absolute error tolerances.

Parameter

`atol` – A `double` array of length `3*xGrid.Length` specifying the absolute error tolerances for the row-wise solutions returned by method `GetSplineCoefficients`. All entries in `atol` must be greater than or equal to zero. Also, not all entries in `atol` and `rtol` are allowed to be equal to 0 simultaneously.

Default: `atol[i]` = 1.0e-5 for `i=0,...,3*xGrid.Length-1`.

SetAbsoluteErrorTolerances

```
public void SetAbsoluteErrorTolerances(double atol)
```

Description

Sets the absolute error tolerances.

Parameter

`atol` – A `double` scalar specifying the absolute error tolerances for the row-wise solutions returned by method `GetSplineCoefficients`. The tolerance value `atol` is applied to all `3*xGrid.Length` solution components. `atol` must be greater than or equal to zero. Also, not all entries in `atol` and `rtol` are allowed to be equal to 0 simultaneously.

Default: `atol` = 1.0e-5.

SetForcingTerm

```
public void SetForcingTerm(Imsl.Math.FeynmanKac.IForcingTerm forceTerm)
```

Description

Sets the user-supplied method that computes approximations to the forcing term $\phi(x)$ and its derivative $\partial\phi/\partial y$ used in the FeynmanKac PDE.

Parameter

`forceTerm` – An `IForcingTerm` object specifying the user defined function used for computation of the forcing term $\phi(f,x,t)$ and its derivative $\partial\phi/\partial y$.

Default: If this member function is not called it is assumed that $\phi(f,x,t)$ is identically zero.

SetInitialData

```
public void SetInitialData(Imsl.Math.FeynmanKac.IInitialData initData)
```

Description

Sets the user-supplied method for adjustment of initial data or as an opportunity for output during the integration steps.

Parameter

`initData` – An `IInitialData` object specifying the user-defined function for adjustment of initial data or as an opportunity for output during the integration steps.

Default: No adjustment of initial data or output during the integration steps is done.

SetRelativeErrorTolerances

```
public void SetRelativeErrorTolerances(double[] rtol)
```

Description

Sets the relative error tolerances.

Parameter

`rtol` – A `double` array of length `3*xGrid.Length` specifying the relative error tolerances for the row-wise solutions returned by method `GetSplineCoefficients`. All entries in `rtol` must be greater than or equal to zero. Also, not all entries in `atol` and `rtol` are allowed to be equal to 0 simultaneously.

Default: `rtol[i]` = 1.0e-5 for `i=0,...,3*xGrid.Length-1`.

SetRelativeErrorTolerances

```
public void SetRelativeErrorTolerances(double rtol)
```

Description

Sets the relative error tolerances.

Parameter

`rtol` – A `double` scalar specifying the relative error tolerances for the row-wise solutions returned by method `GetSplineCoefficients` The tolerance value `rtol` is applied to all `3*xGrid.Length` solution components. `rtol` must be greater than or equal to zero. Also, not all entries in `atol` and `rtol` are allowed to be equal to 0 simultaneously.

Default: `rtol` = 1.0e-5.

SetTimeDependence

```
public void SetTimeDependence(bool[] timeFlag)
```

Description

Sets the time dependence of the coefficients, boundary conditions and function ϕ in the Feynman Kac equation.

Parameter

`timeFlag` – A `bool` vector of length 7 indicating time dependencies in the Feynman-Kac PDE.

Index	*Time dependency of*
0	σ'
1	σ
2	μ
3	κ
4	Left boundary conditions
5	Right boundary conditions
6	ϕ

Remarks

`timeFlag[i] = true` indicates that the associated value is time-dependent, `timeFlag[i] = false` indicates that the associated value is time-independent.

Default: `timeFlag[i] = false` for $i = 0,\ldots,6$.

Example 1: American Option vs. European Option on a Vanilla Put

The value of the American Option on a Vanilla Put can be no smaller than its European counterpart. That is due to the American Option providing the opportunity to exercise at any time prior to expiration. This example compares this difference - or premium value of the American Option - at two time values using the Black-Scholes model. The example is based on Wilmott et al. (1996, p. 176), and uses the non-linear forcing or weighting term described in Hanson, R. (2008), *Integrating Feynman-Kac Equations Using Hermite Quintic Finite Elements*, for evaluating the price of the American Option. The coefficients, payoff, boundary conditions and forcing term for American and European options are defined through interfaces `IPdeCoefficients`, `IBoundaries` and `IForcingTerm`, respectively. One breakpoint is set exactly at the strike price. The sets of parameters in the computation are:

1. Strike price $K = 10.0$
2. Volatility $\sigma = 0.4$
3. Times until expiration = $\{1/4, 1/2\}$
4. Interest rate $r = 0.1$
5. $x_{\min} = 0.0, x_{\max} = 30.0$
6. $nxGrid = 61, n = 3 \times nxGrid = 183$

The payoff function is the "vanilla option", $p(x) = \max(K - x, 0)$.

```
using System;
using Imsl.Math;

public class FeynmanKacEx1 : FeynmanKac.IPdeCoefficients,
   FeynmanKac.IBoundaries
{
   // The coefficient sigma(x)
   public double Sigma(double x, double t)
   {
      double sigma = 0.4;
      return (sigma * x);
   }

   // The coefficient derivative d(sigma) / dx
   public double SigmaPrime(double x, double t)
   {
      double sigma = 0.4;
      return sigma;
   }

   // The coefficient mu(x)
   public double Mu(double x, double t)
   {
      double interestRate = 0.1;
      double dividend = 0.0;
```

```
   return ((interestRate - dividend) * x);
}

// The coefficient kappa(x)
public double Kappa(double x, double t)
{
   double interestRate = 0.1;
   return interestRate;
}

public void LeftBoundaries(double time, double[,] bndCoeffs)
{
   bndCoeffs[0, 0] = 0.0;
   bndCoeffs[0, 1] = 1.0;
   bndCoeffs[0, 2] = 0.0;
   bndCoeffs[0, 3] = -1.0;
   bndCoeffs[1, 0] = 0.0;
   bndCoeffs[1, 1] = 0.0;
   bndCoeffs[1, 2] = 1.0;
   bndCoeffs[1, 3] = 0.0;
}

public void RightBoundaries(double time, double[,] bndCoeffs)
{
   bndCoeffs[0, 0] = 1.0;
   bndCoeffs[0, 1] = 0.0;
   bndCoeffs[0, 2] = 0.0;
   bndCoeffs[0, 3] = 0.0;
   bndCoeffs[1, 0] = 0.0;
   bndCoeffs[1, 1] = 1.0;
   bndCoeffs[1, 2] = 0.0;
   bndCoeffs[1, 3] = 0.0;
   bndCoeffs[2, 0] = 0.0;
   bndCoeffs[2, 1] = 0.0;
   bndCoeffs[2, 2] = 1.0;
   bndCoeffs[2, 3] = 0.0;
}

public double Terminal(double x)
{
   double zero = 0.0;
   // Strike price
   double strikePrice = 10.0;
   // The payoff function
   double val = Math.Max(strikePrice - x, zero);

   return val;
}

class MyForcingTerm : FeynmanKac.IForcingTerm
{
   public void Force(int interval, double[] y, double time, double width,
               double[] xlocal, double[] qw, double[,] u, double[] phi,
               double[,] dphi)
   {
      const int local = 6;
```

```
const double zero = 0.0;
const double one = 1.0;
double[] yl = new double[local];
double[] bf = new double[local];
double val, strikePrice, interestRate;
double rt, mu;
int nxGrid = y.Length / 3;
int ndeg = xlocal.Length;

for (int i = 0; i < local; i++)
{
   yl[i] = y[3 * interval - 3 + i];
   phi[i] = zero;
}
strikePrice = 10.0;
interestRate = 0.1;
val = 1.0e-5;
mu = 2.0;
// This is the local definition of the forcing term
for (int j = 1; j <= local; j++)
{
   for (int l = 1; l <= ndeg; l++)
   {
      bf[0] = u[0, l - 1];
      bf[1] = u[1, l - 1];
      bf[2] = u[2, l - 1];
      bf[3] = u[6, l - 1];
      bf[4] = u[7, l - 1];
      bf[5] = u[8, l - 1];
      rt = 0.0;
      for (int k = 0; k < local; k++)
      {
         rt += yl[k] * bf[k];
      }
      rt = val / (rt + val - (strikePrice - xlocal[l - 1]));
      phi[j - 1] += qw[l - 1] * bf[j - 1] * Math.Pow(rt, mu);
   }
}
for (int i = 0; i < local; i++)
{
   phi[i] = (-phi[i]) * width * interestRate * strikePrice;
}
// This is the local derivative matrix for the forcing term
for (int i = 0; i < local; i++)
{
   for (int j = 0; j < local; j++)
   {
      dphi[i, j] = zero;
   }
}
for (int j = 1; j <= local; j++)
{
   for (int i = 1; i <= local; i++)
   {
      for (int l = 1; l <= ndeg; l++)
      {
```

```
                bf[0] = u[0, l - 1];
                bf[1] = u[1, l - 1];
                bf[2] = u[2, l - 1];
                bf[3] = u[6, l - 1];
                bf[4] = u[7, l - 1];
                bf[5] = u[8, l - 1];
                rt = 0.0;
                for (int k = 0; k < local; k++)
                {
                   rt += yl[k] * bf[k];
                }
                rt = one / (rt + val - (strikePrice - xlocal[l - 1]));
                dphi[j - 1, i - 1] += qw[l - 1] * bf[i - 1] * bf[j - 1]
                                      * Math.Pow(rt, mu + 1.0);
             }
          }
       }
       for (int i = 0; i < local; i++)
       {
          for (int j = 0; j < local; j++)
          {
             dphi[i, j] = mu * dphi[i, j] * width * Math.Pow(val, mu)
                          * interestRate * strikePrice;
          }
       }
       return;
    }
 }

 public static void Main(String[] args)
 {
    // Compute American Option Premium for Vanilla Put
    // The strike price
    double KS = 10.0;
    // The sigma value
    double sigma = 0.4;
    // Time values for the options
    int nt = 2;
    double[] tGrid = { 0.25, 0.5 };
    // Values of the underlying where evaluations are made
    double[] evaluationPoints = { 0.0, 2.0, 4.0, 6.0, 8.0,
                                  10.0, 12.0, 14.0, 16.0
                                };
    // Value of the interest rate
    double r = 0.1;
    // Values of the min and max underlying values modeled
    double xMin = 0.0, xMax = 30.0;
    // Define parameters for the integration step.
    int nxGrid = 61;
    int nv = 9;
    int nint = nxGrid - 1, n = 3 * nxGrid;
    double[] xGrid = new double[nxGrid];
    double dx;
    int nlbcd = 2, nrbcd = 3;
    // Define an equally-spaced grid of points for the
    // underlying price
```

```
dx = (xMax - xMin) / ((double)nint);
xGrid[0] = xMin;
xGrid[nxGrid - 1] = xMax;
for (int i = 2; i <= nxGrid - 1; i++)
{
   xGrid[i - 1] = xGrid[i - 2] + dx;
}

FeynmanKacEx1 ex1 = new FeynmanKacEx1();

FeynmanKac european = new FeynmanKac(ex1);
FeynmanKac american = new FeynmanKac(ex1);

MyForcingTerm forceTerm = new MyForcingTerm();

american.SetForcingTerm(forceTerm);

european.ComputeCoefficients(nlbcd, nrbcd, ex1, xGrid, tGrid);
american.ComputeCoefficients(nlbcd, nrbcd, ex1, xGrid, tGrid);

// Evaluate solutions at vector of points evaluationPoints, at each
// time value prior to expiration.

double[,] europeanCoefficients = european.GetSplineCoefficients();
double[,] americanCoefficients = american.GetSplineCoefficients();

double[] europeanTimeCoeffs = new double[n];
double[] americanTimeCoeffs = new double[n];

double[][] splineValuesEuropean = new double[nt][];
double[][] splineValuesAmerican = new double[nt][];

for (int i = 0; i < nt; i++)
{
   // Exctract spline coefficients for individual times
   for (int j = 0; j < n; j++)
   {
      europeanTimeCoeffs[j] = europeanCoefficients[i + 1, j];
      americanTimeCoeffs[j] = americanCoefficients[i + 1, j];
   }

   splineValuesEuropean[i] =
      european.GetSplineValue(evaluationPoints, europeanTimeCoeffs, 0);
   splineValuesAmerican[i] =
      american.GetSplineValue(evaluationPoints, americanTimeCoeffs, 0);
}

Console.Out.WriteLine();
Console.Out.WriteLine("American Option Premium for Vanilla Put, " +
                      "3 and 6 Months Prior to Expiry");
Console.Out.WriteLine("       Number of equally spaced spline" +
                      " knots:{0,4:d}", nxGrid);
Console.Out.WriteLine("       Number of unknowns:{0,4:d}", n);
Console.Out.WriteLine("       Strike={0,6:f2}, sigma={1,5:f2}, " +
                      "Interest Rate={2,5:f2}", KS, sigma, r);
Console.Out.WriteLine();
```

```
      Console.Out.WriteLine("       Underlying            European" +
                            "             American");
      for (int i = 0; i < nv; i++)
      {
         Console.Out.WriteLine("       {0,10:f4}{1,10:f4}{2,10:f4}" +
                               "{3,10:f4}{4,10:f4}",
                                  evaluationPoints[i],
                                  splineValuesEuropean[0][i],
                                  splineValuesEuropean[1][i],
                                  splineValuesAmerican[0][i],
                                  splineValuesAmerican[1][i]);

      }
   }
}
```

Output

```
American Option Premium for Vanilla Put, 3 and 6 Months Prior to Expiry
       Number of equally spaced spline knots:  61
       Number of unknowns: 183
       Strike= 10.00, sigma= 0.40, Interest Rate= 0.10

       Underlying            European            American
           0.0000    9.7531    9.5123   10.0000   10.0000
           2.0000    7.7531    7.5123    8.0000    8.0000
           4.0000    5.7531    5.5128    6.0000    6.0000
           6.0000    3.7569    3.5583    4.0000    4.0000
           8.0000    1.9024    1.9181    2.0202    2.0954
          10.0000    0.6694    0.8703    0.6923    0.9219
          12.0000    0.1675    0.3477    0.1712    0.3625
          14.0000    0.0326    0.1279    0.0332    0.1321
          16.0000    0.0054    0.0448    0.0055    0.0461
```

Example 2: A diffusion model for Call Options

In Beckers (1980) there is a model for a Stochastic Differential Equation of option pricing. The idea is a "constant elasticity of variance diffusion (or CEV) class"

$$dS = \mu S dt + \sigma S^{\alpha/2} dW,\ 0 \leq \alpha < 2.$$

The Black-Scholes model is the limiting case $\alpha \to 2$. A numerical solution of this diffusion model yields the price of a call option. Various values of the strike price K, time values, σ and power coefficient α are used to evaluate the option price at values of the underlying price. The sets of parameters in the computation are:

1. power $\alpha = 2.0, 1.0, 0.0$
2. strike price $K = 15.0, 20.0, 25.0$
3. volatility $\sigma = 0.2, 0.3, 0.4$
4. times until expiration = $\{1/12,\ 4/12,\ 7/12\}$

5. underlying prices = $\{19.0, 20.0, 21.0\}$
6. interest rate $r = 0.05$
7. $x_{\min} = 0, x_{\max} = 60$
8. $nxGrid = 121, n = 3 \times nxGrid = 363$

With this model the Feynman-Kac differential equation is defined by identifying:

- $x: S$
- $\sigma(x,t): \sigma x^{\alpha/2}; \frac{\partial \sigma}{\partial x} = \frac{\alpha\sigma}{2} x^{\alpha/2-1}$
- $\mu(x,t): rx$
- $\kappa(x,t): r$
- $\phi(f,x,t) \equiv 0$

The payoff function is the "vanilla option", $p(x) = \max(x-K, 0)$.

```
using System;
using Imsl.Math;

public class FeynmanKacEx2
{
   public static void Main(String[] args)
   {
      // Compute Constant Elasticity of Variance Model for Vanilla Call
      // The set of strike prices
      double[] strikePrices = { 15.0, 20.0, 25.0 };
      // The set of sigma values
      double[] sigma = { 0.2, 0.3, 0.4 };
      // The set of model diffusion powers
      double[] alpha = { 2.0, 1.0, 0.0 };
      // Time values for the options
      int nt = 3;
      double[] tGrid = { 1.0 / 12.0, 4.0 / 12.0, 7.0 / 12.0 };
      // Values of the underlying where evaluations are made
      double[] evaluationPoints = { 19.0, 20.0, 21.0 };
      // Value of the interest rate and continuous dividend
      double r = 0.05, dividend = 0.0;
      // Values of the min and max underlying values modeled
      double xMin = 0.0, xMax = 60.0;
      // Define parameters for the integration step. */
      int nxGrid = 121;
      int ntGrid = 3;
      int nv = 3;
      int nint = nxGrid - 1, n = 3 * nxGrid;
      double[] xGrid = new double[nxGrid];
      double dx;
      // Number of left/right boundary conditions
      int nlbcd = 3, nrbcd = 3;
```

```
// Time dependency
bool[] timeDependence = new bool[7];
double[] userData = new double[6];
//int j;
// Define equally-spaced grid of points for the underlying price
dx = (xMax - xMin) / ((double)nint);
xGrid[0] = xMin;
xGrid[nxGrid - 1] = xMax;
for (int i = 2; i <= nxGrid - 1; i++)
{
   xGrid[i - 1] = xGrid[i - 2] + dx;
}

Console.Out.WriteLine("  Constant Elasticity of Variance Model" +
                      " for Vanilla Call");
Console.Out.WriteLine("       Interest Rate:{0,7:f3}   Continuous"
                      + " Dividend:{1,7:f3}", r, dividend);
Console.Out.WriteLine("       Minimum and Maximum Prices of " +
                      "Underlying:{0,7:f2}{1,7:f2}", " ", xMin, xMax);
Console.Out.WriteLine("       Number of equally spaced spline knots:"
                      + "{0,4:d}", nxGrid - 1);
Console.Out.WriteLine("       Number of unknowns:{0,4:d}", n);
Console.Out.WriteLine("       Time in Years Prior to Expiration: " +
                      "{0,7:f4}{1,7:f4}{2,7:f4}",
                      tGrid[0], tGrid[1], tGrid[2]);
Console.Out.WriteLine("       Option valued at Underlying Prices:" +
                      "{0,7:f2}{1,7:f2}{2,7:f2}\n\n",
                       evaluationPoints[0], evaluationPoints[1],
                       evaluationPoints[2]);

for (int i1 = 1; i1 <= 3; i1++)
/* Loop over power */
{
   for (int i2 = 1; i2 <= 3; i2++)
   /* Loop over volatility */
   {
      for (int i3 = 1; i3 <= 3; i3++)
      /* Loop over strike price */
      {
         // Pass data through into evaluation routines.
         userData[0] = strikePrices[i3 - 1];
         userData[1] = xMax;
         userData[2] = sigma[i2 - 1];
         userData[3] = alpha[i1 - 1];
         userData[4] = r;
         userData[5] = dividend;

         FeynmanKac callOption =
                  new FeynmanKac(new MyCoefficients(userData));

         // Right boundary condition is time-dependent
         timeDependence[5] = true;
         callOption.SetTimeDependence(timeDependence);
         callOption.ComputeCoefficients(nlbcd, nrbcd,
                           new MyBoundaries(userData), xGrid, tGrid);
         double[,] optionCoefficients =
```

```
            callOption.GetSplineCoefficients();
         double[] optionTimeCoeffs = new double[n];
         double[][] splineValuesOption = new double[ntGrid][];

         // Evaluate solution at vector evaluationPoints, at each time
         // value prior to expiration.
         for (int i = 0; i < ntGrid; i++)
         {
            for (int j = 0; j < n; j++)
               optionTimeCoeffs[j] = optionCoefficients[i + 1, j];
            splineValuesOption[i] =
                callOption.GetSplineValue(evaluationPoints,
                                          optionTimeCoeffs, 0);
         }

         Console.Out.WriteLine("  Strike={0,5:f2}, Sigma={1,5:f2}, "
                               + "Alpha={2,5:f2}",
                                 strikePrices[i3 - 1],
                                 sigma[i2 - 1],
                                 alpha[i1 - 1]);

         for (int i = 0; i < nv; i++)
         {
            Console.Out.Write("                        Call " +
                              "Option Values  ");
            for (int j = 0; j < nt; j++)
            {
               Console.Out.Write("{0,7:f4} ", splineValuesOption[j][i]);
            }
            Console.Out.WriteLine();
         }
         Console.Out.WriteLine();
      }
   }
}

internal class MyCoefficients : FeynmanKac.IPdeCoefficients
{
   const double zero = 0.0;
   const double half = 0.5;
   private double sigma, strikePrice, interestRate;
   private double alpha, dividend;

   public MyCoefficients(double[] myData)
   {
      this.strikePrice = myData[0];
      this.sigma = myData[2];
      this.alpha = myData[3];
      this.interestRate = myData[4];
      this.dividend = myData[5];
   }

   // The coefficient sigma(x)
   public double Sigma(double x, double t)
   {
```

```
      return (sigma * Math.Pow(x, alpha * half));
   }

   // The coefficient derivative d(sigma) / dx
   public double SigmaPrime(double x, double t)
   {
      return (half * alpha * sigma * Math.Pow(x, alpha * half - 1.0));
   }

   // The coefficient mu(x)
   public double Mu(double x, double t)
   {
      return ((interestRate - dividend) * x);
   }

   // The coefficient kappa(x)
   public double Kappa(double x, double t)
   {
      return interestRate;
   }
}

internal class MyBoundaries : FeynmanKac.IBoundaries
{

   private double xMax, df, interestRate, strikePrice;

   public MyBoundaries(double[] myData)
   {
      this.strikePrice = myData[0];
      this.xMax = myData[1];
      this.interestRate = myData[4];
   }

   public void LeftBoundaries(double time, double[,] bndCoeffs)
   {
      bndCoeffs[0, 0] = 1.0;
      bndCoeffs[0, 1] = 0.0;
      bndCoeffs[0, 2] = 0.0;
      bndCoeffs[0, 3] = 0.0;
      bndCoeffs[1, 0] = 0.0;
      bndCoeffs[1, 1] = 1.0;
      bndCoeffs[1, 2] = 0.0;
      bndCoeffs[1, 3] = 0.0;
      bndCoeffs[2, 0] = 0.0;
      bndCoeffs[2, 1] = 0.0;
      bndCoeffs[2, 2] = 1.0;
      bndCoeffs[2, 3] = 0.0;

      return;
   }

   public void RightBoundaries(double time, double[,] bndCoeffs)
   {
      df = Math.Exp(interestRate * time);
      bndCoeffs[0, 0] = 1.0;
```

```
            bndCoeffs[0, 1] = 0.0;
            bndCoeffs[0, 2] = 0.0;
            bndCoeffs[0, 3] = xMax - df * strikePrice;
            bndCoeffs[1, 0] = 0.0;
            bndCoeffs[1, 1] = 1.0;
            bndCoeffs[1, 2] = 0.0;
            bndCoeffs[1, 3] = 1.0;
            bndCoeffs[2, 0] = 0.0;
            bndCoeffs[2, 1] = 0.0;
            bndCoeffs[2, 2] = 1.0;
            bndCoeffs[2, 3] = 0.0;

            return;
        }

        public double Terminal(double x)
        {
            double zero = 0.0;
            // The payoff function
            double val = Math.Max(x - strikePrice, zero);
            return val;
        }
    }
}
```

Output

```
  Constant Elasticity of Variance Model for Vanilla Call
       Interest Rate:  0.050   Continuous Dividend:  0.000
       Minimum and Maximum Prices of Underlying:          0.00
       Number of equally spaced spline knots: 120
       Number of unknowns: 363
       Time in Years Prior to Expiration:  0.0833 0.3333 0.5833
       Option valued at Underlying Prices:  19.00  20.00  21.00


  Strike=15.00, Sigma= 0.20, Alpha= 2.00
                        Call Option Values   4.0624  4.2576  4.4734
                        Call Option Values   5.0624  5.2505  5.4492
                        Call Option Values   6.0624  6.2486  6.4386

  Strike=20.00, Sigma= 0.20, Alpha= 2.00
                        Call Option Values   0.1312  0.5951  0.9693
                        Call Option Values   0.5024  1.0880  1.5093
                        Call Option Values   1.1980  1.7478  2.1745

  Strike=25.00, Sigma= 0.20, Alpha= 2.00
                        Call Option Values   0.0000  0.0111  0.0751
                        Call Option Values   0.0000  0.0376  0.1630
                        Call Option Values   0.0006  0.1036  0.3150

  Strike=15.00, Sigma= 0.30, Alpha= 2.00
                        Call Option Values   4.0636  4.3405  4.6627
                        Call Option Values   5.0625  5.2951  5.5794
                        Call Option Values   6.0624  6.2712  6.5248
```

```
Strike=20.00, Sigma= 0.30, Alpha= 2.00
                    Call Option Values   0.3107  1.0261  1.5479
                    Call Option Values   0.7317  1.5404  2.0999
                    Call Option Values   1.3762  2.1674  2.7362

Strike=25.00, Sigma= 0.30, Alpha= 2.00
                    Call Option Values   0.0005  0.1124  0.3564
                    Call Option Values   0.0035  0.2184  0.5565
                    Call Option Values   0.0184  0.3869  0.8230

Strike=15.00, Sigma= 0.40, Alpha= 2.00
                    Call Option Values   4.0755  4.5143  4.9673
                    Call Option Values   5.0660  5.4210  5.8328
                    Call Option Values   6.0633  6.3588  6.7306

Strike=20.00, Sigma= 0.40, Alpha= 2.00
                    Call Option Values   0.5109  1.4625  2.1260
                    Call Option Values   0.9611  1.9934  2.6915
                    Call Option Values   1.5807  2.6088  3.3202

Strike=25.00, Sigma= 0.40, Alpha= 2.00
                    Call Option Values   0.0081  0.3302  0.7795
                    Call Option Values   0.0287  0.5178  1.0656
                    Call Option Values   0.0820  0.7690  1.4097

Strike=15.00, Sigma= 0.20, Alpha= 1.00
                    Call Option Values   4.0624  4.2479  4.4312
                    Call Option Values   5.0624  5.2479  5.4312
                    Call Option Values   6.0624  6.2479  6.4312

Strike=20.00, Sigma= 0.20, Alpha= 1.00
                    Call Option Values   0.0000  0.0219  0.1051
                    Call Option Values   0.1497  0.4107  0.6484
                    Call Option Values   1.0832  1.3314  1.5773

Strike=25.00, Sigma= 0.20, Alpha= 1.00
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0000

Strike=15.00, Sigma= 0.30, Alpha= 1.00
                    Call Option Values   4.0624  4.2479  4.4312
                    Call Option Values   5.0624  5.2479  5.4312
                    Call Option Values   6.0624  6.2479  6.4312

Strike=20.00, Sigma= 0.30, Alpha= 1.00
                    Call Option Values   0.0010  0.0786  0.2208
                    Call Option Values   0.1993  0.4997  0.7539
                    Call Option Values   1.0835  1.3444  1.6022

Strike=25.00, Sigma= 0.30, Alpha= 1.00
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0004
```

```
Strike=15.00, Sigma= 0.40, Alpha= 1.00
                    Call Option Values   4.0624  4.2479  4.4312
                    Call Option Values   5.0624  5.2479  5.4312
                    Call Option Values   6.0624  6.2479  6.4312

Strike=20.00, Sigma= 0.40, Alpha= 1.00
                    Call Option Values   0.0072  0.1540  0.3446
                    Call Option Values   0.2498  0.5950  0.8728
                    Call Option Values   1.0868  1.3795  1.6586

Strike=25.00, Sigma= 0.40, Alpha= 1.00
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0005
                    Call Option Values   0.0000  0.0002  0.0057

Strike=15.00, Sigma= 0.20, Alpha= 0.00
                    Call Option Values   4.0624  4.2479  4.4311
                    Call Option Values   5.0624  5.2479  5.4312
                    Call Option Values   6.0624  6.2479  6.4312

Strike=20.00, Sigma= 0.20, Alpha= 0.00
                    Call Option Values   0.0001  0.0001  0.0002
                    Call Option Values   0.0816  0.3316  0.5748
                    Call Option Values   1.0817  1.3308  1.5748

Strike=25.00, Sigma= 0.20, Alpha= 0.00
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0000

Strike=15.00, Sigma= 0.30, Alpha= 0.00
                    Call Option Values   4.0624  4.2479  4.4312
                    Call Option Values   5.0624  5.2479  5.4312
                    Call Option Values   6.0624  6.2479  6.4312

Strike=20.00, Sigma= 0.30, Alpha= 0.00
                    Call Option Values   0.0000  0.0000  0.0026
                    Call Option Values   0.0894  0.3326  0.5753
                    Call Option Values   1.0826  1.3306  1.5749

Strike=25.00, Sigma= 0.30, Alpha= 0.00
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0000
                    Call Option Values   0.0000  0.0000  0.0000

Strike=15.00, Sigma= 0.40, Alpha= 0.00
                    Call Option Values   4.0624  4.2479  4.4312
                    Call Option Values   5.0624  5.2479  5.4312
                    Call Option Values   6.0624  6.2479  6.4312

Strike=20.00, Sigma= 0.40, Alpha= 0.00
                    Call Option Values   0.0000  0.0001  0.0108
                    Call Option Values   0.0985  0.3383  0.5781
                    Call Option Values   1.0830  1.3306  1.5749

Strike=25.00, Sigma= 0.40, Alpha= 0.00
```

```
Call Option Values   0.0000  0.0000  0.0000
Call Option Values   0.0000  0.0000  0.0000
Call Option Values   0.0000  0.0000  0.0000
```

Example 3: European Option with two payoff strategies

This example evaluates the price of a European Option using two payoff strategies: Cash-or-Nothing and Vertical Spread. In the first case the payoff function is

$$p(x) = \begin{cases} 0, & x \le K \\ B, & x > K \end{cases} .$$

The value B is regarded as the bet on the asset price, see Wilmott et al. (1995, p. 39-40). The second case has the payoff function

$$p(x) = \max(x - K_1) - \max(x - K_2), K_2 > K_1.$$

Both problems use the same boundary conditions. Each case requires a separate integration of the Black-Scholes differential equation, but only the payoff function evaluation differs in each case. The sets of parameters in the computation are:

1. Strike and bet prices $K_1 = 10.0,\ K_2 = 15.0, \text{and} B = 2.0.$
2. Volatility $\sigma = 0.4$
3. Times until expiration = $\{1/4,\ 1/2\}$
4. Interest rate $r = 0.1$
5. $x_{\min} = 0.0, x_{\max} = 30.0$
6. $nxGrid = 61, n = 3 \times nxGrid = 183$

```
using System;
using Imsl.Math;

public class FeynmanKacEx3
{

   public static void Main(String[] args)
   {
      int i, j;
      int nxGrid = 61;
      int ntGrid = 2;
      int nv = 12;
      // The strike price
      double strikePrice = 10.0;
      // The spread value
      double spreadValue = 15.0;
      // The Bet for the Cash-or-Nothing Call
      double bet = 2.0;
      // The sigma value
```

```
double sigma = 0.4;
// Time values for the options
double[] tGrid = { 0.25, 0.5 };
// Values of the underlying where evaluations are made
double[] evaluationPoints = new double[nv];
// Value of the interest rate and continuous dividend
double r = 0.1, dividend = 0.0;
// Values of the min and max underlying values modeled
double xMin = 0.0, xMax = 30.0;
// Define parameters for the integration step.
int nint = nxGrid - 1, n = 3 * nxGrid;
double[] xGrid = new double[nxGrid];
double dx;
// Number of left/right boundary conditions.
int nlbcd = 3, nrbcd = 3;
// Structure for the evaluation routines.
int iData;
double[] rData = new double[7];
bool[] timeDependence = new bool[7];

// Define an equally-spaced grid of points for the
// underlying price
dx = (xMax - xMin) / ((double)(nint));
xGrid[0] = xMin;
xGrid[nxGrid - 1] = xMax;
for (i = 2; i <= nxGrid - 1; i++)
{
   xGrid[i - 1] = xGrid[i - 2] + dx;
}
for (i = 1; i <= nv; i++)
{
   evaluationPoints[i - 1] = 2.0 + (i - 1) * 2.0;
}
rData[0] = strikePrice;
rData[1] = bet;
rData[2] = spreadValue;
rData[3] = xMax;
rData[4] = sigma;
rData[5] = r;
rData[6] = dividend;
// Flag the difference in payoff functions
// 1 for the Bet, 2 for the Vertical Spread
// In both cases, no time dependencies for
// the coefficients and the left boundary
// conditions
timeDependence[5] = true;
iData = 1;
FeynmanKac betOption = new FeynmanKac(new MyCoefficients(rData));
betOption.SetTimeDependence(timeDependence);
betOption.ComputeCoefficients(nlbcd, nrbcd,
   new MyBoundaries(rData, iData), xGrid, tGrid);
double[,] betOptionCoefficients = betOption.GetSplineCoefficients();
double[][] splineValuesBetOption = new double[ntGrid][];
double[] betOptionTimeCoeffs = new double[n];

iData = 2;
```

```
   FeynmanKac spreadOption = new FeynmanKac(new MyCoefficients(rData));

   spreadOption.SetTimeDependence(timeDependence);
   spreadOption.ComputeCoefficients(nlbcd, nrbcd,
      new MyBoundaries(rData, iData), xGrid, tGrid);
   double[,] spreadOptionCoefficients =
      spreadOption.GetSplineCoefficients();
   double[][] splineValuesSpreadOption = new double[ntGrid][];
   double[] spreadOptionTimeCoeffs = new double[n];

   // Evaluate solutions at vector evaluationPoints, at each time value
   // prior to expiration.
   for (i = 0; i < ntGrid; i++)
   {
      for (j = 0; j < n; j++)
      {
         betOptionTimeCoeffs[j] = betOptionCoefficients[i + 1, j];
         spreadOptionTimeCoeffs[j] = spreadOptionCoefficients[i + 1, j];
      }
      splineValuesBetOption[i] = betOption.GetSplineValue(
         evaluationPoints, betOptionTimeCoeffs, 0);
      splineValuesSpreadOption[i] = spreadOption.GetSplineValue(
         evaluationPoints, spreadOptionTimeCoeffs, 0);
   }

   Console.Out.WriteLine("  European Option Value for A Bet");
   Console.Out.WriteLine(
      "   and a Vertical Spread, 3 and 6 Months prior to Expiry");
   Console.Out.WriteLine(
      "     Number of equally spaced spline knots:{0,4:d}", nxGrid);
   Console.Out.WriteLine("     Number of unknowns:{0,4:d}", n);
   Console.Out.WriteLine(
      "     Strike={0,5:f2}, Sigma={1,5:f2}, Interest Rate={2,5:f2}",
      strikePrice, sigma, r);
   Console.Out.WriteLine("     Bet={0,5:f2}, Spread Value={1,5:f2}",
      bet, spreadValue);
   Console.Out.WriteLine();
   Console.Out.WriteLine(
      "       Underlying             A Bet   Vertical Spread");
   for (i = 0; i < nv; i++)
   {
      Console.Out.WriteLine(
         "        {0,9:f4}{1,9:f4}{2,9:f4}{3,9:f4}{4,9:f4}",
         evaluationPoints[i], splineValuesBetOption[0][i],
         splineValuesBetOption[1][i], splineValuesSpreadOption[0][i],
         splineValuesSpreadOption[1][i]);
   }
}
// These routines define the coefficients, payoff, boundary conditions
// and forcing term for American and European Options.

class MyCoefficients : FeynmanKac.IPdeCoefficients
{
   const double zero = 0.0;
   double sigma, strikePrice, interestRate;
   double spread, bet, dividend;
```

```
    public MyCoefficients(double[] rData)
    {
       this.strikePrice = rData[0];
       this.bet = rData[1];
       this.spread = rData[2];
       this.sigma = rData[4];
       this.interestRate = rData[5];
       this.dividend = rData[6];
    }

    // The coefficient sigma(x)
    public double Sigma(double x, double t)
    {
       return (sigma * x);
    }

    // The coefficient derivative d(sigma) / dx
    public double SigmaPrime(double x, double t)
    {
       return sigma;
    }

    // The coefficient mu(x)
    public double Mu(double x, double t)
    {
       return ((interestRate - dividend) * x);
    }

    // The coefficient kappa(x)
    public double Kappa(double x, double t)
    {
       return interestRate;
    }
}

class MyBoundaries : FeynmanKac.IBoundaries
{

    double strikePrice, spread, bet, interestRate, df;
    int dataInt;

    public MyBoundaries(double[] rData, int iData)
    {
       this.strikePrice = rData[0];
       this.bet = rData[1];
       this.spread = rData[2];
       this.interestRate = rData[5];
       this.dataInt = iData;
    }

    public void LeftBoundaries(double time, double[,] bndCoeffs)
    {
       bndCoeffs[0, 0] = 1.0;
       bndCoeffs[0, 1] = 0.0;
       bndCoeffs[0, 2] = 0.0;
```

```
      bndCoeffs[0, 3] = 0.0;
      bndCoeffs[1, 0] = 0.0;
      bndCoeffs[1, 1] = 1.0;
      bndCoeffs[1, 2] = 0.0;
      bndCoeffs[1, 3] = 0.0;
      bndCoeffs[2, 0] = 0.0;
      bndCoeffs[2, 1] = 0.0;
      bndCoeffs[2, 2] = 1.0;
      bndCoeffs[2, 3] = 0.0;

      return;
   }

   public void RightBoundaries(double time, double[,] bndCoeffs)
   {
      // This is the discount factor using the risk-free
      // interest rate
      df = Math.Exp(interestRate * time);
      // Use flag passed to decide on boundary condition
      switch (dataInt)
      {

         case 1:
            bndCoeffs[0, 0] = 1.0;
            bndCoeffs[0, 1] = 0.0;
            bndCoeffs[0, 2] = 0.0;
            bndCoeffs[0, 3] = bet * df;
            break;

         case 2:
            bndCoeffs[0, 0] = 1.0;
            bndCoeffs[0, 1] = 0.0;
            bndCoeffs[0, 2] = 0.0;
            bndCoeffs[0, 3] = (spread - strikePrice) * df;
            break;
      }
      bndCoeffs[1, 0] = 0.0;
      bndCoeffs[1, 1] = 1.0;
      bndCoeffs[1, 2] = 0.0;
      bndCoeffs[1, 3] = 0.0;
      bndCoeffs[2, 0] = 0.0;
      bndCoeffs[2, 1] = 0.0;
      bndCoeffs[2, 2] = 1.0;
      bndCoeffs[2, 3] = 0.0;

      return;
   }

   public double Terminal(double x)
   {
      const double zero = 0.0;
      double val = 0.0;
      switch (dataInt)
      {

         // The payoff function - Use flag passed to decide which
```

```
            case 1:
               // After reaching the strike price the payoff jumps
               // from zero to the bet value.
               val = zero;
               if (x > strikePrice)
               {
                  val = bet;
               }
               break;

            case 2:
               /* Function is zero up to strike price.
               Then linear between strike price and spread.
               Then has constant value Spread-Strike Price after
               the value Spread. */
               val = Math.Max(x - strikePrice, zero) -
                  Math.Max(x - spread, zero);
               break;
         }
         return val;
      }
   }
}
```

Output

```
  European Option Value for A Bet
   and a Vertical Spread, 3 and 6 Months prior to Expiry
     Number of equally spaced spline knots:  61
     Number of unknowns: 183
     Strike=10.00, Sigma= 0.40, Interest Rate= 0.10
     Bet= 2.00, Spread Value=15.00

       Underlying             A Bet   Vertical Spread
           2.0000   0.0000   0.0000   0.0000   0.0000
           4.0000   0.0000   0.0013   0.0000   0.0005
           6.0000   0.0112   0.0729   0.0038   0.0452
           8.0000   0.2686   0.4291   0.1486   0.3833
          10.0000   0.9948   0.9781   0.8898   1.1907
          12.0000   1.6103   1.4301   2.1904   2.2267
          14.0000   1.8650   1.6926   3.4267   3.1567
          16.0000   1.9335   1.8171   4.2274   3.8282
          18.0000   1.9477   1.8696   4.6261   4.2499
          20.0000   1.9501   1.8902   4.7903   4.4918
          22.0000   1.9505   1.8979   4.8493   4.6222
          24.0000   1.9506   1.9008   4.8685   4.6901
```

Example 4: Convertible bonds

This example evaluates the price of a convertible bond. Here, convertibility means that the bond may, at any time of the holder's choosing, be converted to a multiple of the specified asset. Thus a convertible bond with price x returns an amount K at time T *unless* the owner has converted the bond to νx, $\nu \geq 1$ units of the asset at some time prior to T. This definition, the differential equation and boundary

conditions are given in Chapter 18 of Wilmott et al. (1996). Using a constant interest rate and volatility factor, the parameters and boundary conditions are:

1. Bond face value $K = 1$, conversion factor $\nu = 1.125$
2. Volatility $\sigma = 0.25$
3. Times until expiration = $\{1/2, 1\}$
4. Interest rate $r = 0.1$, dividend $D = 0.02$
5. $x_{\min} = 0, x_{\max} = 4$
6. $nxGrid = 61, n = 3 \times nxGrid = 183$
7. Boundary conditions $f(0,t) = K\exp(r-(T-t)),\ f(x_{\max},t) = \nu x_{\max}$
8. Terminal data $f(x,T) = \max(K, \nu x)$
9. Constraint for bond holder $f(x,t) \geq \nu x$

Note that the error tolerance is set to a pure absolute error of value 10^{-3}. The free boundary constraint $f(x,t) \geq \nu x$ is achieved by use of a non-linear forcing term in interface `IForcingTerm`. The coefficient values of the Hermite quintic spline representing the approximate solution of the differential equation at the initial time point are provided with the interface `IInitialData`.

```
using System;
using Imsl.Math;

public class FeynmanKacEx4
{
   public static void Main(String[] args)
   {
      int i, j;
      int nxGrid = 61;
      int ntGrid = 2;
      int nv = 13;

      // Compute value of a Convertible Bond
      // The face value
      double KS = 1.0;
      // The sigma or volatility value
      double sigma = 0.25;
      // Time values for the options
      double[] tGrid = { 0.5, 1.0 };
      // Values of the underlying where evaluation are made
      double[] evaluationPoints = new double[nv];
      // Value of the interest rate, continuous dividend and factor
      double r = 0.1, dividend = 0.02, factor = 1.125;
      // Values of the min and max underlying values modeled
      double xMin = 0.0, xMax = 4.0;
      // Define parameters for the integration step.
      int nint = nxGrid - 1, n = 3 * nxGrid;
      double[] xGrid = new double[nxGrid];
```

```
// Array for user-defined data
double[] rData = new double[8];
double dx, atol;
// Number of left/right boundary conditions.
int nlbcd = 3, nrbcd = 3;
bool[] timeDependence = new bool[7];

/*
* Define an equally-spaced grid of points for the
* underlying price
*/
dx = (xMax - xMin) / ((double)nint);
xGrid[0] = xMin;
xGrid[nxGrid - 1] = xMax;
for (i = 2; i <= nxGrid - 1; i++)
{
   xGrid[i - 1] = xGrid[i - 2] + dx;
}
for (i = 1; i <= nv; i++)
{
   evaluationPoints[i - 1] = (i - 1) * 0.25;
}
// Pass the data for evaluation
rData[0] = KS;
rData[1] = xMax;
rData[2] = sigma;
rData[3] = r;
rData[4] = dividend;
rData[5] = factor;
// Use a pure absolute error tolerance for the integration
atol = 1.0e-3;
rData[6] = atol;
// Compute value of convertible bond
FeynmanKac convertibleBond = new FeynmanKac(new MyCoefficients(rData));

MyForcingTerm forceTerm = new MyForcingTerm(rData);
MyInitialData initialData = new MyInitialData(rData);

convertibleBond.SetForcingTerm(forceTerm);
convertibleBond.SetInitialData(initialData);
convertibleBond.SetAbsoluteErrorTolerances(1.0e-3);
convertibleBond.SetRelativeErrorTolerances(0.0);

// Only the left boundary conditions are time dependent
timeDependence[4] = true;
convertibleBond.SetTimeDependence(timeDependence);

convertibleBond.ComputeCoefficients(nlbcd, nrbcd,
   new MyBoundaries(rData), xGrid, tGrid);
double[,] bondCoefficients = convertibleBond.GetSplineCoefficients();
double[] bondTimeCoeffs = new double[n];

double[][] bondSplineValues = new double[ntGrid + 1][];

/*
* Evaluate and display solutions at vector of points XS(:),
```

```
   * at each time value prior to expiration.
   */
   for (i = 0; i <= ntGrid; i++)
   {
      for (j = 0; j < n; j++)
         bondTimeCoeffs[j] = bondCoefficients[i, j];

      bondSplineValues[i] = convertibleBond.GetSplineValue(
         evaluationPoints, bondTimeCoeffs, 0);
   }

   Console.Out.WriteLine("  Convertible Bond Value, 0+, 6 and 12 Months "
      + "Prior to Expiry", " ");
   Console.Out.WriteLine(
      "     Number of equally spaced spline knots:{0,4:d}", nxGrid);
   Console.Out.WriteLine("     Number of unknowns:{0,4:d}", n);
   Console.Out.WriteLine("     Strike={0,5:f2}, Sigma={1,5:f2}",
      KS, sigma);
   Console.Out.WriteLine("     Interest Rate={0,5:f2}, " +
      "Dividend={1,5:f2}, Factor={0,6:f3}", r, dividend, factor);
   Console.Out.WriteLine("     Underlying        Bond Value");
   for (i = 0; i < nv; i++)
   {
      Console.Out.WriteLine("       {0,8:f4}{1,8:f4}{2,8:f4}{3,8:f4}",
                            evaluationPoints[i],
                            bondSplineValues[0][i],
                            bondSplineValues[1][i],
                            bondSplineValues[2][i]);
   }
}
/*
* These classes define the coefficients, payoff, boundary conditions
* and forcing term.
*/

class MyCoefficients : FeynmanKac.IPdeCoefficients
{

   double sigma, strikePrice, interestRate;
   double dividend, factor;

   public MyCoefficients(double[] rData)
   {
      this.strikePrice = rData[0];
      this.sigma = rData[2];
      this.interestRate = rData[3];
      this.dividend = rData[4];
      this.factor = rData[5];
   }

   // The coefficient sigma(x)
   public double Sigma(double x, double t)
   {
      return (sigma * x);
   }
```

```
   // The coefficient derivative d(sigma) / dx
   public double SigmaPrime(double x, double t)
   {
      return sigma;
   }

   // The coefficient mu(x)
   public double Mu(double x, double t)
   {
      return ((interestRate - dividend) * x);
   }

   // The coefficient kappa(x)
   public double Kappa(double x, double t)
   {
      return interestRate;
   }
}

class MyBoundaries : FeynmanKac.IBoundaries
{

   double interestRate, strikePrice, dp, factor, xMax;

   public MyBoundaries(double[] rData)
   {
      this.strikePrice = rData[0];
      this.xMax = rData[1];
      this.interestRate = rData[3];
      this.factor = rData[5];
   }

   public void LeftBoundaries(double time, double[,] bndCoeffs)
   {
      dp = strikePrice * Math.Exp(time * interestRate);
      bndCoeffs[0, 0] = 1.0;
      bndCoeffs[0, 1] = 0.0;
      bndCoeffs[0, 2] = 0.0;
      bndCoeffs[0, 3] = dp;
      bndCoeffs[1, 0] = 0.0;
      bndCoeffs[1, 1] = 1.0;
      bndCoeffs[1, 2] = 0.0;
      bndCoeffs[1, 3] = 0.0;
      bndCoeffs[2, 0] = 0.0;
      bndCoeffs[2, 1] = 0.0;
      bndCoeffs[2, 2] = 1.0;
      bndCoeffs[2, 3] = 0.0;

      return;
   }

   public void RightBoundaries(double time, double[,] bndCoeffs)
   {
      bndCoeffs[0, 0] = 1.0;
      bndCoeffs[0, 1] = 0.0;
      bndCoeffs[0, 2] = 0.0;
```

```
      bndCoeffs[0, 3] = factor * xMax;
      bndCoeffs[1, 0] = 0.0;
      bndCoeffs[1, 1] = 1.0;
      bndCoeffs[1, 2] = 0.0;
      bndCoeffs[1, 3] = factor;
      bndCoeffs[2, 0] = 0.0;
      bndCoeffs[2, 1] = 0.0;
      bndCoeffs[2, 2] = 1.0;
      bndCoeffs[2, 3] = 0.0;

      return;
   }

   public double Terminal(double x)
   {
      // The payoff function
      double val = Math.Max(factor * x, strikePrice);
      return val;
   }
}

class MyForcingTerm : FeynmanKac.IForcingTerm
{
   const double zero = 0.0;
   const double one = 1.0;
   double val, strikePrice, interestRate;
   double rt, mu, factor;

   public MyForcingTerm(double[] rData)
   {
      this.val = rData[6];
      this.strikePrice = rData[0];
      this.interestRate = rData[3];
      this.factor = rData[5];
   }

   public void Force(int interval, double[] y, double time, double width,
      double[] xlocal, double[] qw, double[,] u,
      double[] phi, double[,] dphi)
   {
      const int local = 6;
      int ndeg = xlocal.Length;
      double[] yl = new double[6];
      double[] bf = new double[6];
      for (int i = 0; i < local; i++)
      {
         yl[i] = y[3 * interval - 3 + i];
         phi[i] = zero;
      }
      for (int i = 0; i < local; i++)
      {
         for (int j = 0; j < local; j++)
         {
            dphi[i, j] = zero;
         }
      }
```

```
mu = 2.0;
/*
* This is the local definition of the forcing term -
* It "forces" the constraint f >= factor*x.
*/
for (int j = 1; j <= local; j++)
{
   for (int l = 1; l <= ndeg; l++)
   {
      bf[0] = u[0, l - 1];
      bf[1] = u[1, l - 1];
      bf[2] = u[2, l - 1];
      bf[3] = u[6, l - 1];
      bf[4] = u[7, l - 1];
      bf[5] = u[8, l - 1];
      rt = 0.0;
      for (int k = 0; k < local; k++)
      {
         rt += yl[k] * bf[k];
      }
      rt = val / (rt + val - factor * xlocal[l - 1]);
      phi[j - 1] += qw[l - 1] * bf[j - 1] * Math.Pow(rt, mu);
   }
}
for (int i = 0; i < local; i++)
{
   phi[i] = (-phi[i]) * width * factor * strikePrice;
}
/*
* This is the local derivative matrix for the forcing term
*/
for (int j = 1; j <= local; j++)
{
   for (int i = 1; i <= local; i++)
   {
      for (int l = 1; l <= ndeg; l++)
      {
         bf[0] = u[0, l - 1];
         bf[1] = u[1, l - 1];
         bf[2] = u[2, l - 1];
         bf[3] = u[6, l - 1];
         bf[4] = u[7, l - 1];
         bf[5] = u[8, l - 1];
         rt = 0.0;
         for (int k = 0; k < local; k++)
         {
            rt += yl[k] * bf[k];
         }
         rt = one / (rt + val - factor * xlocal[l - 1]);
         dphi[j - 1, i - 1] += qw[l - 1] * bf[i - 1] * bf[j - 1] *
            Math.Pow(val * rt, mu) * rt;
      }
   }
}
for (int i = 0; i < local; i++)
```

```
         {
            for (int j = 0; j < local; j++)
            {
               dphi[i, j] = (-mu) * dphi[i, j] * width * factor *
                  strikePrice;
            }
         }
         return;
      }
   }

   class MyInitialData : FeynmanKac.IInitialData
   {

      private double[] data;

      public MyInitialData(double[] rData)
      {
         data = new double[rData.Length];
         for (int i = 0; i < rData.Length; i++)
         {
            data[i] = rData[i];
         }
      }

      public void Init(double[] xGrid, double[] tGrid, double tp,
         double[] yprime, double[] y, double[] atol, double[] rtol)
      {
         int nxGrid = xGrid.Length;
         if (tp == 0.0)
         // Set initial data precisely.
         {
            for (int i = 1; i <= nxGrid; i++)
            {
               if (xGrid[i - 1] * data[5] < data[0])
               {
                  y[3 * i - 3] = data[0];
                  y[3 * i - 2] = 0.0;
                  y[3 * i - 1] = 0.0;
               }
               else
               {
                  y[3 * i - 3] = xGrid[i - 1] * data[5];
                  y[3 * i - 2] = data[5];
                  y[3 * i - 1] = 0.0;
               }
            }
         }
         return;
      }
   }
}
```

Output

```
Convertible Bond Value, 0+, 6 and 12 Months Prior to Expiry
   Number of equally spaced spline knots:  61
   Number of unknowns: 183
   Strike= 1.00, Sigma= 0.25
   Interest Rate= 0.10, Dividend= 0.02, Factor= 0.100
   Underlying        Bond Value
       0.0000  1.0000  0.9512  0.9048
       0.2500  1.0000  0.9512  0.9049
       0.5000  1.0000  0.9513  0.9065
       0.7500  1.0000  0.9737  0.9605
       1.0000  1.1250  1.1416  1.1464
       1.2500  1.4063  1.4117  1.4121
       1.5000  1.6875  1.6922  1.6922
       1.7500  1.9688  1.9731  1.9731
       2.0000  2.2500  2.2540  2.2540
       2.2500  2.5313  2.5349  2.5349
       2.5000  2.8125  2.8160  2.8160
       2.7500  3.0938  3.0970  3.0970
       3.0000  3.3750  3.3781  3.3781
```

Example 5: Calculating the Greeks using the Feynman-Kac Class

In this example, the Feynman-Kac (FK) class is used to solve for the Greeks, i.e. various derivatives of FK solutions applicable to the pricing of options and related financial derivatives. In order to illustrate and verify these calculations, the Greeks are calculated by two methods. The first method involves the FK solution to the diffusion model for call options given in Example 2 for the Black-Scholes (BS) case, i.e. $\alpha = 2$. The second method calculates the Greeks using the closed-form BS evaluations which can be found at http://en.wikipedia.org/wiki/The_Greeks.

This example calculates FK and BS solutions $V(S,t)$ to the BS problem and the following Greeks:

- $Delta = \frac{\partial V}{\partial S}$ is the first derivative of the *Value*, $V(S,t)$, of a portfolio of derivative security derived from underlying instrument with respect to the underlying instrument's price S;
- $Gamma = \frac{\partial^2 V}{\partial S^2}$;
- $Theta = -\frac{\partial V}{\partial t}$ is the negative first derivative of V with respect to time t;
- $Charm = \frac{\partial^2 V}{\partial S \partial t}$;
- $Color = \frac{\partial^3 V}{\partial S^2 \partial t}$;
- $Rho = -\frac{\partial V}{\partial r}$ is the first derivative of V with respect to risk free rate r;
- $Vega = \frac{\partial V}{\partial \sigma}$ measures sensitivity to volatility parameter σ of the underlying S;
- $Volga = \frac{\partial^2 V}{\partial \sigma^2}$;
- $Vanna = \frac{\partial^2 V}{\partial S \partial \sigma}$;

- *Speed* $= \frac{\partial^3 V}{\partial S^3}$.

Intrinsic Greeks include derivatives involving only *S* and *t*, the intrinsic FK arguments. In the above list, *Value*, *Delta*, *Gamma*, *Theta*, *Charm*, *Color*, and *Speed* are all intrinsic Greeks. As is discussed in Hanson, R. (2008) *Integrating Feynman-Kac equations Using Hermite Quintic Finite Elements*, the expansion of the FK solution function $V(S,t)$ in terms of quintic polynomial functions defined on *S*-grid subintervals and subject to continuity constraints in derivatives 0, 1, and 2 across the boundaries of these subintervals allows *Value*, *Delta*, *Gamma*, *Theta*, *Charm*, and *Color* to be calculated directly by the FK class methods `GetSplineCoefficients`, `GetSplineCoefficientsPrime`, and `GetSplineValue`.

Non-intrinsic Greeks are derivatives of *V* involving FK parameters other than the intrinsic arguments *S* and *t*, such as *r* and σ. Non-intrinsic Greeks in the above list include *Rho*, *Vega*, *Volga*, and *Vanna*. In order to calculate non-intrinsic Greek (parameter) derivatives or intrinsic Greek *S*-derivatives beyond the second (such as *Speed*) or *t*-derivatives beyond the first, the entire FK solution must be calculated 3 times (for a parabolic fit) or five times (for a quartic fit), at the point where the derivative is to be evaluated and at nearby points located symmetrically on either side.

Using a Taylor series expansion of $f(\sigma+\varepsilon)$ truncated to *m*+1 terms (to allow an *m*-degree polynomial fit of *m*+1 data points):

$$f(\sigma+\varepsilon) \;=\; \sum_{n=0}^{m} \frac{f^{(n)}(\sigma)}{n!}\varepsilon^n$$

we are able to derive the following parabolic (3 point) estimation of first and second derivatives $f^{(1)}(\sigma)$ and $f^{(2)}(\sigma)$ in terms of the three values: $f(\sigma-\varepsilon)$, $f(\sigma)$, and $f(\sigma+\varepsilon)$, where $\varepsilon = \varepsilon_{frac}\sigma$ and $0 < \varepsilon_{frac} << 1$:

$$f^{(1)}(\sigma) \;\equiv\; \frac{\partial f(\sigma)}{\partial \sigma} \;\approx\; f^{[1]}(\sigma,\varepsilon) \;\equiv\; \frac{f(\sigma+\varepsilon)-f(\sigma-\varepsilon)}{2\varepsilon}$$

$$f^{(2)}(\sigma) \;\equiv\; \frac{\partial^2 f(\sigma)}{\partial \sigma^2} \;\approx\; f^{[2]}(\sigma,\varepsilon) \;\equiv\; \frac{f(\sigma+\varepsilon)+f(\sigma-\varepsilon)-2f(\sigma)}{\varepsilon^2}$$

Similarly, the quartic (5 point) estimation of $f^{(1)}(\sigma)$ and $f^{(2)}(\sigma)$ in terms of $f(\sigma-2\varepsilon)$, $f(\sigma-\varepsilon)$, $f(\sigma)$, $f(\sigma+\varepsilon)$, and $f(\sigma+2\varepsilon)$ is:

$$f^{(1)}(\sigma) \;\approx\; \frac{4}{3}f^{[1]}(\sigma,\varepsilon) \;-\; \frac{1}{3}f^{[1]}(\sigma,2\varepsilon)$$

$$f^{(2)}(\sigma) \;\approx\; \frac{4}{3}f^{[2]}(\sigma,\varepsilon) \;-\; \frac{1}{3}f^{[2]}(\sigma,2\varepsilon)$$

For our example, the quartic estimate does not appear to be significantly better than the parabolic estimate, so we have produced only parabolic estimates by setting variable `iquart` to 0. The user may try the example with the quartic estimate simply by setting `iquart` to 1.

As is pointed out in *Integrating Feynman-Kac equations Using Hermite Quintic Finite Elements*, the quintic polynomial expansion function used by Feynman-Kac only allows for continuous derivatives through the second derivative. While up to fifth derivatives can be calculated from the quintic expansion (indeed class `FeynmanKac` method `GetSplineValue` will allow the third derivative to be calculated by setting parameter `ideriv` to 3, as is done in this example), the accuracy is compromised by the inherent lack of continuity across grid points (i.e. subinterval boundaries).

The accurate second derivatives in S returned by FK method `GetSplineValue` can be leveraged into a third derivative estimate by calculating three FK second derivative solutions, the first solution for grid and evaluation point sets $\{S,\ f^{(2)}(S)\}$ and the second and third solutions for solution grid and evaluation point sets $\{S+\varepsilon,\ f^{(2)}(S+\varepsilon)\}$ and $\{S-\varepsilon,\ f^{(2)}(S-\varepsilon)\}$, where the solution grid and evaluation point sets are shifted up and down by ε. In this example, ε is set to $\varepsilon_{frac}\bar{S}$, where $\bar{S}$ is the average value of S over the range of grid values and $0 < \varepsilon_{frac} << 1$. The third derivative solution can then be obtained using the parabolic estimate:

$$f^{(3)}(S) \;=\; \frac{\partial f^{(2)}(S)}{\partial S} \;\approx\; \frac{f^{(2)}(S+\varepsilon) - f^{(2)}(S-\varepsilon)}{2\varepsilon}$$

This procedure is implemented in the current example to calculate the Greek *Speed*. (For comparison purposes, *Speed* is also calculated *directly* by setting the `GetSplineValue` input S derivative parameter `iSDeriv` to 3. The output from this direct calculation is called "*Speed2*".)

The average and maximum relative errors (defined as the absolute value of the difference between the BS and FK values divided by the BS value) for each of the Greeks is given at the end of the output. (These relative error statistics are given for nine combinations of Strike Price and Volatility, but only one of the nine combinations is actually printed in the output.) Both intrinsic and non-intrinsic Greeks have good accuracy (average relative error is in the range 0.01 – 0.0001) except for *Volga*, which has an average relative error of about 0.05. This is probably a result of the fact that *Volga* involves differences of differences, which will more quickly erode accuracy than calculations using only one difference to approximate a derivative. Possible ways to improve upon the 2 to 4 significant digits of accuracy achieved in this example include increasing FK integration accuracy by reducing the initial step size (using property `InitialStepsize`), by choosing more closely spaced S and t grid points (by adjusting method `ComputeCoefficients` input parameter arrays `xGrid` and `tGrid`), and by adjusting ε_{frac} so that the central differences used to calculate the derivatives are not too small to compromise accuracy.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class FeynmanKacEx5
{
   private static double[] strikePrices = { 15.0, 20.0, 25.0 };
   // The set of sigma values
   private static double[] sigma = { 0.2, 0.3, 0.4 };
   // The set of model diffusion powers:  alpha = 2.0 <==> Black Scholes
   private static double[] alpha = { 2.0, 1.0, 0.0 };
   // Time values for the options
   private static double[] tGrid = { 1.0 / 12.0, 4.0 / 12.0, 7.0 / 12.0 };
   // Values of the min and max underlying values modeled
   private static double xMin = 0.0;
```

```
private static double xMax = 60.0;
// Value of the interest rate and continuous dividend
private static double r = 0.05;
private static double dividend = 0.0;
// Define parameters for the integration step.
private static int nXGgrid = 121;
private static int nTGrid = 3;
private static int nv = 3;
private static int nint = nXGgrid - 1;
private static int n = 3 * nXGgrid;
private static double[] xGrid = new double[nXGgrid];
private static double dx;
// Time dependency
private static bool[] timeDependence = new bool[7];
// Number of left/right boundary conditions
private static int nlbcd = 3;
private static int nrbcd = 3;
// Values of the underlying price where evaluations are made
private static double[] evaluationPoints = { 19.0, 20.0, 21.0 };
private static double epsfrac = .001; //used to calc derivatives
private static double dx2 = epsfrac * 0.5 * (xMin + xMax);
private static double sqrt2pi = Math.Sqrt(2.0 * Math.PI);
private static String[] greekName = new String[]{
   " Value", " Delta", " Gamma", " Theta",
   " Charm", " Color", "  Vega", " Volga",
   " Vanna", "   Rho", " Speed", "Speed2"
};

// Time values for the options
private static double[] rex = new double[greekName.Length];
private static double[] reavg = new double[greekName.Length];
private static int[] irect = new int[greekName.Length];

// Compute Constant Elasticity of Variance Model for Vanilla Call
public static void Main(String[] args)
{
   // Define equally-spaced grid of points for the underlying price
   dx = (xMax - xMin) / ((double)nint);
   xGrid[0] = xMin;
   xGrid[nXGgrid - 1] = xMax;

   for (int i = 2; i <= nXGgrid - 1; i++)
   {
      xGrid[i - 1] = xGrid[i - 2] + dx;
   }
   Console.Out.WriteLine("  Constant Elasticity of Variance Model" +
                         " for Vanilla Call Option");
   Console.Out.WriteLine(
      "       Interest Rate:{0,7:f3}   Continuous Dividend:{1,7:f3}",
      r, dividend);
   Console.Out.WriteLine(
      "       Minimum and Maximum Prices of Underlying:{0,7:f2}{1,7:f2}",
      xMin, xMax);
   Console.Out.WriteLine(
      "       Number of equally spaced spline knots:{0,4:d}",
      nXGgrid - 1);
```

```
   Console.Out.WriteLine("       Number of unknowns:{0,4:d}", n);
   Console.Out.WriteLine();
   Console.Out.WriteLine(
      "       Time in Years Prior to Expiration: {0,7:f4}{1,7:f4}{2,7:f4}",
      tGrid[0], tGrid[1], tGrid[2]); // tGrid[] = tau == T - t
   Console.Out.WriteLine(
      "       Option valued at Underlying Prices:{0,7:f2}{1,7:f2}{2,7:f2}",
      evaluationPoints[0], evaluationPoints[1], evaluationPoints[2]);
   Console.Out.WriteLine();

   /*
   * iquart = 0 : derivatives estimated with 3-point fitted parabola
   * iquart = 1 : derivatives estimated with 5-point fitted quartic
   *              polynomial
   */
   int iquart = 0;
   if (iquart == 0)
   {
      Console.Out.WriteLine(
         "       3 point (parabolic) estimate of parameter derivatives;");
   }
   else
   {
      Console.Out.WriteLine(
         "       5 point (quartic) estimate of parameter derivatives");
   }
   Console.Out.WriteLine("       epsfrac = {0,11:f8}", epsfrac);
   //alpha:  Black-Scholes
   for (int i2 = 1; i2 <= 3; i2++)
      /* Loop over volatility */
      for (int i3 = 1; i3 <= 3; i3++)
         /* Loop over strike price */
         calculateGreeks(i2, i3, iquart);

   Console.Out.WriteLine();
   for (int ig = 0; ig < 12; ig++)
   {
      reavg[ig] /= irect[ig];
      Console.Out.WriteLine();
      Console.Out.Write(" Greek: " + greekName[ig] +
         ";  avg rel err: {0,15:f12};  max rel err: {1,15:f12}",
         reavg[ig], rex[ig]);
   }
   Console.Out.WriteLine();
} // end main

private static void calculateGreeks(
   int volatility, int strikePrice, int iquart)
{
   int i1 = 1;
   int nt = 3;
   if ((volatility == 1) && (strikePrice == 1))
   {
      Console.Out.WriteLine();
      Console.Out.WriteLine("                  Strike={0,5:f2}, Sigma=" +
         "{1,5:f2}, Alpha={2,5:f2}:", strikePrices[strikePrice - 1],
```

```
      sigma[volatility - 1], alpha[i1 - 1]);
   Console.Out.WriteLine();
   Console.Out.WriteLine("                years to expiration: " +
                         "   {0,7:f4}       {1,7:f4}       {2,7:f4}",
                         tGrid[0], tGrid[1], tGrid[2]);
   Console.Out.WriteLine();
}
/* Loop over t derivative index */
for (int iTDeriv = 0; iTDeriv < 2; iTDeriv++)
{
   int iSDerivMax = 4 - iTDeriv;
   /* Loop over S derivative index */
   for (int iSDeriv = 0; iSDeriv < iSDerivMax; iSDeriv++)
   {
      int iSigDerivMax = 1;
      if (iTDeriv == 0)
      {
         if (iSDeriv == 0)
            iSigDerivMax = 3;
         if (iSDeriv == 1)
            iSigDerivMax = 2;
      }
      //Loop over sigma deriv index
      for (int iSigDeriv = 0; iSigDeriv < iSigDerivMax; iSigDeriv++)
      {
         int iRDerivMax = 1;
         if (iTDeriv == 0 && iSDeriv == 0 && iSigDeriv == 0)
            iRDerivMax = 2;
         // Loop over r derivative index
         for (int iRDeriv = 0; iRDeriv < iRDerivMax; iRDeriv++)
         {
            int ispeedmin = 0;
            int ispeedmax = 1;
            if (iTDeriv == 0 && iSDeriv == 2)
               ispeedmax = 2;
            if (iTDeriv == 0 && iSDeriv == 3)
            {
               ispeedmin = 2;
               ispeedmax = 3;
            }
            // Loop over speed index
            for (int ispeed = ispeedmin; ispeed < ispeedmax; ispeed++)
            {
               // Pass data through into evaluation routines.
               double[] userData = new double[6];
               userData[0] = strikePrices[strikePrice - 1];
               userData[1] = xMax;
               userData[2] = sigma[volatility - 1];
               userData[3] = alpha[i1 - 1];
               userData[4] = r;
               userData[5] = dividend;
               double[][] splineValuesOption = new double[nTGrid][];
               for (int i = 0; i < nTGrid; i++)
               {
                  splineValuesOption[i] = new double[nv];
               }
```

```
double[][] splineValuesOptionP = new double[nTGrid][];
double[][] splineValuesOptionM = new double[nTGrid][];
double[][] splineValuesOptionPP = new double[nTGrid][];
double[][] splineValuesOptionMM = new double[nTGrid][];
double[] xGridP = new double[nXGgrid];
double[] xGridM = new double[nXGgrid];
double[] evaluationPointsP = new double[nv];
double[] evaluationPointsM = new double[nv];
double[] xGridPP = new double[nXGgrid];
double[] xGridMM = new double[nXGgrid];
double[] evaluationPointsPP = new double[nv];
double[] evaluationPointsMM = new double[nv];
double xMaxP = xMax;
double xMaxM = xMax;
double xMaxPP = xMax, xMaxMM = xMax;

// Evaluate FK solution at vector evaluationPoints, at
// each time value prior to expiration.

if ((iSigDeriv != 1 || iSDeriv == 1 || iRDeriv != 1) &&
   (ispeed == 0 || ispeed == 2))
{

   FeynmanKac callOption =
      new FeynmanKac(new MyCoefficients(userData));
   //Right boundary condition time-dependent
   timeDependence[5] = true;
   callOption.SetTimeDependence(timeDependence);
   callOption.ComputeCoefficients(nlbcd, nrbcd,
      new MyBoundaries(userData), xGrid, tGrid);
   double[] tmpArray = new double[3 * xGrid.Length];
   double[,] optionCoefficients;
   if (iTDeriv == 0)
   {
      optionCoefficients =
         callOption.GetSplineCoefficients();
   }
   else
   {
      optionCoefficients =
         callOption.GetSplineCoefficientsPrime();
   }
   for (int i = 0; i < nTGrid; i++)
   {
      for (int j = 0; j < 3 * xGrid.Length; j++)
         tmpArray[j] = optionCoefficients[i + 1, j];
      // FK option values for tau = tGrid[i]:
      splineValuesOption[i] =
         callOption.GetSplineValue(evaluationPoints,
         tmpArray, iSDeriv);
   }
}
if (iSigDeriv > 0 || iRDeriv > 0 || ispeed == 1)
{
   Array.Copy(xGrid, 0, xGridP, 0, nXGgrid);
   Array.Copy(xGrid, 0, xGridM, 0, nXGgrid);
```

```
Array.Copy(evaluationPoints, 0,
   evaluationPointsP, 0, nv);
Array.Copy(evaluationPoints, 0,
   evaluationPointsM, 0, nv);
if (ispeed == 1)
{
   for (int i = 0; i < nXGgrid; i++)
   {
      xGridP[i] = xGrid[i] + dx2;
      xGridM[i] = xGrid[i] - dx2;
   }
   for (int i = 0; i < nv; i++)
   {
      evaluationPointsP[i] =
         evaluationPoints[i] + dx2;
      evaluationPointsM[i] =
         evaluationPoints[i] - dx2;
   }
   xMaxP = xMax + dx2;
   xMaxM = xMax - dx2;
}

userData[1] = xMaxP;
// calculate spline values for
// sigmaP = sigma[i2-1]*(1. + epsfrac)
// or rP = r*(1. + epsfrac)
if (iSigDeriv > 0)
   userData[2] = sigma[volatility - 1] *
      (1.0 + epsfrac);
else if (iRDeriv > 0)
   userData[4] = r * (1.0 + epsfrac);
FeynmanKac callOptionP =
   new FeynmanKac(new MyCoefficients(userData));
//Right boundary condition time-dependent
timeDependence[5] = true;
callOptionP.SetTimeDependence(timeDependence);
callOptionP.ComputeCoefficients(nlbcd, nrbcd,
   new MyBoundaries(userData), xGridP, tGrid);
double[,] optionCoefficientsP;
double[] tmpArray2 = new double[3 * xGrid.Length];
if (iTDeriv == 0)
{
   optionCoefficientsP =
      callOptionP.GetSplineCoefficients();
}
else
{
   optionCoefficientsP =
      callOptionP.GetSplineCoefficientsPrime();
}
for (int i = 0; i < nTGrid; i++)
{
   for (int j = 0; j < 3 * xGrid.Length; j++)
      tmpArray2[j] = optionCoefficientsP[i + 1, j];
   // FK option values for tau = tGrid[i]:
   splineValuesOptionP[i] =
```

```
      callOptionP.GetSplineValue(evaluationPointsP,
      tmpArray2, iSDeriv);
}

userData[1] = xMaxM;
// calculate spline values for
// sigmaM = sigma[i2-1]*(1. - epsfrac)   or
// rM = r*(1. - epsfrac):
if (iSigDeriv > 0)
   userData[2] = sigma[volatility - 1] *
      (1.0 - epsfrac);
else
   userData[4] = r * (1.0 - epsfrac);
FeynmanKac callOptionM =
   new FeynmanKac(new MyCoefficients(userData));
//Right boundary condition time-dependent
timeDependence[5] = true;
callOptionM.SetTimeDependence(timeDependence);
callOptionM.ComputeCoefficients(nlbcd, nrbcd,
   new MyBoundaries(userData), xGridM, tGrid);
double[,] optionCoefficientsM;
double[] tmpArray3 = new double[3 * xGrid.Length];

if (iTDeriv == 0)
{
   optionCoefficientsM =
      callOptionM.GetSplineCoefficients();
}
else
{
   optionCoefficientsM =
      callOptionM.GetSplineCoefficientsPrime();
}
for (int i = 0; i < nTGrid; i++)
{
   for (int j = 0; j < 3 * xGrid.Length; j++)
      tmpArray3[j] = optionCoefficientsM[i + 1, j];
   // FK option values for tau = tGrid[i]:
   splineValuesOptionM[i] =
      callOptionM.GetSplineValue(evaluationPointsM,
      tmpArray3, iSDeriv);
}
if (iquart == 1)
{
   Array.Copy(xGrid, 0, xGridPP, 0, nXGgrid);
   Array.Copy(xGrid, 0, xGridMM, 0, nXGgrid);
   Array.Copy(evaluationPoints, 0,
      evaluationPointsPP, 0, nv);
   Array.Copy(evaluationPoints, 0,
      evaluationPointsMM, 0, nv);
   if (ispeed == 1)
   {
      for (int i = 0; i < nXGgrid; i++)
      {
         xGridPP[i] = xGrid[i] + 2.0 * dx2;
         xGridMM[i] = xGrid[i] - 2.0 * dx2;
```

```
   }
   for (int i = 0; i < nv; i++)
   {
      evaluationPointsPP[i] =
         evaluationPoints[i] + 2.0 * dx2;
      evaluationPointsMM[i] =
         evaluationPoints[i] - 2.0 * dx2;
   }
   xMaxPP = xMax + 2.0 * dx2;
   xMaxMM = xMax - 2.0 * dx2;
}

userData[1] = xMaxPP;
if (iSigDeriv > 0)
{
   // calculate spline values for
   // sigmaPP = sigma[i2-1]*(1. + 2.*epsfrac):
   userData[2] = sigma[volatility - 1] *
      (1.0 + 2.0 * epsfrac);
}
else if (iRDeriv > 0)
{
   // calculate spline values for
   // rPP = r*(1. + 2.*epsfrac):
   userData[4] = r * (1.0 + 2.0 * epsfrac);
}
FeynmanKac callOptionPP =
   new FeynmanKac(new MyCoefficients(userData));
//Right boundary condition time-dependent
timeDependence[5] = true;
callOptionPP.SetTimeDependence(timeDependence);
callOptionPP.ComputeCoefficients(nlbcd, nrbcd,
   new MyBoundaries(userData), xGridPP, tGrid);

double[] tmpArray4 = new double[3 * xGrid.Length];
double[,] optionCoefficientsPP;

if (iTDeriv == 0)
{
   optionCoefficientsPP =
      callOptionPP.GetSplineCoefficients();
}
else
{
   optionCoefficientsPP =
      callOptionPP.GetSplineCoefficientsPrime();
}
for (int i = 0; i < nTGrid; i++)
{
   for (int j = 0; j < 3 * xGrid.Length; j++)
      tmpArray4[j] =
         optionCoefficientsPP[i + 1, j];
   // FK option values for tau = tGrid[i]:
   splineValuesOptionPP[i] =
      callOptionPP.GetSplineValue(
      evaluationPointsPP, tmpArray4, iSDeriv);
```

```
      }

      userData[1] = xMaxMM;
      // calculate spline values for
      // sigmaMM = sigma[i2-1]*(1. - 2.*epsfrac) or
      //  rMM = r*(1. - 2.*epsfrac)
      if (iSigDeriv > 0)
         userData[2] = sigma[volatility - 1] *
            (1.0 - 2.0 * epsfrac);
      else if (iRDeriv > 0)
         userData[4] = r * (1.0 - 2.0 * epsfrac);
      FeynmanKac callOptionMM =
         new FeynmanKac(new MyCoefficients(userData));
      //Right boundary condition time-dependent
      timeDependence[5] = true;
      callOptionMM.SetTimeDependence(timeDependence);
      callOptionMM.ComputeCoefficients(nlbcd, nrbcd,
         new MyBoundaries(userData), xGridMM, tGrid);
      double[] tmpArray5 = new double[3 * xGrid.Length];
      double[,] optionCoefficientsMM;
      if (iTDeriv == 0)
      {
         optionCoefficientsMM =
            callOptionMM.GetSplineCoefficients();
      }
      else
      {
         optionCoefficientsMM =
            callOptionMM.GetSplineCoefficientsPrime();
      }
      for (int i = 0; i < nTGrid; i++)
      {
         for (int j = 0; j < 3 * xGrid.Length; j++)
            tmpArray5[j] =
               optionCoefficientsMM[i + 1, j];
         // FK option values for tau = tGrid[i]:
         splineValuesOptionMM[i] =
            callOptionMM.GetSplineValue(
            evaluationPointsMM,tmpArray5, iSDeriv);
      }
   }
}

if (iSigDeriv == 1 || iRDeriv == 1 || ispeed == 1)
{
   double eps = 0.0, f11 = 0.0, f12 = 0.0;
   if (iSigDeriv == 1)
      eps = sigma[volatility - 1] * epsfrac;
   if (iRDeriv == 1)
      eps = r * epsfrac;
   if (ispeed == 1)
      eps = dx2;
   for (int i = 0; i < nTGrid; i++)
   {
      for (int j = 0; j < nv; j++)
      {
```

```
            f11 = (splineValuesOptionP[i][j] -
               splineValuesOptionM[i][j]) / (2.0 * eps);
            if (iquart == 0)
               splineValuesOption[i][j] = f11;
            else
            {
               f12 = (splineValuesOptionPP[i][j] -
                  splineValuesOptionMM[i][j]) / (4.0 * eps);
               splineValuesOption[i][j] =
                  (4.0 * f11 - f12) / 3.0;
            }
         }
      }
   }

   if (iSigDeriv == 2)
   {
      double eps = sigma[volatility - 1] * epsfrac;
      double f21 = 0.0;
      double f22 = 0.0;
      for (int i = 0; i < nTGrid; i++)
      {
         for (int j = 0; j < nv; j++)
         {
            f21 = (splineValuesOptionP[i][j] +
               splineValuesOptionM[i][j] - 2.0 *
               splineValuesOption[i][j]) / (eps * eps);
            if (iquart == 0)
               splineValuesOption[i][j] = f21;
            else
            {
               f22 = (splineValuesOptionPP[i][j] +
                  splineValuesOptionMM[i][j] - 2.0 *
                  splineValuesOption[i][j]) /
                  (4.0 * eps * eps);
               splineValuesOption[i][j] =
                  (4.0 * f21 - f22) / 3.0;
            }
         }
      }
   }


   // Evaluate BS solution at vector evaluationPoints,
   // at each time value prior to expiration.

   double[][] BSValuesOption = new double[nTGrid][];
   for (int i7 = 0; i7 < nTGrid; i7++)
   {
      BSValuesOption[i7] = new double[nv];
   }
   for (int i = 0; i < nTGrid; i++)
   {
      /*
      *  Black-Scholes (BS) European call option
      *  value = ValBSEC(S,t) = exp(-q*tau)*S*N01CDF(d1) -
```

```
*                        exp(-r*tau)*K*N01CDF(d2),
*  where:
*    tau = time to maturity = T - t;
*    q = annual dividend yield;
*    r = risk free rate;
*    K = strike price;
*    S = stock price;
*    N01CDF(x) = N(0,1)_CDF(x);
*    d1 = ( log( S/K ) +
*         ( r - q + 0.5*sigma**2 )*tau ) /
*         ( sigma*sqrt(tau) );
*    d2 = d1 - sigma*sqrt(tau)
*/
// BS option values for tau = tGrid[i]:
double tau = tGrid[i];
double sigsqtau =
   Math.Pow(sigma[volatility - 1], 2) * tau;
double sqrt_sigsqtau = Math.Sqrt(sigsqtau);
double sigsq = sigma[volatility - 1] *
   sigma[volatility - 1];
for (int j = 0; j < nv; j++)
{
   // Values of the underlying price where
   // evaluations are made:
   double S = evaluationPoints[j];
   double d1 = (Math.Log(S /
      strikePrices[strikePrice - 1]) + (r - dividend)
      * tau + 0.5 * sigsqtau) / sqrt_sigsqtau;
   double n01pdf_d1 =
      Math.Exp((-0.5) * d1 * d1) / sqrt2pi;
   double nu = Math.Exp((-dividend) * tau) *
      S * n01pdf_d1 * Math.Sqrt(tau);
   if (iTDeriv == 0)
   {
      if (iSDeriv == 0)
      {
         double d2 = d1 - sqrt_sigsqtau;
         if (iSigDeriv == 0)
         {
            if (iRDeriv == 0)
            {
               BSValuesOption[i][j] =
                  Math.Exp((-dividend) * tau) * S *
                  Cdf.Normal(d1) -
                  Math.Exp((-r) * tau) *
                  strikePrices[strikePrice - 1] *
                  Cdf.Normal(d2);
            }
            else
            {
               BSValuesOption[i][j] =
                  Math.Exp((-r) * tau) *
                  strikePrices[strikePrice - 1]
                  * tau * Cdf.Normal(d2);
            }
         }
```

```
            else if (iSigDeriv == 1)
            {
               //greek = Vega
               BSValuesOption[i][j] = nu;
            }
            else if (iSigDeriv == 2)
            {
               //greek = Volga
               BSValuesOption[i][j] = nu * d1 * d2 /
                  sigma[volatility - 1];
            }
         }
         else if (iSDeriv == 1)
         {
            //greek = delta
            if (iSigDeriv == 0)
            {
               BSValuesOption[i][j] =
                  Math.Exp((-dividend) * tau) *
                  Cdf.Normal(d1);
            }
            else if (iSigDeriv == 1)
            {
               //greek = Vanna
               BSValuesOption[i][j] = (nu / S) *
                  (1.0 - d1 / sqrt_sigsqtau);
            }
         }
         else if (iSDeriv == 2)
         {
            if (ispeed == 0)
            {
               //greek = gamma
               BSValuesOption[i][j] =
                  Math.Exp((-dividend) * tau) *
                  n01pdf_d1 /(S * sqrt_sigsqtau);
            }
            else
            {
               //greek = speed
               BSValuesOption[i][j] =
                  (-Math.Exp((-dividend) * tau)) *
                  n01pdf_d1 * (1.0 + d1 / sqrt_sigsqtau)
                  / (S * S * sqrt_sigsqtau);
            }
         }
         else if (iSDeriv == 3 && ispeed == 2)
         {
            //greek = speed
            BSValuesOption[i][j] =
               (-Math.Exp((-dividend) * tau)) *
               n01pdf_d1 * (1.0 + d1 / sqrt_sigsqtau) /
               (S * S * sqrt_sigsqtau);
         }
      }
      else if (iTDeriv == 1)
```

```
      {
         double d2 = d1 - sqrt_sigsqtau;
         if (iSDeriv == 0)
         {
            //greek = theta
            BSValuesOption[i][j] =
               Math.Exp((-dividend) * tau) * S *
               (dividend * Cdf.Normal(d1) - 0.5 * sigsq *
               n01pdf_d1 / sqrt_sigsqtau) - r *
               Math.Exp((-r) * tau) *
               strikePrices[strikePrice - 1] *
               Cdf.Normal(d2);
         }
         else if (iSDeriv == 1)
         {
            //greek = charm
            BSValuesOption[i][j] =
               Math.Exp((-dividend) * tau) * ((-dividend)
               * Cdf.Normal(d1) + n01pdf_d1 *
               ((r - dividend) * tau - 0.5 * d2 *
               sqrt_sigsqtau) / (tau * sqrt_sigsqtau));
         }
         else if (iSDeriv == 2)
         {
            //greek = color
            BSValuesOption[i][j] =
               (-Math.Exp((-dividend) * tau)) * n01pdf_d1 *
               (2.0 * dividend * tau + 1.0 + d1 *
               (2.0 * (r - dividend) * tau - d2 *
               sqrt_sigsqtau) / sqrt_sigsqtau) /
               (2.0 * S * tau * sqrt_sigsqtau);
         }
      }
   }
}

double relerrmax = 0.0;
int gNi = 3 * iTDeriv + iSDeriv;
if (iSigDeriv == 1 && iSDeriv == 0)
   gNi = 6; //vega
if (iSigDeriv == 2 && iSDeriv == 0)
   gNi = 7; //volga
if (iSigDeriv == 1 && iSDeriv == 1)
   gNi = 8; //vanna
if (iRDeriv == 1)
   gNi = 9; //rho
if (ispeed >= 1)
   gNi = 9 + ispeed; //speed

for (int i = 0; i < nv; i++)
{
   for (int j = 0; j < nt; j++)
   {
      double sVo = splineValuesOption[j][i];
      double BSVo = BSValuesOption[j][i];
      //greeks(itd=1) ~ d/dtau = -d/dt for iSDeriv > 0:
```

```
                        if (iTDeriv == 1 && iSDeriv > 0)
                           sVo = -sVo;
                        double relerr = Math.Abs((sVo - BSVo) / BSVo);
                        if (relerr > relerrmax)
                           relerrmax = relerr;
                        reavg[gNi] += relerr;
                        irect[gNi] += 1;
                     }
                  }
                  if (relerrmax > rex[gNi])
                     rex[gNi] = relerrmax;

                  if ((volatility == 1) && (strikePrice == 1))
                  {
                     for (int i = 0; i < nv; i++)
                     {
                        Console.Out.Write(
                           " underlying price: {0,4:f1};  FK " +
                           greekName[gNi] + ": ", evaluationPoints[i]);
                        for (int j = 0; j < nt; j++)
                        {
                           double sVo = splineValuesOption[j][i];
                           //greeks(itd=1) ~ d/dtau = -d/dt for isd > 0:
                           if (iTDeriv == 1 && iSDeriv > 0)
                              sVo = -sVo;
                           Console.Out.Write("{0,13:f10} ", sVo);
                        }
                        Console.Out.WriteLine();
                        Console.Out.Write("                          BS "
                           + greekName[gNi] + ": ");
                        for (int j = 0; j < nt; j++)
                        {
                           Console.Out.Write("{0,13:f10} ",
                              BSValuesOption[j][i]);
                        }
                        Console.Out.WriteLine();
                     }
                  }
               }
            }
         }
      }
   }
}

class MyCoefficients : FeynmanKac.IPdeCoefficients
{
   const double zero = 0.0;
   const double half = 0.5;
   double sigma, strikePrice, interestRate;
   double alpha, dividend;

   public MyCoefficients(double[] myData)
   {
      this.strikePrice = myData[0];
      this.sigma = myData[2];
```

```
      this.alpha = myData[3];
      this.interestRate = myData[4];
      this.dividend = myData[5];
   }

   // The coefficient sigma(x)
   public double Sigma(double x, double t)
   {
      return (sigma * Math.Pow(x, alpha * half));
   }

   // The coefficient derivative d(sigma) / dx
   public double SigmaPrime(double x, double t)
   {
      return (half * alpha * sigma * Math.Pow(x, alpha * half - 1.0));
   }

   // The coefficient mu(x)
   public double Mu(double x, double t)
   {
      return ((interestRate - dividend) * x);
   }

   // The coefficient kappa(x)
   public double Kappa(double x, double t)
   {
      return interestRate;
   }
}

class MyBoundaries : FeynmanKac.IBoundaries
{

   double xMax, df, interestRate, strikePrice;

   public MyBoundaries(double[] myData)
   {
      this.strikePrice = myData[0];
      this.xMax = myData[1];
      this.interestRate = myData[4];
   }

   public void LeftBoundaries(double time, double[,] bndCoeffs)
   {
      bndCoeffs[0, 0] = 1.0;
      bndCoeffs[0, 1] = 0.0;
      bndCoeffs[0, 2] = 0.0;
      bndCoeffs[0, 3] = 0.0;
      bndCoeffs[1, 0] = 0.0;
      bndCoeffs[1, 1] = 1.0;
      bndCoeffs[1, 2] = 0.0;
      bndCoeffs[1, 3] = 0.0;
      bndCoeffs[2, 0] = 0.0;
      bndCoeffs[2, 1] = 0.0;
      bndCoeffs[2, 2] = 1.0;
      bndCoeffs[2, 3] = 0.0;
```

```
         return;
      }

      public void RightBoundaries(double time, double[,] bndCoeffs)
      {
         df = Math.Exp(interestRate * time);
         bndCoeffs[0, 0] = 1.0;
         bndCoeffs[0, 1] = 0.0;
         bndCoeffs[0, 2] = 0.0;
         bndCoeffs[0, 3] = xMax - df * strikePrice;
         bndCoeffs[1, 0] = 0.0;
         bndCoeffs[1, 1] = 1.0;
         bndCoeffs[1, 2] = 0.0;
         bndCoeffs[1, 3] = 1.0;
         bndCoeffs[2, 0] = 0.0;
         bndCoeffs[2, 1] = 0.0;
         bndCoeffs[2, 2] = 1.0;
         bndCoeffs[2, 3] = 0.0;

         return;
      }

      public double Terminal(double x)
      {
         double zero = 0.0;
         // The payoff function
         double val = Math.Max(x - strikePrice, zero);
         return val;
      }
   }
}
```

Output

```
  Constant Elasticity of Variance Model for Vanilla Call Option
       Interest Rate:  0.050   Continuous Dividend:  0.000
       Minimum and Maximum Prices of Underlying:   0.00  60.00
       Number of equally spaced spline knots: 120
       Number of unknowns: 363

       Time in Years Prior to Expiration:  0.0833 0.3333 0.5833
       Option valued at Underlying Prices:  19.00  20.00  21.00

       3 point (parabolic) estimate of parameter derivatives;
       epsfrac =  0.00100000

                  Strike=15.00, Sigma= 0.20, Alpha= 2.00:

                 years to expiration:     0.0833        0.3333        0.5833

 underlying price: 19.0;  FK  Value:  4.0623732450  4.2575924184  4.4733805278
                          BS  Value:  4.0623732509  4.2575929678  4.4733814062
 underlying price: 20.0;  FK  Value:  5.0623700127  5.2505145764  5.4492418798
                          BS  Value:  5.0623700120  5.2505143129  5.4492428547
```

```
underlying price: 21.0;  FK  Value:  6.0623699727  6.2485587059  6.4385718831
                         BS  Value:  6.0623699726  6.2485585270  6.4385720688
underlying price: 19.0;  FK    Rho:  1.2447807022  4.8365676562  8.0884594648
                         BS    Rho:  1.2447806658  4.8365650322  8.0884502627
underlying price: 20.0;  FK    Rho:  1.2448021850  4.8929216544  8.3041708173
                         BS    Rho:  1.2448021908  4.8929245641  8.3041638392
underlying price: 21.0;  FK    Rho:  1.2448024992  4.9107294561  8.4114197621
                         BS    Rho:  1.2448024996  4.9107310444  8.4114199038
underlying price: 19.0;  FK   Vega:  0.0003289870  0.3487168323  1.1153520921
                         BS   Vega:  0.0003295819  0.3487535501  1.1153536190
underlying price: 20.0;  FK   Vega:  0.0000056652  0.1224632724  0.6032458218
                         BS   Vega:  0.0000056246  0.1224675413  0.6033084039
underlying price: 21.0;  FK   Vega:  0.0000000623  0.0376974472  0.3028275296
                         BS   Vega:  0.0000000563  0.0376857196  0.3028629419
underlying price: 19.0;  FK  Volga:  0.0286253687  8.3705172571 16.7944556484
                         BS  Volga:  0.0286064650  8.3691191978 16.8219823169
underlying price: 20.0;  FK  Volga:  0.0007135181  4.2505026165 12.9315443687
                         BS  Volga:  0.0007186004  4.2519372748 12.9612638820
underlying price: 21.0;  FK  Volga:  0.0000100364  1.7613083880  8.6626164020
                         BS  Volga:  0.0000097963  1.7617504949  8.6676581034
underlying price: 19.0;  FK  Delta:  0.9999864098  0.9877532309  0.9652249945
                         BS  Delta:  0.9999863811  0.9877520034  0.9652261127
underlying price: 20.0;  FK  Delta:  0.9999998142  0.9964646548  0.9842482622
                         BS  Delta:  0.9999998151  0.9964644003  0.9842476147
underlying price: 21.0;  FK  Delta:  0.9999999983  0.9990831687  0.9932459040
                         BS  Delta:  0.9999999985  0.9990834124  0.9932451927
underlying price: 19.0;  FK  Vanna: -0.0012418872 -0.3391850563 -0.6388552010
                         BS  Vanna: -0.0012431594 -0.3391932673 -0.6387423326
underlying price: 20.0;  FK  Vanna: -0.0000244490 -0.1366771953 -0.3945466660
                         BS  Vanna: -0.0000244825 -0.1367114682 -0.3945405194
underlying price: 21.0;  FK  Vanna: -0.0000002905 -0.0466333335 -0.2187406645
                         BS  Vanna: -0.0000002726 -0.0466323413 -0.2187858632
underlying price: 19.0;  FK  Gamma:  0.0000543456  0.0144908955  0.0264849216
                         BS  Gamma:  0.0000547782  0.0144911447  0.0264824761
underlying price: 20.0;  FK  Gamma:  0.0000008315  0.0045912854  0.0129288434
                         BS  Gamma:  0.0000008437  0.0045925328  0.0129280372
underlying price: 21.0;  FK  Gamma:  0.0000000080  0.0012817012  0.0058860348
                         BS  Gamma:  0.0000000077  0.0012818272  0.0058865489
underlying price: 19.0;  FK  Speed: -0.0002127758 -0.0157070513 -0.0181086989
                         BS  Speed: -0.0002123854 -0.0156192867 -0.0179536520
underlying price: 20.0;  FK  Speed: -0.0000037305 -0.0056184183 -0.0098403706
                         BS  Speed: -0.0000037568 -0.0055859333 -0.0097472434
underlying price: 21.0;  FK  Speed: -0.0000000385 -0.0017185470 -0.0048615664
                         BS  Speed: -0.0000000378 -0.0017082128 -0.0048130214
underlying price: 19.0;  FK Speed2: -0.0002310655 -0.0156276977 -0.0179516855
                         BS Speed2: -0.0002123854 -0.0156192867 -0.0179536520
underlying price: 20.0;  FK Speed2: -0.0000043215 -0.0055923924 -0.0097502997
                         BS Speed2: -0.0000037568 -0.0055859333 -0.0097472434
underlying price: 21.0;  FK Speed2: -0.0000000475 -0.0017117661 -0.0048153107
                         BS Speed2: -0.0000000378 -0.0017082128 -0.0048130214
underlying price: 19.0;  FK  Theta: -0.7472631891 -0.8301000450 -0.8845209253
                         BS  Theta: -0.7472638978 -0.8301108199 -0.8844992143
underlying price: 20.0;  FK  Theta: -0.7468881086 -0.7706770630 -0.8152217385
                         BS  Theta: -0.7468880640 -0.7706789470 -0.8152097697
underlying price: 21.0;  FK  Theta: -0.7468815742 -0.7479185416 -0.7728950748
                         BS  Theta: -0.7468815673 -0.7479153725 -0.7728982104
```

```
underlying price: 19.0;  FK  Charm: -0.0014382828 -0.0879903285 -0.0843323992
                         BS  Charm: -0.0014397520 -0.0879913927 -0.0843403333
underlying price: 20.0;  FK  Charm: -0.0000284881 -0.0364107814 -0.0547260337
                         BS  Charm: -0.0000285354 -0.0364209077 -0.0547074804
underlying price: 21.0;  FK  Charm: -0.0000003396 -0.0126436426 -0.0313343015
                         BS  Charm: -0.0000003190 -0.0126437838 -0.0313252716
underlying price: 19.0;  FK  Color:  0.0051622176  0.0685064195  0.0299871130
                         BS  Color:  0.0051777484  0.0684737183  0.0300398444
underlying price: 20.0;  FK  Color:  0.0001188761  0.0355826975  0.0274292189
                         BS  Color:  0.0001205713  0.0355891884  0.0274307898
underlying price: 21.0;  FK  Color:  0.0000015431  0.0143174419  0.0190897160
                         BS  Color:  0.0000015141  0.0143247729  0.0190752019


Greek:  Value;  avg rel err:  0.000146170676;  max rel err:  0.009030693731
Greek:  Delta;  avg rel err:  0.000035817236;  max rel err:  0.001158483024
Greek:  Gamma;  avg rel err:  0.001088309906;  max rel err:  0.044839095787
Greek:  Theta;  avg rel err:  0.000054196357;  max rel err:  0.001412847204
Greek:  Charm;  avg rel err:  0.001213365580;  max rel err:  0.064577964461
Greek:  Color;  avg rel err:  0.003323775096;  max rel err:  0.136355625079
Greek:   Vega;  avg rel err:  0.001512805322;  max rel err:  0.106097327138
Greek:  Volga;  avg rel err:  0.058535091480;  max rel err:  1.639568432709
Greek:  Vanna;  avg rel err:  0.001061842143;  max rel err:  0.065654941418
Greek:    Rho;  avg rel err:  0.000146868204;  max rel err:  0.009438785575
Greek:  Speed;  avg rel err:  0.007252489626;  max rel err:  0.123630427780
Greek: Speed2;  avg rel err:  0.008430324710;  max rel err:  0.255782408454
```

FeynmanKac.PDEStepControlMethod Enumeration

```
public enumeration Imsl.Math.FeynmanKac.PDEStepControlMethod
```

Indicates which step control method is used in the integration of the Feynman-Kac PDE.

Fields

MethodOfPetzold

```
public Imsl.Math.FeynmanKac.PDEStepControlMethod MethodOfPetzold
```

Description

Indicates that the step control algorithm of the original Petzold code is used in the integration.

MethodOfSoederlind

```
public Imsl.Math.FeynmanKac.PDEStepControlMethod MethodOfSoederlind
```

Description

Indicates that the step control method by Soederlind is used in the integration.

FeynmanKac.IPdeCoefficients Interface

```
public interface Imsl.Math.FeynmanKac.IPdeCoefficients
```

Interface for computation of coefficients of the Feynman-Kac PDE.

Methods

Kappa

```
abstract public double Kappa(double x, double t)
```

Description

Returns the value of the κ coefficient at the given point.

Parameters

`x` – A `double`, the point in space at which κ is to be evaluated.

`t` – A `double`, the time point at which κ is to be evaluated.

Returns

A `double`, the value of κ at `(x,t)`.

Remarks

Time dependency of κ can be controlled via method `SetTimeDependence`. Use of this method will usually yield a more efficient algorithm because some finite element matrices do not have to be reassembled for each `t` value.

Mu

```
abstract public double Mu(double x, double t)
```

Description

Returns the value of the μ coefficient at the given point.

Parameters

`x` – A `double`, the point in space at which μ is to be evaluated.

`t` – A `double`, the time point at which μ is to be evaluated.

Returns

A `double`, the value of μ at `(x,t)`.

Remarks

Time dependency of μ can be controlled via method `SetTimeDependence`. Use of this method will usually yield a more efficient algorithm because some finite element matrices do not have to be reassembled for each `t` value.

Sigma

```
abstract public double Sigma(double x, double t)
```

Description

Returns the value of the σ coefficient at the given point.

Parameters

`x` – A `double` scalar, the point in space at which σ is to be evaluated.

`t` – A `double` scalar, the time point at which σ is to be evaluated.

Returns

A `double` scalar, the value of σ at `(x,t)`.

Remarks

Time dependency of σ can be controlled via method `SetTimeDependence`. Use of this method will usually yield a more efficient algorithm because some finite element matrices do not have to be reassembled for each `t` value.

SigmaPrime

```
abstract public double SigmaPrime(double x, double t)
```

Description

Returns the value of $\sigma' = \frac{\partial \sigma(x,t)}{\partial x}$ at the given point.

Parameters

`x` – A `double`, the point in space at which σ' is to be evaluated.

`t` – A `double`, the time point at which σ' is to be evaluated.

Returns

A `double`, the value of σ' at `(x,t)`.

Remarks

Time dependency of σ' can be controlled via method `SetTimeDependence`. Use of this method will usually yield a more efficient algorithm because some finite element matrices do not have to be reassembled for each `t` value.

FeynmanKac.IBoundaries Interface

```
public interface Imsl.Math.FeynmanKac.IBoundaries
```

Public interface for user supplied boundary coefficients and terminal condition the PDE must satisfy.

Methods

LeftBoundaries

```
abstract public void LeftBoundaries(double time, double[,] coefficients)
```

Description

Returns the coefficient values of the left boundary conditions. There are `numLeftBounds` conditions specified at the left end, $x_{\min}$. The left boundary conditions are

$$a_i(x,t)f + b_i(x,t)f_x + c_i(x,t)f_{xx} = d_i(x,t),\ x = x_{\min},\ 1 \le i \le \text{numLeftBounds}.$$

Parameters

`time` – A `double`, the time point at which the boundary coefficients are to be evaluated.

`coefficients` – An output `double` matrix of dimension `numLeftBounds` by 4 containing the computed boundary coefficient values. The coefficients are stored row-wise according to the matrix scheme

$$(a_i(x_{\min},t), b_i(x_{\min},t), c_i(x_{\min},t), d_i(x_{\min},t))_{1 \le i \le \text{numLeftBounds}}.$$

Remarks

Time dependency of the boundary coefficients can be controlled via method `SetTimeDependence`. Use of this method will usually yield a more efficient algorithm because some finite element matrices do not have to be reassembled for each `t` value.

RightBoundaries

```
abstract public void RightBoundaries(double time, double[,] coefficients)
```

Description

Returns the coefficient values of the right boundary conditions. There are `numRightBounds` conditions specified at the right end, $x_{\max}$. The right boundary conditions are

$$a_i(x,t)f + b_i(x,t)f_x + c_i(x,t)f_{xx} = d_i(x,t),\ x = x_{\max},\ 1 \le i \le \text{numRightBounds}.$$

Parameters

`time` – A `double`, the time point at which the boundary coefficients are to be evaluated.

`coefficients` – An output `double` matrix of dimension `numRightBounds` by 4, containing the computed boundary coefficient values. The coefficients are stored row-wise according to the matrix scheme

$$(a_i(x_{\max},t), b_i(x_{\max},t), c_i(x_{\max},t), d_i(x_{\max},t))_{1 \le i \le \text{numRightBounds}}.$$

Remarks

Time dependency of the boundary coefficients can be controlled via method `SetTimeDependence`. Use of this method will usually yield a more efficient algorithm because some finite element matrices do not have to be reassembled for each `t` value.

Terminal

```
abstract public double Terminal(double z)
```

Description

Returns the terminal condition value.

Parameter

`z` – A `double` scalar, the point in `x`-direction, where the terminal condition is evaluated.

Returns

A `double` scalar, the value of the terminal condition $p(x)$ at point `z`.

FeynmanKac.IInitialData Interface

```
public interface Imsl.Math.FeynmanKac.IInitialData
```

Public interface for adjustment of initial data or as an opportunity for output during the integration steps.

Method

Init

```
abstract public void Init(double[] xGrid, double[] tGrid, double time, double[]
yprime, double[] y, double[] absoluteErrorTolerance, double[]
relativeErrorTolerance)
```

Description

Method that allows for adjustment of initial data or as an opportunity for output during the integration steps.

Parameters

`xGrid` – A `double` array containing the grid points in the x-direction.

`tGrid` – A `double` array containing the grid points in the t-direction.

`time` – A `double` scalar containing the time point for the evaluation. Possible values are 0 (the initial or “terminal” time point) and all values in array `tGrid`.

`yprime` – An input `double` array of length `3*xGrid.Length` containing the derivatives of the Hermite quintic spline coefficients at time point `time`.

`y` – A `double` array of length `3*xGrid.Length` containing the coefficients of the Hermite quintic spline at time point `time`. For the initial time point `time = 0` this array can be used to reset the Hermite quintic spline coefficients to user defined values. For all other values of `time` array `y` is an input array.

`absoluteErrorTolerance` – An input or output `double` array of length `3*xGrid.Length` containing absolute error tolerances.

`relativeErrorTolerance` – An input or output `double` array of length `3*xGrid.Length` containing relative error tolerances.

FeynmanKac.IForcingTerm Interface

`public interface Imsl.Math.FeynmanKac.IForcingTerm`

Public interface for non-zero forcing term in the Feynman-Kac equation.

Method

Force

`abstract public void Force(int interval, double[] y, double time, double width, double[] xlocal, double[] qw, double[,] u, double[] phi, double[,] dphi)`

Description

Computes approximations to the forcing term $\phi(f,x,t)$ and its derivative $\partial\phi/\partial y$.

Parameters

`interval` – An `int`, the index related to the integration interval `[xGrid[interval-1], xGrid[interval]]`.

`y` – An input `double` array of length `3*xGrid.Length` containing the coefficients of the Hermite quintic spline representing the solution of the Feynman-Kac equation at time point `time`. For each

$$x \in [x_i, x_{i+1}],\ h_i = x_{i+1} - x_i, z_i = (x - x_i)/h_i,\ i = 1, \ldots, \text{xGrid.Length} - 1$$

the approximate solution is locally defined by

$$f(x,t) = f_i b_0(z) + f_{i+1} b_0(1-z) + h_i f_i' b_1(z) - h_i f_{i+1}' b_1(1-z) + h_i^2 f_i'' b_2(z) + h_i^2 f_{i+1}'' b_2(1-z).$$

The values $y_{3i-2} = f_i, y_{3i-1} = f_i', y_{3i} = f_i''$, $i = 1, \ldots,$ xGrid.Length, are stored as successive triplets in `y`.

`time` – A `double` scalar, the time point.

`width` – A `double` scalar, the width of the integration interval, `width=xGrid[interval]-xGrid[interval-1]`.

`xlocal` – An input `double` array containing the Gauss-Legendre points translated and normalized to the interval `[xGrid[interval-1], xGrid[interval]]`. The array length is equal to the degree of the Gauss-Legendre polynomial.

`qw` – An input `double` array containing the Gauss-Legendre weights. The array length is equal to the degree of the Gauss-Legendre polynomial.

`u` – An input `double` matrix of dimension `12 by xlocal.Length` containing the basis function values that define $\tilde{\beta}(x)$ at the Gauss-Legendre points `xlocal`. Setting

$$u_{k,i} := \text{u[k,i]} \quad \text{and} \quad x_i := \text{xlocal[i]},$$

vector $\tilde{\beta}(x_i)$ is defined as

$$\tilde{\beta}(x_i) := (\beta_{3*(\text{interval}-1)}(x_i), \ldots, \beta_{3*\text{interval}+2}(x_i))^T = (u_{0,i}, u_{1,i}, u_{2,i}, u_{6,i}, u_{7,i}, u_{8,i})^T .$$

`phi` – An output `double` array of length 6 containing Gauss-Legendre approximations for the local contributions

$$\phi_t := \int_{\text{xgrid[interval-1]}}^{\text{xgrid[interval]}} \phi(f,x,t)\tilde{\beta}(x)dx,$$

where `t=time` and $\tilde{\beta}(x) := (\beta_{3*(\text{interval}-1)}(x), \ldots, \beta_{3*\text{interval}+2}(x))^T$. Denoting by `degree` the number of Gauss-Legendre points (`xlocal.Length`) and setting $x_j :=$ xlocal[j], vector `phi` contains elements

$$\text{phi[i]} = \text{width} * \sum_{j=0}^{\text{degree-1}} \text{qw}[j]\, \tilde{\beta}_i(x_j)\, \phi(f,x_j,t)$$

for `i=0,...,5`.

`dphi` – An output `double` matrix of dimension 6 by 6 containing a Gauss-Legendre approximation for the Jacobian of the local contributions ϕ_t at time point `t=time`,

$$\frac{\partial \phi_t}{\partial y} = \int_{\text{xgrid[interval-1]}}^{\text{xgrid[interval]}} \frac{\partial \phi(f,x,t)}{\partial f}\, \tilde{\beta}(x)\, \tilde{\beta}^T(x)dx .$$

The approximation to this symmetric matrix is stored row-wise, i.e.

$$\text{dphi[i,j]} = \text{width} * \sum_{k=0}^{\text{degree-1}} \text{qw}[k]\, \tilde{\beta}_i(x_k)\, \tilde{\beta}_j(x_k)\, \left.\frac{\partial \phi}{\partial f}\right|_{x=\text{xlocal}[k],\, t=\text{time}}$$

for `i,j=0,...,5`.

Chapter 6: Transforms

Types

Usage Notes

Fast Fourier Transforms

A fast Fourier transform (FFT) is simply a discrete Fourier transform that is computed efficiently. Basically, the straightforward method for computing the Fourier transform takes approximately n^2 operations where n is the number of points in the transform, while the FFT (which computes the same values) takes approximately
$n \log n$ operations. The algorithms in this chapter are modeled on the Cooley-Tukey (1965) algorithm. Hence, these functions are most efficient for integers that are highly composite; that is, integers that are a product of small primes.

For the two classes, FFT and ComplexFFT, a single instance can be used to transform multiple sequences of the same length. In this situation, the constructor computes the initial setup once. This may result in substantial computational savings. For more information on the use of these classes consult the documentation under the appropriate class name.

Continuous Versus Discrete Fourier Transform

There is, of course, a close connection between the discrete Fourier transform and the continuous Fourier transform. Recall that the continuous Fourier transform is defined (Brigham 1974) as

$$\hat{f}(\omega) = (\Im f)(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt$$

We begin by making the following approximation:

$$\hat{f}(\omega) \approx \int_{-T/2}^{T/2} f(t) e^{-2\pi i \omega t} dt$$

$$= \int_0^T f(t - T/2) e^{-2\pi i \omega (t - T/2)} dt$$

$$= e^{\pi i \omega T} \int_0^T f(t - T/2) e^{-2\pi i \omega t} dt$$

If we approximate the last integral using the rectangle rule with spacing $h = T/n$, we have

$$\hat{f}(\omega) \approx e^{\pi i \omega T} h \sum_{k=0}^{n-1} e^{-2\pi i \omega k h} f(kh - T/2)$$

Finally, setting $\omega = j/T$ for $j = 0, \ldots, n-1$ yields

$$\hat{f}(j/T) \approx e^{\pi i j} h \sum_{k=0}^{n-1} e^{-2\pi i jk/n} f(kh - T/2) = (-1)^j \sum_{k=0}^{n-1} e^{-2\pi i jk/n} f_k^h$$

where the vector $f^h = (f(-T/2), \ldots, f((n-1)h - T/2))$. Thus, after scaling the components by $(-1)^h$, the discrete Fourier transform, as computed in `ComplexFFT` (with input f^h) is related to an approximation of the continuous Fourier transform by the above formula.

FFT Class

`public class Imsl.Math.FFT`

FFT functions.

Class `FFT` computes the discrete Fourier transform of a real vector of size *n*. The method used is a variant of the Cooley-Tukey algorithm, which is most efficient when *n* is a product of small prime factors. If *n* satisfies this condition, then the computational effort is proportional to *n* log *n*.

The `Forward` method computes the forward transform. If *n* is even, then the forward transform is

$$q_{2m-1} = \sum_{k=0}^{n-1} p_k \cos \frac{2\pi km}{n} \quad m = 1, \ldots, n/2$$

$$q_{2m-2} = -\sum_{k=0}^{n-1} p_k \sin \frac{2\pi km}{n} \quad m = 1, \ldots, n/2 - 1$$

$$q_0 = \sum_{k=0}^{n-1} p_k$$

If *n* is odd, q_m is defined as above for *m* from *1* to *(n - 1)/2*.

Let *f* be a real valued function of time. Suppose we sample *f* at *n* equally spaced time intervals of length δ seconds starting at time t_0. That is, we have

$$p_i := f(t_0 + i\Delta)\ i = 0, 1, \ldots, n-1$$

We will assume that *n* is odd for the remainder of this discussion. The class `FFT` treats this sequence as if it were periodic of period *n*. In particular, it assumes that $f(t_0) = f(t_0 + n\Delta)$. Hence, the period of the function is assumed to be $T = n\Delta$. We can invert the above transform for *p* as follows:

$$p_m = \frac{1}{n}\left[q_0 + 2\sum_{k=0}^{(n-3)/2} q_{2k+1}\cos\frac{2\pi(k+1)m}{n} - 2\sum_{k=0}^{(n-3)/2} q_{2k+2}\sin\frac{2\pi(k+1)m}{n}\right]$$

This formula is very revealing. It can be interpreted in the following manner. The coefficients *q* produced by `FFT` determine an interpolating trigonometric polynomial to the data. That is, if we define

$$g(t) = \frac{1}{n}\left[q_0 + 2\sum_{k=0}^{(n-3)/2} q_{2k+1}\cos\frac{2\pi(k+1)(t-t_0)}{n\Delta} - 2\sum_{k=0}^{(n-3)/2} q_{2k+2}\sin\frac{2\pi(k+1)(t-t_0)}{n\Delta}\right]$$

$$= \frac{1}{n}\left[q_0 + 2\sum_{k=0}^{(n-3)/2} q_{2k+1}\cos\frac{2\pi(k+1)(t-t_0)}{T} - 2\sum_{k=0}^{(n-3)/2} q_{2k+2}\sin\frac{2\pi(k+1)(t-t_0)}{T}\right]$$

then we have

$$f(t_0 + (i-1)\Delta) = g(t_0 + (i-1))\Delta$$

Now suppose we want to discover the dominant frequencies, forming the vector *P* of length *(n + 1)/2* as follows:

$$P_0 := |q_0|$$

$$P_k := \sqrt{q_{2k-2}^2 + q_{2k-1}^2} \quad k = 1,\ 2,\ \ldots,\ (n-1)/2$$

These numbers correspond to the energy in the spectrum of the signal. In particular, P_k corresponds to the energy level at frequency

$$\frac{k}{T} = \frac{k}{n\Delta} \quad k = 0,\ 1,\ \ldots,\ \frac{n-1}{2}$$

Furthermore, note that there are only $(n+1)/2 \approx T/(2\Delta)$ resolvable frequencies when n observations are taken. This is related to the Nyquist phenomenon, which is induced by discrete sampling of a continuous signal. Similar relations hold for the case when n is even.

If the `Backward` method is used, then the backward transform is computed. If `n` is even, then the backward transform is

$$q_m = p_0 + (-1)^m p_{n-1} + 2\sum_{k=0}^{n/2-1} p_{2k+1} \cos\frac{2\pi(k+1)m}{n} - 2\sum_{k=0}^{n/2-2} p_{2k+2} \sin\frac{2\pi(k+1)m}{n}$$

If n is odd,

$$q_m = p_0 + 2\sum_{k=0}^{(n-3)/2} p_{2k+1} \cos\frac{2\pi(k+1)m}{n} - 2\sum_{k=0}^{(n-3)/2} p_{2k+2} \sin\frac{2\pi(k+1)m}{n}$$

The backward Fourier transform is the unnormalized inverse of the forward Fourier transform.

`FFT` is based on the real FFT in FFTPACK, which was developed by Paul Swarztrauber at the National Center for Atmospheric Research.

Constructor

FFT

```
public FFT(int n)
```

Description

Constructs an FFT object.

Parameter

`n` – A `int` which specifies the array size that this object can handle.

Methods

Backward

```
public double[] Backward(double[] coef)
```

Description

Compute the real periodic sequence from its Fourier coefficients.

Parameter

`coef` – A `double` array containing the Fourier coefficients.

Returns

A `double` array containing the periodic sequence.

Forward

```
public double[] Forward(double[] seq)
```

Description

Compute the Fourier coefficients of a real periodic sequence.

Parameter

`seq` – A `double` array containing the sequence to be transformed.

Returns

A `double` array containing the transformed sequence.

Example: Fast Fourier Transform

The Fourier coefficients of a periodic sequence are computed. The coefficients are then used to reproduce the periodic sequence.

```
using System;
using Imsl.Math;

public class FFTEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{1, 2, 3, 4, 5, 6, 7, 8};
        FFT fft = new FFT(x.Length);

        double[] y = fft.Forward(x);
        double[] z = fft.Backward(y);
        for (int i = 0; i < x.Length; i++)
        {
            z[i] = z[i] / x.Length;
        }

        new PrintMatrix("x").Print(x);
```

```
        new PrintMatrix("y").Print(y);
        new PrintMatrix("z").Print(z);
    }
}
```

Output

```
  x
   0
0  1
1  2
2  3
3  4
4  5
5  6
6  7
7  8

          y
           0
0  36
1  -4
2   9.65685424949238
3  -4
4   4
5  -4
6   1.65685424949238
7  -4

  z
   0
0  1
1  2
2  3
3  4
4  5
5  6
6  7
7  8
```

ComplexFFT Class

`public class Imsl.Math.ComplexFFT`

Complex FFT.

Class `ComplexFFT` computes the discrete complex Fourier transform of a complex vector of size *N*. The method used is a variant of the Cooley-Tukey algorithm, which is most efficient when *N* is a product of

small prime factors. If N satisfies this condition, then the computational effort is proportional to $N \log N$. This considerable savings has historically led people to refer to this algorithm as the "fast Fourier transform" or FFT.

Specifically, given an N-vector x, method `Forward` returns

$$c_m = \sum_{n=0}^{N-1} x_n e^{-2\pi inm/N}$$

Furthermore, a vector of Euclidean norm S is mapped into a vector of norm

$$\sqrt{N}S$$

Finally, note that we can invert the Fourier transform as follows:

$$x_n = \frac{1}{N} \sum_{j=0}^{N-1} c_m e^{2\pi inj/N}$$

This formula reveals the fact that, after properly normalizing the Fourier coefficients, one has the coefficients for a trigonometric interpolating polynomial to the data. An unnormalized inverse is implemented in `Backward`. `ComplexFFT` is based on the complex FFT in FFTPACK. The package, FFTPACK was developed by Paul Swarztrauber at the National Center for Atmospheric Research.

Specifically, given an N-vector c, `Backward` returns

$$s_m = \sum_{n=0}^{N} c_n e^{2\pi inm/N}$$

Furthermore, a vector of Euclidean norm S is mapped into a vector of norm

$$\sqrt{N}S$$

Finally, note that we can invert the inverse Fourier transform as follows:

$$c_n = \frac{1}{N} \sum_{m=0}^{N-1} s_m e^{-2\pi inm/N}$$

This formula reveals the fact that, after properly normalizing the Fourier coefficients, one has the coefficients for a trigonometric interpolating polynomial to the data. `Backward` is based on the complex inverse FFT in FFTPACK. The package, FFTPACK was developed by Paul Swarztrauber at the National Center for Atmospheric Research.

Constructor

ComplexFFT

```
public ComplexFFT(int n)
```

Description

Constructs a complex FFT object.

Parameter

`n` – A `int` which specifies the array size that this object can handle.

Methods

Backward

```
public Imsl.Math.Complex[] Backward(Imsl.Math.Complex[] coef)
```

Description

Compute the complex periodic sequence from its Fourier coefficients.

Parameter

`coef` – A `Complex` array of Fourier coefficients.

Returns

A `Complex` array containing the periodic sequence.

Forward

```
public Imsl.Math.Complex[] Forward(Imsl.Math.Complex[] seq)
```

Description

Compute the Fourier coefficients of a complex periodic sequence.

Parameter

`seq` – A `Complex` array containing the sequence to be transformed.

Returns

A `Complex` array containing the transformed sequence.

Example: Complex FFT

The Fourier coefficients of a complex periodic sequence are computed. Then the coefficients are used to try to reproduce the periodic sequence.

```
using System;
using Imsl.Math;
```

```
public class ComplexFFTEx1
{
    public static void  Main(String[] args)
    {
        Complex[] x = new Complex[]{
            new Complex(1, 8), new Complex(2, 7), new Complex(3, 6),
            new Complex(4, 5), new Complex(5, 4), new Complex(6, 3),
            new Complex(7, 2), new Complex(8, 1)
        };
        ComplexFFT fft = new ComplexFFT(x.Length);

        Complex[] y = fft.Forward(x);
        Complex[] z = fft.Backward(y);
        for (int i = 0; i < x.Length; i++)
        {
            z[i] /= x.Length;
        }

        new PrintMatrix("x").Print(x);
        new PrintMatrix("y").Print(y);
        new PrintMatrix("z").Print(z);
    }
}
```

Output

```
    x
    0
0  1+8i
1  2+7i
2  3+6i
3  4+5i
4  5+4i
5  6+3i
6  7+2i
7  8+1i

                  y
                   0
0                 36+36i
1   5.65685424949238+13.6568542494924i
2                   +8i
3  -2.34314575050762+5.65685424949238i
4                 -4+4i
5  -5.65685424949238+2.34314575050762i
6                 -8
7  -13.6568542494924-5.65685424949238i

    z
    0
0  1+8i
1  2+7i
2  3+6i
3  4+5i
4  5+4i
```

```
5  6+3i
6  7+2i
7  8+1i
```

Chapter 7: Nonlinear Equations

Types

Usage Notes

Zeros of a Polynomial

A polynomial function of degree n can be expressed as follows:

$$p(z) = a_n z^n n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_n \neq 0$. The `ZeroPolynomial` class finds zeros of a polynomial with real or complex coefficients using Aberth's method.

Zeros of a Function

The `ZeroFunction` class uses Muller's method to find the real zeros of a real-valued function.

Root of System of Equations

A system of equations can be stated as follows:

$$f_i(x) = 0, \text{for } i = 1, 2, \ldots, n$$

where $x \in \mathbf{R}^n$, and $f_i : \mathbf{R^n} \rightarrow \mathbf{R}$. The `ZeroSystem` class uses a modified hybrid method due to M.J.D. Powell to find the zero of a system of nonlinear equations.

ZeroPolynomial Class

```
public class Imsl.Math.ZeroPolynomial
```

The ZeroPolynomial class computes the zeros of a polynomial with complex coefficients, Aberth's method.

This class is a translation of a Fortran code written by Dario Andrea Bini, University of Pisa, Italy (bini@dm.unipi.it). Numerical computation of polynomial zeros by means of Aberth's method, Numerical Algorithms, 13 (1996), pp. 179-200.

The original Fortran code includes the following notice.

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Property

MaximumIterations

```
public int MaximumIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` which specifies the maximum number of iterations allowed. The default value is 30.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is less than or equal to zero

Constructor

ZeroPolynomial

```
public ZeroPolynomial()
```

Description

Creates an instance of the solver.

Methods

ComputeRoots

```
public Imsl.Math.Complex[] ComputeRoots(double[] coef)
```

Description

Computes the roots of the polynomial with real coefficients.

$$p(x) = \text{coef}[n] \times x^n + \text{coef}[n-1] \times x^{n-1} + \ldots + \text{coef}[0]$$

Parameter

`coef` – A `double` array containing the polynomial coefficients.

Returns

A `Complex` array containing the roots of the polynomial.

Exception

`Imsl.Math.DidNotConvergeException` is thrown if the iteration did not converge.

ComputeRoots

```
public Imsl.Math.Complex[] ComputeRoots(Imsl.Math.Complex[] coef)
```

Description

Computes the roots of the polynomial with Complex coefficients.

$$p(x) = \text{coef}[n] \times x^n + \text{coef}[n-1] \times x^{n-1} + \ldots + \text{coef}[0]$$

Parameter

`coef` – A `Complex` array containing the polynomial coefficients.

Returns

A `Complex` array containing the roots of the polynomial.

Exception

`Imsl.Math.DidNotConvergeException` is thrown if if the iteration for the zeros did not converge.

GetRadius

```
public double GetRadius(int index)
```

Description

Returns an a-posteriori absolute error bound on the root.

Parameter

`index` – An `int` specifying the (0-based) index of the root whose error bound is to be returned.

Returns

A `double` representing the error bound on the index-th root.

Remarks

`NaN` is returned if the corresponding root cannot be represented as floating point due to overflow or underflow or if the roots have not yet been computed.

GetRoot

```
public Imsl.Math.Complex GetRoot(int index)
```

Description

Returns a zero of the polynomial.

Parameter

`index` – An `int` which specifies the (0-based) index of the root to be returned.

Returns

A `Complex` which represents the index-th root of the polynomial.

GetRoots

```
public Imsl.Math.Complex[] GetRoots()
```

Description

Returns the zeros of the polynomial.

Returns

A `Complex` array containing the roots of the polynomial.

GetStatus

```
public bool GetStatus(int index)
```

Description

Returns the error status of a root.

Parameter

`index` – An `int` representing the (0-based) index of the root whose error status is to be returned.

Returns

A `bool` representing the error status on the index-th root.

Remarks

It is `false` if the approximation of the index-th root has been carried out successfully, for example, the computed approximation can be viewed as the exact root of a slightly perturbed polynomial. It is true if more iterations are needed for the index-th root.

Example 1: Zeros of a Polynomial

The zeros of a polynomial with real coefficients are computed.

```
using System;
using Imsl.Math;

public class ZeroPolynomialEx1
{
    public static void  Main(String[] args)
    {
        double[] coef = new double[]{- 2, 4, - 3, 1};

        ZeroPolynomial zp = new ZeroPolynomial();
        Complex[] root = zp.ComputeRoots(coef);

        for (int k = 0; k < root.Length; k++)
        {
            Console.Out.WriteLine("root = " + root[k]);
            Console.Out.WriteLine("    radius = " + zp.GetRadius(k));
            Console.Out.WriteLine("    status = " + zp.GetStatus(k));
        }
    }
}
```

Output

```
root = 0.99999999999999978-0.99999999999999978i
    radius = 1.99006775678924E-14
    status = False
root = 1.0000000000000004+1.0000000000000002i
    radius = 1.96185227616234E-14
    status = False
root = 0.99999999999999989-1.6543612251060553E-24i
    radius = 2.04503081135961E-14
    status = False
```

Example 2: Zeros of a Polynomial with Complex Coefficients

The zeros of a polynomial with Complex coefficients are computed.

```
using System;
using Imsl.Math;
```

```
public class ZeroPolynomialEx2
{
    public static void  Main(String[] args)
    {
        // Find zeros of z^3-(3+6i)*z^2+(-8+12i)*z+10
        Complex[] coef = new Complex[]{
            new Complex(10),
            new Complex(-8, 12),
            new Complex(- 3, - 6),
            new Complex(1)};

        ZeroPolynomial zp = new ZeroPolynomial();
        Complex[] root = zp.ComputeRoots(coef);

        for (int k = 0; k < root.Length; k++)
        {
            Console.Out.WriteLine("root = " + root[k]);
            Console.Out.WriteLine("    radius = " +
            zp.GetRadius(k).ToString("0.00e+0"));
            Console.Out.WriteLine("    status = " + zp.GetStatus(k));
        }
    }
}
```

Output

```
root = 1.0000000000000004+1.0000000000000004i
    radius = 6.30e-14
    status = False
root = 0.99999999999999944+2.0000000000000018i
    radius = 1.96e-13
    status = False
root = 0.999999999999999+2.9999999999999987i
    radius = 1.47e-13
    status = False
```

ZerosFunction Class

`public class Imsl.Math.ZerosFunction`

Finds the real zeros of a real, continuous, univariate function, *f(x)*.

`ZerosFunction` computes *n* real zeros of a real, continuous, univariate function *f*. The search for the zeros of the function can be limited to a specified interval, or extended over the entire real line. The algorithm is generally more efficient if an interval is specified. The user supplied function, *f(x)*, must return valid results for all values in the specified interval. If no interval is given, the user-supplied function must return valid results for all real numbers.

The function has two convergence criteria. The first criterion accepts a root, x, if

$$|f(x)| \leq \tau$$

where τ = `Error`, see property `Error`.

The second criterion accepts a root if it is known to be inside of an interval of length at most `AbsoluteError`, see property `AbsoluteError`.

A root is accepted if it satisfies either criteria and is not within `MinimumSeparation` of another accepted root, see property `MinimumSeparation`.

If initial guesses for the roots are given, Müller's method (Müller 1956) is used for each of these guesses. For each guess, the Müller iteration is stopped if the next step would be outside of the bound, if given. The iteration is also stopped if the algorithm cannot make further progress in finding a root.

If no guesses for the zeros were given, or if Müller's method with the guesses did not find the requested number of roots, a meta-algorithm, combining Müller's and Brent's methods, is used. Müller's method is used primarily to find the roots of functions, such as $f(x) = x^2$, where the function does not cross the y=0 line. Brent's method is used to find other types of roots.

The meta-algorithm successively refines the interval using a one-dimensional Faure low-discrepancy sequence. The Faure sequence may be scaled by setting the bound interval [a,b] using the `SetBounds` method. The Faure sequence will be scaled from (0,1) to (a,b).

If no bound on the function's domain is given, the entire real line must be searched for roots. In this case the Faure sequence is scaled from (0, 1) to $(-\infty, +\infty)$ using the mapping

$$h(u) = \text{XScale} \cdot \tan(\pi(u - 1/2))$$

where `XScale` is set by the `XScale` property.

At each step of the iteration, the next point in the Faure sequence is added to the list of breakpoints defining the subintervals. Call the points $x_0 = a, x_1 = b, x_2, x_3, \ldots$. The new point, x_s splits an existing subinterval, $[x_p, x_q]$.

The function is evaluated at x_s. If its value is small enough, specifically if

$$|f(x_s)| < \text{MullerTolerance}$$

then Müller's method is used with x_p, x_q and x_s as starting values. If a root is found, it is added to the list of roots. If more roots are required, the new Faure point is used.

If Müller's method did not find a root using the new point, the function value at the point is compared with the function values at the endpoints of the subinterval it divides. If $f(x_p) f(x_s) < 0$ and no root has previously been found in $[x_p, x_s]$, then Brent's method is used to find a root in this interval. Similarly, if the function changes sign over the interval $[x_s, x_q]$, and a root has not already been found in the subinterval, Brent's method is used.

Properties

AbsoluteError

```
virtual public double AbsoluteError {get; set; }
```

Description

The second convergence criterion.

Property Value

A `double` value specifying the second convergence criterion. A root is accepted if the absolute value of the function at the point is less than or equal to `AbsoluteError`. `AbsoluteError` must be greater than or equal to 0.0.

Default: `AbsoluteError` = 2.22e-14

Remarks

The second criterion accepts a root if the root is known to be inside an interval of length at most `AbsoluteError`.

AllConverged

```
virtual public bool AllConverged {get; }
```

Description

Returns true if the iterations for all of the roots have converged.

Property Value

A `bool` representing status of the roots convergence. `AllConverged` = `true` if the iterations for all of the roots converged.

Error

```
virtual public double Error {get; set; }
```

Description

First convergence criterion.

Property Value

A `double` containing the first convergence criterion. `Error` must be greater than or equal to 0.0.

Default: `Error` = 2.0e-8/`XScale`.

Remarks

A root is accepted if it is bracketed within an interval of length `Error`, $|f(x)| \leq \tau$, where τ = `Error`.

MaxEvaluations

```
virtual public int MaxEvaluations {get; set; }
```

Description

The maximum number of function evaluations allowed.

Property Value

An `int` containing the maximum number of function evaluations allowed. Once this limit is reached, the roots found are returned.

Default: `MaxEvaluations`= 100.

Remarks

Methods `AllConverged` and property `NumberOfRootsFound` can be used to confirm whether or not the number of roots requested were found within the maximum evaluations specified. `MaxEvaluations` must be greater than or equal to 0.

MinimumSeparation

```
virtual public double MinimumSeparation {get; set; }
```

Description

Sets the minimum separation between accepted roots.

Property Value

A `double` containing the minimum separation between accepted roots. If two points satisfy the convergence criteria, but are within `MinimumSeparation` of each other, only one of the roots is accepted. `MinimumSeparation` must be greater than or equal to 0.0.

Default: `MinimumSeparation` = 1.0e-8/`XScale`.

MullerTolerance

```
virtual public double MullerTolerance {get; set; }
```

Description

Tolerance used during refinement to determine if Müllers method is started.

Property Value

A `double` containing the tolerance used during refinement to determine when the Müller's method is used.

Default: `MullerTolerance` = 1.0e-8/`AbsoluteError`

Remarks

Müller's method is started if, during refinement, a point is found for which the absolute value of the function is less than `MullerTolerance` and the point is not near an already discovered root. If `MullerTolerance` is less than or equal to zero, Müller's method is never used.

NumberOfEvaluations

```
virtual public int NumberOfEvaluations {get; }
```

Description

The actual number of function evaluations performed.

Property Value

An `int` containing the actual number of function evaluations performed.

NumberOfRoots

```
virtual public int NumberOfRoots {get; set; }
```

Description

The requested number of roots to be found.

Property Value

An `int` containing the number of roots to be found. `NumberOfRoots` must be greater than or equal to zero.

Default: `NumberOfRoots`=1.

NumberOfRootsFound

```
virtual public int NumberOfRootsFound {get; }
```

Description

Returns the number of zeros found.

XScale

```
virtual public double XScale {get; set; }
```

Description

Sets the the scaling in the x-coordinate.

Property Value

A `double` containing the scaling in the x-coordinate. The absolute value of the roots divided by `XScale` should be about one. `XScale` must be greater than 0.0.

Default: `XScale`=1.0.

Remarks

If no bound on the function's domain is given, the entire real line must be searched for roots. In this case the Faure sequence is scaled from (0, 1) to $(-\infty, \infty)$ using the mapping

$$h(u) = \text{XScale} \cdot \tan(\pi(u - 1/2))$$

.

Constructor

ZerosFunction

```
public ZerosFunction()
```

Description

Creates an instance of the solver.

Methods

ComputeZeros

```
virtual public double[] ComputeZeros(Imsl.Math.ZerosFunction.IFunction objectF)
```

Description

Returns the zeros of a univariate function.

Parameter

`objectF` – Contains the function for which the zeros will be found

Returns

A `double` array containing the zeros of the univariate function.

SetBounds

```
virtual public void SetBounds(double lowerBound, double upperBound)
```

Description

Sets the closed interval in which to search for the roots.

Parameters

`lowerBound` – a `double` containing the lower interval bound. `lowerBound` cannot be greater than or equal to `upperBound`.

`upperBound` – a `double` containing the upper interval bound.

Remarks

The function must be defined for all values in this interval.

Default: The search for the roots is not bounded.

SetGuess

```
virtual public void SetGuess(double[] guess)
```

Description

Sets the initial guess for the zeros.

Parameter

`guess` – A `double` array containing the initial guesses for the number of zeros to be found. If a bound on the zeros is also given, the guesses must satisfy the bound condition.

Example: Zeros of a Univariate Function

In this example 3 zeros of the sin function are found.

```
using System;
using Imsl.Math;

public class ZerosFunctionEx1 : ZerosFunction.IFunction
```

```
{
    public virtual double F(double x)
    {
        return Math.Sin(x);
    }

    public static void  Main(String[] args)
    {

        ZerosFunction.IFunction fcn = new ZerosFunctionEx1();

        ZerosFunction zf = new ZerosFunction();
        double[] guess = new double[]{5, 18, - 6};
        zf.SetGuess(guess);
        double[] zeros = zf.ComputeZeros(fcn);
        for (int k = 0; k < zeros.Length; k++)
        {
            Console.Out.WriteLine(zeros[k] + " = " + (zeros[k] / Math.PI) + " pi");
        }
    }
}
```

Output

```
-2.953824405026E-22 = -9.40231510170729E-23 pi
3.14159265358979 = 1 pi
6.28318530717959 = 2 pi
```

ZerosFunction.IFunction Interface

`public interface Imsl.Math.ZerosFunction.IFunction`

Public interface for the user supplied function to `ZerosFunction`. The user supplied function, *f(x)*, must return valid results for all values in the specified interval. If no interval is given, the user-supplied function must return valid results for all real numbers.

Method

F

`abstract public double F(double x)`

Description

Returns the value of the function at the given point.

Parameter

x – A `double` specifying the point at which the function is to be evaluated.

Returns

A `double` containing the value of the function at x.

ZeroSystem Class

`public class Imsl.Math.ZeroSystem`

Solves a system of n nonlinear equations f(x) = 0 using a modified Powell hybrid algorithm.

`ZeroSystem` is based on the MINPACK subroutine `HYBRD1`, which uses a modification of M.J.D. Powell's hybrid algorithm. This algorithm is a variation of Newton's method, which uses a finite-difference approximation to the Jacobian and takes precautions to avoid large step sizes or increasing residuals. For further description, see More et al. (1980).

A finite-difference method is used to estimate the Jacobian. Whenever the exact Jacobian can be easily provided, `f` should implement `ZeroSystem.IJacobian`.

Note that one can use logging to generate intermediate output for the solver. Accumulated levels of detail correspond to `Config`, `Fine`, `Finer`, and `Finest` logging levels with `Config` yielding the smallest amount of information and `Finest` yielding the most. The levels of output yield the following:

Level	*Output*
Config	Iteration increments are printed.
Fine	Prints convergence tests.
Finer	Intermediate solution values are provided.
Finest	Tracks progress through internal methods.

Properties

Logger

`public Imsl.Logger Logger {get; set; }`

Description

The `Logger` associated with this object.

Property Value

The `Logger` associated with this object.

Remarks

The Logger object enables logging.

Default: Logger = `null`

MaximumIterations

```
public int MaximumIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` specifying the maximum number of iterations allowed.

Remarks

The default value is 200.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is less than or equal to zero

RelativeError

```
public double RelativeError {get; set; }
```

Description

The relative error tolerance.

Property Value

A `double` specifying the relative error tolerance.

Remarks

The root is accepted if the relative error between two successive approximations to this root is within errorRelative. The default is the square root of the precision, about 1.0e-08.

Exception

`System.ArgumentException` is thrown if `RelativeError` is less than 0 or greater than 1

Constructor

ZeroSystem

```
public ZeroSystem(int n)
```

Description

Creates an object to find the zeros of a system of n equations.

Parameter

`n` – The number of equations that the solver handles.

Methods

SetGuess

```
public void SetGuess(double[] guess)
```

Description

Sets initial guess for the the solution.

Parameter

`guess` – A `double` array containing the initial guess.

Solve

```
public double[] Solve(Imsl.Math.ZeroSystem.IFunction f)
```

Description

Solve a system of nonlinear equations using the Levenberg-Marquardt algorithm.

Parameter

`f` – Defines a `ZeroSystem.IFunction` whose zero is to be found. If `f` implements a `ZeroSystem.IJacobian` then its Jacobian is used. Otherwise finite difference is used.

Returns

A `double` array containing the solution.

Exceptions

`Imsl.Math.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded

`Imsl.Math.ToleranceTooSmallException` is thrown if the error tolerance is too small

`Imsl.Math.DidNotConvergeException` is thrown if the algorithm does not converge

See Also: Imsl.Math.ZeroSystem.IJacobian (p. 333)

Example 1: Solve a System of Nonlinear Equations

A system of nonlinear equations is solved.

```
using System;
using Imsl.Math;

public class ZeroSystemEx1 : ZeroSystem.IFunction
{
    public void  F(double[] x, double[] f)
    {
        f[0] = x[0] + System.Math.Exp(x[0] - 1.0) + (x[1] + x[2]) *
            (x[1] + x[2]) - 27.0;
        f[1] = System.Math.Exp(x[1] - 2.0) / x[0] + x[2] * x[2] - 10.0;
        f[2] = x[2] + System.Math.Sin(x[1] - 2.0) + x[1] * x[1] - 7.0;
    }
```

```
    public static void  Main(String[] args)
    {
        ZeroSystem zf = new ZeroSystem(3);
        zf.SetGuess(new double[]{4, 4, 4});
        new PrintMatrix("zeros").Print(zf.Solve(new ZeroSystemEx1()));
    }
}
```

Output

```
        zeros
          0
0  0.99999999995498
1  2.00000000000656
2  2.99999999999468
```

Example 2: Solve a System of Nonlinear Equations with Logging

A system of nonlinear equations is solved with reduced accuracy and logging enabled.

```
using System;
using Imsl.Math;

public class ZeroSystemEx2 : ZeroSystem.IFunction
{
    public void  F(double[] x, double[] f)
    {
        f[0] = 0.5 * x[0] + x[1] + 0.5 * x[2] - x[5] / x[6];
        f[1] = x[2] + x[3] + 2 * x[4] - 2.0 / x[6];
        f[2] = x[0] + x[1] + x[4] - 1 / x[6];
        f[3] = -28837 * x[0] - 139009 * x[1] - 78213 * x[2] + 18927 *
                x[3] + 8427 * x[4] + 13492 / x[6] - 10690 * x[5] / x[6];
        f[4] = x[0] + x[1] + x[2] + x[3] + x[4] - 1;
        f[5] = 400 * x[0] * x[3] * x[3] * x[3] - 178370.0 * x[2] * x[4];
        f[6] = x[0] * x[2] - 2.6058 * x[1] * x[3];
    }


    public static void  Main(String[] args)
    {
        ZeroSystem zf = new ZeroSystem(7);
        zf.RelativeError = 0.01;        // reduced accuracy
        double[] guess = { 0.5, 0.0, 0.0, 0.5, 0.0, 0.5, 2.0 };
        zf.SetGuess(guess);

        // enable logging to the console
        zf.Logger = new Imsl.Logger();
        zf.Logger.LogLevel = Imsl.Logger.Level.Finer;
```

```
        new PrintMatrix("zeros").Print(zf.Solve(new ZeroSystemEx2()));
    }
}
```

Output

```
ZeroSystem: Iteration 1
ZeroSystem: Current solution:
0.5
ZeroSystem: Convergence if 0.214305934943747 <= 0.0217944947177034
ZeroSystem: Convergence if 881.854579848628 == 0.0
ZeroSystem: Convergence if 0.428611869887493 <= 0.0237140589837329
ZeroSystem: Convergence if 62.2023634352405 == 0.0
ZeroSystem: Iteration 2
ZeroSystem: Current solution:
0.451941633824387
ZeroSystem: Convergence if 0.214305934943747 <= 0.0237140589837329
ZeroSystem: Convergence if 62.2023634352405 == 0.0
ZeroSystem: Convergence if 0.214305934943747 <= 0.0256689283849008
ZeroSystem: Convergence if 48.6311502628328 == 0.0
ZeroSystem: Iteration 3
ZeroSystem: Current solution:
0.410355683664779
ZeroSystem: Convergence if 0.107152967471873 <= 0.0256689283849008
ZeroSystem: Convergence if 48.6311502628328 == 0.0
ZeroSystem: Convergence if 0.214305934943747 <= 0.026655240629938
ZeroSystem: Convergence if 24.594369376895 == 0.0
ZeroSystem: Iteration 4
ZeroSystem: Current solution:
0.389763013577616
ZeroSystem: Convergence if 0.107152967471873 <= 0.026655240629938
ZeroSystem: Convergence if 24.594369376895 == 0.0
ZeroSystem: Convergence if 0.214305934943747 <= 0.0276551655584895
ZeroSystem: Convergence if 11.2333356734159 == 0.0
ZeroSystem: Iteration 5
ZeroSystem: Current solution:
0.371417874014809
ZeroSystem: Convergence if 0.107152967471873 <= 0.0276551655584895
ZeroSystem: Convergence if 11.2333356734159 == 0.0
ZeroSystem: Convergence if 0.214305934943747 <= 0.0286672864029196
ZeroSystem: Convergence if 4.27372353401102 == 0.0
ZeroSystem: Iteration 6
ZeroSystem: Current solution:
0.355229191830464
ZeroSystem: Convergence if 0.107152967471873 <= 0.0286672864029196
ZeroSystem: Convergence if 4.27372353401102 == 0.0
ZeroSystem: Convergence if 0.0535764837359367 <= 0.0286672864029196
ZeroSystem: Convergence if 4.27372353401102 == 0.0
ZeroSystem: Iteration 6
ZeroSystem: Current solution:
0.355229191830464
ZeroSystem: Convergence if 0.107152967471873 <= 0.0291782082184281
ZeroSystem: Convergence if 1.33974481695402 == 0.0
ZeroSystem: Iteration 7
ZeroSystem: Current solution:
```

```
0.347853157507777
ZeroSystem: Convergence if 0.0535764837359367 <= 0.0291782082184281
ZeroSystem: Convergence if 1.33974481695402 == 0.0
ZeroSystem: Convergence if 0.0535764837359367 <= 0.0296908704988974
ZeroSystem: Convergence if 1.21901505512876 == 0.0
ZeroSystem: Iteration 8
ZeroSystem: Current solution:
0.341044692677201
ZeroSystem: Convergence if 0.0267882418679683 <= 0.0296908704988974
ZeroSystem: Convergence if 1.21901505512876 == 0.0
         zeros
            0
0  0.341044692677201
1  0.0080070341885894
2  0.0406562347121202
3  0.605298573689315
4  0.00418754444673453
5  0.564491500544133
6  2.83061739420516
```

ZeroSystem.IFunction Interface

`public interface Imsl.Math.ZeroSystem.IFunction`

Public interface for user supplied function to `ZeroSystem` object.

Method

F

`abstract public void F(double[] x, double[] fvalue)`

Description

On return, `fvalue` contains the function value at the given point.

Parameters

`x` – A `double` array which contains the point at which the function is to be evaluated. The contents of this array must not be altered by this function.

`fvalue` – A `double` array which, on return, contains the value of the function at `x`.

ZeroSystem.IJacobian Interface

```
public interface Imsl.Math.ZeroSystem.IJacobian :
Imsl.Math.ZeroSystem.IFunction
```

Public interface for user supplied function to `ZeroSystem` object.

Method

Jacobian

```
abstract public void Jacobian(double[] x, double[,] jac)
```

Description

On return, `jac` contains the value of the Jacobian at the given point.

Parameters

`x` – A `double` array which contains the point at which the Jacobian is to be evaluated. The contents of this array must not be altered by this function.

`jac` – A `double` matrix which, on return, contains the value of the Jacobian at `x`. The value of `jac[i,j]` is the derivative of f[i] with respect to `x[j]`.

Chapter 8: Optimization

Types

Usage Notes

Unconstrained Minimization

The unconstrained minimization problem can be stated as follows:

$$\min_{x \in R^n} f(x)$$

where $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is continuous and has derivatives of all orders required by the algorithms. The functions for unconstrained minimization are grouped into three categories: univariate functions, multivariate functions, and nonlinear least-squares functions.

For the univariate functions, it is assumed that the function is unimodal within the specified interval. For discussion on unimodality, see Brent (1973).

The class `MinUnconMultiVar` finds the minimum of a multivariate function using a quasi-Newton method. The default is to use a finite-difference approximation of the gradient of *f(x)*. Here, the gradient is defined to be the vector

$$\nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right]$$

However, when the exact gradient can be easily provided, the gradient should be provided by implementing the interface `MinUnconMultiVar.Gradient`. The `NumericalDerivatives` class can also be used in computing the gradient for `MinUnconMultiVar` as well as other classes. For an example, see `NumericalDerivatives` Example 6.

The nonlinear least-squares function uses a modified Levenberg-Marquardt algorithm. The most common application of the function is the nonlinear data-fitting problem where the user is trying to fit the data with a nonlinear model.

These functions are designed to find only a local minimum point. However, a function may have many local minima. Try different initial points and intervals to obtain a better local solution.

Linearly Constrained Minimization

The linearly constrained minimization problem can be stated as follows:

$$\begin{aligned} &\min_{x \in R^n} f(x) \\ &\text{subject to } A_1 x = b_1 \end{aligned}$$

where $f : \mathbf{R}^n \rightarrow \mathbf{R}$, A_1 is a coefficient matrix, and b_1 is a vector. If *f(x)* is linear, then the problem is a linear programming problem. If *f(x)* is quadratic, the problem is a quadratic programming problem.

The class `DenseLP` can be used to solve small- to medium-sized linear programming problems. No sparsity is assumed since the coefficients are stored in full matrix form.

The class `QuadraticProgramming` is designed to solve convex quadratic programming problems using a dual quadratic programming algorithm. If the given Hessian is not positive definite, then

QuadraticProgramming modifies it to be positive definite. In this case, output should be interpreted with care because the problem has been changed slightly. Here, the Hessian of *f(x)* is defined to be the *n* x *n* matrix

$$\nabla^2 f(x) = \left[\frac{\partial^2}{\partial x_i \partial x_j} f(x) \right]$$

Nonlinearly Constrained Minimization

The nonlinearly constrained minimization problem can be stated as follows:

$$\begin{aligned} &\min_{x \in R^n} f(x) \\ &\text{subject to } g_i(x) = 0 \text{ for } i = 1, 2, \ldots, m_1 \\ &g_i(x) \geq 0 \text{ for } i = m_1 + 1, \ldots, m \end{aligned}$$

where $f : \mathbf{R}^n \rightarrow \mathbf{R}$ and $g_i : \mathbf{R}^n \rightarrow \mathbf{R}$, for $i = 1, 2, \ldots, m$.

The class MinConNLP uses a sequential equality constrained quadratic programming algorithm to solve this problem. A more complete discussion of this algorithm can be found in the documentation.

Return Values from User-Supplied Functions

All values returned by user-supplied functions must be valid real numbers. It is the user's responsibility to check that the values returned by a user-supplied function do not contain NaN, infinity, or negative infinity values.

Example: Minimum of a smooth function

The minimum of $e^x - 5x$ is found using function evaluations only.

```
using System;
using Imsl.Math;

public class OptimizationIntroEx1 : MinUncon.IFunction
{
   public double F(double x)
   {
      double y = Math.Exp(x) - 5.0 * x;
      if (!(Double.IsNaN(y)))
      {
         return y;
      }
      else
      {
         return 0.0;
      }
   }

   public static void Main(String[] args)
   {
      MinUncon zf = new MinUncon();
```

```
      zf.Guess = 0.0;
      zf.Accuracy = 0.001;
      MinUncon.IFunction fcn = new OptimizationIntroEx1();
      Console.Write("Minimum is " + zf.ComputeMin(fcn));
   }
}
```

MinUncon Class

`public class Imsl.Math.MinUncon`

Finds the minimum point for a smooth univariate function using function and optionally first derivative evaluations.

`MinUncon` uses two separate algorithms to compute the minimum depending on what the user supplies as the function `f`.

If `f` defines the function whose minimum is to be found `MinUncon` uses a safeguarded quadratic interpolation method to find a minimum point of a univariate function. Both the code and the underlying algorithm are based on the routine `ZXLSF` written by M.J.D. Powell at the University of Cambridge.

`MinUncon` finds the least value of a univariate function, *f*, where *f* is a `MinUncon.IFunction`. Optional data include an initial estimate of the solution, and a positive number specified by the `Bound` property. Let $x_0 = Guess$ where `Guess` is specified by the `Guess` property and $b = Bound$, then x is restricted to the interval $[x_0 - b, x_0 + b]$. Usually, the algorithm begins the search by moving from x_0 to $x = x_0 + s$, where $s = Step$. `Step` is set by the `Step` property. If `Step` is not called then `Step` is set to *0.1*. `Step` may be positive or negative. The first two function evaluations indicate the direction to the minimum point, and the search strides out along this direction until a bracket on a minimum point is found or until x reaches one of the bounds $x_0 \pm b$. During this stage, the step length increases by a factor of between two and nine per function evaluation; the factor depends on the position of the minimum point that is predicted by quadratic interpolation of the three most recent function values.

When an interval containing a solution has been found, we will have three points, x_1, x_2, and x_3, with $x_1 < x_2 < x_3$ and $f(x_2) \leq f(x_1)$ and $f(x_2) \leq f(x_3)$. There are three main ingredients in the technique for choosing the new x from these three points. They are (i) the estimate of the minimum point that is given by quadratic interpolation of the three function values, (ii) a tolerance parameter ε, that depends on the closeness of *f* to a quadratic, and (iii) whether x_2 is near the center of the range between x_1 and x_3 or is relatively close to an end of this range. In outline, the new value of x is as near as possible to the predicted minimum point, subject to being at least ε from x_2, and subject to being in the longer interval between x_1 and x_2 or x_2 and x_3 when x_2 is particularly close to x_1 or x_3. There is some elaboration, however, when the distance between these points is close to the required accuracy; when the distance is close to the machine precision; or when ε is relatively large.

The algorithm is intended to provide fast convergence when *f* has a positive and continuous second derivative at the minimum and to avoid gross inefficiencies in pathological cases, such as

$$f(x) = x + 1.001|x|$$

The algorithm can make ε large automatically in the pathological cases. In this case, it is usual for a new value of x to be at the midpoint of the longer interval that is adjacent to the least calculated function value. The midpoint strategy is used frequently when changes to f are dominated by computer rounding errors, which will almost certainly happen if the user requests an accuracy that is less than the square root of the machine precision. In such cases, the routine claims to have achieved the required accuracy if it knows that there is a local minimum point within distance δ of x, where $\delta = xacc$, specified by the `Accuracy` property even though the rounding errors in f may cause the existence of other local minimum points nearby. This difficulty is inevitable in minimization routines that use only function values, so high precision arithmetic is recommended.

If f is a `MinUncon.IDerivative` then `MinUncon` uses a descent method with either the secant method or cubic interpolation to find a minimum point of a univariate function. It starts with an initial guess and two endpoints. If any of the three points is a local minimum point and has least function value, the routine terminates with a solution. Otherwise, the point with least function value will be used as the starting point.

From the starting point, say x_c, the function value $f_c = f(x_c)$, the derivative value $g_c = g(x_c)$, and a new point x_n defined by $x_n = x_c - g_c$ are computed. The function $f_n = f(x_n)$, and the derivative $g_n = g(x_n)$ are then evaluated. If either $f_n \geq f_c$ or g_n has the opposite sign of g_c, then there exists a minimum point between x_c and x_n; and an initial interval is obtained. Otherwise, since x_c is kept as the point that has lowest function value, an interchange between x_n and x_c is performed. The secant method is then used to get a new point

$$x_s = x_c - g_c\left(\frac{g_n - g_c}{x_n - x_c}\right)$$

Let $x_n \leftarrow x_s$ and repeat this process until an interval containing a minimum is found or one of the convergence criteria is satisfied. The convergence criteria are as follows:

Criterion 1:

$$|x_c - x_n| \leq \varepsilon_c$$

Criterion 2:

$$|g_c| \leq \varepsilon_g$$

where $\varepsilon_c = \max\{1.0, |x_c|\}\varepsilon$, ε is a relative error tolerance and ε_c is a gradient tolerance.

When convergence is not achieved, a cubic interpolation is performed to obtain a new point. The function and derivative are then evaluated at that point; and accordingly, a smaller interval that contains a

minimum point is chosen. A safeguarded method is used to ensure that the interval reduces by at least a fraction of the previous interval. Another cubic interpolation is then performed, and this procedure is repeated until one of the stopping criteria is met.

Properties

Accuracy

```
public double Accuracy {get; set; }
```

Description

The required absolute accuracy in the final value returned by the `ComputeMin` method.

Property Value

A `double` scalar value specifying the required absolute accuracy in the final value returned by the `ComputeMin` method.

Remarks

By default, the required accuracy is set to 1.0e-8.

Bound

```
public double Bound {get; set; }
```

Description

The amount by which X may be changed from its initial value, `Guess`.

Property Value

A `double` scalar value specifying the amount by which X may be changed from its initial value. In other words, X is restricted to the interval [`Guess-Bound`, `Guess+Bound`].

Remarks

By default, `Bound` is set to 100.

DerivTolerance

```
public double DerivTolerance {get; set; }
```

Description

The derivative tolerance used by member method `ComputeMin` to decide if the current point is a local minimum.

Property Value

A `double` scalar value specifying the derivative tolerance used by member method `ComputeMin`.

Remarks

This is the second stopping criterion. *x* is returned as a solution when *G(x)* is less than or equal to `DerivTolerance`. `DerivTolerance` should be nonnegative, otherwise zero will be used. By default, `DerivTolerance` is set to 1.0e-8.

Guess

```
public double Guess {get; set; }
```

Description

The initial guess of the minimum point of the input function.

Property Value

A `double` scalar value specifying the initial guess of the minimum point of the input function.

Remarks

By default, an initial guess of 0.0 is used.

Step

```
public double Step {get; set; }
```

Description

The stepsize to use when changing *x*.

Property Value

A `double` scalar value specifying the order of magnitude estimate of the required change in *x* when stepping towards the minimum.

Remarks

By default, `Step` is set to 0.1.

Constructor

MinUncon

```
public MinUncon()
```

Description

Unconstrained minimum constructor for a smooth function of a single variable of type `double`.

Method

ComputeMin

```
public double ComputeMin(Imsl.Math.MinUncon.IFunction f)
```

Description

Return the minimum of a smooth function of a single variable of type `double` using function values only or using function values and derivatives.

Parameter

`f` – The `MinUncon.IFunction` whose minimum is to be found. An attempt to find the minimum is made using function values only.

Returns

A `double` scalar value containing the minimum of the input function.

Example 1: Minimum of a smooth function

The minimum of $e^x - 5x$ is found using function evaluations only.

```
using System;
using Imsl.Math;

public class MinUnconEx1 : MinUncon.IFunction
{
    public double F(double x)
    {
        return Math.Exp(x) - 5.0 * x;
    }


    public static void  Main(String[] args)
    {
        MinUncon zf = new MinUncon();
        zf.Guess = 0.0;
        zf.Accuracy = 0.001;
        MinUncon.IFunction fcn = new MinUnconEx1();
        Console.Out.WriteLine("Minimum is " + zf.ComputeMin(fcn));
    }
}
```

Output

```
Minimum is 1.60941759992002
```

Example 2: Minimum of a smooth function

The minimum of $e^x - 5x$ is found using function evaluations and first derivative evaluations.

```
using System;
using Imsl.Math;

public class MinUnconEx2 : MinUncon.IDerivative
{
    public double F(double x)
    {
        return Math.Exp(x) - 5.0 * x;
    }

    public double Derivative(double x)
```

```
    {
        return Math.Exp(x) - 5.0;
    }


    public static void  Main(String[] args)
    {
        MinUncon zf = new MinUncon();
        zf.Guess = 0.0;
        zf.Accuracy = .001;
        double x = zf.ComputeMin(new MinUnconEx2());
        Console.Out.WriteLine("x = " + x);
    }
}
```

Output

```
x = 1.61001131622703
```

MinUncon.IFunction Interface

`public interface Imsl.Math.MinUncon.IFunction`

Interface for the user supplied function for the smooth function of a single variable to be minimized.

Method

F

`abstract public double F(double x)`

Description

Smooth function of a single variable to be minimized.

Parameter

`x` – A `double`, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at x.

MinUncon.IDerivative Interface

```
public interface Imsl.Math.MinUncon.IDerivative :  Imsl.Math.MinUncon.IFunction
```

Interface for the smooth function of a single variable to be minimized and its derivative.

Method

Derivative

```
abstract public double Derivative(double x)
```

Description

Derivative of the smooth function of a single variable to be minimized.

Parameter

`x` – A `double`, the point at which the derivative of the function is to be evaluated.

Returns

A `double`, the value of the derivative of the function at `x`.

MinUnconMultiVar Class

```
public class Imsl.Math.MinUnconMultiVar
```

Minimizes a multivariate function using a quasi-Newton method.

Class `MinUnconMultivar` uses a quasi-Newton method to find the minimum of a function *f(x)* of *n* variables. The problem is stated as follows:

$$\min_{x \in R^n} f(x)$$

Given a starting point x_c, the search direction is computed according to the formula

$$d = -B^{-1} g_c$$

where B is a positive definite approximation of the Hessian, and g_c is the gradient evaluated at x_c. A line search is then used to find a new point

$$x_n = x_c + \lambda d, \lambda > 0$$

such that

$$f(x_n) \leq f(x_c) + \alpha g^T d, \;\; \alpha \in (0, 0.5)$$

Finally, the optimality condition $||g(x)|| \leq \varepsilon$ where ε is a gradient tolerance.

When optimality is not achieved, B is updated according to the BFGS formula

$$B \leftarrow B - \frac{Bss^T B}{s^T Bs} + \frac{yy^T}{y^T s}$$

where $s = x_n - x_c$ and $y = g_n - g_c$. Another search direction is then computed to begin the next iteration. For more details, see Dennis and Schnabel (1983, Appendix A).

In this implementation, the first stopping criterion for `MinUnconMultivar` occurs when the norm of the gradient is less than the given gradient tolerance property, `GradientTolerance`. The second stopping criterion for `MinUnconMultivar` occurs when the scaled distance between the last two steps is less than the step tolerance property, `StepTolerance`.

Since by default, a finite-difference method is used to estimate the gradient. An inaccurate estimate of the gradient may cause the algorithm to terminate at a noncritical point. Supply the gradient for a more accurate gradient evaluation (`MinConMultiVar.IGradient`).

Properties

Digits

```
public double Digits {get; set; }
```

Description

The number of good digits in the function.

Property Value

A `double` scalar value specifying the number of good digits in the user supplied function.

Remarks

By default, `Digits` is set to 15.75.

Exception

`System.ArgumentException` is thrown if `Digits` is less than or equal to 0

ErrorStatus

```
public int ErrorStatus {get; }
```

Description

The non-fatal error status.

Property Value

An `int` specifying the non-fatal error status:

ErrorStatus	Meaning
1	The last global step failed to locate a lower point than the current *x* value. The current *x* may be an approximate local minimizer and no more accuracy is possible or the step tolerance may be too large.
2	Relative function convergence; both the actual and predicted relative reductions in the function are less than or equal to the relative function convergence tolerance.
3	Scaled step tolerance satisfied; the current point may be an approximate local solution, or the algorithm is making very slow progress and is not near a solution, or the step tolerance is too big.

Fscale

```
public double Fscale {get; set; }
```

Description

The function scaling value for scaling the gradient.

Property Value

A `double` scalar specifying the function scaling value for scaling the gradient.

Remarks

By default, the value of this scalar is set to 1.0.

Exception

`System.ArgumentException` is thrown if `Fscale` is less than or equal to 0

GradientTolerance

```
public double GradientTolerance {get; set; }
```

Description

The gradient tolerance used to compute the gradient.

Property Value

A `double` specifying the gradient tolerance used to compute the gradient.

Remarks

By default, the cube root of machine precision squared is used to compute the gradient.

Exception

`System.ArgumentException` is thrown if `GradientTolerance` is less than or equal to 0

Ihess

```
public int Ihess {get; set; }
```

Description

The Hessian initialization parameter.

Property Value

An `int` scalar value specifying the Hessian initialization parameter.

Remarks

By default, `Ihess` is set to 0.0 and the Hessian is initialized to the identity matrix. If this member function is called and `Ihess` is set to anything other than 0.0, the Hessian is initialized to the diagonal matrix containing

max(abs(f(xguess)),fscale)*xscale*xscale

where xguess is the initial guess of the computed solution and xscale is the scaling vector for the variables.

Iterations

```
public int Iterations {get; }
```

Description

The number of iterations used to compute a minimum.

Property Value

An `int` specifying the number of iterations used to compute the minimum.

MaximumStepsize

```
public double MaximumStepsize {get; set; }
```

Description

The maximum allowable stepsize to use.

Property Value

A nonnegative `double` value specifying the maximum allowable stepsize.

Remarks

By default, maximum stepsize is set to a value based on a scaled `Guess`.

Exception

`System.ArgumentException` is thrown if `MaximumStepsize` is less than or equal to 0

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` specifying the maximum number of iterations allowed.

Remarks

By default, the maximum number of iterations is set to 100.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is less than or equal to 0

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Parallel

```
public bool Parallel {get; set; }
```

Description

Enable or disable performing `MinUnconMultiVar.IFunction.F` in parallel.

Property Value

A `bool` indicating whether or not the `MinUnconMultiVar.IFunction.F` calculations are to be performed in parallel.

Remarks

By default, `Parallel` = `true`. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

StepTolerance

```
public double StepTolerance {get; set; }
```

Description

The scaled step tolerance to use when changing x.

Property Value

A `double` scalar value specifying the scaled step tolerance.

Remarks

The i-th component of the scaled step between two points x and y is computed as

abs(x(i)-y(i))/max(abs(x(i)),1/xscale(i))

where xscale is the scaling vector for the variables.

By default, the scaled step tolerance is set to 3.66685e-11.

Exception

`System.ArgumentException` is thrown if `StepTolerance` is less than or equal to 0

Constructor

MinUnconMultiVar

```
public MinUnconMultiVar(int n)
```

Description

Unconstrained minimum constructor for a function of n variables of type `double`.

Parameter

`n` – An `int` scalar value which defines the number of variables of the function whose minimum is to be found.

Methods

ComputeMin

```
public double[] ComputeMin(Imsl.Math.MinUnconMultiVar.IFunction f)
```

Description

Return the minimum point of a function of n variables of type `double` using a finite-difference gradient or using a user-supplied gradient.

Parameter

`f` – The `MinUnconMultiVar.IFunction` whose minimum is to be found.

Returns

A `double` array containing the point at which the minimum of the input function occurs.

Remarks

`f` can be used to supply a gradient of the function. If `f` implements `IGradient` then the user-supplied gradient is used. Otherwise, an attempt to find the minimum is made using a finite-difference gradient.

Exceptions

`Imsl.Math.FalseConvergenceException` is thrown if the iterates appear to be converging to a noncritical point

`Imsl.Math.MaxIterationsException` is thrown if the maximum number of iterations is exceeded

`Imsl.Math.UnboundedBelowException` is thrown if five consecutive steps of the maximum allowable stepsize have been taken, either the function is unbounded below, or has a finite asymptote in some direction or the maximum allowable step size is too small

SetGuess

```
public void SetGuess(double[] guess)
```

Description

Sets the initial guess of the minimum point of the input function.

Parameter

`guess` – A `double` array specifying the initial guess of the minimum point of the input function.

Remarks

By default, the elements of this array are set to 0.0.

SetXscale

```
public void SetXscale(double[] xscale)
```

Description

Sets the diagonal scaling matrix for the variables.

Parameter

`xscale` – A `double` array specifying the diagonal scaling matrix for the variables.

Remarks

By default, the elements of this array are set to 1.0.

Exception

`System.ArgumentException` is thrown if any of the elements of `Xscale` is less than or equal to or equal to 0

Example 1: Minimum of a multivariate function

The minimum of $100(x_2 - x_1^2)^2 + (1 - x_1)^2$ is found using function evaluations only.

```
using System;
using Imsl.Math;

public class MinUnconMultiVarEx1 : MinUnconMultiVar.IFunction
{
    public double F(double[] x)
    {
```

```
        return 100.0 * ((x[1] - x[0] * x[0]) * (x[1] - x[0] * x[0])) +
            (1.0 - x[0]) * (1.0 - x[0]);
    }


    public static void  Main(String[] args)
    {
        MinUnconMultiVar solver = new MinUnconMultiVar(2);
        solver.SetGuess(new double[]{- 1.2, 1.0});
        double[] x = solver.ComputeMin(new MinUnconMultiVarEx1());
        Console.Out.WriteLine
            ("Minimum point is (" + x[0] + ", " + x[1] + ")");
    }
}
```

Output

```
Minimum point is (0.99999996726513, 0.99999993304521)
```

Example 2: Minimum of a multivariate function

The minimum of $100(x_2 - x_1^2)^2 + (1 - x_1)^2$ is found using function evaluations and a user supplied gradient.

```
using System;
using Imsl.Math;

public class MinUnconMultiVarEx2 : MinUnconMultiVar.IGradient
{
    public double F(double[] x)
    {
        return 100.0 * ((x[1] - x[0] * x[0]) * (x[1] - x[0] * x[0])) +
            (1.0 - x[0]) * (1.0 - x[0]);
    }

    public void Gradient(double[] x, double[] gp)
    {
        gp[0] = - 400.0 * (x[1] - x[0] * x[0]) * x[0] - 2.0 *
            (1.0 - x[0]);
        gp[1] = 200.0 * (x[1] - x[0] * x[0]);
    }


    public static void  Main(String[] args)
    {
        MinUnconMultiVar solver = new MinUnconMultiVar(2);
        solver.SetGuess(new double[]{- 1.2, 1.0});

        double[] x = solver.ComputeMin(new MinUnconMultiVarEx2());
        Console.Out.WriteLine
            ("Minimum point is (" + x[0] + ", " + x[1] + ")");
    }
}
```

Output

```
Minimum point is (0.999999966882301, 0.999999932254245)
```

MinUnconMultiVar.IFunction Interface

```
public interface Imsl.Math.MinUnconMultiVar.IFunction
```

Interface for the user supplied multivariate function to be minimized.

Method

F

```
abstract public double F(double[] x)
```

Description

Multivariate function to be minimized.

Parameter

x – A `double` array, the point at which the function is to be evaluated.

Returns

A `double`, the value of the function at `x`.

MinUnconMultiVar.IGradient Interface

```
public interface Imsl.Math.MinUnconMultiVar.IGradient :
Imsl.Math.MinUnconMultiVar.IFunction
```

Interface for the user supplied multivariate function to be minimized and its gradient.

Method

Gradient

```
abstract public void Gradient(double[] x, double[] gvalue)
```

Description

On return, `gvalue` contains the value of the gradient, of the function, at `x`.

Parameters

`x` – A `double` array, the point at which the gradient of the function is to be evaluated.

`gvalue` – A `double` array which, on return, contains the value of the gradient, of the function, at `x`.

NonlinLeastSquares Class

`public class Imsl.Math.NonlinLeastSquares`

Solves a nonlinear least squares problem using a modified Levenberg-Marquardt algorithm.

`NonlinLeastSquares` is based on the MINPACK routine `LMDIF` by More et al. (1980). It uses a modified Levenberg-Marquardt method to solve nonlinear least squares problems. The problem is stated as follows:

$$\min_{x \in R^n} \frac{1}{2} F(x)^T F(x) = \frac{1}{2} \sum_{i=1}^{m} f_i(x)^2$$

where $m \geq n$, $F: R^n \rightarrow R^m$, and $f_i(x)$ is the i-th component function of *F(x)*. From a current point, the algorithm uses the trust region approach:

$$\min_{x_n \in R^n} \|F(x_c) + J(x_c)(x_n - x_c)\|_2$$

subject to

$$\|x_n - x_c\|_2 \leq \delta_c$$

to get a new point x_n, which is computed as

$$x_n = x_c - \left(J(x_c)^T J(x_c) + \mu_c I\right)^{-1} J(x_c)^T F(x_c)$$

where $\mu_c = 0$ if $\delta_c \geq \left\| \left(J(x_c)^T J(x_c)\right)^{-1} J(x_c)^T F(x_c) \right\|_2$ and $\mu_c > 0$ otherwise. $F(x_c)$ and $J(x_c)$ are the function values and the Jacobian evaluated at the current point x_c. This procedure is repeated until the stopping criteria are satisfied. The first stopping criteria occurs when the norm of the function is less than the property `AbsoluteTolerance`. The second stopping criteria occurs when the norm of the

scaled gradient is less than the proprty `GradientTolerance`. The third stopping criteria occurs when the scaled distance between the last two steps is less than the property `StepTolerance`. For more details, see Levenberg (1944), Marquardt (1963), or Dennis and Schnabel (1983, Chapter 10).

A finite-difference method is used to estimate the Jacobian when the user supplied function, `f`, defines the least-squares problem. Whenever the exact Jacobian can be easily provided, `f` should implement `NonlinLeastSquares.Jacobian`.

Properties

AbsoluteTolerance

```
public double AbsoluteTolerance {get; set; }
```

Description

The absolute function tolerance.

Property Value

A `double` scalar value specifying the absolute function tolerance.

Remarks

By default, 1.0e-32 is used as the absolute function tolerance.

Exception

`System.ArgumentException` is thrown if `AbsoluteTolerance` is less than or equal to 0

Digits

```
virtual public int Digits {get; set; }
```

Description

The number of good digits in the function.

Property Value

An `int` specifying the number of good digits in the user supplied function which defines the least-squares problem.

Remarks

By default, the number of good digits is set to 7.

Exception

`System.ArgumentException` is thrown if `Digits` is less than or equal to 0

ErrorStatus

```
public int ErrorStatus {get; }
```

Description

Get information about the performance of NonlinLeastSquares.

Property Value

An `int` specifying information about convergence.

Remarks

Value	Meaning
0	All convergence tests were met.
1	Scaled step tolerance was satisfied. The current point may be an approximate local solution, or the algorithm is making very slow progress and is not near a solution, or `StepTolerance` is too big.
2	Scaled actual and predicted reductions in the function are less than or equal to the relative function convergence tolerance `RelativeTolerance`.
3	Iterates appear to be converging to a noncritical point. Incorrect gradient information, a discontinuous function, or stopping tolerances being too tight may be the cause.
4	Five consecutive steps with the maximum stepsize have been taken. Either the function is unbounded below, or has a finite asymptote in some direction, or the maximum stepsize is too small.

See Also: Imsl.Math.NonlinLeastSquares.RelativeTolerance (p. 357), Imsl.Math.NonlinLeastSquares.StepTolerance (p. 358)

FalseConvergenceTolerance

```
public double FalseConvergenceTolerance {get; set; }
```

Description

The false convergence tolerance.

Property Value

A `double` scalar value specifying the false convergence tolerance.

Remarks

By default, 100.0e-16 is used as the false convergence tolerance.

Exception

`System.ArgumentException` is thrown if `FalseConvergenceTolerance` is less than or equal to 0

GradientTolerance

```
public double GradientTolerance {get; set; }
```

Description

The scaled gradient tolerance used to compute the gradient.

Property Value

A `double` specifying the scaled gradient tolerance used to compute the gradient. The *i*-th component of the scaled gradient at x is calculated as

$$\frac{|g_i| * \max(|x_i|, \frac{1}{s_i})}{\frac{1}{2}||F(x)||_2^2}$$

where $g = \nabla F(x)$, s = `xscale`, and

$$||F(x)||_2^2 = \sum_{i=1}^{m} f_i(x)^2$$

where f = `fscale`.

Remarks

By default, the cube root of machine precision squared is used to compute the gradient.

Exception

`System.ArgumentException` is thrown if `GradientTolerance` is less than or equal to 0

InitialTrustRegion

```
public double InitialTrustRegion {get; set; }
```

Description

The initial trust region radius.

Property Value

A `double` scalar value specifying the initial trust region radius.

Remarks

By default, `InitialTrustRegion` is set based on the initial scaled Cauchy step.

Exception

`System.ArgumentException` is thrown if `InitialTrustRegion` is less than or equal to 0

MaximumIterations

```
public int MaximumIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` specifying the maximum number of iterations allowed.

Remarks

By default, the maximum number of iterations is set to 100.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is less than or equal to 0

MaximumStepsize

```
public double MaximumStepsize {get; set; }
```

Description

The maximum allowable stepsize to use.

Property Value

A nonnegative `double` value specifying the maximum allowable stepsize.

Remarks

By default, the maximum stepsize is set to a default value based on a scaled `Guess`.

Exception

`System.ArgumentException` is thrown if `MaximumStepsize` is less than or equal to 0

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Parallel

```
public bool Parallel {get; set; }
```

Description

Enable or disable performing `NonlinLeastSquares.IFunction.F` in parallel.

Property Value

A `bool` indicating whether or not the `NonlinLeastSquares.IFunction.F` calculations are to be performed in parallel.

Remarks

By default, `Parallel` = `true`. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

RelativeTolerance

```
public double RelativeTolerance {get; set; }
```

Description

The relative function tolerance.

Property Value

A `double` scalar value specifying the relative function tolerance.

Remarks

By default, 1.0e-20 is used as the relative function tolerance.

Exception

`System.ArgumentException` is thrown if `RelativeTolerance` is less than or equal to 0

StepTolerance

```
public double StepTolerance {get; set; }
```

Description

The scaled step tolerance used to step between two points.

Property Value

A `double` scalar value specifying the scaled step tolerance used to step between two points. The *i*-th component of the scaled step between two points *x* and *y* is computed as

$$\frac{|x_i - y_i|}{\max(|x_i|, \frac{1}{s_i})}$$

where *s* = `xscale`

Remarks

By default, the cube root of machine precision is used as the step tolerance.

Exception

`System.ArgumentException` is thrown if `StepTolerance` is less than or equal to 0

Constructor

NonlinLeastSquares

```
public NonlinLeastSquares(int m, int n)
```

Description

Creates an object to solve a nonlinear least squares problem.

Parameters

`m` – The number of functions

`n` – The number of variables. n must be less than or equal to m.

Methods

SetFscale

```
public void SetFscale(double[] fscale)
```

Description

Sets the diagonal scaling matrix for the functions.

Parameter

`fscale` – A `double` array specifying the diagonal scaling matrix for the functions. The *i*-th component of `fscale` is a positive scalar specifying the reciprocal magnitude of the *i*-th component function of the problem.

Remarks

By default, the identity is used.

Exception

`System.ArgumentException` is thrown if any of the elements of `fscale` is less than or equal to 0

SetGuess

```
public void SetGuess(double[] guess)
```

Description

Sets the initial guess of the minimum point of the input function.

Parameter

`guess` – A `double` array specifying the initial guess of the minimum point of the input function.

Remarks

By default, an initial guess of 0.0 is used.

SetXscale

```
public void SetXscale(double[] xscale)
```

Description

Set the diagonal scaling matrix for the variables.

Parameter

`xscale` – A `double` array specifying the diagonal scaling matrix for the variables. `xscale` is used in scaling the gradient and the distance between two points. See properties GradientTolerance and StepTolerance for more detail.

Remarks

By default, the identity is used.

Exception

`System.ArgumentException` is thrown if any of the elements of `xscale` is less than or equal to 0

Solve

```
public double[] Solve(Imsl.Math.NonlinLeastSquares.IFunction f)
```

Description

Solve a nonlinear least-squares problem using a modified Levenberg-Marquardt algorithm and a Jacobian.

Parameter

`f` – User supplied `NonlinLeastSquares.IFunction` that defines the least-squares problem. If `f` implements `IJacobian` then its Jacobian is used. Otherwise, a finite difference Jacobian is used.

Returns

A `double` array of length n containing the approximate solution.

Exceptions

`Imsl.Math.TooManyIterationsException` is thrown if the number of iterations exceeds `MaximumIterations`, `MaximumIterations` is set to 100 by default

`Imsl.Math.NoProgressException` is thrown if the algorithm is not making any progress.

Example 1: Nonlinear least-squares problem

A nonlinear least-squares problem is solved using a finite-difference Jacobian.

```
using System;
using Imsl.Math;

public class NonlinLeastSquaresEx1 : NonlinLeastSquares.IFunction
{
    public void F(double[] x, double[] f)
    {
        f[0] = 10.0 * (x[1] - x[0] * x[0]);
        f[1] = 1.0 - x[0];
    }


    public static void  Main(String[] args)
    {
        int m = 2;
        int n = 2;
        double[] x = new double[m];
        NonlinLeastSquares zs = new NonlinLeastSquares(m, n);
        zs.SetGuess(new double[]{- 1.2, 1.0});
        zs.SetXscale(new double[]{1.0, 1.0});
        zs.SetFscale(new double[]{1.0, 1.0});
        x = zs.Solve(new NonlinLeastSquaresEx1());

        for (int k = 0; k < n; k++)
        {
            Console.Out.WriteLine("x[" + k + "] = " + x[k]);
        }
    }
}
```

Output

```
x[0] = 1
x[1] = 1
```

Example 2: Nonlinear least-squares problem

A nonlinear least-squares problem is solved using a user-supplied Jacobian.

```
using System;
using Imsl.Math;

public class NonlinLeastSquaresEx2 : NonlinLeastSquares.IJacobian
{
    public void F(double[] x, double[] f)
    {
        f[0] = 10.0 * (x[1] - x[0] * x[0]);
        f[1] = 1.0 - x[0];
    }

    public void Jacobian(double[] x, double[,] fjac)
    {
        fjac[0,0] = - 20.0 * x[0];
        fjac[1,0] = 10.0;
        fjac[0,1] = - 1.0;
        fjac[1,1] = 0.0;
    }


    public static void  Main(String[] args)
    {
        int m = 2;
        int n = 2;
        double[] x = new double[n];
        NonlinLeastSquares zs = new NonlinLeastSquares(m, n);
        zs.SetGuess(new double[]{- 1.2, 1.0});
        zs.SetXscale(new double[]{1.0, 1.0});
        zs.SetFscale(new double[]{1.0, 1.0});
        x = zs.Solve(new NonlinLeastSquaresEx2());

        for (int k = 0; k < n; k++)
        {
            Console.Out.WriteLine("x[" + k + "] = " + x[k]);
        }
    }
}
```

Output

```
x[0] = 1
x[1] = 1
```

NonlinLeastSquares.IFunction Interface

```
public interface Imsl.Math.NonlinLeastSquares.IFunction
```

Interface for the user supplied nonlinear least-squares function.

Method

F

```
abstract public void F(double[] x, double[] fvalue)
```

Description

User supplied nonlinear least-squares function.

Parameters

`x` – A `double` array containing the point at which the function is to be evaluated. The contents of this array must not be altered by this function.

`fvalue` – A `double` array which, on return, contains the function value at `x`.

NonlinLeastSquares.IJacobian Interface

```
public interface Imsl.Math.NonlinLeastSquares.IJacobian :
Imsl.Math.NonlinLeastSquares.IFunction
```

Interface for the user supplied nonlinear least squares function and its Jacobian.

Method

Jacobian

```
abstract public void Jacobian(double[] x, double[,] jvalue)
```

Description

Jacobian of the user supplied nonlinear least squares function.

Parameters

`x` – A `double` array containing the point at which the Jacobian of the function is to be evaluated.

`jvalue` – A `double` matrix which, on return, contains the value of the Jacobian, of the function, at `x`.

DenseLP Class

`public class Imsl.Math.DenseLP`

Solves a linear programming problem using an active set strategy.

Class `DenseLP` uses an active set strategy to solve linear programming problems, i.e., problems of the form

$$\min_{x \in R^n} c^T x$$

subject to

$$b_l \leq Ax \leq b_u$$

$$x_l \leq x \leq x_u$$

where c is the objective coefficient vector, A is the coefficient matrix, and the vectors b_l, b_u, x_l, and x_u are the lower and upper bounds on the constraints and the variables, respectively.

If the linear constraints are infeasible an L_1 solution to the constraints are used as a replacement for the stated constraints. An exception is thrown but a generalized solution is computed and available using methods `GetSolution` or `GetDualSolution`. Similar comments hold for any of the three additional conditions:

1. There are multiple solutions;
2. some constraints are discarded, or
3. cycling in the algorithm is identified.

Refer to the following paper for further information: Krogh, Fred, T. (2005), An Algorithm for Linear Programming, http://mathalacarte.com/fkrogh/pub/lp.pdf , Tujunga, CA.

Properties

IterationCount

```
public int IterationCount {get; }
```

Description

Returns the number of iterations used.

Property Value

An `int` containing the number of iterations used during the Solve() step.

ObjectiveValue

```
public double ObjectiveValue {get; }
```

Description

Returns the optimal value of the objective function.

Property Value

A `double` scalar containing the optimal value of the objective function. If a solution has not been computed, `Double.NaN` is returned.

RefinementType

```
public int RefinementType {get; set; }
```

Description

The type of refinement used, if any.

Property Value

A `int` specifying the type of refinement used.

Remarks

The possible settings are:

Value	Action
0	No refinement. Always compute dual. This is the default.
1	Iterative refinement.
2	Use extended refinement. Iterate until no more progress.

If refinement is used, the coefficient matrices and other data are saved at the beginning of the computation. When finished this data together with the solution obtained is checked for consistency. If the discrepancy is too large, the solution process is restarted using the problem data just after processing the equalities, but with the final x values and final active set.

Constructors

DenseLP

```
public DenseLP(Imsl.Math.MPSReader mps)
```

Description

Constructor using an MPSReader object.

Parameter

`mps` – A `MPSReader` specifying the Linear Programming problem.

Exception

`System.ArgumentException` is thrown if the problem dimensions are not consistent.

DenseLP

```
public DenseLP(double[,] a, double[] b, double[] c)
```

Description

Constructor variables of type `double`.

Parameters

`a` – A `double` matrix with coefficients of the constraints

`b` – A `double` array containing the right-hand side of the constraints.

`c` – A `double` array containing the coefficients of the objective function.

Exception

`System.ArgumentException` is thrown if the dimensions of `a`, `b.length`, and `c.length` are not consistent

Methods

GetDualSolution

```
public double[] GetDualSolution()
```

Description

Returns the dual solution.

Returns

A `double` array containing the dual solution of the linear programming problem.

GetSolution

```
public double[] GetSolution()
```

Description

Returns the solution x of the linear programming problem.

Returns

A `double` array containing the solution x of the linear programming problem.

SetConstraintType

```
public void SetConstraintType(int[] constraintType)
```

Description

Sets the types of general constraints in the matrix a.

Parameter

`constraintType` – A `int` array containing the types of general constraints.

Remarks

Let $r_i = a_{i1}x_1 + \cdots + a_{in}x_n$

constraintType	Constraint
0	$r_i = b_i$
1	$r_i \leq b_{u_i}$
2	$r_i \geq b_i$
3	$b_i \leq r_i \leq b_{u_i}$
4	Ignore this constraint

SetLowerBound

```
public void SetLowerBound(double[] lowerBound)
```

Description

Sets the lower bound, x_l on the variables.

Parameter

`lowerBound` – A `double` array containing the lower bounds on the variables.

Remarks

If there is no lower bound on a variable, then 10e30 should be set as the lower bound. By default, `lowerBound`= 0.

SetUpperBound

```
public void SetUpperBound(double[] upperBound)
```

Description

Sets the upper bound, x_u on the variables.

Parameter

`upperBound` – A `double` array containing the upper bound on the variables.

Remarks

If there is no upper bound on a variable, then -10e30 should be set as the upper bound. By default there is no upper bound on a variable.

SetUpperLimit

```
public void SetUpperLimit(double[] upperLimit)
```

Description

Sets the upper limit of the constraints.

Parameter

`upperLimit` – A `double` array containing the upper limit, b_u, of the constraints that have both the lower and the upper bounds.

Solve

```
public void Solve()
```

Description

Solves the problem using an active set strategy.

Remarks

Solve must be invoked prior to any of the "get" methods.

Exceptions

`Imsl.Math.BoundsInconsistentException` is thrown if the bounds are inconsistent.

`Imsl.Math.NoAcceptablePivotException` is thrown if an acceptable pivot could not be found.

`Imsl.Math.ProblemUnboundedException` is thrown if there is no finite solution to the problem.

`Imsl.Math.MultipleSolutionsException` is thrown if the problem has multiple solutions producing essentially the same minimum.

`Imsl.Math.SomeConstraintsDiscardedException` is thrown if some constraints are too linearly dependent on other active constraints.

`Imsl.Math.AllConstraintsNotSatisfiedException` is thrown if some constraints are not satisfied.

`Imsl.Math.CyclingOccurringException` is thrown if the algorithm appears to be cycling.

Example 1: Dense Linear Programming

The linear programming problem in the standard form

$$\min f(x) = -x_1 - 3x_2$$

subject to:

$x_1 + x_2 + x_3 = 1.5$
$x_1 + x_2 - x_4 = 0.5$

$x_1 + x_5 = 1.0$
$x_2 + x_6 = 1.0$
$x_i \geq 0, \text{ for } i = 1, \ldots, 6$

is solved.

```
using System;
using Imsl.Math;

public class DenseLPEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {{1.0, 1.0, 1.0, 0.0, 0.0, 0.0},
                       {1.0, 1.0, 0.0, - 1.0, 0.0, 0.0},
                       {1.0, 0.0, 0.0, 0.0, 1.0, 0.0},
                       {0.0, 1.0, 0.0, 0.0, 0.0, 1.0}};
        double[] b = new double[]{1.5, 0.5, 1.0, 1.0};
        double[] c = new double[]{- 1.0, - 3.0, 0.0, 0.0, 0.0, 0.0};

        DenseLP zf = new DenseLP(a, b, c);

        zf.Solve();
        new PrintMatrix("Solution").Print(zf.GetSolution());
    }
}
```

Output

```
Solution
    0
0  0.5
1  1
2  0
3  1
4  0.5
5  0
```

Example 2: Dense Linear Programming

The linear programming problem

$$\min f(x) = -x_1 - 3x_2$$

subject to:

$0.5 \leq x_1 + x_2 \leq 1.5$
$0 \leq x_1 \leq 1.0$
$0 \leq x_2 \leq 1.0$

```
using System;
```

```
using Imsl.Math;

public class DenseLPEx2
{
    public static void  Main(String[] args)
    {
        int[] constraintType = new int[]{3};
        double[] upperBound = new double[]{1.0, 1.0};
        double[,] a = {{1.0, 1.0}};
        double[] b = new double[]{0.5};
        double[] upperLimit = new double[]{1.5};
        double[] c = new double[]{- 1.0, - 3.0};

        DenseLP zf = new DenseLP(a, b, c);

        zf.SetUpperLimit(upperLimit);
        zf.SetConstraintType(constraintType);
        zf.SetUpperBound(upperBound);
        zf.Solve();
        new PrintMatrix("Solution").Print(zf.GetSolution());
        new PrintMatrix("Dual Solution").Print(zf.GetDualSolution());
        Console.Out.WriteLine("Optimal Value = " + zf.ObjectiveValue);
    }
}
```

Output

```
Solution
    0
0  0.5
1  1

Dual Solution
   0
0  -1

Optimal Value = -3.5
```

Example 3: Linear Programming Maximize Example

Maximize the linear programming problem

$$\max f(x) = 10x_1 + 15x_2 + 15x_3 + 13x_4 + 9x_5$$

subject to the following set of restrictions:

$100x_1 + 50x_2 + 50x_3 + 40x_4 + 120x_5 \leq 300$
$40x_1 + 50x_2 + 50x_3 + 15x_4 + 30x_5 \leq 40$
$0 \leq x_1 \leq 1.0$; $0 \leq x_2 \leq 1.0$; $0 \leq x_3 \leq 1.0$; $0 \leq x_4 \leq 1.0$; $0 \leq x_5 \leq 1.0$

Since DenseLP *minimizes*, the sign of the objective function must be changed to compute this solution. The signs of the dual solution components and the optimal value must also be changed. Because x_2 and

x_3 are not uniquely determined within the bounds, this problem has a convex family of solutions. DenseLP issues an exception, `MultipleSolutionsException`. A particular solution is available and retrieved in the `finally` block.

```
using System;
using Imsl.Math;

public class DenseLPEx3
{

    public static void  Main(System.String[] args)
    {

        int[] constraintType = new int[]{1, 1}; /* Ax <= b */
        double[] lowerVariableBound =
            new double[]{0.0, 0.0, 0.0, 0.0, 0.0};
        double[] upperVariableBound =
            new double[]{1.0, 1.0, 1.0, 1.0, 1.0};


        double[,] A = new double[,]{
            {100.0, 50.0, 50.0, 40.0, 120.0},
            {40.0, 50.0, 50.0, 15.0, 30.0}
        };
        /* constraint type Ax <= b */
        double[] b = new double[]{300.0, 40.0};
        double[] c = new double[]{10.0, 15.0, 15.0, 13.0, 9.0};

        /*  Since DenseLP minimizes, change signs of the
            objective coefficients. */
        double[] negC = new double[c.Length];
        for (int i = 0; i < c.Length; i++)
            negC[i] = - c[i];

        DenseLP zf = new DenseLP(A, b, negC);
        zf.SetLowerBound(lowerVariableBound);
        zf.SetConstraintType(constraintType);
        zf.SetUpperBound(upperVariableBound);


        try
        {
            zf.Solve();
        }
        catch (MultipleSolutionsException e)
        {
            /* x_2 and x_3 are not uniquely determined, expect multiple
             * solutions. Catch the exception, but continue to print
             * result found. */
            System.Console.Out.WriteLine(e.Message);
        }
        finally
        {

            double[] dSolution = zf.GetDualSolution();
```

```
            /* Change the sign of the dual solution and optimal value
             * since DenseLP minimizes. */
            for (int i = 0; i < dSolution.Length; i++)
                dSolution[i] = - dSolution[i];
            double optimalValue = -zf.ObjectiveValue;

            new PrintMatrix("Solution").Print(zf.GetSolution());
            new PrintMatrix("Dual Solution").Print(dSolution);
            System.Console.Out.WriteLine(
                "Optimal Value = " + optimalValue);
        }
    }
}
```

Output

```
Multiple solutions giving essentially the same minimum exist.
       Solution
           0
0  0
1  0.184987694831829
2  0.315012305168171
3  1
4  0

Dual Solution
    0
0  0
1  0.3

Optimal Value = 20.5
```

MPSReader Class

`public class Imsl.Math.MPSReader`

Reads a linear programming problem from an MPS file.

An MPS file defines a linear or quadratic programming problem. Linear programming problems read using this class are assumed to be of the form:

$$\min_{x \in R^n} c^T x$$

subject to

$$b_l \leq Ax \leq b_u$$

$$x_l \leq x \leq x_u$$

where c is the objective coefficient vector, A is the coefficient matrix, and the vectors b_l, b_u, x_l, and x_u are the lower and upper bounds on the constraints and the variables, respectively.

The following table helps map this notation into use of `MPSReader`.

C	Objective
A	Constraint matrix
b_l	Lower Range
b_u	Upper Range
x_l	Lower Bound
x_u	Upper Bound

If the MPS file specifies an equality constraint or bound, the corresponding lower and upper values will be exactly equal.

The problem formulation assumes that the constraints and bounds are two-sided. If a particular constraint or bound has no lower limit, then the corresponding entry in the structure is set to negative machine infinity. If the upper limit is missing, then the corresponding entry in the structure is set to positive machine infinity.

MPS File Format

There is some variability in the MPS format. This section describes the MPS format accepted by this reader.

An MPS file consists of a number of sections. Each section begins with a name in column 1. With the exception of the NAME section, the rest of this line is ignored. Lines with a '*' or '$' in column 1 are considered comment lines and are ignored.

The body of each section consists of lines divided into fields, as follows:

Field Number	Columns	Content
1	2-3	Indicator
2	5-12	Name
3	15-22	Name
4	25-36	Value
5	40-47	Name
6	50-61	Value

The format limits MPS names to 8 characters and values to 12 characters. The names in fields 2, 3 and 5 are case sensitive. Leading and trailing blanks are ignored, but internal spaces are significant.

The sections in an MPS file are as follows:

NAME

ROWS

COLUMNS

RHS

RANGES (optional)

BOUNDS (optional)

QUADRATIC (optional)

ENDATA

Sections must occur in the above order.

MPS keywords, section names and indicator values, are case insensitive. Row, column and set names are case sensitive.

NAME Section

The NAME section contains the single line. A problem name can occur anywhere on the line after NAME and before columns 62. The problem name is truncated to 8 characters.

ROWS Section

The ROWS section defines the name and type for each row. Field 1 contains the row type and field 2 contains the row name. Row type values are not case sensitive. Row names are case sensitive. The following row types are allowed:

Row Type	Meaning
E	Equality constraint
L	Less than or equal constraint
G	Greater than or equal constraint
N	Objective of a free row

COLUMNS Section

The COLUMNS section defines the nonzero entries in the objective and the constraint matrix. The row names here must have been defined in the ROWS section.

Field	Contents
2	Column name
3	Row name
4	Value for the entry whose row and column are given by fields 2 and 3
5	Row name
6	Value for the entry whose row and column are given by fields 2 and 5

Note: Fields 5 and 6 are optional.

The COLUMNS section can also contain markers. These are indicated by the name 'MARKER' (with the quotes) in field 3 and the marker type in field 4 or 5.

Marker type 'INTORG' (with the quotes) begins an integer group. The marker type 'INTEND' (with the quotes) ends this group. The variables corresponding to the columns defined within this group are required to be integer.

RHS Section

The RHS section defines the right-hand side of the constraints. An MPS file can contain more than one RHS set, distinguished by the RHS set name. The row names here must be defined in the ROWS section.

Field	Contents
2	RHS name
3	Row name
4	Value for the entry whose row and column are given by fields 2 and 3
5	Row name
6	Value for the entry whose row and column are given by fields 2 and 5

Note: Fields 5 and 6 are optional.

RANGES Section

The optional RANGES section defines two-sided constraints. An MPS file can contain more than one range set, distinguished by the range set name. The row names here must have been defined in the ROWS section.

Field	Contents
2	Range set name
3	Row name
4	Value for the entry whose row and column are given by fields 2 and 3
5	Row name
6	Value for the entry whose row and column are given by fields 2 and 5

Note: Fields 5 and 6 are optional.

Ranges change one-sided constraints, defined in the RHS section, into two-sided constraints. The two-sided constraint for row i depends on the range value, r_i, defined in this section. The right-hand side value, b_i, is defined in the RHS section. The two sided constraints for row i are given in the following table:

Row Type	Lower Constraint	Upper Constraint
G	b_i	$b_i + \|r_i\|$
L	$b_i - \|r_i\|$	b_i
E	$b_i + min(0, r_i)$	$b_i + max(0, r_i)$

BOUNDS Section

The optional BOUNDS section defines bounds on the variables. By default, the bounds are $0 \leq x_i \leq \infty$. The bounds can also be used to indicate that a variable must be an integer.

More than one bound can be set for a single variable. For example, to set $2 \leq x_i \leq 6$ use a LO bound with value 2 to set $2 \leq x_i$ and an UP bound with value 6 to add the condition $x_i \leq 6$.

An MPS file can contain more than one bounds set, distinguished by the bound set name.

Field	Contents
1	Bounds type
2	Bounds set name
3	Column name
4	Value for the entry whose set and column are given by fields 2 and 3
5	Column name
6	Value for the entry whose set and column are given by fields 2 and 5

Note: Fields 5 and 6 are optional.

The bound types are as follows. Here b_i are the bound values defined in this section, the x_i are the variables, and I is the set of integers.

Bound Type	Definintion	Formula
LO	Lower bound	$b_i \leq x_i$
UP	Upper bound	$x_i \leq b_i$
FX	Fixed Variable	$x_i = b_i$
FR	Free variable	$-\infty \leq x_i \leq \infty$
MI	Lower bound is minus infinity	$-\infty \leq x_i$
PL	Upper bound is positive infinity	$x_i \leq \infty$
BV	Binary variable (variable must be 0 or 1)	$x_i \in \{0,1\}$
UI	Upper bound and integer	$x_i \leq b_i$ and $x_i \in I$
LI	Lower bound and integer	$b_i \leq x_i$ and $x_i \in I$
SC	Semicontinuous	0 or $b_i \leq x_i$

The bound type names are not case sensitive.

If the bound type is UP or UI and $b_i \leq x_i$ then the lower bound is set to $-\infty$.

ENDATA Section

The ENDATA section ends the MPS file.

Fields

BINARY_VARIABLE

```
public int BINARY_VARIABLE
```

Description

Variable must be either 0 or 1.

CONTINUOUS_VARIABLE

```
public int CONTINUOUS_VARIABLE
```

Description

Variable is a real number.

INTEGER_VARIABLE

```
public int INTEGER_VARIABLE
```

Description

Variable must be an integer.

Properties

Name

```
virtual public string Name {get; }
```

Description

Returns the name of the MPS problem.

Property Value

A `String` containing the value of the name field.

Remarks

This is the value of the NAME field.

NameBounds

```
virtual public string NameBounds {get; set; }
```

Description

The name of the BOUNDS set.

Property Value

A `String` containing the name of the Bounds set.

Remarks

An MPS file can contain multiple sets of BOUNDS, but only one is retained by this reader. If not set, then the first set in the file is used.

NameObjective

```
virtual public string NameObjective {get; set; }
```

Description

The name of the free row containing the objective.

Property Value

A `String` containing the name of the free row containing the objective.

Remarks

An MPS file can contain free rows, but only one is retained by this reader as the objective. If not set, then the first free row in the file is used as the objective.

NameRanges

```
virtual public string NameRanges {get; set; }
```

Description

The name of the RANGES set.

Property Value

A `String` containing the name of the Ranges set.

Remarks

An MPS file can contain multiple sets of RANGES, but only one is retained by this reader. If not set, then the first setin the file is used.

NameRHS

```
virtual public string NameRHS {get; set; }
```

Description

The name of the RHS set used.

Property Value

A `String` containging the name of the RHS set used.

Remarks

An MPS file can contain multiple sets of RHS values, but only one is retained by this reader. If not set, then the first set in the file is used.

NumberOfBinaryConstraints

```
virtual public int NumberOfBinaryConstraints {get; }
```

Description

The number of binary constraints.

Property Value

An `int` containing the number of binary constraints.

Remarks

An binary constraint is the requirement that a variable be either 0 or 1. Binary constraints are also integer contraints.

NumberOfColumns

```
virtual public int NumberOfColumns {get; }
```

Description

The number of columns in the constraint matrix.

Property Value

An `int` containing the number of columns in the constraint matrix.

NumberOfIntegerConstraints

```
virtual public int NumberOfIntegerConstraints {get; }
```

Description

The number of integer constraints.

Property Value

An `int` containing the number of integer constraints.

Remarks

An integer constraint is the requirement that a variable be an integer.

NumberOfNonZeros

```
virtual public int NumberOfNonZeros {get; }
```

Description

The number of nonzeros in the constraint matrix.

Property Value

An `int` specifying the number of nonzeros in the constraint matrix.

NumberOfRows

```
virtual public int NumberOfRows {get; }
```

Description

The number of rows in the constraint matrix.

Property Value

An `int` containing the number of rows in the constraint matrix.

Objective

```
virtual public Imsl.Math.MPSReader.Row Objective {get; }
```

Description

The objective as a `Row`.

Property Value

A `Row` containing a representation of the objective.

ObjectiveCoefficients

```
virtual public double[] ObjectiveCoefficients {get; }
```

Description

The coefficents of the objective row.

Property Value

A `double[]` containing the coefficents of the objective.

Constructor

MPSReader

`public MPSReader()`

Description

constructor for MPSReader

Methods

GetLowerBound

`virtual public double GetLowerBound(int iVariable)`

Description

Returns the lower bound for a variable.

Parameter

`iVariable` – An `int` specifying the number of the variable.

Returns

A `double` containing the lower bound for a variable.

GetLowerRange

`virtual public double GetLowerRange(int iRow)`

Description

Returns the lower range value for a constraint equation.

Parameter

`iRow` – An `int` specifying the row number of the equation.

Returns

A `double` containing the lower range value for a constraint equation.

GetNameColumn

`virtual public string GetNameColumn(int iColumn)`

Description

Returns the name of a constraint column. Constraint column names are also variable names.

Parameter

`iColumn` – An `int` specifying the column for which a name is to be returned.

Returns

A `String` containing the name of a constraint column.

GetNameRow

```
virtual public string GetNameRow(int iRow)
```

Description

Returns the name of a constraint row.

Parameter

`iRow` – An `int` specifying the row for which a name is to be returned.

Returns

A `String` containing the name of a constraint row.

GetRow

```
virtual public Imsl.Math.MPSReader.Row GetRow(int iRow)
```

Description

Returns a row of the constraint matrix or a free row.

Parameter

`iRow` – An `int` specifying the number of the row that is to be returned.

Returns

A `Row` associated with the indicated row number, `iRow`.

GetRowCoefficients

```
virtual public double[] GetRowCoefficients(int iRow)
```

Description

Returns the coefficients of a row.

Parameter

`iRow` – An `int` specifying the number of the row that is to be returned.

Returns

A `double[]` containing the coefficients associated with the indicated row number, `iRow`.

GetTypeVariable

```
virtual public int GetTypeVariable(int iVariable)
```

Description

Returns the type of a variable. The variable types are `CONTINUOUS_VARIABLE`, `BINARY_VARIABLE` or `INTEGER_VARIABLE`.

Parameter

`iVariable` – An `int` specifying the number of the variable.

Returns

An `int` containing the variable type.

GetUpperBound

```
virtual public double GetUpperBound(int iVariable)
```

Description

Returns the upper bound for a variable.

Parameter

`iVariable` – An `int` specifying the number of the variable.

Returns

A `double` containing the upper bound for a variable.

GetUpperRange

```
virtual public double GetUpperRange(int iRow)
```

Description

Returns the upper range value for a constraint equation.

Parameter

`iRow` – An `int` specifying the row number of the equation.

Returns

A `double` containing the row number of the equation.

ProcessCommand

```
virtual protected internal string ProcessCommand(string command, string line)
```

Description

Process a section of the MPS file.

Parameters

`command` – A `String` specifying the data file section to be processed.

`line` – A `String` specifying the next line to be processed.

Returns

A `String` containing the next line to be processed. This line was read, but was not part of the section being processed.

Read

```
virtual public void Read(System.IO.StreamReader reader)
```

Description

Reads and parses the MPS file.

Parameter

`reader` – The `StreamReader` that has been associated with the data file.

Example: Reading an MPS file

This example reads the data for a linear programming problem from an MPS file.

```
using System;
using System.IO;
using Imsl.Math;

public class MPSReaderEx1
{
    public static void  Main(String[] args)
    {
        FileStream aFile = File.OpenRead("testprob.mps");
        StreamReader sr = new StreamReader(aFile);
        MPSReader mps = new MPSReader();
        mps.Read(sr);

        Console.Out.WriteLine(mps.Name);
        Console.Out.WriteLine(mps.NameRHS);
        Console.Out.WriteLine(mps.NameBounds);
        Console.Out.WriteLine(mps.NameRanges);

        int nRows = mps.NumberOfRows;
        System.Console.Out.WriteLine("NumberOfConstraints " + nRows);
        for (int i = 0; i < nRows; i++)
        {
            System.Console.Out.WriteLine("   " + mps.GetLowerRange(i) +
                                " <= row[" + i + "] = " + mps.GetNameRow(i) +
                                " <= " + mps.GetUpperRange(i));
        }

        int nColumns = mps.NumberOfColumns;
        System.Console.Out.WriteLine("NumberOfColumns " + nColumns);
        for (int i = 0; i < nColumns; i++)
        {
            System.Console.Out.WriteLine("   " + mps.GetLowerBound(i) +
                                " <= var[" + i + "] = " + mps.GetNameColumn(i) +
                                " <= " + mps.GetUpperBound(i));
        }

        System.Console.Out.WriteLine("NumberOfNonZeros " + mps.NumberOfNonZeros);
        for (int iRow = 0; iRow < nRows; iRow++)
        {
            System.Console.Out.WriteLine("       row " + mps.GetNameRow(iRow));
            System.Collections.IEnumerator iter = mps.GetRow(iRow).Iterator();
            while (iter.MoveNext())
            {
                MPSReader.Element elem = (MPSReader.Element) iter.Current;
```

```
                int iColumn = elem.Column;
                System.String nameColumn = mps.GetNameColumn(iColumn);
                System.Console.Out.WriteLine("          " +
               nameColumn + ": " + elem.Value);
            }
        }
    }
}
```

Output

```
TESTPROB
RHS1
BND1

NumberOfConstraints 3
   -Infinity <= row[0] = LIM1 <= 5
   10 <= row[1] = LIM2 <= Infinity
   7 <= row[2] = MYEQN <= 7
NumberOfColumns 3
   0 <= var[0] = XONE <= 4
   -1 <= var[1] = YTWO <= 1
   0 <= var[2] = ZTHREE <= Infinity
NumberOfNonZeros 6
       row LIM1
          XONE: 1
          YTWO: 1
       row LIM2
          XONE: 1
          ZTHREE: 1
       row MYEQN
          YTWO: -1
          ZTHREE: 1
```

MPSReader.Element Class

```
public class Imsl.Math.MPSReader.Element
```

An element in the sparse contraint matrix.

Properties

Column

```
virtual public int Column {get; }
```

Description

The column index.

Property Value

An `int` specifying the column index.

Value

```
virtual public double Value {get; }
```

Description

The value of the element.

Property Value

A `double` specifying the value of the element.

MPSReader.Row Class

```
public class Imsl.Math.MPSReader.Row
```

A row either in the constraint matrix or a free row.

Properties

Coefficients

```
virtual public double[] Coefficients {get; }
```

Description

The coeffients of this row as a dense array.

Property Value

A `double[]` containing the coefficients of this row.

Name

```
virtual public string Name {get; }
```

Description

The name of this row.

Property Value

A `String` containing the name of this row.

NumberOfNonZeros

```
virtual public int NumberOfNonZeros {get; }
```

Description

The number of nonzero elements in this row.

Property Value

An `int` containing the number of nonzero elements in this row.

Method

Iterator

```
virtual public System.Collections.IEnumerator Iterator()
```

Description

Returns an iterator over the elements in this row.

Returns

An `IEnumerator` object that can be used as an iterator over the elements in this row.

Remarks

This is used to retrieve the coeffients in a sparse form.

See Also: (p. 383)

QuadraticProgramming Class

```
public class Imsl.Math.QuadraticProgramming
```

Solves the convex quadratic programming problem subject to equality or inequality constraints.

Class `QuadraticProgramming` is based on M.J.D. Powell's implementation of the Goldfarb and Idnani dual quadratic programming (QP) algorithm for convex QP problems subject to general linear equality/inequality constraints (Goldfarb and Idnani 1983); i.e., problems of the form

$$\min_{x \in R^n} g^T x + \frac{1}{2} x^T H x$$

subject to

$$A_1 x = b_1$$

$$A_2 x \geq b_2$$

given the vectors b_1, b_2, and g, and the matrices H, A_1, and A_2. H is required to be positive definite. In this case, a unique x solves the problem or the constraints are inconsistent. If H is not positive definite, a positive definite perturbation of H is used in place of H. For more details, see Powell (1983, 1985).

If a perturbation of H, $H + \alpha I$, is used in the QP problem, then $H + \alpha I$ also should be used in the definition of the Lagrange multipliers.

If the constraints are infeasible an exception is thrown. See Example 3 where the exception is caught and printed.

Property

NoMoreProgress

```
public bool NoMoreProgress {get; }
```

Description

Contains status of `true` or `false` if computer rounding error is inhibiting improvement in the objective function.

Property Value

Is `true` if due to computer rounding error, a change in the variables fails to improve the objective function.

Remarks

Usually the solution is close to optimum.

Constructor

QuadraticProgramming

```
public QuadraticProgramming(double[,] h, double[] g, double[,] aEquality,
double[] bEquality, double[,] aInequality, double[] bInequality)
```

Description

Solve a quadratic programming problem.

Parameters

`h` – A square array containing the Hessian. It must be positive definite.

`g` – A `double` array containing the coefficients of the linear term of the objective function.

`aEquality` – A rectangular matrix containing the equality constraints. It can be null if there are no equality constraints.

`bEquality` – A `double` array containing the right-side of the equality constraints. It can be null if there are no equality constraints.

`aInequality` – A rectangular matrix containing the inequality constraints. It can be null if there are no inequality constraints.

`bInequality` – A `double` array containing the right-side of the inequality constraints. It can be null if there are no inequality constraints.

Exception

`Imsl.Math.InconsistentSystemException` is thrown if the problem is inconsistent.

Methods

GetDualSolution

```
public double[] GetDualSolution()
```

Description

Returns the dual (Lagrange multipliers).

Returns

A `double` array containing the dual.

GetSolution

```
public double[] GetSolution()
```

Description

Returns the solution.

Returns

A `double` array containing the unique solution.

Example 1: Solve a Quadratic Programming Problem

The quadratic programming problem is to minimize

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_2 - 2x_3x_4 - 2x_0$$

subject to

$$x_0 + x_1 + x_2 + x_3 + x_4 = 5$$

$$x_2 - 2x_3 - 2x_4 = -3$$

```
using System;
using Imsl.Math;
```

```
public class QuadraticProgrammingEx1
{
    public static void  Main(String[] args)
    {
        double[,] h = {
            {2, 0, 0, 0, 0},
            {0, 2, - 2, 0, 0},
            {0, - 2, 2, 0, 0},
            {0, 0, 0, 2, - 2},
            {0, 0, 0, - 2, 2}
        };
        double[,] aeq = {
            {1, 1, 1, 1, 1},
            {0, 0, 1, - 2, - 2}
        };
        double[] beq = new double[]{5, - 3};
        double[] g = new double[]{- 2, 0, 0, 0, 0};

        QuadraticProgramming qp =
            new QuadraticProgramming(h, g, aeq, beq, null, null);

        // Print the solution and its dual
        new PrintMatrix("x").Print(qp.GetSolution());
        new PrintMatrix("dual").Print(qp.GetDualSolution());
    }
}
```

Output

```
  x
   0
0  1
1  1
2  1
3  1
4  1

           dual
             0
0   0
1  -1.18329135783152E-32
```

Example 2: Solve a Quadratic Programming Problem

The quadratic programming problem is to minimize

$$x_0^2 + x_1^2 + x_2^2$$

subject to

$$x_0 + 2x_1 - x_2 = 4$$

$$x_0 - x_1 + x_2 = -2$$

```
using System;
using Imsl.Math;

public class QuadraticProgrammingEx2
{
    public static void  Main(String[] args)
    {
        double[,] h = {
            {2, 0, 0},
            {0, 2, 0},
            {0, 0, 2}
        };
        double[,] aeq = {
            {1, 2, - 1},
            {1, - 1, 1}
        };
        double[] beq = new double[]{4, - 2};
        double[] g = new double[]{0, 0, 0};

        QuadraticProgramming qp =
            new QuadraticProgramming(h, g, aeq, beq, null, null);

        // Print the solution and its dual
        new PrintMatrix("x").Print(qp.GetSolution());
        new PrintMatrix("dual").Print(qp.GetDualSolution());
    }
}
```

Output

```
           x
           0
0   0.285714285714286
1   1.42857142857143
2  -0.857142857142857

         dual
           0
0   1.14285714285714
1  -0.571428571428572
```

Example 3: Solve a Quadratic Programming Problem with Inconsistent System Constraints

In the quadratic programming problem variables 2 and 6 are fixed at the value zero by the equality constraints. The inequalities propose that the sums of the variables are at least 5.1 and no more than 4.9. These last two are inconsistent conditions, causing the NoLPSolutionException to be thrown.

```
using System;
```

```
using Imsl.Math;

public class QuadraticProgrammingEx3
{

   public static void Main(System.String[] args)
   {
      double[,] h = {
            {2.000, 0.000, 0.000, 0.000, 0.000, 0.000},
            {0.000, 2.000, 0.000, 0.000, 0.000, 0.000},
            {0.000, 0.000, 2.000, 0.000, 0.000, 0.000},
            {0.000, 0.000, 0.000, 2.000, 0.000, 0.000},
            {0.000, 0.000, 0.000, 0.000, 2.000, 0.000},
            {0.000, 0.000, 0.000, 0.000, 0.000, 2.000}};
      double[] g = { 5.000, 5.000, 5.000, 5.000, 5.000, 5.000 };
      double[,] aEquality = {
            {0.000, 1.000, 0.000, 0.000, 0.000, 0.000},
            {0.000, 0.000, 0.000, 0.000, 0.000, 1.000}};
      double[] bEquality = { 0.000, 0.000 };

      double[,] aInequality = {
            {1.000, 1.000, 1.000, 1.000, 1.000, 1.000},
            {-1.000, -1.000, -1.000, -1.000, -1.000, -1.000}};
      double delta = 0.1; // change to 0.0 to pass
      double[] bInequality = { 5 + delta, -5 + delta };

      try
      {
         QuadraticProgramming qp = new QuadraticProgramming(h, g,
             aEquality, bEquality, aInequality, bInequality);
         double[] x = qp.GetSolution();
         new Imsl.Math.PrintMatrix("Solution").Print(x);
      }
      catch (Imsl.Math.NoLPSolutionException e)
      {
         WriteCutString(e.Message, 72);

      }
      catch (System.Exception e)
      {
         WriteCutString(e.Message, 72);
      }
   }

   public static void WriteCutString(string value, int interval)
   {
      int rem = value.Length % interval;
      int result = value.Length / interval + rem /
          (1 > rem ? 1 : rem);
      for (int i = 0; i < result; i++)
      {
         Console.WriteLine(value.Substring(i * interval,
            (interval < (value.Length - i * interval)) ? interval :
            (value.Length - i * interval)));
      }
   }
```

```
}
```

Output

```
No solution for the LP problem was found. All constraints are not satisf
ied. L1 minimization was applied to all constraints (including bounds an
d simple variables) but the equalities, to approximate violated non-equa
lities as well as possible.  If a feasible solution is possible then try
 using refinement.
```

MinConGenLin Class

`public class Imsl.Math.MinConGenLin`

Minimizes a general objective function subject to linear equality/inequality constraints.

The class `MinConGenLin` is based on M.J.D. Powell's TOLMIN, which solves linearly constrained optimization problems, i.e., problems of the form

$$\min f(x)$$

subject to

$$A_1 x = b_1$$

$$A_2 x \leq b_2$$

$$x_l \leq x \leq x_u$$

given the vectors b_1, b_2, x_l, and x_u and the matrices A_1 and A_2.

The algorithm starts by checking the equality constraints for inconsistency and redundancy. If the equality constraints are consistent, the method will revise x^0, the initial guess, to satisfy

$$A_1 x = b_1$$

Next, x^0 is adjusted to satisfy the simple bounds and inequality constraints. This is done by solving a sequence of quadratic programming subproblems to minimize the sum of the constraint or bound violations.

Now, for each iteration with a feasible x^k, let J_k be the set of indices of inequality constraints that have small residuals. Here, the simple bounds are treated as inequality constraints. Let I_k be the set of indices of active constraints. The following quadratic programming problem

$$\min f\left(x^k\right) + d^T \nabla f\left(x^k\right) + \frac{1}{2} d^T B^k d$$

subject to

$$a_j d = 0,\ j \in I_k$$

$$a_j d \leq 0,\ j \in J_k$$

is solved to get (d^k, λ^k) where a_j is a row vector representing either a constraint in A_1 or A_2 or a bound constraint on x. In the latter case, the $a_j = e_j$ for the bound constraint $x_i \leq (x_u)_i$ and $a_j = -e_i$ for the constraint $-x_i \leq (x_l)_i$. Here, e_i is a vector with 1 as the i-th component, and zeros elsewhere. Variables λ^k are the Lagrange multipliers, and B^k is a positive definite approximation to the second derivative $\nabla^2 f(x^k)$.

After the search direction d^k is obtained, a line search is performed to locate a better point. The new point $x^{k+1} = x^k + \alpha^k d^k$ has to satisfy the conditions

$$f(x^k + \alpha^k d^k) \leq f(x^k) + 0.1\alpha^k (d^k)^T \nabla f(x^k)$$

and

$$(d^k)^T \nabla f(x^k + \alpha^k d^k) \geq 0.7 (d^k)^T \nabla f(x^k)$$

The main idea in forming the set J_k is that, if any of the equality constraints restricts the step-length α^k, then its index is not in J_k. Therefore, small steps are likely to be avoided.

Finally, the second derivative approximation B^K, is updated by the BFGS formula, if the condition

$$\left(d^K\right)^T \nabla f\left(x^k + \alpha^k d^k\right) - \nabla f\left(x^k\right) > 0$$

holds. Let $x^k \leftarrow x^{k+1}$, and start another iteration.

The iteration repeats until the stopping criterion

$$\left\| \nabla f(x^k) - A^k \lambda^K \right\|_2 \leq \tau$$

is satisfied. Here τ is the supplied tolerance. For more details, see Powell (1988, 1989).

Properties

FinalActiveConstraintsNum

```
public int FinalActiveConstraintsNum {get; }
```

Description

Returns the final number of active constraints.

Property Value

An `int` scalar containing the final number of active constraints.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

ObjectiveValue

```
public double ObjectiveValue {get; }
```

Description

Returns the value of the objective function.

Property Value

A `double` scalar containing the value of the objective function.

Parallel

```
public bool Parallel {get; set; }
```

Description

Enable or disable performing `MinConGenLin.IFunctionn.F` in parallel.

Property Value

A `bool` indicating whether or not the `MinConGenLin.IFunction.F` calculations are to be performed in parallel.

Remarks

By default, `Parallel` = `true`. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Tolerance

```
public double Tolerance {get; set; }
```

Description

The nonnegative tolerance on the first order conditions at the calculated solution.

Property Value

A `double` scalar containing the tolerance.

Constructor

MinConGenLin

```
public MinConGenLin(Imsl.Math.MinConGenLin.IFunction fcn, int nvar, int ncon,
int neq, double[] a, double[] b, double[] lowerBound, double[] upperBound)
```

Description

Constructor for `MinConGenLin`.

Parameters

`fcn` – The user-supplied `MinConGenLin.IFunction` to be minimized.

`nvar` – An `int` scalar containing the number of variables.

`ncon` – An `int` scalar containing the number of linear constraints (excluding simple bounds).

`neq` – An `int` scalar containing the number of linear equality constraints.

`a` – A `double` array containing the equality constraint gradients in the first neq rows followed by the inequality constraint gradients. `a.length` = `ncon` * `nvar`.

`b` – A `double` array containing the right-hand sides of the linear constraints.

`lowerBound` – A `double` array containing the lower bounds on the variables. `lowerBound.length` = `nvar`.

`upperBound` – A `double` array containing the upper bounds on the variables. `upperBound.length` = `nvar`.

Exception

`System.ArgumentException` is thrown if the dimensions of `nvar`, `ncon`, `neq`, `a.length` , `b.length`, `lowerBound.length` and `upperBound.length` are not consistent

Methods

GetFinalActiveConstraints

```
public int[] GetFinalActiveConstraints()
```

Description

Returns the indices of the final active constraints.

Returns

An `int` array containing the indices of the final active constraints.

GetLagrangeMultiplierEstimate

```
public double[] GetLagrangeMultiplierEstimate()
```

Description

Returns the Lagrange multiplier estimates of the final active constraints.

Returns

A `double` array containing the Lagrange multiplier estimates of the final active constraints.

GetSolution

```
public double[] GetSolution()
```

Description

Returns the computed solution.

Returns

A `double` array containing the computed solution.

SetGuess

```
public void SetGuess(double[] guess)
```

Description

Sets an initial guess of the solution.

Parameter

`guess` – A `double` array containing an initial guess.

Solve

```
public void Solve()
```

Description

Minimizes a general objective function subject to linear equality/inequality constraints.

Exceptions

`Imsl.Math.ConstraintsInconsistentException` is thrown if the constraints are inconsistent.

`Imsl.Math.VarBoundsInconsistentException` is thrown if the bounds on the variables are inconsistent.

`Imsl.Math.ConstraintsNotSatisfiedException` is thrown if a solution satisfying the constraints could not be found.

`Imsl.Math.EqualityConstraintsException` is thrown if the variables are determined by the constraints.

Example 1: Linear Constrained Optimization

The problem

$$\min f(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 2x_2x_3 - 2x_4x_5 - 2x_1$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 5$$

$$x_3 - 2x_4 - 2x_5 = -3$$

$$0 \leq x \leq 10$$

is solved.

```
using System;
using Imsl.Math;

public class MinConGenLinEx1 : MinConGenLin.IFunction
{
    public double F(double[] x)
    {
        return x[0] * x[0] + x[1] * x[1] + x[2] * x[2] + x[3] * x[3] +
            x[4] * x[4] - 2.0 * x[1] * x[2] - 2.0 * x[3] *
            x[4] - 2.0 * x[0];
    }

    public static void  Main(String[] args)
    {
        int neq = 2;
```

```
        int ncon = 2;
        int nvar = 5;
        double[] a = new double[]{1.0, 1.0, 1.0, 1.0, 1.0,
                                  0.0, 0.0, 1.0, - 2.0, - 2.0};
        double[] b = new double[]{5.0, - 3.0};
        double[] xlb = new double[]{0.0, 0.0, 0.0, 0.0, 0.0};
        double[] xub = new double[]{10.0, 10.0, 10.0, 10.0, 10.0};

        MinConGenLin.IFunction fcn = new MinConGenLinEx1();
        MinConGenLin zf = new MinConGenLin(fcn, nvar, ncon, neq, a, b,
            xlb, xub);
        zf.Solve();
        new PrintMatrix("Solution").Print(zf.GetSolution());
    }
}
```

Output

```
Solution
   0
0  1
1  1
2  1
3  1
4  1
```

Example 2: Linear Constrained Optimization

The problem

$$\min f(x) = -x_0 x_1 x_2$$

subject to

$$-x_0 - 2x_1 - 2x_2 \leq 0$$

$$x_0 + 2x_1 + 2x_2 \leq 72$$

$$0 \leq x_0 \leq 20$$

$$0 \leq x_1 \leq 11$$

$$0 \leq x_2 \leq 42$$

is solved with an initial guess of $x_0 = 10$, $x_1 = 10$ and $x_2 = 10$.

```
using System;
using Imsl.Math;

public class MinConGenLinEx2 : MinConGenLin.IGradient
{
    public double F(double[] x)
    {
        return - x[0] * x[1] * x[2];
    }

    public void Gradient(double[] x, double[] g)
    {
        g[0] = - x[1] * x[2];
        g[1] = - x[0] * x[2];
        g[2] = - x[0] * x[1];
    }

    public static void  Main(String[] args)
    {
        int neq = 0;
        int ncon = 2;
        int nvar = 3;
        double[] a = new double[]{- 1.0, - 2.0, - 2.0, 1.0, 2.0, 2.0};
        double[] xlb = new double[]{0.0, 0.0, 0.0};
        double[] xub = new double[]{20.0, 11.0, 42.0};
        double[] b = new double[]{0.0, 72.0};

        MinConGenLin.IGradient fcn = new MinConGenLinEx2();
        MinConGenLin zf = new MinConGenLin(fcn, nvar, ncon, neq, a, b,
            xlb, xub);
        zf.SetGuess(new double[]{10.0, 10.0, 10.0});
        zf.Solve();
        new PrintMatrix("Solution").Print(zf.GetSolution());
        Console.Out.WriteLine("Objective value = " +
            zf.ObjectiveValue);
    }
}
```

Output

```
Solution
   0
0  20
1  11
2  15

Objective value = -3300
```

MinConGenLin.IFunction Interface

```
public interface Imsl.Math.MinConGenLin.IFunction
```

Public interface for the user-supplied function to evaluate the function to be minimized.

Method

F

```
abstract public double F(double[] x)
```

Description

Public interface for the function to be minimized.

Parameter

x – A `double` array, the point at which the function is evaluated. `x.length` equals the number of variables.

Returns

A `double` scalar, the function value at `x`.

MinConGenLin.IGradient Interface

```
public interface Imsl.Math.MinConGenLin.IGradient :
Imsl.Math.MinConGenLin.IFunction
```

Public interface for the user-supplied function to compute the gradient.

Method

Gradient

```
abstract public void Gradient(double[] x, double[] g)
```

Description

Public interface for the user-supplied function to compute the gradient at point `x`.

Parameters

`x` – A `double` array, the point at which the gradient is evaluated. `x.length` equals the number of variables.

`g` – A `double` array which, on return, contains the values of the gradient of the objective function.

BoundedLeastSquares Class

`public class Imsl.Math.BoundedLeastSquares`

Solves a nonlinear least-squares problem subject to bounds on the variables using a modified Levenberg-Marquardt algorithm.

Class `BoundedLeastSquares` uses a modified Levenberg-Marquardt method and an active set strategy to solve nonlinear least-squares problems subject to simple bounds on the variables. The problem is stated as follows:

$$min\frac{1}{2}F(x)^T F(x) = \frac{1}{2}\sum_{i=1}^{m} f_i(x)^2$$

subject to

$$l \leq x \leq u$$

here m $\geq$ n, $F : \mathrm{R}^n \rightarrow \mathrm{R}^m$, and $f_i(x)$ is the *i*-th component function of *F(x)*. From a given starting point, an active set `IA`, which contains the indices of the variables at their bounds, is built. A variable is called a "free variable" if it is not in the active set. The routine then computes the search direction for the free variables according to the formula

$$d = -\left(J^T J + \mu I\right)^{-1} J^T F$$

where μ is the Levenberg-Marquardt parameter, *F = F(x)*, and *J* is the Jacobian with respect to the free variables. The search direction for the variables in `IA` is set to zero. The trust region approach discussed by Dennis and Schnabel (1983) is used to find the new point. Finally, the optimality conditions are checked. The conditions are:

$$\|g(x_i)\| \leq \varepsilon, l_i < x_i < u_i$$
$$g(x_i) < 0, x_i = u_i$$
$$g(x_i) > 0, x_i = l_i$$

where ε is a gradient tolerance. This process is repeated until the optimality criterion is achieved.

The active set is changed only when a free variable hits its bounds during an iteration or the optimality condition is met for the free variables but not for all variables in `IA`, the active set. In the latter case, a variable that violates the optimality condition will be dropped out of `IA`. For more details on the Levenberg-Marquardt method, see Levenberg (1944) or Marquardt (1963). For more detail on the active set strategy, see Gill and Murray (1976).

Properties

AbsoluteTolerance

```
public double AbsoluteTolerance {get; set; }
```

Description

The absolute function tolerance.

Property Value

A `double` scalar containing the absolute function tolerance.

Digits

```
public int Digits {get; set; }
```

Description

The number of good digits in the function.

Property Value

A `int` scalar containing the number of good digits.

GradientTolerance

```
public double GradientTolerance {get; set; }
```

Description

The scaled gradient tolerance.

Property Value

A `double` scalar containing the scaled gradient tolerance.

MaximumFunctionEvals

```
public int MaximumFunctionEvals {get; set; }
```

Description

The maximum number of function evaluations.

Property Value

A `int` scalar containing the maximum number of function evaluations.

MaximumIterations

```
public int MaximumIterations {get; set; }
```

Description

The maximum number of iterations.

Property Value

A `int` scalar containing the maximum number of iterations.

MaximumJacobianEvals

```
public int MaximumJacobianEvals {get; set; }
```

Description

The maximum number of Jacobian evaluations.

Property Value

A `int` scalar containing the maximum number of Jacobian evaluations.

MaximumStepsize

```
public double MaximumStepsize {get; set; }
```

Description

The maximum allowable step size.

Property Value

A `double` scalar containing the maximum allowable step size.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Parallel

```
public bool Parallel {get; set; }
```

Description

Enable or disable performing `MinUnconMultiVar.IFunction.F` in parallel.

Property Value

A `bool` indicating whether or not the `MinUnconMultiVar.IFunction.F` calculations are to be performed in parallel.

Remarks

By default, `Parallel` = `true`. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

RelativeTolerance

```
public double RelativeTolerance {get; set; }
```

Description

The relative function tolerance.

Property Value

A `double` scalar containing the relative function tolerance.

ScaledStepTolerance

```
public double ScaledStepTolerance {get; set; }
```

Description

The scaled step tolerance.

Property Value

A `double` scalar containing the scaled step tolerance.

TrustRegion

```
public double TrustRegion {get; set; }
```

Description

The size of initial trust region radius.

Property Value

A `double` scalar containing the initial trust region radius.

Constructor

BoundedLeastSquares

```
public BoundedLeastSquares(Imsl.Math.BoundedLeastSquares.IFunction f, int
mFunctions, int nVariables, int boundType, double[] lowerBound, double[]
upperBound)
```

Description

Constructor for `BoundedLeastSquares`.

Parameters

`f` – The user-supplied `BoundedLeastSquares.IFunction` to be minimized.

`mFunctions` – A `int` scalar containing the number of functions.

`nVariables` – A `int` scalar containing the number of variables.

`boundType` – A `int` scalar containing the types of bounds on the variable.

boundType	Action
0	User will supply all the bounds.
1	All variables are nonnegative.
2	All variables are nonpositive.
3	User supplies only the bounds on first variable, all other variables will have the same bounds.

`lowerBound` – A `double` array containing the lower bounds on the variables.

`upperBound` – A `double` array containing the upper bounds on the variables.

Exception

`System.ArgumentException` is thrown if the dimensions of `mFunctions`, `nVariables`, `boundType`, `lowerBound.length` and `upperBound.length` are not consistent

Methods

GetJacobianSolution

```
public double[,] GetJacobianSolution()
```

Description

Returns the Jacobian at the approximate solution.

Returns

A mFunctions x nVariables `double` matrix containing the Jacobian at the approximate solution.

GetResiduals

```
public double[] GetResiduals()
```

Description

Returns the residuals at the approximate solution.

Returns

A `double` array containing the residuals at the approximate solution.

GetSolution

```
public double[] GetSolution()
```

Description

Returns the solution.

Returns

A `double` array containing the computed solution.

SetFscale

```
public void SetFscale(double[] fscale)
```

Description

Sets the diagonal scaling matrix for the functions.

Parameter

`fscale` – A `double` array containing the diagonal scaling for the functions.

Remarks

The i-th component of fscale is a positive scalar specifying the reciprocal magnitude of the i-th component function of the problem. By default, `fscale[] = {1}`.

SetGuess

```
public void SetGuess(double[] guess)
```

Description

Sets the initial guess of the solution.

Parameter

`guess` – A `double` array containing an initial guess.

SetInternalScale

```
public void SetInternalScale()
```

Description

The internal variable scaling option.

Remarks

With this option, the values for `xscale` are set internally.

SetXscale

```
public void SetXscale(double[] xscale)
```

Description

The scaling vector for the variables.

Parameter

`xscale` – A `double` array containing the scaling vector for the variables.

Remarks

Argument `xscale` is used mainly in scaling the gradient and the distance between two points. See `GradientTolernce` and `ScaledStepTolerance` for more details. By default, `xscale[] = {1}`.

Solve

```
public void Solve()
```

Description

Solves a nonlinear least-squares problem subject to bounds on the variables using a modified Levenberg-Marquardt algorithm.

Exceptions

`Imsl.Math.FalseConvergenceException` is thrown if there is a problem with convergence.

`Imsl.Math.NoProgressException` is thrown if the algorithm is not making any progress.

Example 1: Bounded Least Squares

The nonlinear least-squares problem

$$\min \frac{1}{2} \sum_{i=0}^{1} f_i(x)^2$$

$$-2 \leq x_0 \leq 0.5$$

$$-1 \leq x_1 \leq 2$$

where

$$f_0(x) = 10(x_1 - x_0^2) \text{ and } f_1(x) = (1 - x_0)$$

is solved.

```
using System;
using Imsl.Math;

public class BoundedLeastSquaresEx1 : BoundedLeastSquares.IFunction
{
    public void F(double[] x, double[] f)
    {
        f[0] = 10.0 * (x[1] - x[0] * x[0]);
        f[1] = 1.0 - x[0];
    }

    public static void Main(String[] args)
    {
        int m = 2;
        int n = 2;
        int ibtype = 0;
        double[] xlb = new double[] { -2.0, -1.0 };
        double[] xub = new double[] { 0.5, 2.0 };

        BoundedLeastSquares.IFunction rosbck =
            new BoundedLeastSquaresEx1();
        BoundedLeastSquares zf =
            new BoundedLeastSquares(rosbck, m, n, ibtype, xlb, xub);
        zf.Solve();
        new PrintMatrix("Solution").Print(zf.GetSolution());
    }
}
```

Output

```
       Solution
           0
```

```
0  0.5
1  0.250000000009201
```

Example 2: Bounded Least Squares

The nonlinear least-squares problem

$$\min \frac{1}{2} \sum_{i=0}^{1} f_i(x)^2$$

$$-2 \leq x_0 \leq 0.5$$

$$-1 \leq x_1 \leq 2$$

where

$$f_0(x) = 10(x_1 - x_0^2) \text{ and } f_1(x) = (1 - x_0)$$

is solved. An initial guess (-1.2, 1.0) is supplied, as well as the analytic Jacobian. The residual at the approximate solution is returned.

```
using System;
using Imsl.Math;

public class BoundedLeastSquaresEx2 : BoundedLeastSquares.IJacobian
{
    public void F(double[] x, double[] f)
    {
        f[0] = 10.0 * (x[1] - x[0] * x[0]);
        f[1] = 1.0 - x[0];
    }

    public void Jacobian(double[] x, double[] fjac)
    {
        fjac[0] = -20.0 * x[0];
        fjac[1] = 10.0;
        fjac[2] = -1.0;
        fjac[3] = 0.0;
    }


    public static void Main(String[] args)
    {
        int m = 2;
        int n = 2;
        int ibtype = 0;
        double[] xlb = { -2.0, -1.0 };
        double[] xub = { 0.5, 2.0 };
```

```
        BoundedLeastSquares.IJacobian rosbck =
            new BoundedLeastSquaresEx2();
        BoundedLeastSquares zf =
            new BoundedLeastSquares(rosbck, m, n, ibtype, xlb, xub);
        zf.SetGuess(new double[] { -1.2, 1.0 });
        zf.Solve();
        new PrintMatrix("Solution").Print(zf.GetSolution());
        new PrintMatrix("Residuals").Print(zf.GetResiduals());
    }
}
```

Output

```
Solution
    0
0  0.5
1  0.25

Residuals
    0
0  0
1  0.5
```

BoundedLeastSquares.IFunction Interface

`public interface Imsl.Math.BoundedLeastSquares.IFunction`

Public interface for the user-supplied function to evaluate the function that defines the least-squares problem.

Method

F

`abstract public void F(double[] x, double[] fvalue)`

Description

Public interface for the user-supplied function to evaluate the function that defines the least-squares problem.

Parameters

`x` – A `double` array, the point at which the function is to evaluated. `x.length = nVariables`.

`fvalue` – A `double` array, the function values at point `x`. `f.Length` = mFunctions.

BoundedLeastSquares.IJacobian Interface

```
public interface Imsl.Math.BoundedLeastSquares.IJacobian :
Imsl.Math.BoundedLeastSquares.IFunction
```

Public interface for the user-supplied function to compute the Jacobian.

Method

Jacobian

```
abstract public void Jacobian(double[] x, double[] fjac)
```

Description

Public interface for the user-supplied function to compute the Jacobian.

Parameters

`x` – A `double` array, the point at which the Jacobian is to evaluated. `x.length` = nVariables.

`fjac` – A `double` array which, on return, contains the computed Jacobian at the point `x`. `fjac.length` = mFunctions x nVariables.

BoundedVariableLeastSquares Class

```
public class Imsl.Math.BoundedVariableLeastSquares
```

Solve a linear least-squares problem with bounds on the variables.

`BoundedVariableLeastSquares` solves the least-squares problem

$$\min_x ||Ax - b||^2$$

subject to the conditions

$$\alpha_k \leq x_k \leq \beta_k$$

for all k.

This algorithm is a generalization of Imsl.Math.NonNegativeLeastSquares (p. 413), that solves the least-squares problem, $Ax = b$, subject to all $x_j \geq 0$. `NonNegativeLeastSquares` is based on the subroutine NNLS which appeared in Lawson and Hanson (1974). The additional work on bounded variable least squares was published in a later reprint (Lawson and Hanson, 1995).

Properties

Iterations

```
public int Iterations {get; }
```

Description

The number of iterations used to find the solution.

Property Value

An `int` containing the number of iterations.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations.

Property Value

An `int` containing the maximum number of iterations.

Default: MaxIterations = `3*a.GetLength(1)`.

ResidualNorm

```
virtual public double ResidualNorm {get; }
```

Description

The euclidean norm of the residual vector, $||Ax - b||^2$.

Property Value

A `double` containing the euclidean norm of the residual vector.

Tolerance

```
public double Tolerance {get; set; }
```

Description

The internal tolerance used to determine the relative linear dependence of a column vector for a variable moved from its initial value.

Property Value

A `double` value specifying the tolerance.

Default: Tolerance = 1.0e-7.

Constructor

BoundedVariableLeastSquares

```
public BoundedVariableLeastSquares(double[,] a, double[] b, double[]
lowerBound, double[] upperBound)
```

Description

Construct a new BoundedVariableLeastSquares instance to solve *Ax*-*b* subject to bounds on the variables. Each upper bound must be greater than or equal to the corresponding lower bound.

Parameters

`a` – The `double` input matrix.

`b` – A `double` array of length `a.GetLength(0)`.

`lowerBound` – A `double` array of length `a.GetLength(0)` containing lower bounds. Use `Double.NEGATIVE_INFINITY` for variables which are not bounded below.

`upperBound` – A `double` array of length `a.GetLength(0)` containing upper bounds. Use `Double.POSITIVE_INFINITY` for variables which are not bounded above.

Methods

GetDualSolution

```
public double[] GetDualSolution()
```

Description

Returns the dual solution vector, *w*.

Returns

A `double` array containing the dual solution vector, *w*.

Remarks

If x_j is at neither its upper nor lower bound then $w_j = 0$. If x_j is at its lower bound then $w_j \leq 0$. If x_j is at its upper bound then $w_j \geq 0$. If the upper and lower bound for the *j*-th variable are equal, fixing the value of x_j, then the value of w_j is arbitrary.

GetSolution

```
public double[] GetSolution()
```

Description

Returns the solution to the problem.

Returns

A `double` array containing the solution.

Solve

```
virtual public void Solve()
```

Description

Find the solution *x* to the problem for the current constraints.

Example 1: Bounded Variable Least Squares

The following example solves a linear least squares problem with bounds on the variables and compares the result to its unbounded solution. The normal of the residuals is 0.0 for the exact solution.

```
using System;
using Imsl.Math;

public class BoundedVariableLeastSquaresEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {{1, - 3, 2}, {- 3, 10, - 5},{2, - 5, 6}};
        double[] b = {27, - 78, 64};
        double[] xlb = {- 1, - 1, - 1};
        double[] xub = {5, 5, 5};
        double[] xl_ub = {
            Double.NegativeInfinity,
            Double.NegativeInfinity,
            Double.NegativeInfinity
        };
        double[] xu_ub = {
            Double.PositiveInfinity,
            Double.PositiveInfinity,
            Double.PositiveInfinity
        };

        // compute the bounded solution
        BoundedVariableLeastSquares bvls =
            new BoundedVariableLeastSquares(a, b, xlb, xub);
        bvls.Solve();
        double[] x_bounded = bvls.GetSolution();
        new PrintMatrix("Bounded Solution").Print(x_bounded);
        Console.WriteLine("Norm of the Residuals = " +
            bvls.ResidualNorm);

        // compute the unbounded solution
        bvls = new BoundedVariableLeastSquares(a, b, xl_ub, xu_ub);
        bvls.Solve();
        double[] x_unbounded = bvls.GetSolution();
        new PrintMatrix("\nUnbounded Solution").Print(x_unbounded);
        Console.WriteLine("Norm of the Residuals = " +
            bvls.ResidualNorm);
    }
}
```

Output

```
Bounded Solution
   0
0   5
1  -1
2   5

Norm of the Residuals = 35.0142828000232
```

```
Unbounded Solution
           0
0   1.00000000000004
1  -3.99999999999999
2   7

Norm of the Residuals = 0
```

NonNegativeLeastSquares Class

`public class Imsl.Math.NonNegativeLeastSquares`

Solves a linear least squares problem with nonnegativity constraints.

`NonNegativeLeastSquares` solves the problem

$$\min_x ||Ax - b||^2$$

subject to the condition $x \geq 0$.

If a starting point x_0 is provided, those entries of x_0 that are > 0 are first combined with a descent gradient component. The start point is the origin. When x_0 is not provided the algorithm uses only the gradient to verify that an optimum has been found. The algorithm completes using only the gradient components to reach an optimum. For more information, see Lawson and Hanson (1974).

See Also

Imsl.Math.BoundedVariableLeastSquares (p. 409)

Properties

DualTolerance

`virtual public double DualTolerance {get; set; }`

Description

The dual tolerance controlling when the computation stops.

Property Value

A `double` containing the dual tolerance. The computation stops if the largest gradient is smaller than this.

Default: DualTolerance = 0.

Iterations

```
virtual public int Iterations {get; }
```

Description

The number of iterations used to find the solution.

Property Value

An `int` containing the number of iterations.

MaximumTime

```
virtual public long MaximumTime {get; set; }
```

Description

The maximum time allowed for the solve step.

Property Value

A `long` value specifying the maximum time, in milliseconds, to be allowed for the solve step.

Default: There is no time limit.

Remarks

If `MaximumTime` is less than or equal to zero, then no time limit is imposed.

MaxIterations

```
virtual public int MaxIterations {get; set; }
```

Description

The maximum number of iterations.

Property Value

An `int` specifying the maximum number of iterations.

Default: MaxIterations = 3*`a.GetLength(1)`.

NormTolerance

```
virtual public double NormTolerance {get; set; }
```

Description

The residual norm tolerance.

Property Value

A `double` containing the residual norm tolerance. The computation stops if $||\text{rnorm}||^2 \leq$ NormTolerance $\times ||b||$, where *rnorm* is the residual norm.

Default: NormTolerance = 0.

RankTolerance

```
virtual public double RankTolerance {get; set; }
```

Description

The tolerance used for the incoming column rank deficient check.

Property Value

A `double` value used to check for rank deficiency.

Default: RankTolerance = 2.220e-016.

ResidualNorm

```
virtual public double ResidualNorm {get; }
```

Description

The euclidean norm of the residual vector, $||Ax - b||^2$.

Property Value

A `double` containing the euclidean norm of the residual vector.

Constructor

NonNegativeLeastSquares

```
public NonNegativeLeastSquares(double[,] a, double[] b)
```

Description

Construct a new NonNegativeLeastSquares instance to solve *Ax-b* where *x* is a vector of *n* unknowns.

Parameters

`a` – The `double` input matrix.

`b` – A `double` array of length `a.GetLength(0)`.

Methods

GetDualSolution

```
virtual public double[] GetDualSolution()
```

Description

Returns the dual solution vector, *w*. If $x_j = 0$ then $w_j \leq 0$, otherwise $w_j = 0$.

Returns

A `double` array containing the dual solution vector, *w*.

GetSolution

```
virtual public double[] GetSolution()
```

Description

Returns the solution to the problem, *x*.

Returns

A `double` array containing the solution.

SetGuess

```
virtual public void SetGuess(double[] guess)
```

Description

Sets the initial guess.

Parameter

`guess` – A `double` array containing the initial guess.

Remarks

If set, the guess is used in a two-phase algorithm where the positive components are matched with positive gradients to choose an incoming column. If not set, the algorithm uses only the gradient to verify that an optimum has been found.

Solve

```
virtual public void Solve()
```

Description

Finds the solution to the problem for the current constraints.

Exceptions

`Imsl.Math.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Math.TooMuchTimeException` is thrown if the maximum time allowed for the `Solve` method is exceeded.

Example 1: Non-negative Least Squares

Consider the following problem:

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 10 & -5 \\ 2 & -5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ -78 \\ 64 \end{bmatrix}$$

Subject to the constraint $x \geq 0$. The NonNegativeLeastSquares class is used to compute a solution, which is compared to the exact solution of $\{1, -4, 7\}$.

```
using System;
using Imsl.Math;
public class NonNegativeLeastSquaresEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {{1, - 3, 2}, {- 3, 10, - 5} , {2, - 5, 6}};
        double[] b = {27, - 78, 64};

        NonNegativeLeastSquares nnls =
```

```
            new NonNegativeLeastSquares(a, b);
        nnls.Solve();
        double[] x = nnls.GetSolution();

        new PrintMatrix("Solution").Print(x);

        // compare solution with exact answer
        double[,] compare = new double[2, 3];
        double[] tmp1 = Matrix.Multiply(a, x);
        double[] tmp2 = Matrix.Multiply(a, new double[]{1, - 4, 7});
        for (int i = 0; i < compare.GetLength(1); i++)
        {
            compare[0, i] = tmp1[i];
            compare[1, i] = tmp2[i];
        }

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.SetColumnLabels(new String[]{"x >= 0", "exact"});
        PrintMatrix pm = new PrintMatrix("Comparison of 'b'");
        pm.Print(pmf, Matrix.Transpose(compare));
    }
}
```

Output

```
      Solution
          0
0  18.4492753623188
1   0
2   4.5072463768116

      Comparison of 'b'
        x >= 0        exact
0   27.463768115942     27
1  -77.8840579710145   -78
2   63.9420289855073    64
```

MinConNLP Class

`public class Imsl.Math.MinConNLP`

General nonlinear programming solver.

`MinConNLP` is based on the FORTRAN subroutine, `DONLP2`, by Peter Spellucci and licensed from TU Darmstadt. `MinConNLP` uses a sequential equality constrained quadratic programming method with an active set technique, and an alternative usage of a fully regularized mixed constrained subproblem in case of nonregular constraints (i.e. linear dependent gradients in the "working sets"). It uses a slightly modified version of the Pantoja-Mayne update for the Hessian of the Lagrangian, variable dual scaling

and an improved Armjijo-type stepsize algorithm. Bounds on the variables are treated in a gradient-projection like fashion. Details may be found in the following two papers:

P. Spellucci: *An SQP method for general nonlinear programs using only equality constrained subproblems.* Math. Prog. 82, (1998), 413-448.

P. Spellucci: *A new technique for inconsistent problems in the SQP method.* Math. Meth. of Oper. Res. 47, (1998), 355-500. (published by Physica Verlag, Heidelberg, Germany).

The problem is stated as follows:

$$\min_{x \in R^n} f(x)$$

subject to

$$g_j(x) = 0, \text{for } j = 1, \ldots, m_e$$
$$g_j(x) \geq 0, \text{for j} = \text{m}_\text{e} + 1, \ldots, \text{m}$$
$$x_l \leq x \leq x_u$$

where all problem functions are assumed to be continuously differentiable. Although default values are provided for optional input arguments, it may be necessary to adjust these values for some problems. Through the use of member functions, `MinConNLP` allows for several parameters of the algorithm to be adjusted to account for specific characteristics of problems. The DONLP2 Users Guide provides detailed descriptions of these parameters as well as strategies for maximizing the performance of the algorithm. In addition, the following are a number of guidelines to consider when using `MinConNLP`:

- A good initial starting point is very problem specific and should be provided by the calling program whenever possible. See method `SetGuess`.
- Gradient approximation methods can have an effect on the success of `MinConNLP`. Selecting a higher order approximation method may be necessary for some problems. See property `DifferentiationType`.
- If a two sided constraint $l_i \leq g_i(x) \leq u_i$ is transformed into two constraints, $g_{2i}(x) \geq 0$ and $g_{2i+1}(x) \geq 0$, then choose $BindingThreshold < 1/2\,(u_i - l_i)\,/max\{1, \|\nabla g_i(x)\|\}$, or at least try to provide an estimate for that value. This will increase the efficiency of the algorithm. See property `BindingThreshold`.
- The parameter `ierr` provided in the interface to the user supplied function `F` can be very useful in cases when evaluation is requested at a point that is not possible or reasonable. For example, if evaluation at the requested point would result in a floating point exception, then setting `ierr` to `true` and returning without performing the evaluation will avoid the exception. `MinConNLP` will then reduce the stepsize and try the step again. Note, if `ierr` is set to `true` for the initial guess, then an error is issued.

Properties

BindingThreshold

```
public double BindingThreshold {get; set; }
```

Description

The binding threshold for constraints.

Property Value

A `double` scalar value specifying the binding threshold for constraints.

Remarks

In the initial phase of minimization a constraint is considered binding if
$\frac{g_i(x)}{max(1,\|\nabla g_i(x)\|)} \leq BindingThreshold \quad i = M_e + 1, \ldots, M$

Good values are between .01 and 1.0. If `BindingThreshold` is chosen too small then identification of the correct set of binding constraints may be delayed. Contrary, if `BindingThreshold` is too large, then the method will often escape to the full regularized SQP method, using individual slack variables for any active constraint, which is quite costly. For well scaled problems `BindingThreshold` = 1.0 is reasonable. By default, `BindingThreshold` is set to .5 * `PenaltyBound`.

Exception

`System.ArgumentException` is thrown if `BindingThreshold` is less than or equal to 0.0

BoundViolationBound

```
public double BoundViolationBound {get; set; }
```

Description

The amount by which bounds may be violated during numerical differentiation.

Property Value

A `double` scalar value specifying the amount by which bounds may be violated during numerical differentiation.

Remarks

By default, `BoundViolationBound` is set to 1.0.

Exception

`System.ArgumentException` is thrown if `BoundViolationBound` is set to a value less than or equal to 0.0

DifferentiationType

```
public int DifferentiationType {get; set; }
```

Description

The type of numerical differentiation to be used.

Property Value

An `int` scalar value specifying the type of numerical differentiation to be used. By default, `DifferentiationType` is set to 1.

Value	Action
1	Use a forward difference quotient with discretization stepsize $0.1\left(FunctionPrecision^{1/2}\right)$ componentwise relative. This is the default value used.
2	Use the symmetric difference quotient with discretization stepsize $0.1\left(FunctionPrecision^{1/3}\right)$ componentwise relative.
3	Use the sixth order approximation computing a Richardson extrapolation of three symmetric difference quotient values. This uses a discretization stepsize $0.01\left(FunctionPrecision^{1/7}\right)$.

Exception

`System.ArgumentException` is thrown if `DifferentiationType` is set to a value less than or equal to 0, or greater than or equal to 4

FunctionPrecision

```
public double FunctionPrecision {get; set; }
```

Description

The relative precision of the function evaluation routine.

Property Value

A `double` scalar value specifying the relative precision of the function evaluation routine.

Remarks

By default, `FunctionPrecision` is set to 2.2e-16.

Exception

`System.ArgumentException` is thrown if `FunctionPrecision` is set to a value less than or equal to 0.0

GradientPrecision

```
public double GradientPrecision {get; set; }
```

Description

The relative precision in gradients.

Property Value

A `double` scalar value specifying the relative precision in gradients.

Remarks

By default, `GradientPrecision` is set to 2.2e-16.

Exception

`System.ArgumentException` is thrown if less than or equal to 0.0

MaximumIterations

```
public int MaximumIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` specifying the maximum number of iterations allowed.

Remarks

By default, `MaximumIterations` is set to 200.

Exception

`System.ArgumentException` is thrown if `MaximumIterations` is less than or equal to 0

MultiplierError

```
public double MultiplierError {get; set; }
```

Description

The error allowed in the multipliers.

Property Value

A `double` scalar value specifying the error allowed in the multipliers.

Remarks

A negative multiplier of an inequality constraint is accepted (as zero) if its absolute value is less than `MultiplierError`. By default, `MultiplierError` is set to $e^{2\log\varepsilon/3}$.

Exception

`System.ArgumentException` is thrown if `MultiplierError` is less than or equal to 0.0

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Parallel

```
public bool Parallel {get; set; }
```

Description

Enable or disable performing `MinConNLP.IFunction.F` in parallel.

Property Value

A `bool` indicating whether or not the `MinConNLP.IFunction.F` calculations are to be performed in parallel.

Remarks

By default, `Parallel` = `true`. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

PenaltyBound

```
public double PenaltyBound {get; set; }
```

Description

The universal bound for describing how much the unscaled penalty-term may deviate from zero.

Property Value

A `double` scalar value specifying the universal bound for describing how much the unscaled penalty-term may deviate from zero.

Remarks

A small `PenaltyBound` diminishes the efficiency of the solver because the iterates then will follow the boundary of the feasible set closely. Conversely, a large `PenaltyBound` may degrade the reliability of the code. By default, `PenaltyBound` is set to 1.0.

Exception

`System.ArgumentException` is thrown if `PenaltyBound` is less than or equal to 0.0

ScalingBound

```
public double ScalingBound {get; set; }
```

Description

The scaling bound for the internal automatic scaling of the objective function.

Property Value

A `double` scalar value specifying the scaling variable for the problem function.

Remarks

By default, `ScalingBound` is set to 1.0e4.

Exception

`System.ArgumentException` is thrown if `ScalingBound` is less than or equal to 0.0

ViolationBound

```
public double ViolationBound {get; set; }
```

Description

Defines allowable constraint violations of the final accepted result.

Property Value

A `double` scalar value specifying the allowable constraint violations of the final accepted result.

Remarks

Constraints are satisfied if $|g_i(x)| \leq ViolationBound$, and $g_i(x) \geq -ViolationBound$ respectively. By default, `ViolationBound` is set to $min(\texttt{BindingThreshold}/10, max(epsdif, min(\texttt{BindingThreshold}/10, max((1.e-6)\texttt{BindingThreshold}, small_w))))$.

Exception

`System.ArgumentException` is thrown if `ViolationBound` is less than or equal to 0.0

Constructor

MinConNLP

```
public MinConNLP(int mTotalConstraints, int mEqualityConstraints, int
nVariables)
```

Description

Nonlinear programming solver constructor.

Parameters

`mTotalConstraints` – An `int` scalar value which defines the total number of constraints.

`mEqualityConstraints` – An `int` scalar value which defines the number of equality constraints.

`nVariables` – An `int` scalar value which defines the number of variables.

Methods

GetConstraintResiduals

```
public double[] GetConstraintResiduals()
```

Description

Returns the constraint residuals.

Returns

A `double` array containing the constraint residuals.

GetLagrangeMultiplierEst

```
public double[] GetLagrangeMultiplierEst()
```

Description

Returns the Lagrange multiplier estimates of the constraints.

Returns

A `double` array containing the Lagrange multiplier estimates of the constraints.

GetSolution

```
public double[] GetSolution()
```

Description

Returns the last computed solution.

Returns

A `double` array containing the solution.

Remarks

This is the same solution as returned by the `solve` method.

SetGuess

```
public void SetGuess(double[] guess)
```

Description

Sets the initial guess of the minimum point of the input function.

Parameter

`guess` – A `double` array specifying the initial guess of the minimum point of the input function.

Remarks

By default, the elements of this array are set to x, (with the smallest value of $||x||_2$) that satisfies the bounds.

SetXlowerBound

```
public void SetXlowerBound(double[] lower)
```

Description

Sets the lower bounds on the variables.

Parameter

`lower` – A `double` array specifying the lower bounds on the variables.

Remarks

By default, the elements of this array are set to -1.79e308.

SetXscale

```
public void SetXscale(double[] scale)
```

Description

The internal scaling of the variables.

Parameter

`scale` – A `double` array specifying the internal scaling of the variables.

Remarks

The initial value given and the objective function and gradient evaluations, however, are always given in the original unscaled variables. The first internal variable is obtained by dividing the values `x[i]` by `Xscale[i]`. By default, `Xscale[i]` is set to 1.0.

Exception

`System.ArgumentException` is thrown if `Xscale[i]` is less than or equal to 0.0

SetXupperBound

```
public void SetXupperBound(double[] upper)
```

Description

Sets the upper bounds on the variables.

Parameter

`upper` – A `double` array specifying the upper bounds on the variables.

Remarks

By default, the elements of this array are set to 1.79e308.

Solve

```
public double[] Solve(Imsl.Math.MinConNLP.IFunction f)
```

Description

Solve a general nonlinear programming problem using the successive quadratic programming algorithm with a finite-difference gradient or with a user-supplied gradient.

Parameter

`f` – Defines the user-supplied `MinConNLP.IFunction` to be evaluated at a given point. `f` can be used to supply a `MinConNLP.IGradient` of the function. If `f` implements `IGradient` the user-supplied gradient is used. Otherwise, an attempt to solve the problem is made using a finite-difference gradient.

Returns

A `double` array containing the last computed solution of the nonlinear programming problem.

Exceptions

`Imsl.Math.ConstraintEvaluationException` is thrown if a constraint evaluation returns an error.

`Imsl.Math.ObjectiveEvaluationException` is thrown if objective evaluation returns an error.

`Imsl.Math.WorkingSetSingularException` is thrown if

`Imsl.Math.QPInfeasibleException` is thrown if the working set is singular in dual extended QP.

`Imsl.Math.PenaltyFunctionPointInfeasibleException` is thrown if the penalty function point infeasible.

`Imsl.Math.LimitingAccuracyException` is thrown if limiting accuracy reached for a singular problem.

`Imsl.Math.TooManyIterationsException` is thrown if maximum number of iterations exceeded.

`Imsl.Math.NoAcceptableStepsizeException` is thrown if there is no acceptable stepsize.

`Imsl.Math.BadInitialGuessException` is thrown if the penalty function point infeasible for original problem.

`Imsl.Math.IllConditionedException` is thrown if the problem is singular or ill-conditioned.

`Imsl.Math.SingularException` is thrown ifthe problem is singular.

`Imsl.Math.LinearlyDependentGradientsException` is thrown if the working set gradients are linearly dependent.

`Imsl.Math.TerminationCriteriaNotSatisfiedException` is thrown if termination criteria are not satisfied.

Example 1: Solving a general nonlinear programming problem

A general nonlinear programming problem is solved using a finite difference gradient. The problem

$$\min F(x) = (x_1 - 2)^2 + (x_2 - 1)^2$$

subject to

$$g_1(x) = x_1 - 2x_2 + 1 = 0$$

$$g_2(x) = -x_1^2/4 - x_2^2 + 1 \geq 0$$

is solved.

```
using System;
using Imsl.Math;

public class MinConNLPEx1 : MinConNLP.IFunction
{
    public double F(double[] x, int iact, bool[] ierr)
    {
        double result;
        ierr[0] = false;
        if (iact == 0)
        {
            result = (x[0] - 2.0) * (x[0] - 2e0) +
                (x[1] - 1.0) * (x[1] - 1.0);
            return result;
        }
```

```
        else
        {
            switch (iact)
            {

                case 1:
                    result = (x[0] - 2.0 * x[1] + 1.0);
                    return result;

                case 2:
                    result = (-(x[0] * x[0]) / 4.0 - (x[1] * x[1])
                        + 1.0);
                    return result;

                default:
                    ierr[0] = true;
                    return 0.0;

            }
        }
    }


    public static void  Main(String[] args)
    {
        int m = 2;
        int me = 1;
        int n = 2;
        MinConNLP minconnon = new MinConNLP(m, me, n);
        minconnon.SetGuess(new double[]{2.0, 2.0});
        double[] x = minconnon.Solve(new MinConNLPEx1());
        new PrintMatrix("x").Print(x);
    }
}
```

Output

```
          x
           0
0  0.822875655532512
1  0.911437827766256
```

Example 2: Solving a general nonlinear programming problem

A general nonlinear programming problem is solved using a user-supplied gradient. The problem

$$\min F(x) = (x_1 - 2)^2 + (x_2 - 1)^2$$

subject to

$$g_1(x) = x_1 - 2x_2 + 1 = 0$$

$$g_2(x) = -x_1^2/4 - x_2^2 + 1 \geq 0$$

is solved.

```
using System;
using Imsl.Math;

public class MinConNLPEx2 : MinConNLP.IGradient
{
    public double F(double[] x, int iact, bool[] ierr)
    {
        double result;
        ierr[0] = false;
        if (iact == 0)
        {
            result = (x[0] - 2.0) * (x[0] - 2.0) +
                (x[1] - 1.0) * (x[1] - 1.0);
            return result;
        }
        else
        {
            switch (iact)
            {

                case 1:
                    result = (x[0] - 2.0 * x[1] + 1.0);
                    return result;

                case 2:
                    result = (-(x[0] * x[0]) / 4.0 -
                        (x[1] * x[1]) + 1.0);
                    return result;

                default:
                    ierr[0] = true;
                    return 0.0;

            }
        }
    }


    public void Gradient(double[] x, int iact, double[] result)
    {
        if (iact == 0)
        {
            result[0] = 2.0 * (x[0] - 2.0);
            result[1] = 2.0 * (x[1] - 1.0);
            return;
        }
```

```
        else
        {
            switch (iact)
            {
                case 1:
                    result[0] = 1.0;
                    result[1] = - 2.0;
                    return;

                case 2:
                    result[0] = - 0.5 * x[0];
                    result[1] = - 2.0 * x[1];
                    return;
                }
        }
    }


    public static void  Main(String[] args)
    {
        int m = 2;
        int me = 1;
        int n = 2;
        MinConNLP minconnon = new MinConNLP(m, me, n);
        minconnon.SetGuess(new double[]{2.0, 2.0});
        double[] x = minconnon.Solve(new MinConNLPEx2());
        new PrintMatrix("x").Print(x);
    }
}
```

Output

```
          x
           0
0  0.822875655532512
1  0.911437827766256
```

MinConNLP.IFunction Interface

`public interface Imsl.Math.MinConNLP.IFunction`

Public interface for the user supplied function to the `MinConNLP` object.

Method

F

```
abstract public double F(double[] x, int iact, bool[] ierr)
```

Description

Compute the value of the function at the given point.

Parameters

`x` – An input `double` array, the point at which the objective function or constraint is to be evaluated.

`iact` – An input `int` value indicating whether evaluation of the objective function is requested or evaluation of a constraint is requested. If iact is zero, then an objective function evaluation is requested. If iact is nonzero then the value of iact indicates the index of the constraint to evaluate. (1 indicates the first constraint, 2 indicates the second, etc.)

`ierr` – An input/output `bool` array of length 1. On input ierr[0] is set to false. If an error or other undesirable condition occurs during evaluation, then ierr[0] should be set to true. Setting ierr[0] to true will result in the step size being reduced and the step being tried again. (If ierr[0] is set to true for xguess, then an error is issued.)

Returns

A `double`. If iact is zero, then the value of the objective function at x is returned. If iact is nonzero, then the computed constraint value at the point x is returned.

MinConNLP.IGradient Interface

```
public interface Imsl.Math.MinConNLP.IGradient :  Imsl.Math.MinConNLP.IFunction
```

Public interface for the user supplied function to compute the gradient for `MinConNLP` object.

Method

Gradient

```
abstract public void Gradient(double[] x, int iact, double[] result)
```

Description

Computes the value of the gradient of the function at the given point.

Parameters

`x` – An input `double` array, the point at which the gradient of the objective function or gradient of a constraint is to be evaluated.

`iact` – An input `int` value indicating whether evaluation of the objective function gradient is requested or evaluation of a constraint gradient is requested. If iact is zero, then an objective function gradient evaluation is requested. If iact is nonzero then the value of iact indicates the index of the constraint gradient to evaluate. (1 indicates the first constraint, 2 indicates the second, etc.)

`result` – A `double` array. If iact is zero, then the value of the objective function gradient at x is returned in result. If iact is nonzero, then the computed gradient of the requested constraint value at the point x is returned in result.

NumericalDerivatives Class

```
public class Imsl.Math.NumericalDerivatives
```

Compute the Jacobian matrix for a function $f(y)$ with *m* components in *n* independent variables.

`NumericalDerivatives` uses divided finite differences to compute the Jacobian. This class is designed for use in numerical methods for solving nonlinear problems where a Jacobian is evaluated repeatedly at neighboring arguments. For example, this occurs in a Gauss-Newton method for solving non-linear least squares problems or a non-linear optimization method.

`NumericalDerivatives` is suited for applications where the Jacobian is a dense matrix. All cases $m < n$, $m = n$, or $m > n$ are allowed. Both one-sided and central divided differences can be used.

The design allows for computation of derivatives in a variety of contexts. Note that a gradient should be considered as the special case with $m = 1$, $n \geq 1$. A derivative of a single function of one variable is the case $m = 1$, $n = 1$. Any non-linear solving routine that optionally requests a Jacobian or gradient can use `NumericalDerivatives`. This should be considered if there are special properties or scaling issues associated with $f(y)$. Use the method `SetDifferencingMethods` to specify different differencing options for numerical differentiation. These can be combined with some analytic subexpressions or other known relationships.

The divided differences are computed using values of the independent variables at the initial point $y_e = y$, and differenced points $y_e = y + del \times e_j$. Here the e_j, $j = 1, \ldots, n$, are the unit coordinate vectors. The value for each difference *del* depends on the variable *j*, the differencing method, and the scaling for that variable. This difference is computed internally. See `SetPercentageFactor` for computational details. The evaluation of $f(y_e)$ is normally done by the user-provided method `NumericalDerivatives.IFunction.F`, using the values y_e. The index *j* and values y_e are arguments to `NumericalDerivatives.IFunction.F`.

The computational kernel of `EvaluateJ` performs the following steps:

1. evaluate the equations at the point y using `NumericalDerivatives.IFunction.F`.
2. compute the Jacobian.
3. compute the difference at y_e.

By default, EvaluateJ uses NumericalDerivatives.IFunction.F in step 3. The user may choose to override the EvaluateF method to extend the capability of the class beyond the default.

There are six examples provided which illustrate various ways to use NumericalDerivatives. A discussion of the expected errors for these difference methods is found in *A First Course in Numerical Analysis*, Anthony Ralston, McGraw-Hill, NY, (1965).

Constructor

NumericalDerivatives

```
public NumericalDerivatives(Imsl.Math.NumericalDerivatives.IFunction fcn)
```

Description

Constructor for NumericalDerivatives.

Parameter

fcn – An IFunction object which is a user-supplied function to evaluate the equations at the point y.

Methods

EvaluateF

```
virtual public double[] EvaluateF(int varIndex, double[] y)
```

Description

This method is provided by the user to compute the function values at the current independent variable values y.

Parameters

varIndex – An int which indicates the index of the variable to perturb.

y – A double array of length *n*, the point at which the function is to be evaluated.

Returns

A double array of length *m*. The equations evaluated at the point y.

Remarks

If the user does not override the EvaluateF method, then NumericalDerivatives.IFunction.F is used to compute the function values.

EvaluateJ

```
public double[,] EvaluateJ(double[] y)
```

Description

Evaluates the Jacobian for a system of (*m*) equations in (*n*) variables.

Parameter

y – A `double` array of length *n*, the point at which the Jacobian is to be evaluated.

Returns

A `double` matrix containing the Jacobian. Columns that are accumulated must have the additive term defined on entry or else be set to zero. Columns that are skipped can be defined either before or after the `EvaluateJ` method is invoked.

GetPercentageFactor

```
public double[] GetPercentageFactor()
```

Description

Returns the percentage factor for differencing.

Returns

A `double` array containing the percentage factor for differencing. See `SetPercentageFactor` for more detail.

GetScalingFactors

```
public double[] GetScalingFactors()
```

Description

Returns the scaling factors for the `y` values.

Returns

A `double` array containing the scaling factors.

GetStats

```
public int[] GetStats()
```

Description

Returns status information. This information might prove useful to the user wanting to gain better control over the differencing parameters. This information can often be ignored.

Returns

An `int` array containing the ten diagnostic values described in the following table. These values can be used to monitor the progress or expense of the Jacobian computation.

index	**Description**
0	the number of times a function evaluation was computed.
1	the number of columns in which three attempts were made to increase a percentage factor for differencing (i.e. a component in the `factor` array) but the computed *del* remained unacceptably small relative to `y[`*j*`-1]` or `scale[`*j*`-1]`. In such cases the percentage factor is set to 1.4901161193847656e-8, which is the square root of machine precision
2	the number of columns in which the computed *del* was zero to machine precision because `y[`*j*`-1]` or `scale[`*j*`-1]` was zero. In such cases *del* is set to 1.4901161193847656e-8, which is the square root of machine precision
3	the number of Jacobian columns which had to be recomputed because the largest difference formed in the column was close to zero relative to scale, where $$scale = \max\left(\left\|f_i(y)\right\|, \left\|f_i(y + del \times e_j)\right\|\right)$$ and *i* denotes the row index of the largest difference in the column currently being processed. *index* = 9 gives the last column where this occurred.
4	the number of columns whose largest difference is close to zero relative to scale after the column has been recomputed.
5	the number of times scale information was not available for use in the roundoff and truncation error tests. This occurs when $$\min\left(\left\|f_i(y)\right\|, \left\|f_i(y + del \times e_j)\right\|\right) = 0$$ where i is the index of the largest difference for the column currently being processed.
6	the number of times the increment for differencing (*del*) was computed and had to be increased because (`scale[`*j*`-1]`+*del*) - `scale[`*j*`-1]`) was too small relative to `y[`*j*`-1]` or `scale[`*j*`-1]`.
7	the number of times a component of the `factor` array was reduced because changes in function values were large and excess truncation error was suspected. *index* = 8 gives the last column in which this occurred.
8	the index of the last column where the corresponding component of the `factor` array had to be reduced because excessive truncation error was suspected.
9	the index of the last column where the difference was small and the column had to be recomputed with an adjusted increment (see *index* = 3). The largest derivative in this column may be inaccurate due to excessive roundoff error.

SetDifferencingMethods

```
public void
SetDifferencingMethods(Imsl.Math.NumericalDerivatives.DifferencingMethod[]
options)
```

Description

Sets the methods used to compute the derivatives.

Parameter

`options` – A `DifferencingMethod` array of length *n*, containing the methods used to compute the derivatives. `options[i]` is the method to be used for the *i*-th variable. `options[i]` can be one of the values in the table which follows.

Entry	Description
`OneSided`	Indicates one sided differences.
`Central`	Indicates central differences.
`Accumulate`	Indicates the accumulation of the result from whatever type of differences have been specified previously into initial values of the Jacobian.
`Skip`	Indicates a variable to be skipped.

Default: `OneSided` differences are used for each variable.

SetInitialF

```
public void SetInitialF(double[] valueF)
```

Description

Set the initial function values.

Parameter

`valueF` – A `double` array of length *m* containing the initial function values, y_0.

Default: All values are 0.0.

Remarks

Use the values $f(y_0)$, where y_0 is the initial value of the independent variables located in array `y`.

SetPercentageFactor

```
public void SetPercentageFactor(double[] factor)
```

Description

Sets the percentage factor for differencing

Parameter

`factor` – A `double` array of length *n* containing the percentage factor for differencing. Except for initialization, the `factor` array should not be altered in the `EvaluateF` method. The elements of `factor` must be such that

$$1.8189894035458565\text{e-}12 \leq \text{factor}[j-1] \leq 0.1$$

where 1.8189894035458565e-12 is machine precision to the three-fourths power.
Default: all elements of `factor` are set to 1.4901161193847656e-8, which is the square root of machine precision.

Remarks

For each divided difference for variable *j* the increment used is *del*. The value of *del* is computed as follows: First define $\sigma = sign(\text{scale}[j-1])$. If the user has set the elements of array `scale` to non-default values, then define $y_a = |\text{scale}[j-1]|$. Otherwise, $y_a = |y[j-1]|$ and $\sigma = 1$. Finally, compute $del = \sigma y_a \text{ factor}[j-1]$. By changing the sign of `scale[j-1]`, the difference *del* can have any desired orientation, such as staying within bounds on variable *j*. For central differences, a reduced factor is used for *del* that normally results in relative errors as small as machine precision to the 2/3 power.

SetScalingFactors

```
public void SetScalingFactors(double[] scale)
```

Description

Sets the scaling factors for the `y` values. The user can also use `scale` to provide appropriate signs for the increments.

Parameter

`scale` – A `double` array of length *n* containing the scaling factors.

Default: All values are 1.0.

Example 1: One-Sided Differences

A simple use of `NumericalDerivatives` is shown. The gradient of the function $f(y_1, y_2) = a\exp(by_1) + cy_1y_2^2$ is required for values $a = 2.5e6$, $b = 3.4$, $c = 4.5$, $y_1 = 2.1$, $y_2 = 3.2$ The numerical gradient is compared to the analytic gradient, cast as a 1 by 2 Jacobian:

$$grad(f) = \left[\begin{array}{cc} ab\exp(by_1) + cy_2^2, & 2cy_1y_2 \end{array} \right]$$

This analytic gradient is expected to approximately agree with the numerical differentiation gradient. Relative agreement should be approximately the square root of machine precision. That is achieved here. Generally this is the most accuracy one can expect using one-sided divided differences.

```
using System;
using Imsl.Math;

public class NumericalDerivativesEx1 : NumericalDerivatives.IFunction
{
    static int m = 1, n = 2;
    static double a, b, c;

    // This sets the function value used in forming one-sided
    // differences.
    public double[] F(int varIndex, double[] y)
    {
        double[] tmp = new double[m];
        tmp[0] = a * Math.Exp(b * y[0]) + c * y[0] * y[1] * y[1];
        return tmp;
    }


    public static void Main(String[] args)
```

```
    {
        double u;
        double[] y = new double[n];
        double[] scale = new double[n];
        double[,] actual = new double[m, n];
        double[] re = new double[2];

        // Define data and point of evaluation:
        a = 2.5e6;
        b = 3.4e0;
        c = 4.5e0;
        y[0] = 2.1e0;
        y[1] = 3.2e0;

        // Precision for measuring errors
        u = Math.Sqrt(2.220446049250313e-016);

        // Set scaling:
        scale[0] = 1.0e0;
        // Increase scale to account for large value of a.
        scale[1] = 8.0e3;

        // Compute true values of partials.
        actual[0, 0] = a * b * Math.Exp(b * y[0]) + c * y[1] * y[1];
        actual[0, 1] = 2 * c * y[0] * y[1];

        NumericalDerivatives derv =
            new NumericalDerivatives(new NumericalDerivativesEx1());
        derv.SetScalingFactors(scale);
        double[,] jacobian = derv.EvaluateJ(y);

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.00";
        new PrintMatrix("Numerical gradient:").Print(pmf, jacobian);
        new PrintMatrix("Analytic gradient:").Print(pmf, actual);

        // Check the relative accuracy of one-sided differences.
        // They should be good to about half-precision.
        jacobian[0, 0] = (jacobian[0, 0] - actual[0, 0]) / actual[0, 0];
        jacobian[0, 1] = (jacobian[0, 1] - actual[0, 1]) / actual[0, 1];
        re[0] = jacobian[0, 0];
        re[1] = jacobian[0, 1];

        Console.Out.WriteLine("Relative accuracy:");
        Console.Out.WriteLine("df/dy_1       df/dy_2");
        Console.Out.WriteLine(" {0:F2}u        {1:F2}u",
            re[0]/u, re[1]/u);
        Console.Out.WriteLine("({0:#.###e+00})  ({1:#.###e+00})",
            re[0], re[1]);
    }
}
```

Output

```
   Numerical gradient:
```

```
         0           1
0  10722141696.00  60.48

    Analytic gradient:
         0           1
0  10722141353.42  60.48

Relative accuracy:
df/dy_1       df/dy_2
 2.14u        0.00u
(3.195e-08)  (-1.175e-16)
```

Example 2: Skipping A Gradient Component

This example uses the same data as in the One-Sided Differences example. Here we assume that the second component of the gradient is analytically known. Therefore only the first gradient component needs numerical approximation. The input values of array `options` specify that numerical differentiation with respect to y_2 is skipped.

```
using System;
using Imsl.Math;

public class NumericalDerivativesEx2 : NumericalDerivatives.IJacobian
{
    static int m = 1, n = 2;
    static double a, b, c;


    public double[] F(int varIndex, double[] y)
    {
        double[] tmp = new double[m];
        tmp[0] = a * Math.Exp(b * y[0]) + c * y[0] * y[1] * y[1];
        return tmp;
    }


    public double[,] Jacobian(double[] y)
    {
        double[,] tmp = new double[m, n];

        // The second component partial is skipped,
        // since it is known analytically
        tmp[0, 1] = 2.0e0 * c * y[0] * y[1];

        return tmp;
    }


    public static void Main(String[] args)
    {
        NumericalDerivatives.DifferencingMethod[] options =
            new NumericalDerivatives.DifferencingMethod[n];
        double u;
        double[] y = new double[n];
        double[] valueF = new double[m];
```

```
        double[] scale = new double[n];
        double[,] actual = new double[m, n];
        double[] re = new double[2];

        // Define data and point of evaluation:
        a = 2.5e6;
        b = 3.4e0;
        c = 4.5e0;
        y[0] = 2.1e0;
        y[1] = 3.2e0;

        // Precision, for measuring errors
        u = Math.Sqrt(2.220446049250313e-016);

        // Set scaling:
        scale[0] = 1.0e0;
        // Increase scale to account for large value of a.
        scale[1] = 8.0e3;

        // Compute true values of partials.
        actual[0, 0] = a * b * Math.Exp(b * y[0]) + c * y[1] * y[1];
        actual[0, 1] = 2 * c * y[0] * y[1];

        options[0] = NumericalDerivatives.DifferencingMethod.OneSided;
        options[1] = NumericalDerivatives.DifferencingMethod.Skip;

        valueF[0] = a * Math.Exp(b * y[0]) + c * y[0] * y[1] * y[1];

        NumericalDerivatives derv =
            new NumericalDerivatives(new NumericalDerivativesEx2());
        derv.SetDifferencingMethods(options);
        derv.SetScalingFactors(scale);
        derv.SetInitialF(valueF);
        double[,] jacobian = derv.EvaluateJ(y);

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.00";
        new PrintMatrix("Numerical gradient:").Print(pmf, jacobian);
        new PrintMatrix("Analytic gradient:").Print(pmf, actual);

        jacobian[0, 0] = (jacobian[0, 0] - actual[0, 0]) / actual[0, 0];
        jacobian[0, 1] = (jacobian[0, 1] - actual[0, 1]) / actual[0, 1];
        re[0] = jacobian[0, 0];
        re[1] = jacobian[0, 1];

        Console.Out.WriteLine("Relative accuracy:");
        Console.Out.WriteLine("df/dy_1       df/dy_2");
        Console.Out.WriteLine(" {0:F2}u        {1:F2}u",
            re[0]/u, re[1]/u);
        Console.Out.WriteLine("({0:#.###e+00})  ({1:#.###e+00})",
            re[0], re[1]);
    }
}
```

Output

```
   Numerical gradient:
         0           1
0  10722141696.00  60.48

    Analytic gradient:
         0           1
0  10722141353.42  60.48

Relative accuracy:
df/dy_1       df/dy_2
 2.14u        0.00u
(3.195e-08)  (0e+00)
```

Example 3: Accumulation Of A Component

This example uses the same data as in the One-Sided Differences example. An alternate examination of the function $f(y_1, y_2) = a\exp(by_1) + cy_1y_2^2$ shows that the first term on the right-hand side need be evaluated just when computing the first partial. The additive term cy_2^2 occurs when computing the partial with respect to y_1. Also the first term does not depend on the second variable. Thus the first term can be left out of the function evaluation when computing the partial with respect to y_2, potentially avoiding cancellation errors. The input values of array `options` allow `NumericalDerivatives` to use these facts and obtain greater accuracy using a minimum number of computations of the exponential function.

```
using System;
using Imsl.Math;

public class NumericalDerivativesEx3 : NumericalDerivatives.IJacobian
{
    static int m = 1, n = 2;
    static double a, b, c;


    public double[] F(int varIndex, double[] y)
    {
        double[] tmp = new double[m];

        if (varIndex != 2)
        {
            tmp[0] = a * Math.Exp(b * y[0]);
        }
        else
        {
            // This is the function value for the partial wrt y_2.
            tmp[0] = c * y[0] * y[1] * y[1];
        }

        return tmp;
    }


    public double[,] Jacobian(double[] y)
    {
```

```
    double[,] tmp = new double[m, n];

    // Start with part of the derivative that is known.
    tmp[0, 0] = c * y[1] * y[1];

    return tmp;
}


public static void Main(String[] args)
{
    NumericalDerivatives.DifferencingMethod[] options =
        new NumericalDerivatives.DifferencingMethod[n];
    double u;
    double[] y = new double[n];
    double[] valueF = new double[m];
    double[] scale = new double[n];
    double[,] actual = new double[m, n];
    double[] re = new double[2];

    // Define data and point of evaluation:
    a = 2.5e6;
    b = 3.4e0;
    c = 4.5e0;
    y[0] = 2.1e0;
    y[1] = 3.2e0;

    // Precision, for measuring errors
    u = Math.Sqrt(2.220446049250313e-016);

    // Set scaling:
    scale[0] = 1.0e0;
    // Increase scale to account for large value of a.
    scale[1] = 8.0e3;

    // Compute true values of partials.
    actual[0, 0] = a * b * Math.Exp(b * y[0]) + c * y[1] * y[1];
    actual[0, 1] = 2 * c * y[0] * y[1];

    options[0] = NumericalDerivatives.DifferencingMethod.Accumulate;
    options[1] = NumericalDerivatives.DifferencingMethod.OneSided;

    valueF[0] = a * Math.Exp(b * y[0]);
    scale[1] = 1.0e0;

    NumericalDerivatives derv =
        new NumericalDerivatives(new NumericalDerivativesEx3());
    derv.SetDifferencingMethods(options);
    derv.SetScalingFactors(scale);
    derv.SetInitialF(valueF);
    double[,] jacobian = derv.EvaluateJ(y);

    PrintMatrixFormat pmf = new PrintMatrixFormat();
    pmf.NumberFormat = "0.00";
    new PrintMatrix("Numerical gradient:").Print(pmf, jacobian);
    new PrintMatrix("Analytic gradient:").Print(pmf, actual);
```

```
        jacobian[0, 0] = (jacobian[0, 0] - actual[0, 0]) / actual[0, 0];
        jacobian[0, 1] = (jacobian[0, 1] - actual[0, 1]) / actual[0, 1];
        re[0] = jacobian[0, 0];
        re[1] = jacobian[0, 1];

        Console.Out.WriteLine("Relative accuracy:");
        Console.Out.WriteLine("df/dy_1       df/dy_2");
        Console.Out.WriteLine(" {0:F2}u        {1:F2}u",
            re[0]/u, re[1]/u);
        Console.Out.WriteLine("({0:#.###e+00})  ({1:#.###e+00})",
            re[0], re[1]);
    }
}
```

Output

```
   Numerical gradient:
         0           1
0  10722141710.08  60.48

    Analytic gradient:
         0           1
0  10722141353.42  60.48

Relative accuracy:
df/dy_1       df/dy_2
 2.23u        -0.51u
(3.326e-08)  (-7.569e-09)
```

Example 4: Central Differences

This example uses the same data as in the One-Sided Differences example. Agreement should be approximately the two-thirds power of machine precision. That agreement is achieved here. Generally this is the *most* accuracy one can expect using central divided differences. Note that using central differences requires essentially twice the number of evaluations of the function compared with obtaining one-sided differences. This can be a significant issue for functions that are expensive to evaluate. This example shows how to override `EvaluateF`.

```
using System;
using Imsl.Math;


public class NumericalDerivativesEx4 : NumericalDerivatives
{
    static int m = 1, n = 2;
    static double a, b, c, v = 0.0;


    class NumericalDerivativesFcn : NumericalDerivatives.IFunction
    {
        public double[] F(int varIndex, double[] y)
        {
```

```
        return new double[m];
    }
}


public NumericalDerivativesEx4(NumericalDerivatives.IFunction fcn) :
    base(fcn)
{
}


// Override EvaluateF.
public override double[] EvaluateF(int varIndex, double[] y)
{
    double[] valueF = new double[m];

    valueF[0] = a * Math.Exp(b * y[0]) + c * y[0] * y[1] * y[1];
    return valueF;
}


public static void Main(String[] args)
{
    NumericalDerivatives.DifferencingMethod[] options =
        new NumericalDerivatives.DifferencingMethod[n];
    double u;
    double[] y = new double[n];
    double[] scale = new double[n];
    double[,] actual = new double[m, n];
    double[] re = new double[2];

    // Define data and point of evaluation:
    a = 2.5e6;
    b = 3.4e0;
    c = 4.5e0;
    y[0] = 2.1e0;
    y[1] = 3.2e0;

    // Machine precision, for measuring errors
    u = 2.220446049250313e-016;
    v = Math.Pow(3.0e0 * u, 2.0e0 / 3.0e0);

    // Set scaling:
    scale[0] = 1.0e0;
    // Increase scale to account for large value of a.
    scale[1] = 8.0e3;

    // Compute true values of partials.
    actual[0, 0] = a * b * Math.Exp(b * y[0]) + c * y[1] * y[1];
    actual[0, 1] = 2 * c * y[0] * y[1];

    options[0] = NumericalDerivatives.DifferencingMethod.Central;
    options[1] = NumericalDerivatives.DifferencingMethod.Central;

    // Set the increment used at the default value.
    scale[1] = 8.0e3;
```

```
        NumericalDerivativesEx4 derv =
            new NumericalDerivativesEx4(new NumericalDerivativesFcn());
        derv.SetDifferencingMethods(options);
        derv.SetScalingFactors(scale);
        double[,] jacobian = derv.EvaluateJ(y);

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.00";
        new PrintMatrix("Numerical gradient:").Print(pmf, jacobian);
        new PrintMatrix("Analytic gradient:").Print(pmf, actual);

        // Since the function is never evaluated at the
        // initial point, hold back until the request is made.

        // Check the relative accuracy of central differences.
        // They should be good to about two thirds-precision.
        jacobian[0, 0] = (jacobian[0, 0] - actual[0, 0]) / actual[0, 0];
        jacobian[0, 1] = (jacobian[0, 1] - actual[0, 1]) / actual[0, 1];
        re[0] = jacobian[0, 0];
        re[1] = jacobian[0, 1];

        Console.Out.WriteLine("Relative accuracy:");
        Console.Out.WriteLine("df/dy_1       df/dy_2");
        Console.Out.WriteLine(" {0:F2}v        {1:F2}v",
            re[0]/v, re[1]/v);
        Console.Out.WriteLine("({0:#.###e+00})  ({1:#.###e+00})",
            re[0], re[1]);
    }
}
```

Output

```
   Numerical gradient:
         0           1
0  10722141354.39  60.48

    Analytic gradient:
         0           1
0  10722141353.42  60.48

Relative accuracy:
df/dy_1       df/dy_2
 1.19v        0.27v
(9.1e-11)  (2.045e-11)
```

Example 5: Hessian Approximation

This example uses the same data as in the One-Sided Differences example. In this example numerical differentiation is used to approximate the Hessian matrix of $f(y_1, y_2)$. This symmetric Hessian matrix is

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial y_1^2} & \frac{\partial^2 f}{\partial y_1 \partial y_2} \\ \frac{\partial^2 f}{\partial y_1 \partial y_2} & \frac{\partial^2 f}{\partial y_2^2} \end{bmatrix}$$

Our method is based on casting the matrix Hf as the 2 by 2 Jacobian matrix of the gradient function. Each inner evaluation of the gradient function is itself computed using `NumericalDerivatives`. Central differences are used for both the inner and outer numerical differentiation. Because of the inherent error in both processes, the expected accuracy is about the $4/9 = (2/3)^2$ power of machine precision. Note that the approximation obtained is not symmetric. However, the difference between the off-diagonal elements provides an error estimate of that term.

```
using System;
using Imsl.Math;

public class NumericalDerivativesEx5 : NumericalDerivatives
{
    static int m = 1, n = 2;
    static double a, b, c, v = 0.0;


    class NumericalDerivativesFcn : NumericalDerivatives.IFunction
    {
        private int num;

        public NumericalDerivativesFcn(int num)
        {
            this.num = num;
        }

        public double[] F(int varIndex, double[] y)
        {
            return new double[num];
        }
    }


    class InnerNumericalDerivatives : NumericalDerivatives
    {
        public InnerNumericalDerivatives(NumericalDerivatives.IFunction fcn) :
            base(fcn)
        {
        }

        // Override EvaluateF.
        public override double[] EvaluateF(int varIndex, double[] y)
        {
            double[] valueF = new double[m];

            valueF[0] = a * Math.Exp(b * y[0]) + c * y[0] * y[1] * y[1];
            return valueF;
        }
    }


    public NumericalDerivativesEx5(NumericalDerivatives.IFunction fcn) :
        base(fcn)
    {
    }
```

```
// Override EvaluateF.
public override double[] EvaluateF(int varIndex, double[] y)
{
    NumericalDerivatives.DifferencingMethod[] iopt =
        new NumericalDerivatives.DifferencingMethod[n];
    double[] valueF = new double[m];
    double[] scale = new double[n];
    double[] fac = new double[n];

    // This is the analytic gradient.  for comparison only.
    //        stateh(1)=a*b*exp(b*y(1))+c*y(2)**2
    //        stateh(2)=2*c*y(1)*y(2)
    //
    // Each request for a gradient evaluation uses the
    // functionality of numerical evaluation, but with
    // the same numerical code.
    iopt[0] = NumericalDerivatives.DifferencingMethod.Central;
    iopt[1] = NumericalDerivatives.DifferencingMethod.Central;

    // Set the increment used at the default value.

    // Set defaults for increments and scaling:
    fac[0] = 1.4901161193847656E-8;
    fac[1] = 1.4901161193847656E-8;
    // Change scale to account for large value of a.
    switch (varIndex)
    {
        case 1:
            scale[0] = 1.0e0;
            scale[1] = 8.0e8;
            break;
        case 2:
            scale[0] = 1.0e4;
            scale[1] = 8.0e8;
            break;
    }

    InnerNumericalDerivatives derv =
        new InnerNumericalDerivatives(new NumericalDerivativesFcn(m));
    derv.SetPercentageFactor(fac);
    derv.SetDifferencingMethods(iopt);
    derv.SetScalingFactors(scale);
    derv.SetInitialF(valueF);
    double[,] fjac = derv.EvaluateJ(y);

    // Since the function is never evaluated at the
    // initial point, hold back until the request is made.

    // Copy gradient value into array expected by
    // outer loop computing the Hessian matrix.
    double[] tmp = new double[n];
    for (int i = 0; i < n; i++)
    {
        tmp[i] = fjac[0, i];
    }
```

```
    return tmp;
}


public static void Main(String[] args)
{
    NumericalDerivatives.DifferencingMethod[] iopth =
        new NumericalDerivatives.DifferencingMethod[n];
    double u;
    double[] fach = new double[n];
    double[] stateh = new double[n];
    double[] scaleh = new double[n];
    double[,] actual = new double[n, n];
    double[] y = new double[n];

    // Define data and point of evaluation:
    a = 2.5e6;
    b = 3.4e0;
    c = 4.5e0;
    y[0] = 2.1e0;
    y[1] = 3.2e0;

    // Machine precision, for measuring errors
    u = 2.220446049250313e-016;

    // Compute expected relative error using two applications
    // of central differences.
    v = Math.Pow(3.0e0 * u, 2.0e0 / 3.0e0);
    v = Math.Pow(3 * v, 2.0 / 3.0);

    // Set increments and scaling:
    fach[0] = 1.4901161193847656E-8;
    fach[1] = 1.4901161193847656E-8;
    iopth[0] = NumericalDerivatives.DifferencingMethod.Central;
    iopth[1] = NumericalDerivatives.DifferencingMethod.Central;

    // Compute true values of partials.
    actual[0, 0] = a * b * b * Math.Exp(b * y[0]);
    actual[1, 0] = 2 * c * y[1];
    actual[0, 1] = 2 * c * y[1];
    actual[1, 1] = 2 * c * y[0];

    // Set the increment used at the default value.
    scaleh[0] = 1;
    scaleh[1] = 8.0e5;

    NumericalDerivativesEx5 derv2 =
        new NumericalDerivativesEx5(new NumericalDerivativesFcn(n));
    derv2.SetPercentageFactor(fach);
    derv2.SetDifferencingMethods(iopth);
    derv2.SetScalingFactors(scaleh);
    derv2.SetInitialF(stateh);
    double[,] h = derv2.EvaluateJ(y);

    PrintMatrixFormat pmf = new PrintMatrixFormat();
    pmf.NumberFormat = "0.00";
```

```
        new PrintMatrix("Numerical Hessian:").Print(pmf, h);
        new PrintMatrix("Analytic Hessian:").Print(pmf, actual);

        // Since the function is never evaluated at the
        // initial point, hold back until the request is made.

        // Subtract the actual hessian matrix values and check.
        h[0, 0] = (h[0, 0] - actual[0, 0]) / h[0, 0] / v;
        h[1, 0] = (h[1, 0] - actual[1, 0]) / h[1, 0] / v;
        h[0, 1] = (h[0, 1] - actual[0, 1]) / h[0, 1] / v;
        h[1, 1] = (h[1, 1] - actual[1, 1]) / h[1, 1] / v;

        pmf.NumberFormat = "0.000";
        new PrintMatrix("Hessian Matrix, Expected Normalized " +
            "Relative Error, |all entries|").Print(pmf, h);
    }
}
```

Output

```
    Numerical Hessian:
          0           1
0  36455292905.82  28.80
1  28.80           18.90

    Analytic Hessian:
          0           1
0  36455280444.94  28.80
1  28.80           18.90

Hessian Matrix, Expected Normalized Relative Error, |all entries|
     0      1
0  0.914  0.037
1  0.036  0.000
```

Example 6: Usage With Class `MinUnconMultiVar`

The minimum of $100(x_2 - x_1^2)^2 + (1 - x_1)^2$ is found using `MinUnconMultiVar`. `NumericalDerivatives` is used to compute the numerical gradients.

```
using System;
using Imsl.Math;

public class NumericalDerivativesEx6 : MinUnconMultiVar.IGradient
{
    static int m = 1, n = 2;

    class NumericalDerivativesFcn : NumericalDerivatives.IFunction
    {
        public double[] F(int varIndex, double[] y)
        {
            double[] tmp = new double[m];
```

```
            tmp[0] = NumericalDerivativesEx6.FcnEvaluation(y);
            return tmp;
        }
    }

    static public double FcnEvaluation(double[] x)
    {
        return 100.0 * ((x[1] - x[0] * x[0]) * (x[1] - x[0] * x[0])) +
            (1.0 - x[0]) * (1.0 - x[0]);
    }

    public double F(double[] x)
    {
        return FcnEvaluation(x);
    }

    public void Gradient(double[] x, double[] gp)
    {
        NumericalDerivatives nderv =
            new NumericalDerivatives(new NumericalDerivativesFcn());
        double[,] jacobian = nderv.EvaluateJ(x);

        gp[0] = jacobian[0, 0];
        gp[1] = jacobian[0, 1];
    }


    public static void  Main(String[] args)
    {
        MinUnconMultiVar solver = new MinUnconMultiVar(n);
        solver.SetGuess(new double[]{- 1.2, 1.0});

        double[] x = solver.ComputeMin(new NumericalDerivativesEx6());
        Console.Out.WriteLine
            ("Minimum point is (" + x[0] + ", " + x[1] + ")");
    }
}
```

Output

```
Minimum point is (0.999998611858024, 0.999997274648157)
```

NumericalDerivatives.IFunction Interface

```
public interface Imsl.Math.NumericalDerivatives.IFunction
```

Public interface function.

Method

F

```
abstract public double[] F(int varIndex, double[] y)
```

Description

Returns the equations evaluated at the point `y`. If the user does not override the `EvaluateF` method, then `F` is also used to compute the function values at the current independent variable values y_e.

Parameters

`varIndex` – An `int` indicating the index of the variable to perturb. `varIndex` = 1 indicates variable 1 in `y[0]`.

`y` – A `double` array of length *n*, the point at which the Jacobian is to be evaluated.

Returns

A `double` array of length *m*. The equations evaluated at the point `y`.

NumericalDerivatives.IJacobian Interface

```
public interface Imsl.Math.NumericalDerivatives.IJacobian :
Imsl.Math.NumericalDerivatives.IFunction
```

Public interface for the user-supplied function to compute the Jacobian.

Method

Jacobian

```
abstract public double[,] Jacobian(double[] y)
```

Description

User-supplied function to compute the Jacobian.

Parameter

`y` – A `double` array of length *n*, the point at which the Jacobian is to be evaluated.

Returns

A `double` *m* by *n* matrix containing the Jacobian. Columns that are accumulated must have the analytic part defined on entry or else be set to zero. Columns that are skipped can be defined either before or after the `EvaluateJ` method is invoked.

NumericalDerivatives.DifferencingMethod Enumeration

```
public enumeration Imsl.Math.NumericalDerivatives.DifferencingMethod
```

Specifies the differencing method.

Fields

Accumulate

```
public Imsl.Math.NumericalDerivatives.DifferencingMethod Accumulate
```

Description

Indicates the accumulation of the result from whatever type of differences have been specified previously into initial values of the Jacobian.

Central

```
public Imsl.Math.NumericalDerivatives.DifferencingMethod Central
```

Description

Indicates central differences.

OneSided

```
public Imsl.Math.NumericalDerivatives.DifferencingMethod OneSided
```

Description

Indicates one sided differences.

Skip

```
public Imsl.Math.NumericalDerivatives.DifferencingMethod Skip
```

Description

Indicates a variable to be skipped.

Chapter 9: Special Functions

Types

Sfun Class

`public class Imsl.Math.Sfun`

Collection of special functions.

Fields

EpsilonLarge

`public double EpsilonLarge`

Description

The largest relative spacing for `doubles`.

EpsilonSmall

`public double EpsilonSmall`

Description

The smallest relative spacing for `doubles`.

Methods

Asinh

```
static public double Asinh(double x)
```

Description

Returns the hyperbolic arc sine of a `double`.

Parameter

`x` – A `double` value for which the hyperbolic arc sine is desired.

Returns

A `double` specifying the hyperbolic arc sine value.

Beta

```
static public double Beta(double a, double b)
```

Description

Returns the value of the Beta function.

Parameters

`a` – A `double` value.

`b` – A `double` value.

Returns

A `double` value specifying the Beta function.

Remarks

The Beta function is defined to be

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

See `Gamma` for the definition of $\Gamma(x)$.

The method `Beta` requires that both arguments be positive.

BetaIncomplete

```
static public double BetaIncomplete(double x, double p, double q)
```

Description

Returns the incomplete Beta function ratio.

Parameters

`x` – A `double` value specifying the upper limit of integration It must be in the interval [0,1] inclusive.

`p` – A `double` value specifying the first Beta parameter. It must be positive.

`q` – A `double` value specifying the second Beta parameter. It must be positive.

Returns

A `double` value specifying the incomplete Beta function ratio.

Remarks

The incomplete beta function is defined to be

$$I_x(p,\, q) = \frac{\beta_x(p,\, q)}{\beta(p,\, q)} = \frac{1}{\beta(p,\, q)} \int_0^x t^{p-1}(1-t)^{q-1} dt \text{ for } 0 \le x \le 1,\, p > 0,\, q > 0$$

See `Beta` for the definition of $\beta\,(p, q)$.

The parameters p and q must both be greater than zero. The argument x must lie in the range 0 to 1. The incomplete beta function can underflow for sufficiently small x and large p; however, this underflow is not reported as an error. Instead, the value zero is returned as the function value.

The method `BetaIncomplete` is based on the work of Bosten and Battiste (1974).

Cot

```
static public double Cot(double x)
```

Description

Returns the cotangent of a `double`.

Parameter

`x` – A `double` value

Returns

A `double` value specifying the cotangent of x. If x is NaN, the result is NaN.

Erf

```
static public double Erf(double x)
```

Description

Returns the error function of a `double`.

Parameter

`x` – A `double` value.

Returns

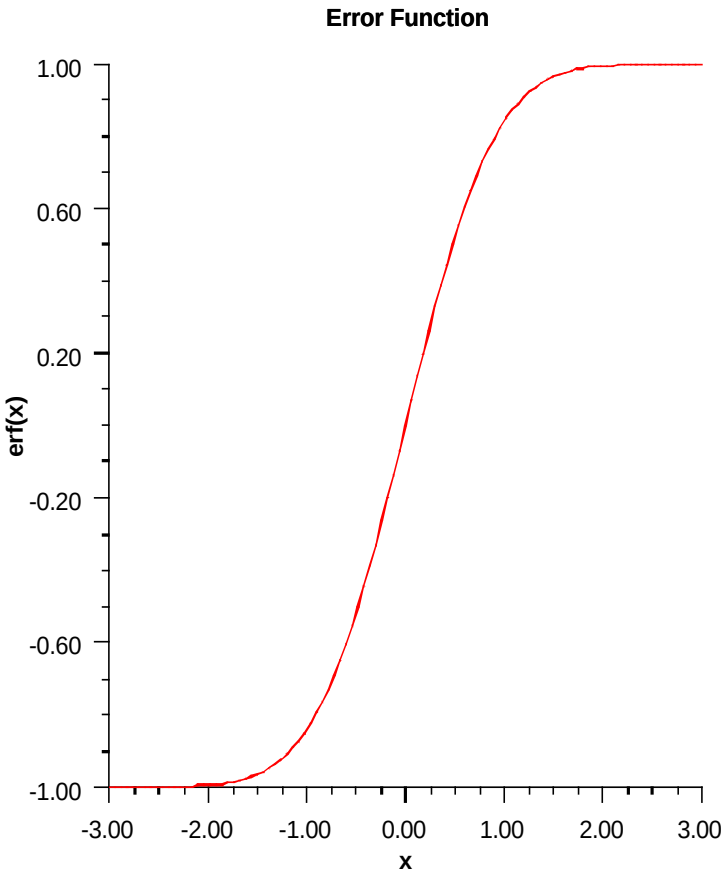
A `double` value specifying the error function of x.

Remarks

The error function method, `Erf`(x), is defined to be

$$\text{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

All values of x are legal.



Erfc

```
static public double Erfc(double x)
```

Description

Returns the complementary error function of a `double`.

Parameter

`x` – A `double` value.

Returns

A `double` value specifying the complementary error function of x.

Remarks

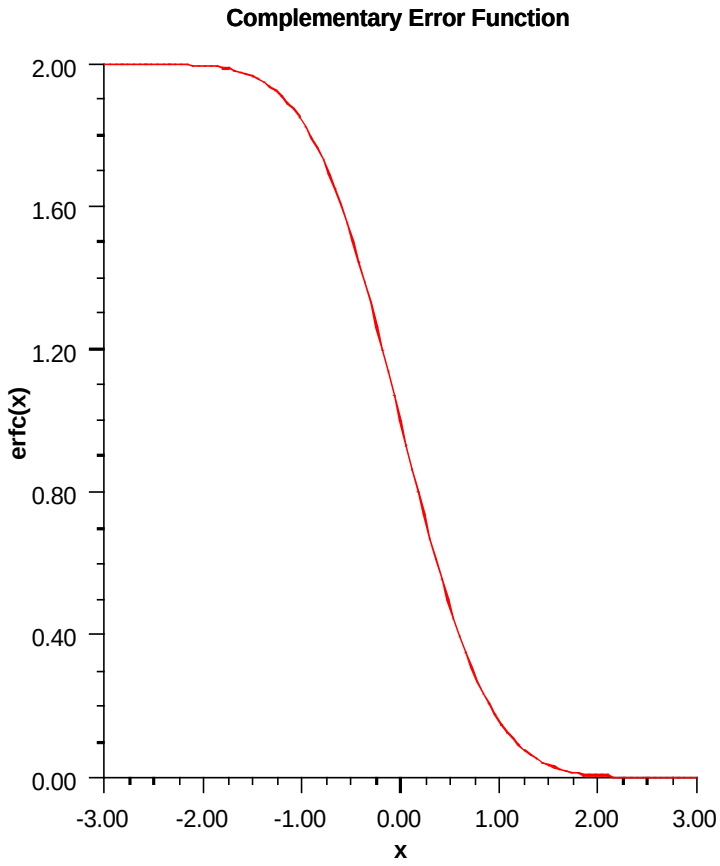
The complementary error function method, `Erfc` (*x*), is defined to be

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

The argument *x* must not be so large that the result underflows. Approximately, *x* should be less than

$$\left[-ln\left(\sqrt{\pi}s\right)\right]^{1/2}$$

where *s* = *Double.Epsilon* is the smallest representable positive floating-point number.



Erfce

```
static public double Erfce(double x)
```

Description

Returns the exponentially scaled complementary error function.

Parameter

`x` – A `double` value for which the function value is desired.

Returns

A `double` value specifying the exponentially scaled complementary error function of `x`.

Remarks

The exponentially scaled complementary error function is defined as

$$e^{x^2}\operatorname{erfc}(x)$$

where erfc(x) is the complementary error function. See Erfc (p. 456) for its definition.

To prevent the answer from underflowing, x must be greater than

$$x_{\min} \simeq -\sqrt{\ln(b/2)} = -26.618735713751487$$

where b = `Double.MaxValueM` is the largest representable double precision number.

ErfcInverse

```
static public double ErfcInverse(double x)
```

Description

Returns the inverse of the complementary error function.

Parameter

`x` – A `double` value, $0 \leq x \leq 2$.

Returns

A `double` value specifying the inverse of the error function of `x`.

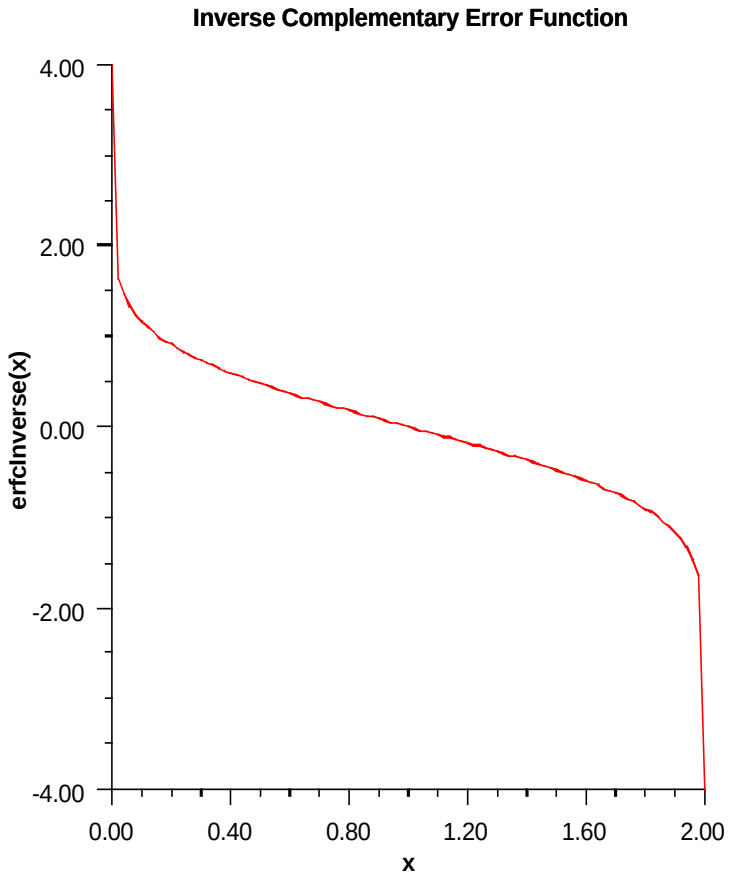
Remarks

The `Erfcinverse(x)` method computes the inverse of the complementary error function erfc x, defined in `Erfc`.

`Erfcinverse(x)` is defined for $0 < x < 2$. If $x_{\max} < x < 2$, then the answer will be less accurate than half precision. Very approximately,

$$x_{max} \approx 2 - \sqrt{\varepsilon/(4\pi)}$$

where ε = machine precision (approximately 1.11e-16).



ErfInverse

```
static public double ErfInverse(double x)
```

Description

Returns the inverse of the error function.

Parameter

`x` – A `double` value.

Returns

A `double` value specifying the inverse of the error function of x.

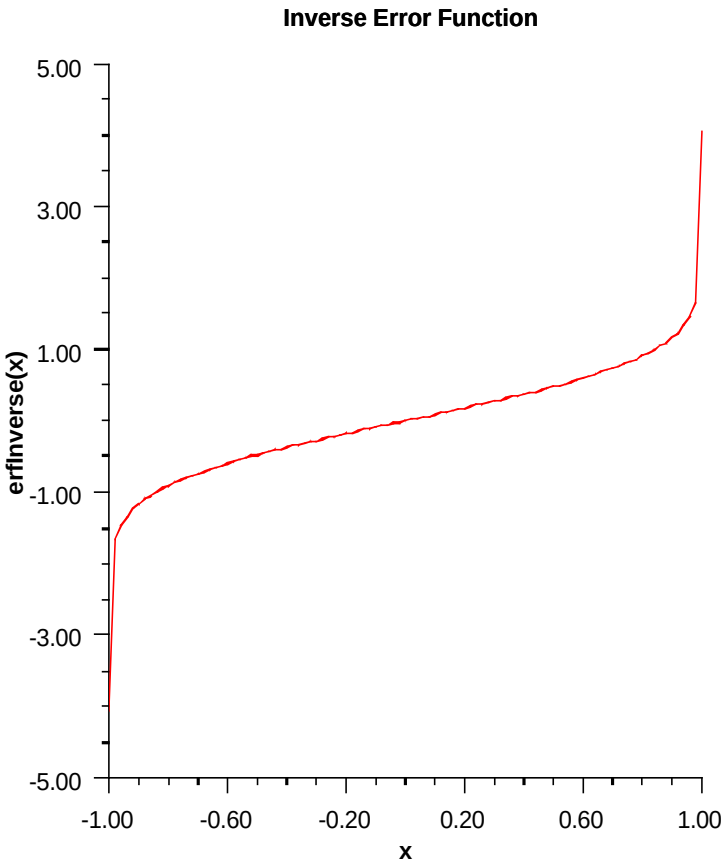
Remarks

`ErfInverse(X)` method computes the inverse of the error function erf *x*, defined in `Erf`.

The method `ErfInverse(X)` is defined for $x_{max} < |x| < 1$, then the answer will be less accurate than half precision. Very approximately,

$$x_{\max} \approx 1 - \sqrt{\varepsilon/(4\pi)}$$

where ε is the machine precision (approximately 1.11e-16).



Fact

```
static public double Fact(int n)
```

Description

Returns the factorial of an integer.

Parameter

`n` – An `int` value.

Returns

A `double` value specifying the factorial of n, n!. If x is negative, the result is NaN.

Gamma

```
static public double Gamma(double x)
```

Description

Returns the Gamma function of a `double`.

Parameter

`x` – A `double` value.

Returns

A `double` value specifying the Gamma function of x. If x is a negative integer, the result is NaN.

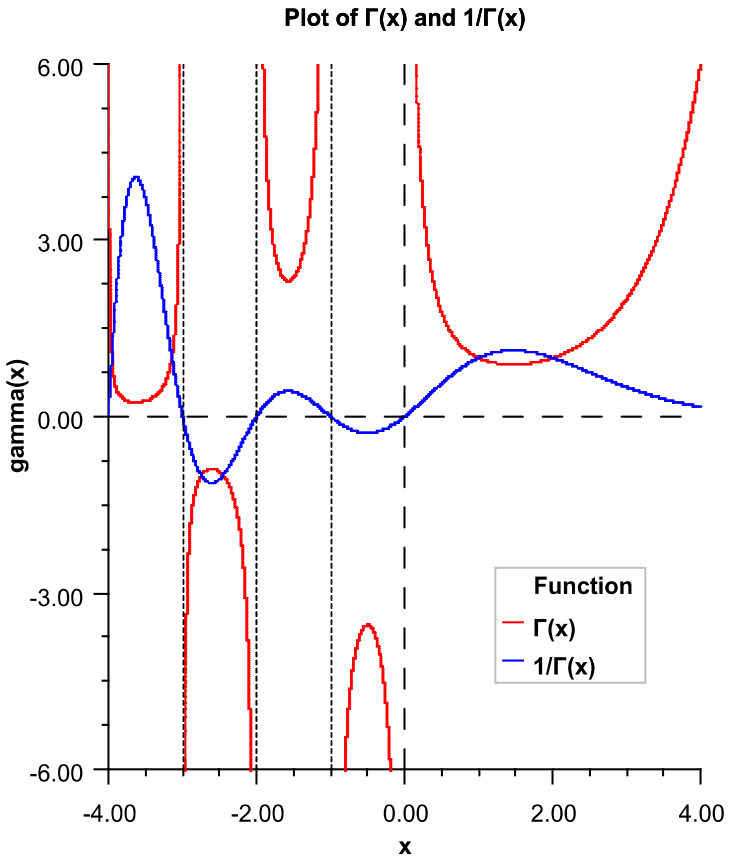
Remarks

The Gamma function, $\Gamma(x)$, is defined to be

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad for x > 0$$

For $x < 0$, the above definition is extended by analytic continuation.

The Gamma function is not defined for integers less than or equal to zero. Also, the argument x must be greater than -170.56 so that $\Gamma(x)$ does not underflow, and x must be less than *171.64* so that $\Gamma(x)$ does not overflow. The underflow limit occurs first for arguments that are close to large negative half integers. Even though other arguments away from these half integers may yield machine-representable values of $\Gamma(x)$, such arguments are considered illegal. Users who need such values should use the Log Gamma. Finally, the argument should not be so close to a negative integer that the result is less accurate than half precision.



GammaIncomplete

```
static public double GammaIncomplete(double a, double x)
```

Description

Evaluates the incomplete gamma function.

Parameters

`a` – A `double` value representing the integrand exponent parameter of the incomplete gamma function. If `a` is less than zero a `Double.NaN` is returned. equal to zero.

`x` – A `double` value specifying the point at which the incomplete gamma function is to be evaluated. If `x` is less than zero or equal to zero, `Double.NaN` or `0.0` respectively is returned. nonnegative.

Returns

A `double` value specifying the incomplete gamma function.

Remarks

The lower limit of integration of the incomplete gamma function, $\gamma(a,x)$, is defined to be

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt \quad \text{for } x \geq 0 \text{ and } a > 0$$

Although $\gamma(a,x)$ is well defined for $x > -\infty$, this algorithm does not calculate $\gamma(a,x)$ for negative x. For large a and sufficiently large x, $\gamma(a,x)$ may overflow. $\gamma(a,x)$ is bounded by $\Gamma(a)$, and users may find this bound a useful guide in determining legal values for a.

Note that the upper limit of integration of the incomplete gamma, $\Gamma(a,x)$, is defined to be

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$$

Therefore, by definition, the two incomplete gamma function forms satisfy the relationship

$$\Gamma(a,x) + \gamma(a,x) = \Gamma(a)$$

Log10

```
static public double Log10(double x)
```

Description

Returns the common (base 10) logarithm of a `double`.

Parameter

`x` – A `double` value.

Returns

A `double` value specifying the common logarithm of x.

Log1p

```
static public double Log1p(double x)
```

Description

Returns log(1+x), the logarithm of (x plus 1).

Parameter

`x` – A `double` value representing the argument.

Returns

A `double` value representing Log(1+x).

Remarks

Specifically:

Log1p(± 0) returns ± 0.

Log1p(-1) returns $-\infty$.

Log1p(x) returns NaN, if $x < -1$.

Log1p($\pm\infty$) returns $\pm\infty$.

LogBeta

```
static public double LogBeta(double a, double b)
```

Description

Returns the logarithm of the Beta function.

Parameters

`a` – A `double` value.

`b` – A `double` value.

Returns

A `double` value specifying the natural logarithm of the Beta function.

Remarks

Method `LogBeta` computes $\ln \beta(a,b) = \ln \beta(b,a)$. See `Beta` for the definition of $\beta(a,b)$.

`LogBeta` is defined for $a > 0$ and $b > 0$. It returns accurate results even when a or b is very small. It can overflow for very large arguments; this error condition is not detected except by the computer hardware.

LogGamma

```
static public double LogGamma(double x)
```

Description

Returns the logarithm of the Gamma function of the absolute value of a `double`.

Parameter

`x` – A `double` value.

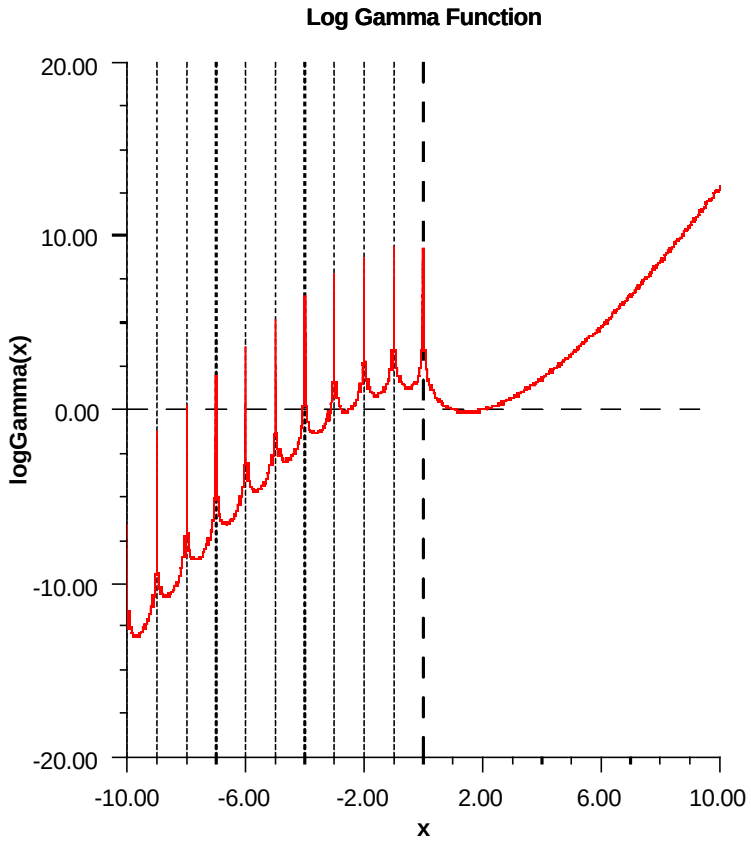
Returns

A `double` value specifying the natural logarithm of the Gamma function of $|x|$. If x is a negative integer, the result is NaN.

Remarks

Method `LogGamma` computes $\ln|\Gamma(x)|$. See `Gamma` for the definition of $\Gamma(x)$.

The Gamma function is not defined for integers less than or equal to zero. Also, $|x|$ must not be so large that the result overflows. Neither should x be so close to a negative integer that the accuracy is worse than half precision.



Poch

```
static public double Poch(double a, double x)
```

Description

Returns a generalization of Pochhammer's symbol.

Parameters

`a` – A `double` value specifying the first argument.

`x` – A `double` value specifying the second, differential argument.

Returns

A `double` value specifying the generalized Pochhammer symbol, Gamma(a+x)/Gamma(a).

Remarks

Method `Poch` evaluates Pochhammer's symbol $(a)_n = (a)(a-1)\dots(a-n+1)$ for n a nonnegative integer. Pochhammer's generalized symbol is defined to be

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}$$

See `Gamma` for the definition of $\Gamma(x)$.

Note that a straightforward evaluation of Pochhammer's generalized symbol with either Gamma or Log Gamma functions can be especially unreliable when a is large or x is small.

Substantial loss can occur if $a + x$ or a are close to a negative integer unless $|x|$ is sufficiently small. To insure that the result does not overflow or underflow, one can keep the arguments a and $a + x$ well within the range dictated by the Gamma function method Gamma or one can keep $|x|$ small whenever a is large. `Poch` also works for a variety of arguments outside these rough limits, but any more general limits that are also useful are difficult to specify.

Psi

```
static public double Psi(double x)
```

Description

Returns the derivative of the log gamma function, also called the digamma function.

Parameter

`x` – A `double` value, the point at which the digamma function is to be evaluated.

Returns

A `double` value specifying the logarithmic derivative of the gamma function of `x`. If `x` is a zero or a negative integer, the result is `NaN`. If `x` is too close to a negative integer the accuracy of the result will be less than half precision.

Remarks

The psi function is defined to be

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

See Gamma (p. 461) for the definition of $\Gamma(x)$.

The argument `x` must not be exactly zero or a negative integer, or $\psi(x)$ is undefined. Also, `x` must not be too close to a negative integer such that the accuracy of the result is less than half precision.

Psi1

```
static public double Psi1(double x)
```

Description

Returns the ψ_1 function, also known as the trigamma function.

Parameter

`x` – A `double` value, the point at which the trigamma function is to be evaluated.

Returns

A `double` value specifying the trigamma function of `x`. If `x` is a negative integer or zero, the result is `NaN`.

Remarks

The trigamma function, $\psi_1(x)$, is defined to be

$$\psi_1(x) = \frac{d^2}{dx^2} \ln \Gamma(x)$$

The trigamma function is not defined for integers less than or equal to zero.

R9lgmc

```
static public double R9lgmc(double x)
```

Description

Returns the Log Gamma correction term for argument values greater than or equal to 10.0.

Parameter

`x` – A `double` value.

Returns

A `double` value specifying the Log Gamma correction term.

Sign

```
static public double Sign(double x, double y)
```

Description

Returns the value of x with the sign of y.

Parameters

`x` – A `double` value.

`y` – A `double` value.

Returns

A `double` value specifying the absolute value of x and the sign of y.

Example: The Special Functions

Various special functions are exercised. Their use in this example typifies the manner in which other special functions in the Sfun class would be used.

```
using System;
using Imsl.Math;

public class SfunEx1
{
    public static void  Main(String[] args)
    {
        double result;

        // Log base 10 of x
```

```
        double x = 100.0;
        result = Sfun.Log10(x);
        Console.Out.WriteLine("The log base 10 of 100. is " + result);

        // Factorial of 10
        int n = 10;
        result = Sfun.Fact(n);
        Console.Out.WriteLine("10 factorial is " + result);

        // Gamma of 5.0
        double x1 = 5.0;
        result = Sfun.Gamma(x1);
        Console.Out.WriteLine
            ("The Gamma function at 5.0 is " + result);

        // LogGamma of 1.85
        double x2 = 1.85;
        result = Sfun.LogGamma(x2);
        Console.Out.WriteLine
            ("The logarithm of the absolute value of the " +
            "Gamma function \n    at 1.85 is " + result);

        // Beta of (2.2, 3.7)
        double a = 2.2;
        double b = 3.7;
        result = Sfun.Beta(a, b);
        Console.Out.WriteLine("Beta(2.2, 3.7) is " + result);

        // LogBeta of (2.2, 3.7)
        double a1 = 2.2;
        double b1 = 3.7;
        result = Sfun.LogBeta(a1, b1);
        Console.Out.WriteLine("logBeta(2.2, 3.7) is " + result + "\n");
    }
}
```

Output

```
The log base 10 of 100. is 2
10 factorial is 3628800
The Gamma function at 5.0 is 24
The logarithm of the absolute value of the Gamma function
    at 1.85 is -0.0559238130196572
Beta(2.2, 3.7) is 0.0453759834847081
logBeta(2.2, 3.7) is -3.09277231203789
```

Bessel Class

`public class Imsl.Math.Bessel`

Collection of Bessel functions.

Methods

I

`static public double[] I(double x, int n)`

Description

Evaluates a sequence of modified Bessel functions of the first kind with integer order and real argument.

Parameters

`x` – A `double` representing the argument of the Bessel functions to be evaluated.

`n` – The `int` order of the last element in the sequence.

Returns

A `double` array of length n+1 containing the values of the function through the series.

Remarks

Bessel.I[i] contains the value of the Bessel function of order i. The Bessel function $I_n(x)$ is defined to be

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos(n\theta)\,d\theta$$

The input x must satisfy $|x| \leq \log(b)$ where b is the largest representable floating-point number. The algorithm is based on a code due to Sookne (1973b), which uses backward recursion.

I

`static public double[] I(double xnu, double x, int n)`

Description

Evaluates a sequence of modified Bessel functions of the first kind with real order and real argument.

Parameters

`xnu` – A `double` representing the lowest order desired. `xnu` must be at least zero and less than 1.

`x` – A `double` representing the argument of the Bessel functions to be evaluated.

`n` – The `int` order of the last element in the sequence.

Returns

A `double` array of length `n` + 1 containing the values of the function through the series.

Remarks

Bessel.I[i] contains the value of the Bessel function of order `i+xnu`. The Bessel function $I_\nu(x)$, is defined to be

$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos(\nu\theta)d\,\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-x\cosh t-\nu t}dt$$

Here, argument `xnu` is represented by ν in the above equation.

The input `x` must be nonnegative and less than or equal to log(b) (b is the largest representable number). The argument ν = `xnu` must satisfy $0 \le \nu \le 1$.

This function is based on a code due to Cody (1983), which uses backward recursion.

J

```
static public double[] J(double x, int n)
```

Description

Evaluates a sequence of Bessel functions of the first kind with integer order and real argument.

Parameters

`x` – A `double` representing the argument for which the sequence of Bessel functions is to be evaluated.

`n` – A `int` which specifies the order of the last element in the sequence.

Returns

A `double` array of length `n` + 1 containing the values of the function through the series.

Remarks

Bessel.J[i] contains the value of the Bessel function of order i at `x` for i = 0 to `n`. The Bessel function $J_n(x)$, is defined to be

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\,d\,\theta$$

The algorithm is based on a code due to Sookne (1973b) that uses backward recursion with strict error control.

J

```
static public double[] J(double xnu, double x, int n)
```

Description

Evaluate a sequence of Bessel functions of the first kind with real order and real positive argument.

Parameters

`xnu` – A `double` representing the lowest order desired. `xnu` must be at least zero and less than 1.

`x` – A `double` representing the argument for which the sequence of Bessel functions is to be evaluated.

`n` – A `int` representing the order of the last element in the sequence. If order is the highest order desired, set `n` to `int`(order).

Returns

A `double` array of length n+1 containing the values of the function through the series. Bessel.J[I] contains the value of the Bessel function of order I + v at `x` for I=0 to `n`.

Remarks

The Bessel function $J_v(x)$, is defined to be

$$J_v(x) = \frac{(x/2)^v}{\sqrt{\pi}\Gamma(v+1/2)} \int_0^\pi \cos(x\cos\theta)\sin^{2v}\theta \, d\,\theta$$

This code is based on the work of Gautschi (1964) and Skovgaard (1975). It uses backward recursion.

K

```
static public double[] K(double x, int n)
```

Description

Evaluates a sequence of modified Bessel functions of the third kind with integer order and real argument.

Parameters

`x` – A `double` representing the argument for which the sequence of Bessel functions is to be evaluated.

`n` – A `int` which specifies the order of the last element in the sequence.

Returns

A `double` array of length `n` + 1 containing the values of the function through the series.

Remarks

This function uses $e^x K_{v+k-1}$ for $k = 1, \ldots, n$ and $v = 0$. For the definition of $K_v(x)$, see below.

K

```
static public double[] K(double xnu, double x, int n)
```

Description

Evaluates a sequence of modified Bessel functions of the third kind with fractional order and real argument.

Parameters

`xnu` – A `double` representing the fractional order of the function. `xnu` must be less than one in absolute value.

`x` – A `double` representing the argument for which the sequence of Bessel functions is to be evaluated.

`n` – A `int` representing the order of the last element in the sequence. If order is the highest order desired, set `n` to `int`(order).

Returns

A `double` array of length n+1 containing the values of the function through the series.

Remarks

Bessel.K[I] contains the value of the Bessel function of order I + v at x for I = 0 to n. The Bessel function $K_v(x)$ is defined to be

$$K_v(x) = \frac{\pi}{2} e^{v\pi i/2} [i J_v(ix) - Y_v(ix)] \quad \text{for} - \pi < \arg x \leq \frac{\pi}{2}$$

Currently, xnu (represented by v in the above equation) is restricted to be less than one in absolute value. A total of n values is stored in the result, K.

$K[0] = K_v(x)$, $K[1] = K_{v+1}(x)$, ..., $K[n-1] = K_{v+n-1}(x)$.

This method is based on the work of Cody (1983).

ScaledK

```
static public double[] ScaledK(double v, double x, int n)
```

Description

Evaluate a sequence of exponentially scaled modified Bessel functions of the third kind with fractional order and real argument.

Parameters

v – A double representing the fractional order of the function. v must be less than one in absolute value.

x – A double representing the argument for which the sequence of Bessel functions is to be evaluated.

n – A int representing the order of the last element in the sequence. If order is the highest order desired, set n to int(order).

Returns

A double array of length *n+1* containing the values of the function through the series.

Remarks

If n is positive, Bessel.K[I] contains e^x times the value of the Bessel function of order I + v at x for I = 0 to n.

If n is negative, Bessel.K[I] contains e^x times the value of the Bessel function of order v - I at x for I = 0 to n. This function evaluates $e^x K_{v+i-1}(x)$, for i=1,...,n where K is the modified Bessel function of the third kind. Currently, v is restricted to be less than 1 in absolute value. A total of $|n| + 1$ elements are returned in the array. This code is particularly useful for calculating sequences for large x provided n = x. (Overflow becomes a problem if $n << x$.) n must not be zero, and x must be greater than zero. $|v|$ must be less than 1. Also, when $|n|$ is large compared with x, $|v+n|$ must not be so large that

$$e^x K_{v+n}(x) = e^x \frac{\Gamma(|v+n|)}{2(x/2)^{v+n}}$$

overflows. The code is based on work of Cody (1983).

Y

```
static public double[] Y(double xnu, double x, int n)
```

Description

Evaluate a sequence of Bessel functions of the second kind with real nonnegative order and real positive argument.

Parameters

`xnu` – A `double` representing the lowest order desired. `xnu` must be at least zero and less than 1.

`x` – A `double` representing the argument for which the sequence of Bessel functions is to be evaluated.

`n` – A `int` which specifies that `n` + 1 elements will be evaluated in the sequence.

Returns

A `double` array of length `n` + 1 containing the values of the function through the series.

Remarks

Bessel.K[I] contains the value of the Bessel function of order I + v at `x` for I=0 to `n`. The Bessel function $Y_v(x)$ is defined to be

$$Y_v(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - v\theta)d\,\theta$$

$$-\frac{1}{\pi}\int_0^{\infty} \left[e^{vt} + e^{-vt}\cos(v\pi)\right] e^{-x\sinh t}\,dt$$

The variable `xnu` (represented by v in the above equation) must satisfy $0 \le v < 1$. If this condition is not met, then `Y` is set to `NaN`. In addition, `x` must be in $[x_m, x_M]$ where $x_m = 6(16^{-32})$ and $x_m = 16^9$. If $x < x_m$, then the largest representable number is returned; and if $x < x_M$, then zero is returned.

The algorithm is based on work of Cody and others, (see Cody et al. 1976; Cody 1969; NATS FUNPACK 1976). It uses a special series expansion for small arguments. For moderate arguments, an analytic continuation in the argument based on Taylor series with special rational minimax approximations providing starting values is employed. An asymptotic expansion is used for large arguments.

Example: The Bessel Functions

The Bessel functions I, J, and K are exercised for orders 0, 1, 2, and 3 at argument 10.e0.

```
using System;
using Imsl.Math;

public class BesselEx1
{
    public static void  Main(String[] args)
    {
        double x = 10e0;
        int hiorder = 4;
        //    Exercise some of the Bessel functions with argument 10.0
```

```
        double[] bi = Bessel.I(x, hiorder);
        double[] bj = Bessel.J(x, hiorder);
        double[] bk = Bessel.K(x, hiorder);

        Console.Out.WriteLine("Order     Bessel.I               " +
                              "Bessel.J               Bessel.K");
        for (int i = 0; i < 4; i++)
        {
            Console.Out.WriteLine(i + "     " + bi[i] + "     " + bj[i]
                                  + "     " + bk[i]);
        }
        Console.Out.WriteLine();
    }
}
```

Output

```
Order     Bessel.I              Bessel.J              Bessel.K
0     2815.71662846626     -0.245935764451348     1.77800623161677E-05
1     2670.98830370126     0.0434727461688615     1.86487734538256E-05
2     2281.518967726     0.254630313685121     2.15098170069328E-05
3     1758.38071661085     0.0583793793051867     2.72527002565987E-05
```

Chapter 10: Miscellaneous

Types

Complex Structure

```
public structure Imsl.Math.Complex :  System.IComparable, System.IFormattable
```

Set of mathematical functions for complex numbers. It provides the basic operations (addition, subtraction, multiplication, division) as well as a set of complex functions.

The binary operations have the form, where op is `Add`, `Subtract`, `Multiply` or `Divide`.

```
public static Complex op(Complex x, Complex y)   // x op y
public static Complex op(Complex x, double y)    // x op y
public static Complex op(double x, Complex y)    // x op y
```

`Complex` objects are immutable. Once created there is no way to change their value. The functions in this class follow the rules for complex arithmetic as defined C9x *Annex G: IEC 559-compatible complex arithmetic*. The API is not the same, but handling of infinities, NaNs, and positive and negative zeros is intended to follow the same rules.

Field

I

```
public Imsl.Math.Complex I
```

Description

The imaginary unit.

Remarks

This constant is set to new `Complex(0,1)`.

Constructors

Complex

```
public Complex(Imsl.Math.Complex z)
```

Description

Constructs a `Complex` equal to the argument.

Parameter

`z` – A `Complex` object.

Exception

`System.NullReferenceException` is thrown if `z` is null

Complex

```
public Complex(double re, double im)
```

Description

Constructs a `Complex` with real and imaginary parts given by the input arguments.

Parameters

`re` – A `double` value equal to the real part of the `Complex` object.

`im` – A `double` value equal to the imaginary part of the `Complex` object.

Complex

```
public Complex(double re)
```

Description

Constructs a `Complex` with a zero imaginary part.

Parameter

`re` – A `double` value equal to the real part of the `Complex` object.

Operators

Operator +

```
static public operator x + y
```

Description

Returns the sum of two `Complex` objects, *x+y*.

Parameters

`x` – A `Complex` object.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to *x+y*.

Operator +

```
static public operator x + y
```

Description

Returns the sum of a `Complex` and a `double`, *x+y*.

Parameters

`x` – A `Complex` object.

`y` – A `double` value.

Returns

A newly constructed `Complex` initialized to *x+y*.

Operator +

```
static public operator x + y
```

Description

Returns the sum of a `double` and a `Complex`, *x+y*.

Parameters

`x` – A `double` value.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to *x+y*.

Operator /

```
static public operator x / y
```

Description

Returns the result of a `Complex` object divided by a `Complex` object, *x/y*.

Parameters

`x` – A `Complex` object representing the numerator.

`y` – A `Complex` object representing the denominator.

Returns

A newly constructed `Complex` initialized to *x/y*.

Operator /

```
static public operator x / y
```

Description

Returns the result of a `Complex` object divided by a `double`, *x/y*.

Parameters

`x` – A `Complex` object representing the numerator.

`y` – A `double` representing the denominator.

Returns

A newly constructed `Complex` initialized to *x/y*.

Operator /

```
static public operator x / y
```

Description

Returns the result of a `double` divided by a `Complex` object, *x/y*.

Parameters

`x` – A `double` value.

`y` – A `Complex` object representing the denominator.

Returns

A newly constructed `Complex` initialized to *x/y*.

Operator ==

```
static public operator x == y
```

Description

Returns `true` if `x` and `y` are equal.

Parameters

`x` – A `Complex` value.

`y` – A `Complex` value.

Returns

`true` if `x` and `y` are equal.

Operator ==

```
static public operator x == y
```

Description

Returns `true` if `x` and `y` are equal.

Parameters

x – A `Complex` value.

y – A `double` value.

Returns

`true` if `x` and `y` are equal.

Operator ==

```
static public operator x == y
```

Description

Returns `true` if `x` and `y` are equal.

Parameters

x – A `double` value.

y – A `Complex` value.

Returns

`true` if `x` and `y` are equal.

Operator !=

```
static public operator x != y
```

Description

Returns `true` if `x` and `y` are not equal.

Parameters

x – A `Complex` value.

y – A `double` value.

Returns

`true` if `x` and `y` are equal.

Operator !=

```
static public operator x != y
```

Description

Returns `true` if `x` and `y` are not equal.

Parameters

x – A `double` value.

y – A `Complex` value.

Returns

`true` if `x` and `y` are not equal.

Operator !=

```
static public operator x != y
```

Description

Returns `true` if `x` and `y` are not equal.

Parameters

`x` – A `Complex` value.

`y` – A `Complex` value.

Returns

`true` if xx and `y` are equal.

Operator *

```
static public operator x * y
```

Description

Returns the product of two `Complex` objects, $x * y$.

Parameters

`x` – A `Complex` object.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to $x \times y$.

Operator *

```
static public operator x * y
```

Description

Returns the product of a `Complex` object and a `double`, $x * y$.

Parameters

`x` – A `Complex` object.

`y` – A `double` value.

Returns

A newly constructed `Complex` initialized to $x \times y$.

Operator *

```
static public operator x * y
```

Description

Returns the product of a `double` and a `Complex` object, $x * y$.

Parameters

`x` – A `double` value.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to $x \times y$.

Operator -

```
static public operator x - y
```

Description

Returns the difference of two `Complex` objects, *x-y*.

Parameters

`x` – A `Complex` object.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to *x-y*.

Operator -

```
static public operator x - y
```

Description

Returns the difference of a `Complex` object and a `double`, *x-y*.

Parameters

`x` – A `Complex` object.

`y` – A `double` value.

Returns

A newly constructed `Complex` initialized to *x-y*.

Operator -

```
static public operator x - y
```

Description

Returns the difference of a `double` and a `Complex` object, *x-y*.

Parameters

`x` – A `double` value.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to *x-y*.

Operator -

```
static public operator - x
```

Description

Returns the negative of a `Complex` object, -x.

Parameter

x – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the negative of the `Complex` argument, `x`.

Methods

Abs

```
static public double Abs(Imsl.Math.Complex z)
```

Description

Returns the absolute value (modulus) of a `Complex`, $|z|$.

Parameter

z – A `Complex` object.

Returns

A `double` value equal to the absolute value of the argument.

Acos

```
static public Imsl.Math.Complex Acos(Imsl.Math.Complex z)
```

Description

Returns the inverse cosine (arc cosine) of a `Complex`, with branch cuts outside the interval [-1,1] along the real axis.

Parameter

z – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the inverse (arc) cosine of the argument. The real part of the result is in the interval $[0, \pi]$.

Remarks

Specifically, if $z = x+iy$,

$\text{acos}(\overline{z}) = \overline{\text{acos}(z)}$.

$\text{acos}(\pm 0 + i0)$ returns $\pi/2 - i0$.

$\text{acos}(-\infty + i\infty)$ returns $3\pi/4 - i\infty$.

$\text{acos}(+\infty + i\infty)$ returns $\pi/4 - i\infty$.

$\text{acos}(x + i\infty)$ returns $\pi/2 - i\infty$, for finite x.

$\text{acos}(-\infty + iy)$ returns $\pi - i\infty$, for positive-signed finite y.

$\text{acos}(+\infty + iy)$ returns $+0 - i\infty$, for positive-signed finite y.

$\text{acos}(\pm\infty + i\text{NaN})$ returns $\text{NaN} \pm i\infty$ (where the sign of the imaginary part of the result is unspecified).

$\text{acos}(\pm 0 + i\text{NaN})$ returns $\pi/2 + i\text{NaN}$.

$\text{acos}(\text{NaN} + i\infty)$ returns $\text{NaN} - i\infty$.

$\text{acos}(x + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$, for nonzero finite x.

$\text{acos}(\text{NaN} + iy)$ returns $\text{NaN} + i\text{NaN}$, for finite y.

$\text{acos}(\text{NaN} + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$.

Acosh

```
static public Imsl.Math.Complex Acosh(Imsl.Math.Complex z)
```

Description

Returns the inverse hyperbolic cosine (arc cosh) of a `Complex`, with a branch cut at values less than one along the real axis.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the inverse (arc) hyperbolic cosine of the argument. The real part of the result is non-negative and its imaginary part is in the interval $[-i\pi, i\pi]$.

Remarks

Specifically, if $z = x+iy$,

$\text{acosh}(\overline{z}) = \overline{\text{acosh}(z)}$.

$\text{acosh}(\pm 0 + i0)$ returns $+0 + i\pi/2$.

$\text{acosh}(-\infty + i\infty)$ returns $+\infty + i3\pi/4$.

$\text{acosh}(+\infty + i\infty)$ returns $+\infty + i\pi/4$.

$\text{acosh}(x + i\infty)$ returns $+\infty + i\pi/2$, for finite x.

$\text{acosh}(-\infty + iy)$ returns $+\infty + i\pi$, for positive-signed finite y.

$\text{acosh}(+\infty + iy)$ returns $+\infty + i0$, for positive-signed finite y.

$\text{acosh}(\text{NaN} + i\infty)$ returns $+\infty + i\text{NaN}$.

$\text{acosh}(\pm\infty + i\text{NaN})$ returns $+\infty + i\text{NaN}$.

$\text{acosh}(x + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$, for finite x.

$\text{acosh}(\text{NaN} + iy)$ returns $\text{NaN} + i\text{NaN}$, for finite y.

$\text{acosh}(\text{NaN} + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$.

Add

```
static public Imsl.Math.Complex Add(Imsl.Math.Complex x, Imsl.Math.Complex y)
```

Description

Returns the sum of two `Complex` objects, $x+y$.

Parameters

x – A `Complex` object.

y – A `Complex` object.

Returns

A newly constructed `Complex` initialized to $x+y$.

Add

```
static public Imsl.Math.Complex Add(Imsl.Math.Complex x, double y)
```

Description

Returns the sum of a `Complex` and a `double`, $x+y$.

Parameters

x – A `Complex` object.

y – A `double` value.

Returns

A newly constructed `Complex` initialized to $x+y$.

Add

```
static public Imsl.Math.Complex Add(double x, Imsl.Math.Complex y)
```

Description

Returns the sum of a `double` and a `Complex`, $x+y$.

Parameters

x – A `double` value.

y – A `Complex` object.

Returns

A newly constructed `Complex` initialized to $x+y$.

Argument

```
static public double Argument(Imsl.Math.Complex z)
```

Description

Returns the argument (phase) of a `Complex`, in radians, with a branch cut along the negative real axis.

Parameter

z – A `Complex` object.

Returns

A `double` value equal to the argument (or phase) of a `Complex`. It is in the interval $[-\pi, \pi]$.

Asin

```
static public Imsl.Math.Complex Asin(Imsl.Math.Complex z)
```

Description

Returns the inverse sine (arc sine) of a `Complex`, with branch cuts outside the interval [-1,1] along the real axis.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the inverse (arc) sine of the argument. The real part of the result is in the interval $[-\pi/2, +\pi/2]$.

Remarks

The value of Asin is defined in terms of the function Asinh, by $\text{asin}(z) = -i\,\text{asinh}(iz)$.

See Also: Imsl.Math.Complex.Asinh(Imsl.Math.Complex) (p. 485)

Asinh

```
static public Imsl.Math.Complex Asinh(Imsl.Math.Complex z)
```

Description

Returns the inverse hyperbolic sine (arc sinh) of a `Complex`, with branch cuts outside the interval [-*i*,*i*].

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the inverse (arc) hyperbolic sine of the argument. Its imaginary part is in the interval $[-i\pi/2, i\pi/2]$.

Remarks

Specifically, if $z = x+iy$,

$\text{asinh}(\overline{z}) = \overline{\text{asinh}(z)}$ and asinh is odd.

$\text{asinh}(+0+i0)$ returns $0+i0$.

$\text{asinh}(\infty+i\infty)$ returns $+\infty+i\pi/4$.

$\text{asinh}(x+i\infty)$ returns $+\infty+i\pi/2$ for positive-signed finite x.

$\text{asinh}(+\infty+iy)$ returns $+\infty+i0$ for positive-signed finite y.

$\text{asinh}(\text{NaN}+i\infty)$ returns $\pm\infty+i\text{NaN}$ (where the sign of the real part of the result is unspecified).

$\text{asinh}(+\infty+i\text{NaN})$ returns $+\infty+i\text{NaN}$.

$\text{asinh}(\text{NaN}+i0)$ returns $\text{NaN}+i0$.

$\text{asinh}(\text{NaN}+iy)$ returns $\text{NaN}+i\text{NaN}$, for finite nonzero y.

$\text{asinh}(x+i\text{NaN})$ returns $\text{NaN}+i\text{NaN}$, for finite x.

$\text{asinh}(\text{NaN}+i\text{NaN})$ returns $\text{NaN}+i\text{NaN}$.

Atan

```
static public Imsl.Math.Complex Atan(Imsl.Math.Complex z)
```

Description

Returns the inverse tangent (arc tangent) of a `Complex`, with branch cuts outside the interval [-*i*,*i*] along the imaginary axis.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the inverse (arc) tangent of the argument. Its real part is in the interval $[-\pi/2, \pi/2]$.

Remarks

The value of Atan is defined in terms of the function Atanh, by $\text{atan}(z) = -i\,\text{atanh}(iz)$.

See Also: Imsl.Math.Complex.Atanh(Imsl.Math.Complex) (p. 486)

Atanh

```
static public Imsl.Math.Complex Atanh(Imsl.Math.Complex z)
```

Description

Returns the inverse hyperbolic tangent (arc tanh) of a `Complex`, with branch cuts outside the interval [-1,1] on the real axis.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the inverse (arc) hyperbolic tangent of the argument. The imaginary part of the result is in the interval $[-i\pi/2, i\pi/2]$.

Remarks

Specifically, if $z = x+iy$,

$\text{atanh}(\overline{z}) = \overline{\text{atanh}(z)}$ and atanh is odd.

$\text{atanh}(+0 + i0)$ returns $+0 + i0$.

$\text{atanh}(+\infty + i\infty)$ returns $+0 + i\pi/2$.

$\text{atanh}(+\infty + iy)$ returns $+0 + i\pi/2$, for finite positive-signed y.

$\text{atanh}(x + i\infty)$ returns $+0 + i\pi/2$, for finite positive-signed x.

$\text{atanh}(+0 + i\text{NaN})$ returns $+0 + i\text{NaN}$.

$\text{atanh}(\text{NaN} + i\infty)$ returns $\pm 0 + ipi/2$ (where the sign of the real part of the result is unspecified).

$\text{atanh}(+\infty + i\text{NaN})$ returns $+0 + i\text{NaN}$.

$\text{atanh}(\text{NaN} + iy)$ returns $\text{NaN} + i\text{NaN}$, for finite y.

$\text{atanh}(x + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$, for nonzero finite x.

$\text{atanh}(\text{NaN} + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$.

CompareTo

```
Final public int CompareTo(object obj)
```

Description

Compares this `Complex` to another Object.

Parameter

`obj` – An `Object` to be compared.

Returns

An `int`, 0 if `obj` is equal to this `Complex`; a value less than 0 if this `Complex` is less than `obj`; and a value greater than 0 if this `Complex` is greater than `obj`.

Remarks

If the Object is a `Complex`, this function behaves like compareTo(`Complex`). Otherwise, it throws a `InvalidCastException` (as `Complex` objects are comparable only to other `Complex` objects).

Exception

`System.InvalidCastException` is thrown if obj is not a `Complex` object

CompareTo

```
public int CompareTo(Imsl.Math.Complex z)
```

Description

Compares two `Complex` objects.

Parameter

`z` – A `Complex` to be compared.

Returns

The value 0 if `z` is equal to this `Complex`; a value less than 0 if this `Complex` is less than `z`; and a value greater than 0 if this `Complex` is greater than `z`.

Remarks

A lexigraphical ordering is used. First the real parts are compared in the sense of Double.compareTo. If the real parts are unequal this is the return value. If the return parts are equal then the comparison of the imaginary parts is returned.

Conjugate

```
static public Imsl.Math.Complex Conjugate(Imsl.Math.Complex z)
```

Description

Returns the complex conjugate of a `Complex` object.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the complex conjugate of `Complex` argument, `z`.

Cos

```
static public Imsl.Math.Complex Cos(Imsl.Math.Complex z)
```

Description

Returns the cosine of a `Complex`.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the cosine of the argument.

Remarks

The value of Cos is defined in terms of the function Cosh, by $\cos(z) = \cosh(iz)$.

See Also: Imsl.Math.Complex.Cosh(Imsl.Math.Complex) (p. 488)

Cosh

```
static public Imsl.Math.Complex Cosh(Imsl.Math.Complex z)
```

Description

Returns the hyperbolic cosh of a `Complex`.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the hyperbolic cosine of the argument.

Remarks

If $z = x+iy$,

$\cosh(\overline{z}) = \overline{\cosh(z)}$ and cosh is even.

$\cosh(+0+i0)$ returns $1+i0$.

$\cosh(+0+i\infty)$ returns $\text{NaN} \pm i0$ (where the sign of the imaginary part of the result is unspecified).

$\cosh(+\infty+i0)$ returns $+\infty+i0$.

$\cosh(+\infty+i\infty)$ returns $+\infty+i\text{NaN}$.

$\cosh(x+i\infty)$ returns $\text{NaN}+i\text{NaN}$, for finite nonzero x.

$\cosh(+\infty+iy)$ returns $+\infty[\cos(y)+i\sin(y)]$, for finite nonzero y.

$\cosh(+0+i\text{NaN})$ returns $\text{NaN} \pm i0$ (where the sign of the imaginary part of the result is unspecified).

$\cosh(+\infty+i\text{NaN})$ returns $+\infty+i\text{NaN}$.

$\cosh(x+i\text{NaN})$ returns $\text{NaN}+i\text{NaN}$, for finite nonzero x.

$\cosh(\text{NaN}+i0)$ returns $\text{NaN} \pm i0$ (where the sign of the imaginary part of the result is unspecified).

$\cosh(\text{NaN}+iy)$ returns $\text{NaN}+i\text{NaN}$, for all nonzero numbers y.

$\cosh(\text{NaN}+i\text{NaN})$ returns $\text{NaN}+i\text{NaN}$.

Divide

```
static public Imsl.Math.Complex Divide(Imsl.Math.Complex x, Imsl.Math.Complex
y)
```

Description

Returns the result of a `Complex` object divided by a `Complex` object, *x/y*.

Parameters

`x` – A `Complex` object representing the numerator.

`y` – A `Complex` object representing the denominator.

Returns

A newly constructed `Complex` initialized to *x/y*.

Divide

```
static public Imsl.Math.Complex Divide(Imsl.Math.Complex x, double y)
```

Description

Returns the result of a `Complex` object divided by a `double`, *x/y*.

Parameters

`x` – A `Complex` object representing the numerator.

`y` – A `double` representing the denominator.

Returns

A newly constructed `Complex` initialized to *x/y*.

Divide

```
static public Imsl.Math.Complex Divide(double x, Imsl.Math.Complex y)
```

Description

Returns the result of a `double` divided by a `Complex` object, *x/y*.

Parameters

`x` – A `double` value.

`y` – A `Complex` object representing the denominator.

Returns

A newly constructed `Complex` initialized to *x/y*.

Equals

```
override public bool Equals(object x)
```

Description

Compares this object against the specified object.

Parameter

`x` – The object to compare with.

Returns

`true` if the objects are the same; `false` otherwise.

Exp

```
static public Imsl.Math.Complex Exp(Imsl.Math.Complex z)
```

Description

Returns the exponential of a `Complex` z, exp(z).

Parameter

z – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the exponential of the argument.

Remarks

Specifically, if $z = x+iy$,

$\exp(\overline{z}) = \overline{\exp(z)}$.

$\exp(\pm 0 + i0)$ returns $1 + i0$.

$\exp(+\infty + i0)$ returns $+\infty + i0$.

$\exp(-\infty + i\infty)$ returns $\pm 0 \pm i0$ (where the signs of the real and imaginary parts of the result are unspecified).

$\exp(+\infty + i\infty)$ returns $\pm\infty + i\text{NaN}$ (where the sign of the real part of the result is unspecified).

$\exp(x + i\infty)$ returns $\text{NaN} + i\text{NaN}$, for finite x.

$\exp(-\infty + iy)$ returns $+0[\cos(y) + i\sin(y)]$, for finite y.

$\exp(+\infty + iy)$ returns $+\infty[\cos(y) + i\sin(y)]$, for finite nonzero y.

$\exp(-\infty + i\text{NaN})$ returns $\pm 0 \pm i0$ (where the signs of the real and imaginary parts of the result are unspecified).

$\exp(+\infty + i\text{NaN})$ returns $\pm\infty + i\text{NaN}$ (where the sign of the real part of the result is unspecified).

$\exp(\text{NaN} + i0)$ returns $\text{NaN} + i0$.

$\exp(\text{NaN} + iy)$ returns $\text{NaN} + i\text{NaN}$, for all non-zero numbers y.

$\exp(x + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$, for finite x.

GetHashCode

```
override public int GetHashCode()
```

Description

Returns a hashcode for this `Complex`.

Returns

A hash code value for this object.

Imag

```
public double Imag()
```

Description

Returns the imaginary part of a `Complex` object.

Returns

A `double` representing the imaginary part of a `Complex` object, z.

Imag

```
static public double Imag(Imsl.Math.Complex z)
```

Description

Returns the imaginary part of a `Complex` object.

Parameter

`z` – A `Complex` object.

Returns

A `double` representing the imaginary part of the `Complex` object, `z`.

Log

```
static public Imsl.Math.Complex Log(Imsl.Math.Complex z)
```

Description

Returns the logarithm of a `Complex` z, with a branch cut along the negative real axis.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the logarithm of the argument. Its imaginary part is in the interval $[-i\pi, i\pi]$.

Remarks

Specifically, if $z = x+iy$,

$\log(\overline{z}) = \overline{\log(z)}$.

$\log(0+i0)$ returns $-\infty+i\pi$.

$\log(+0+i0)$ returns $-\infty+i0$.

$\log(-\infty+i\infty)$ returns $+\infty+i3\pi/4$.

$\log(+\infty+i\infty)$ returns $+\infty+i\pi/4$.

$\log(x+i\infty)$ returns $+\infty+i\pi/2$, for finite x.

$\log(-\infty+iy)$ returns $+\infty+i\pi$, for finite positive-signed y.

$\log(+\infty+iy)$ returns $+\infty+i0$, for finite positive-signed y.

$\log(\pm\infty+i\text{NaN})$ returns $+\infty+i\text{NaN}$.

$\log(\text{NaN}+i\infty)$ returns $+\infty+i\text{NaN}$.

$\log(x+i\text{NaN})$ returns $\text{NaN}+i\text{NaN}$, for finite x.

$\log(\text{NaN} + iy)$ returns $\text{NaN} + i\text{NaN}$, for finite y.

$\log(\text{NaN} + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$.

Multiply

```
static public Imsl.Math.Complex Multiply(Imsl.Math.Complex x, Imsl.Math.Complex
y)
```

Description

Returns the product of two `Complex` objects, $x * y$.

Parameters

`x` – A `Complex` object.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to $x \times y$.

Multiply

```
static public Imsl.Math.Complex Multiply(Imsl.Math.Complex x, double y)
```

Description

Returns the product of a `Complex` object and a `double`, $x * y$.

Parameters

`x` – A `Complex` object.

`y` – A `double` value.

Returns

A newly constructed `Complex` initialized to $x \times y$.

Multiply

```
static public Imsl.Math.Complex Multiply(double x, Imsl.Math.Complex y)
```

Description

Returns the product of a `double` and a `Complex` object, $x * y$.

Parameters

`x` – A `double` value.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to $x \times y$.

MultiplyImag

```
static public Imsl.Math.Complex MultiplyImag(Imsl.Math.Complex x, double y)
```

Description

Returns the product of a `Complex` object and a pure imaginary `double`, x * iy.

Parameters

`x` – A `Complex` object.

`y` – A `double` value representing a pure imaginary.

Returns

A newly constructed `Complex` initialized to *x * iy*.

MultiplyImag

```
static public Imsl.Math.Complex MultiplyImag(double x, Imsl.Math.Complex y)
```

Description

Returns the product of a pure imaginary `double` and a `Complex` object, *ix * y*.

Parameters

`x` – A `double` value representing a pure imaginary.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to *ix * y*.

Negate

```
static public Imsl.Math.Complex Negate(Imsl.Math.Complex z)
```

Description

Returns the negative of a `Complex` object, -z.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the negative of the `Complex` argument, `z`.

Parse

```
static public Imsl.Math.Complex Parse(string s)
```

Description

Converts the string representation of a number in a specified style to its Complex number equivalent.

Parameter

`s` – A `string` containing a number to convert.

Returns

A `Complex` number represented in the String.

Parse

```
static public Imsl.Math.Complex Parse(string s, System.IFormatProvider
formatProvider)
```

Description

Converts the string representation of a number in a specified style to its Complex number equivalent.

Parameters

`s` – A `string` containing a number to convert.

`formatProvider` – An `IFormatProvider` that supplies culture-specific formatting information about s.

Returns

A `Complex` number represented in the String.

Pow

```
static public Imsl.Math.Complex Pow(Imsl.Math.Complex z, double x)
```

Description

Returns the `Complex` z raised to the x power, with a branch cut for the first parameter (z) along the negative real axis.

Parameters

`z` – A `Complex` object.

`x` – A `double` value.

Returns

A newly constructed `Complex` initialized to z to the power x.

Pow

```
static public Imsl.Math.Complex Pow(Imsl.Math.Complex x, Imsl.Math.Complex y)
```

Description

Returns the `Complex` x raised to the `Complex` y power.

Parameters

`x` – A `Complex` object.

`y` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to x^y.

Remarks

The value of Pow is defined in terms of the functions Exp and Log, by $\text{pow}(x,y) = \exp(y\log(x))$.

See Also: Imsl.Math.Complex.Exp(Imsl.Math.Complex) (p. 490),
Imsl.Math.Complex.Log(Imsl.Math.Complex) (p. 491)

Real

```
public double Real()
```

Description

Returns the real part of a `Complex` object.

Returns

A `double` representing the real part of a `Complex` object, z.

Real

```
static public double Real(Imsl.Math.Complex z)
```

Description

Returns the real part of a `Complex` object.

Parameter

`z` – A `Complex` object.

Returns

A `double` representing the real part of the `Complex` object, `z`.

Sin

```
static public Imsl.Math.Complex Sin(Imsl.Math.Complex z)
```

Description

Returns the sine of a `Complex`.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the sine of the argument.

Remarks

The value of Sin is defined in terms of the function Sinh, by $\sin(z) = -i\sinh(iz)$.

See Also: Imsl.Math.Complex.Sinh(Imsl.Math.Complex) (p. 495)

Sinh

```
static public Imsl.Math.Complex Sinh(Imsl.Math.Complex z)
```

Description

Returns the hyperbolic sine of a `Complex`.

Parameter

z – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the hyperbolic sine of the argument.

Remarks

If $z = x+iy$,

$\sinh(\overline{z}) = \overline{\sinh(z)}$ and sinh is odd.

$\sinh(+0+i0)$ returns $+0+i0$.

$\sinh(+0+i\infty)$ returns $\pm 0+i\text{NaN}$ (where the sign of the real part of the result is unspecified).

$\sinh(+\infty+i0)$ returns $+\infty+i0$.

$\sinh(+\infty+i\infty)$ returns $\pm\infty+i\text{NaN}$ (where the sign of the real part of the result is unspecified).

$\sinh(+\infty+iy)$ returns $+\infty[\cos(y)+i\sin(y)]$, for positive finite y.

$\sinh(x+i\infty)$ returns $\text{NaN}+i\text{NaN}$, for positive finite x.

$\sinh(+0+i\text{NaN})$ returns $\pm 0+i\text{NaN}$ (where the sign of the real part of the result is unspecified).

$\sinh(+\infty+i\text{NaN})$ returns $\pm\infty+i\text{NaN}$ (where the sign of the real part of the result is unspecified).

$\sinh(x+i\text{NaN})$ returns $\text{NaN}+i\text{NaN}$, for finite nonzero x.

$\sinh(\text{NaN}+i0)$ returns $\text{NaN}+i0$.

$\sinh(\text{NaN}+iy)$ returns $\text{NaN}+i\text{NaN}$, for all nonzero numbers y.

$\sinh(\text{NaN}+i\text{NaN})$ returns $\text{NaN}+i\text{NaN}$.

Sqrt

```
static public Imsl.Math.Complex Sqrt(Imsl.Math.Complex z)
```

Description

Returns the square root of a `Complex`, with a branch cut along the negative real axis.

Parameter

z – A `Complex` object.

Returns

A newly constructed `Complex` initialized to square root of z.

Remarks

Specifically, if $z = x+iy$,

$\text{sqrt}(\overline{z}) = \overline{\text{sqrt}(z)}$.

$\text{sqrt}(\pm 0+i0)$ returns $+0+i0$.

$\text{sqrt}(-\infty+iy)$ returns $+0+i\infty$, for finite positive-signed y.

$\text{sqrt}(+\infty+iy)$ returns $+\infty+i0$, for finite positive-signed y.

$\text{sqrt}(x+i\infty)$ returns $+\infty+i\infty$, for all x (including NaN).

sqrt$(-\infty + i\text{NaN})$ returns $\text{NaN} \pm i\infty$ (where the sign of the imaginary part of the result is unspecified).

sqrt$(+\infty + i\text{NaN})$ returns $+\infty + i\text{NaN}$.

sqrt$(x + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$ and optionally raises the invalid exception, for finite x.

sqrt$(\text{NaN} + iy)$ returns $\text{NaN} + i\text{NaN}$ and optionally raises the invalid exception, for finite y.

sqrt$(\text{NaN} + i\text{NaN})$ returns $\text{NaN} + i\text{NaN}$.

Subtract

```
static public Imsl.Math.Complex Subtract(Imsl.Math.Complex x, Imsl.Math.Complex
y)
```

Description

Returns the difference of two `Complex` objects, *x-y*.

Parameters

x – A `Complex` object.

y – A `Complex` object.

Returns

A newly constructed `Complex` initialized to *x-y*.

Subtract

```
static public Imsl.Math.Complex Subtract(Imsl.Math.Complex x, double y)
```

Description

Returns the difference of a `Complex` object and a `double`, *x-y*.

Parameters

x – A `Complex` object.

y – A `double` value.

Returns

A newly constructed `Complex` initialized to *x-y*.

Subtract

```
static public Imsl.Math.Complex Subtract(double x, Imsl.Math.Complex y)
```

Description

Returns the difference of a `double` and a `Complex` object, *x-y*.

Parameters

x – A `double` value.

y – A `Complex` object.

Returns

A newly constructed `Complex` initialized to *x*-*y*.

Tan

```
static public Imsl.Math.Complex Tan(Imsl.Math.Complex z)
```

Description

Returns the tangent of a `Complex`.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the tangent of the argument.

Remarks

The value of Tan is defined in terms of the function Tanh, by $\tan(z) = -i\tanh(iz)$.

See Also: Imsl.Math.Complex.Tanh(Imsl.Math.Complex) (p. 498)

Tanh

```
static public Imsl.Math.Complex Tanh(Imsl.Math.Complex z)
```

Description

Returns the hyperbolic tanh of a `Complex`.

Parameter

`z` – A `Complex` object.

Returns

A newly constructed `Complex` initialized to the hyperbolic tangent of the argument.

Remarks

If $z = x+iy$,

$\tanh(\overline{z}) = \overline{\tanh(z)}$ and tanh is odd.

$\tanh(+0+i0)$ returns $+0+i0$.

$\tanh(+\infty+iy)$ returns $1+i0$, for all positive-signed numbers y.

$\tanh(x+i\infty)$ returns $\mathrm{NaN}+i\mathrm{NaN}$, for finite x.

$\tanh(+\infty+i\mathrm{NaN})$ returns $1\pm i0$ (where the sign of the imaginary part of the result is unspecified).

$\tanh(\mathrm{NaN}+i0)$ returns $\mathrm{NaN}+i0$.

$\tanh(\mathrm{NaN}+iy)$ returns $\mathrm{NaN}+i\mathrm{NaN}$, for all nonzero numbers y.

$\tanh(x+i\mathrm{NaN})$ returns $\mathrm{NaN}+i\mathrm{NaN}$, for finite x.

$\tanh(\mathrm{NaN}+i\mathrm{NaN})$ returns $\mathrm{NaN}+i\mathrm{NaN}$.

ToString

```
override public string ToString()
```

Description

Returns a `String` representation for the specified `Complex`.

Returns

A `String` containing the Round-trip representation of this object. Round-trip guarantees that a numeric value converted to a `String` will be parsed back into the same numeric value.

ToString

```
Final public string ToString(string format, System.IFormatProvider
formatProvider)
```

Description

Formats the value of the current instance using the specified format.

Parameters

`format` – The `String` specifying the format to use.

`formatProvider` – The `IFormatProvider` to use to format the value.

Example: Roots of a Quadratic Equation

The two roots of the quadratic equation $ax^2 + bx + c$ are computed using the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

```
using System;
using Imsl.Math;

public class ComplexEx1
{
    public static void  Main(String[] args)
    {
        Complex a = new Complex(2.0, 3.0);
        double  b = 4.0;
        Complex c = new Complex(1.0, -2.0);

        Complex disc = Complex.Sqrt(b*b - 4.0*a*c);
        Complex root1 = (-b + disc) / (2.0*a);
        Complex root2 = (-b - disc) / (2.0*a);

        Console.Out.WriteLine("Root1 = " + root1);
        Console.Out.WriteLine("Root2 = " + root2);
    }
}
```

Output

```
Root1 = 0.19555270402037395+0.71433567154613054i
Root2 = -0.81093731940498925+0.20874125153079251i
```

Physical Structure

`public structure Imsl.Math.Physical`

Return the value of various mathematical and physical constants.

The case of the `String` specifying the name of the physical constant does not matter. The names 'PI', 'Pi', 'pI' and 'pi' are equivalent.

The units of the physical constants are in SI units, (meter-kilogram-second).

The names allowed are as follows:

Name	Description	Value	Reference
AMU	Atomic mass unit	1.6605402E-27 kg	[1]
ATM	Standard atm pressure	1.01325E+5 N/m^2	E[2]
AU	Astronomical unit	1.496E+11 m	[ ]
Avogadro	Avogadro's number	6.0221367E+23 1/mole	[1]
Boltzman	Boltzman's constant	1.380658E-23 J/K	[1]
C	Speed of light	2.997924580E+8 m/s	E[1]
Catalan	Catalan's constant	0.915965...	E[3]
E	Base of natural logs	2.718...	E[3]
ElectronCharge	Electron change	1.60217733E-19 C	[1]
ElectronMass	Electron mass	9.1093897E-31 kg	[1]
ElectronVolt	Electron volt	1.60217733E-19 J	[1]
Euler	Euler's constant gamma	0.577...	E[3]
Faraday	Faraday constant	9.6485309E+4 C/mole	[1]
FineStructure	Fine structure	7.29735308E-3	[1]
Gamma	Euler's constant	0.577...	E[3]
Gas	Gas constant	8.314510 J/mole/K	[1]
Gravity	Gravitational constant	6.67259E-11 Nm^2/kg^2	[1]
Hbar	Planck constant / 2 pi	1.05457266E-34 J*s	[1]
PerfectGasVolume	Std vol ideal gas	2.241383E-2 $m^3/mole$	[*]
Pi	Pi	3.141...	E[3]
Planck	Planck's constant h	6.6260755E-34 J*s	[1]
ProtonMass	Proton mass	1.6726231E-27 kg	[1]
Rydberg	Rydberg's constant	1.0973731534E+7 /m	[1]
SpeedLight	Speed of light	2.997924580E+8 m/s	E[1]
StandardGravity	Standard g	9.80665 m/s^2	E[2]
StandardPressure	Standard atm pressure	1.01325E+5 N/m^2	E[2]
StefanBoltzmann	Stefan-Boltzman	5.67051E-8 $W/K^4/m^2$	[1]
WaterTriple	Triple point of water	2.7316E+2 K	E[2]

The reference for constants are indicated by the code in the [] comment above.

[1]	Cohen and Taylor (1986)
[2]	Liepman (1964)
[3]	Precomputed mathematical constants

The constants marked with an E before the [] are exact (to machine precision).

Units strings have the form U1*U2*...*Um/V1/.../Vn, where Ui and Vi are the names of basic units or are the names of basic units raised to a power. Examples are, 'METER*KILOGRAM/SECOND', 'M*KG/S', 'METER', or 'M/KG^2'. These strings are case insensitive.

The basic unit names allowed are as follows.

Units of time

day, hour = hr, min = minute, s = sec = second, year

Units of frequency

Hertz = Hz

Units of mass

AMU, g = gram, lb = pound, ounce = oz, slug

Units of distance

Angstrom, AU, ft = feet = foot, in = inch, m = meter = metre, micron, mile, mill, parsec, yard

Units of area

acre

Units of volume

l = liter = litre

Units of force

dyne, N = Newton, poundal

Units of energy

BTU(thermochemical), Erg, J = Joule

Units of work

W = watt

Units of pressure

ATM = atomosphere, bar, Pascal

Units of temperature

degC = Celsius, degF = Fahrenheit, degK = Kelvin

Units of viscosity

poise, stoke

Units of charge

Abcoulomb, C = Coulomb, statcoulomb

Units of current

A = ampere, abampere, statampere

Units of voltage

Abvolt, V = volt

Units of magnetic induction

T = Tesla, Wb = Weber

Other units

1, farad, mole, Gauss, Henry, Maxwell, Ohm

The following metric prefixes may be used with the above units. Note that the one or two letter prefixes may only be used with one letter unit abbreviations.

A = atto = 1.E-18

F = femto = 1.E-15

P = pico = 1.E-12

N = nano = 1.E-9

U = micro = 1.E-6

M = milli = 1.E-3

C = centi = 1.E-2

D = deci = 1.E-1

DK = deca = 1.E+1

K = kilo = 1.E+3

myria = 1.E+4 (no single letter prefix; M means milli)

mega = 1.E+6 (no single letter prefix; M means milli)

G = giga = 1.E+9

T = tera = 1.E+12

Constructors

Physical

```
public Physical(double magnitude, string units)
```

Description

Constructs a new `Physical` object and initializes this object to a `double` value.

Parameters

`magnitude` – A `double` value to which the copy of the object is initialized.

`units` – A `String` specifying the unit.

Physical

```
public Physical(double magnitude, int length, int mass, int time, int current,
int temperature)
```

Description

Constructs a new `Physical` object and initializes this object to a `double` value along with `int` values for length, mass, time, current, and temperature.

Parameters

`magnitude` – A `double` value to which this object is initialized.

`length` – An `int` value assigned to this object's length.

`mass` – An `int` value assigned to this object's mass.

`time` – An `int` value assigned to this object's time.

`current` – An `int` value assigned to this object's current.

`temperature` – An `int` value assigned to this object's temperature.

Operators

Operator +

```
static public operator x + y
```

Description

Add two compatible `Physical` objects.

Exception

`System.ArgumentException` is thrown if x and y are not compatible

Parameters

`x` – A `Physical` object which is to be added.

`y` – A `Physical` object which is to be added.

Returns

A `Physical` object which is the sum of x + y.

Operator /

```
static public operator x / y
```

Description

Divide two `Physical` objects.

Parameters

`x` – A `Physical` object which is the numerator.

`y` – A `Physical` object which is the divisor.

Returns

A `Physical` object which is the result of x/y.

Operator /

```
static public operator x / y
```

Description

Divide a `Physical` object by a `double`.

Parameters

`x` – A `Physical` object which is the numerator.

`y` – A `double` object which is the divisor.

Returns

A `Physical` object which is the result of x/y.

Operator /

```
static public operator x / y
```

Description

Divide a `double` by a `Physical` object.

Parameters

`x` – A `double` which is the numerator.

`y` – A `Physical` object which is the divisor.

Returns

A `Physical` object which is the result of x/y.

Operator ==

```
static public operator x == y
```

Description

Returns `true` if x and y are equal.

Parameters

`x` – A `Physical` object.

`y` – A `Physical` object.

Returns

A `bool` value of `true` if x and y are equal.

Operator ()

```
static public operator () x
```

Description

Returns the value of this dimensionless object.

Exception

`System.ArgumentException` is thrown if the this object is not dimensionless

Parameter

`x` – A `Physical` object.

Returns

The `double` value of the dimensionless object.

Operator !=

```
static public operator x != y
```

Description

Returns `true` if x and y are not equal.

Parameters

`x` – A `Physical` object.

`y` – A `Physical` object.

Returns

A `bool` value of `true` if x and y are not equal.

Operator *

```
static public operator x * y
```

Description

Multiply two `Physical` objects.

Parameters

`x` – A `Physical` object which is to be multiplied.

`y` – A `Physical` object which is to be multiplied.

Returns

A `Physical` object which is the product of x and y.

Operator *

```
static public operator x * y
```

Description

Multiply a `Physical` object and a `double`.

Parameters

`x` – A `Physical` object which is to be multiplied.

`y` – A `double` which is to be multiplied.

Returns

A `Physical` object which is the product of x and y.

Operator *

```
static public operator x * y
```

Description

Multiply a `double` and a `Physical` object

Parameters

`x` – A `double` which is to be multiplied.

`y` – A `Physical` object which is to be multiplied.

Returns

A `Physical` object which is the product of x and y.

Operator -

```
static public operator x - y
```

Description

Subtract two compatible `Physical` objects.

Exception

`System.ArgumentException` is thrown if x and y are not compatible

Parameters

`x` – A `Physical` object.

`y` – A `Physical` object which is to be subtracted from x.

Returns

A `Physical` object which is the result of x - y.

Operator -

```
static public operator - x
```

Description

Negate a `Physical` object.

Parameter

`x` – A `Physical` object which is to be negated.

Returns

A `Physical` object which has been negated.

Methods

Add

```
static public Imsl.Math.Physical Add(Imsl.Math.Physical x, Imsl.Math.Physical
y)
```

Description

Adds two compatible `Physical` objects.

Parameters

`x` – A `Physical` object which is to be added.

`y` – A `Physical` object which is to be added.

Returns

A `Physical` object which is the sum of x + y.

Exception

`System.ArgumentException` is thrown if x and y are not compatible

CheckCompatibility

```
static public void CheckCompatibility(Imsl.Math.Physical x, Imsl.Math.Physical
y)
```

Description

Checks the compatibility of two `Physical` objects.

Parameters

`x` – A `Physical` object.

`y` – A `Physical` object to be checked against x.

Exception

`System.ArgumentException` is thrown if the two `Physical` objects are incompatible

Constant

```
static public Imsl.Math.Physical Constant(string name)
```

Description

Returns the value of a constant, given its name.

Parameter

`name` – A `String` representing the name of the constant to be returned.

Returns

The `Physical` object containing the value of the constant, in its default units.

Exception

`System.ArgumentException` is thrown when the name given is undefined

Constant

```
static public double Constant(string name, string units)
```

Description

Returns the value of a constant, given its name, in the specified units.

Parameters

`name` – A `String` representing the name of the constant to be returned.

`units` – A `String` representing the units in which the constant is to be returned.

Returns

A `double` containing the value of the constant in the specified units.

Exception

`System.ArgumentException` is thrown if the constant name is undefined

Convert

```
static public Imsl.Math.Physical Convert(Imsl.Math.Physical physical, string
unitsNew)
```

Description

Converts a value to a different set of units.

Parameters

`physical` – A `Physical` object specifying the value to be converted.

`unitsNew` – A `String` specifying the units to which `physical` is to be converted.

Returns

A `Physical` object containing the value of `physical` converted to the new units.

Exception

`System.ArgumentException` is thrown if the new and old units are incompatible

DefineConstant

```
static public void DefineConstant(string name, Imsl.Math.Physical magnitude)
```

Description

Defines a new constant.

Parameters

`name` – A `String` specifying the name of the constant to be defined.

`magnitude` – A `Physical` object defining the value of the new constant.

DefinePrefix

```
static public void DefinePrefix(string name, double magnitude)
```

Description

Defines a new prefix.

Parameters

`name` – A `String` specifying the name of the prefix to be defined.

`magnitude` – The `double` value of the prefix.

DefineUnit

```
static public void DefineUnit(string name, Imsl.Math.Physical magnitude)
```

Description

Defines a new unit.

Parameters

`name` – A `String` specifying the name of the unit to be defined.

`magnitude` – A `Physical` object defining the value of one unit in terms of SI units.

Divide

```
static public Imsl.Math.Physical Divide(Imsl.Math.Physical x,
Imsl.Math.Physical y)
```

Description

Divides two `Physical` objects.

Parameters

`x` – A `Physical` object which is the numerator.

`y` – A `Physical` object which is the divisor.

Returns

A `Physical` object which is the result of x/y.

Divide

```
static public Imsl.Math.Physical Divide(Imsl.Math.Physical x, double y)
```

Description

Divides a `Physical` object by a `double`.

Parameters

`x` – A `Physical` object which is the numerator.

`y` – A `double` object which is the divisor.

Returns

A `Physical` object which is the result of x/y.

Divide

```
static public Imsl.Math.Physical Divide(double x, Imsl.Math.Physical y)
```

Description

Divides a `double` by a `Physical` object.

Parameters

`x` – A `double` which is the numerator.

`y` – A `Physical` object which is the divisor.

Returns

A `Physical` object which is the result of x/y.

DoubleValue

```
public double DoubleValue()
```

Description

Returns the value of this dimensionless object.

Returns

The `double` value of the dimensionless object.

Exception

`System.ArgumentException` is thrown if the this object is not dimensionless

Equals

```
override public bool Equals(object x)
```

Description

Returns `true` if x equals this value.

Parameter

`x` – An `Object` to be tested for equality.

Returns

A `bool` value of `true` if x and this `Physical` object are equal.

Equals

```
static public bool Equals(Imsl.Math.Physical x, Imsl.Math.Physical y)
```

Description

Returns `true` if x and y are equal.

Parameters

x – A Physical object.

y – A Physical object.

Returns

A bool value of true if x and y are equal.

GetHashCode

```
override public int GetHashCode()
```

Description

Serves as a hash function for a particular type, suitable for use in hashing algorithms and data structures like a hash table.

Returns

A hash code for the current Object.

Multiply

```
static public Imsl.Math.Physical Multiply(Imsl.Math.Physical x,
Imsl.Math.Physical y)
```

Description

Multiply two Physical objects.

Parameters

x – A Physical object which is to be multiplied.

y – A Physical object which is to be multiplied.

Returns

A Physical object which is the product of x and y.

Multiply

```
static public Imsl.Math.Physical Multiply(Imsl.Math.Physical x, double y)
```

Description

Multiply a Physical object and a double.

Parameters

x – A Physical object which is to be multiplied.

y – A double which is to be multiplied.

Returns

A Physical object which is the product of x and y.

Multiply

```
static public Imsl.Math.Physical Multiply(double x, Imsl.Math.Physical y)
```

Description

Multiply a `double` and a `Physical` object.

Parameters

`x` – A `double` which is to be multiplied.

`y` – A `Physical` object which is to be multiplied.

Returns

A `Physical` object which is the product of x and y.

Negate

```
static public Imsl.Math.Physical Negate(Imsl.Math.Physical x)
```

Description

Negate a `Physical` object.

Parameter

`x` – A `Physical` object which is to be negated.

Returns

A `Physical` object which has been negated.

Subtract

```
static public Imsl.Math.Physical Subtract(Imsl.Math.Physical x,
Imsl.Math.Physical y)
```

Description

Subtract two compatible `Physical` objects.

Parameters

`x` – A `Physical` object.

`y` – A `Physical` object which is to be subtracted from x.

Returns

A `Physical` object which is the result of x - y.

Exception

`System.ArgumentException` is thrown if x and y are not compatible

ToString

```
override public string ToString()
```

Description

Returns a `String` containing the value and units, if any.

Returns

A `String` specifying the value and units, if any, of this `Physical` object.

UnitsString

```
public string UnitsString()
```

Description

Returns a `String` containing the units only.

Returns

A `String` specifying the units of this `Physical` object.

Example: Compute Kinetic Energy

The kinetic energy of a mass in motion is given by

$$T = \frac{1}{2}mv^2$$

where m is the mass and v is the velocity. In this example the mass is 2.4 pounds and the velocity is 6.7 meters per second. The infix operators defined by `Physical` automatically handle the unit convertions and computes the current units for the result.

```
using System;
using Imsl.Math;

public class PhysicalEx1
{
    public static void  Main(String[] args)
    {
        Physical mass = new Physical(2.4, "pound");
        Physical velocity = new Physical(6.7, "m/s");
        Physical energy = 0.5*mass*velocity*velocity;
        Console.Out.WriteLine("Kinetic energy is " + energy);
    }
}
```

Output

```
Kinetic energy is 24.43411378716 m^2*kg/s^2
```

EpsilonAlgorithm Class

```
public class Imsl.Math.EpsilonAlgorithm
```

The class is used to determine the limit of a sequence of approximations, by means of the Epsilon algorithm of P. Wynn.

An estimate of the absolute error is also given. The condensed Epsilon table is computed. Only those elements needed for the computation of the next diagonal are preserved.

Property

ErrorEstimate

```
public double ErrorEstimate {get; }
```

Description

Returns the current error estimate.

Property Value

A `double` containing the current error estimate.

Constructors

EpsilonAlgorithm

```
public EpsilonAlgorithm()
```

Description

The class is used to determine the limit of a sequence of approximations, by means of the Epsilon algorithm of P. Wynn.

Remarks

An estimate of the absolute error is also given. The condensed Epsilon table is computed. Only those elements needed for the computation of the next diagonal are preserved.

EpsilonAlgorithm

```
public EpsilonAlgorithm(int maxTableSize)
```

Description

The class is used to determine the limit of a sequence of approximations, by means of the Epsilon algorithm of P. Wynn.

Parameter

`maxTableSize` – A `int` which specifies the maximum size of Episilon Table to be computed.

Remarks

An estimate of the absolute error is also given. The condensed Epsilon table is computed. Only those elements needed for the computation of the next diagonal are preserved.

Method

Extrapolate

```
public double Extrapolate(double x)
```

Description

Extrapolates the convergence limit of a sequence.

Parameter

`x` – A `double` which specifies the next point in the original series.

Returns

A `double` containing the estimate of the limit of the series.

Example: The Epsilon Algorithm

The Epsilon algorithm is used to accelerate a series of partial sums of an infinite sum. The error shows that the Epsilon algorithm is effective.

```
using System;
using Imsl.Math;

public class EpsilonAlgorithmEx1
{
    static public void  Main(System.String[] arg)
    {
        EpsilonAlgorithm eps = new EpsilonAlgorithm();

        int n = 100;
        double sum = 0.0;
        for (int i = 1; i < n; i++)
        {
            sum += 1.0 / (i * i);
            eps.Extrapolate(sum);
        }
        sum += 1.0 / (n * n);
        double extrapolated = eps.Extrapolate(1.0 / (n * n));
        double expected = System.Math.PI * System.Math.PI / 6.0;
        double exError = expected - extrapolated;
        double sumError = expected - sum;
        Console.Out.WriteLine("Expected     {0:0.00000}", expected);
        Console.Out.WriteLine("Extrapolated {0:0.00000}   error = {1:0.00000}",
            extrapolated, exError);
        Console.Out.WriteLine("Sum          {0:0.00000}   error = {1:0.00000}",
            sum, sumError);
    }
}
```

Output

```
Expected     1.64493
```

```
Extrapolated 1.64278   error = 0.00216
Sum          1.63498   error = 0.00995
```

Chapter 11: Printing Functions

Types

PrintMatrix Class

`public class Imsl.Math.PrintMatrix`

Matrix printing utilities.

Constructors

PrintMatrix

`public PrintMatrix()`

Description

Creates an instance of the `PrintMatrix` class without a title and directs it to the default output stream.

Remarks

The matrix is printed without a title to `System.Console.Out`.

PrintMatrix

```
public PrintMatrix(System.IO.TextWriter writer)
```

Description

Creates an instance of the `PrintMatrix` class without a title and directs it to a specified output stream.

Parameter

`writer` – The `TextWriter` to which the matrix is to be written.

PrintMatrix

```
public PrintMatrix(string title)
```

Description

Creates a `PrintMatrix` object with a title directed to the default output stream.

Parameter

`title` – A `String` which specifies the title to be printed above the matrix.

Remarks

The matrix is printed without a title to `System.Console.Out`.

PrintMatrix

```
public PrintMatrix(System.IO.TextWriter writer, string title)
```

Description

Creates a `PrintMatrix` object with a title directed to a specified output stream.

Parameters

`writer` – A `String` which specifies the `TextWriter` to which the matrix is to be written.

`title` – The title to be printed above the matrix.

Methods

Print

```
virtual public void Print(string text)
```

Description

Prints a string.

Parameter

`text` – The `String` to be printed.

Remarks

This function can be overridden to print to something other than a `PrintStream`.

Print

```
virtual public void Print(object array)
```

Description

Prints an nRow by nColumn matrix with the default format.

Parameter

`array` – A two-dimensional, non-empty, rectangular `Object` array.

Print

```
virtual public void Print(Imsl.Math.PrintMatrixFormat pmf, object array)
```

Description

Prints an nRow by nColumn matrix with specified format.

Parameters

`pmf` – A `PrintMatrixFormat` matrix format.

`array` – A two-dimensional, non-empty, rectangular `Object` array.

PrintHTML

```
public void PrintHTML(Imsl.Math.PrintMatrixFormat pmf, object array, int nRows,
int nColumns)
```

Description

Prints an nRow by nColumn matrix with specified format for HTML output.

Parameters

`pmf` – A `PrintMatrixFormat` matrix format.

`array` – The Matrix to be printed.

`nRows` – An `int` specifying the number of rows in the matrix.

`nColumns` – An `int` specifying the number of columns in the matrix.

Println

```
virtual public void Println()
```

Description

Prints a newline.

Remarks

This function can be overridden to print to something other than a `PrintStream`.

SetColumnSpacing

```
public Imsl.Math.PrintMatrix SetColumnSpacing(int columnSpacing)
```

Description

Sets the number of spaces between columns.

Parameter

`columnSpacing` – An `int` specifying the number of spaces between columns.

Returns

The `PrintMatrix` object.

Remarks

The default value is 2.

SetEqualColumnWidths

```
public Imsl.Math.PrintMatrix SetEqualColumnWidths(bool equalColumnWidths)
```

Description

Force all of the columns to have the same width.

Parameter

`equalColumnWidths` – A `bool` which specifies that all column widths will be equal.

Returns

The `PrintMatrix` object.

SetMatrixType

```
public Imsl.Math.PrintMatrix SetMatrixType(Imsl.Math.PrintMatrix.MatrixType
matrixType)
```

Description

Set matrix type.

Parameter

`matrixType` – An `int` specifying the matrix type.

Returns

The `PrintMatrix` object.

Remarks

Values for `matrixType` are:

Value	Enumeration
0	MatrixType.Full
1	MatrixType.UpperTriangular
2	MatrixType.LowerTriangular
3	MatrixType.StrictUpperTriangular
4	MatrixType.StrictLowerTriangular

SetPageWidth

```
public Imsl.Math.PrintMatrix SetPageWidth(int pageWidth)
```

Description

Sets the page width.

Parameter

`pageWidth` – An `int` specifying the page width.

Returns

The `PrintMatrix` object.

Remarks

The default value is the largest possible integer.

SetTitle

```
public Imsl.Math.PrintMatrix SetTitle(string title)
```

Description

Sets the matrix title.

Parameter

`title` – A `String` specifying the title of the matrix.

Returns

The `PrintMatrix` object.

Example: Matrix and PrintMatrix

The 1 norm of a matrix is found using a method from the Matrix class. The matrix is printed using the PrintMatrix class.

```
using System;
using Imsl.Math;

public class PrintMatrixEx1
{
    public static void  Main(String[] args)
    {
        double nrm1;
        double[,] a = {{0.0, 1.0, 2.0, 3.0},
                       {4.0, 5.0, 6.0, 7.0},
                       {8.0, 9.0, 8.0, 1.0},
                       {6.0, 3.0, 4.0, 3.0}};

        //    Get the 1 norm of matrix a
        nrm1 = Matrix.OneNorm(a);

        //    Construct a PrintMatrix object with a title
        PrintMatrix p = new PrintMatrix("A Simple Matrix");

        //    Print the matrix and its 1 norm
        p.Print(a);
        Console.Out.WriteLine("The 1 norm of the matrix is " + nrm1);
```

```
    }
}
```

Output

```
A Simple Matrix
   0  1  2  3
0  0  1  2  3
1  4  5  6  7
2  8  9  8  1
3  6  3  4  3

The 1 norm of the matrix is 20
```

PrintMatrixFormat Class

`public class Imsl.Math.PrintMatrixFormat`

This class can be used to customize the actions of PrintMatrix.

By default, entries are formatted using the data type's `ToString` method.

See Also

Imsl.Math.PrintMatrix (p. 517)

Properties

FirstColumnNumber

`public int FirstColumnNumber {get; set; }`

Description

Turns on column labeling with index numbers and sets the index for the label of the first column.

Property Value

The number for the first column label.

Remarks

This is usually 0 or 1. The default is 0.

FirstRowNumber

`public int FirstRowNumber {get; set; }`

Description

Turns on row labeling with index numbers and sets the index for the label of the first row.

Property Value

The number for the first row label.

Remarks

This is usually 0 or 1. The default is 0.

NumberFormat

```
public string NumberFormat {get; set; }
```

Description

The NumberFormat to be used in formatting `double` and Complex (p. 475) entries.

Property Value

A `String` containing the format to be used in Complex entries.

Constructor

PrintMatrixFormat

```
public PrintMatrixFormat()
```

Description

Constructs a `PrintMatrixFormat` object.

Methods

Format

```
virtual public string Format(Imsl.Math.PrintMatrixFormat.FormatType type,
object entry, int row, int col, Imsl.Math.PrintMatrixFormat.ParsePosition pos)
```

Description

Returns a formatted string. This method is used by the methods Imsl.Math.PrintMatrix.Print(System.String) (p. 518) and Imsl.Math.PrintMatrix.PrintHTML(Imsl.Math.PrintMatrixFormat,System.Object,System.Int32,System.Int32) (p. 519). This method can be overridden to gain finer control over printing.

Parameters

`type` – The type of string requested. See `PrintMatrixFormat.FormatType` Enumeration.

`entry` – The entry to be formatted. This is only used if type equals Imsl.Math.PrintMatrixFormat.FormatType.Entry (p. 530). For other values of type, this can be set to null.

`row` – The (0-based) row number of the element to be formatted. This is -1 if there is no row number associated with this request.

`col` – The (0-based) column number of the element to be formatted. This is -1 if there is no column number associated with this request.

`pos` – A `ParsePosition` object used to indicate the alignment center of the return string. This is used only if `type` is Imsl.Math.PrintMatrixFormat.FormatType.Entry (p. 530).

Returns

A `String` to be put into the printed table.

Remarks

Note, if `type` is not `FormatType.Entry`, `pos` will be set based on the following criteria.

entry	behavior
`double`	The index is the position of the decimal point.
`int`	The index is the position of the end of the formatted integer.

See Also: Imsl.Math.PrintMatrixFormat.FormatType (p. 528)

SetColumnLabels

```
public void SetColumnLabels(string[] columnLabels)
```

Description

Turns on column labeling using the given labels.

Parameter

`columnLabels` – An array of `String`s to be used as column labels. If there are more columns than labels, the labels are reused.

SetNoColumnLabels

```
virtual public void SetNoColumnLabels()
```

Description

Turns off column labels.

SetNoRowLabels

```
virtual public void SetNoRowLabels()
```

Description

Turns off row labels.

Example: Matrix Formatting

A simple matrix is printed using the default format with the PrintMatrix class. The PrintMatrixFormat class is then used to change the default format.

```
using System;
using Imsl.Math;

public class PrintMatrixFormatEx1
{
    public static void  Main(String[] args)
    {
        double[,] a = {{0.0, 1.0, 2.0, 3.0},
                       {4.0, 5.0, 6.0, 7.0},
                       {8.0, 9.0, 8.0, 1.0},
                       {6.0, 3.0, 4.0, 3.0}};

        //    Construct a PrintMatrix object with a title
        PrintMatrix p = new PrintMatrix("A Simple Matrix");

        //    Print the matrix
        p.Print(a);

        //    Turn row and column labels off
        PrintMatrixFormat mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();

        //    Print the matrix
        p.Print(mf, a);
    }
}
```

Output

```
A Simple Matrix
   0  1  2  3
0  0  1  2  3
1  4  5  6  7
2  8  9  8  1
3  6  3  4  3

A Simple Matrix

0  1  2  3
4  5  6  7
8  9  8  1
6  3  4  3
```

Example: Matrix Formatting

A matrix is printed in CSV (comma separated value) format. This is done by overriding the `format` method of `PrintMatrixFormat` to add commas after all but the last number in each row.

```
using System;
using Imsl.Math;

public class PrintMatrixFormatEx2 : PrintMatrixFormat
```

```
{
    private int ncols;

    public PrintMatrixFormatEx2(int ncols)
    {
        this.ncols = ncols;
    }

    public override String Format(FormatType type, System.Object entry, int row,
      int col, ParsePosition pos)
    {
        String text = base.Format(type, entry, row, col, pos);
        if (type == FormatType.Entry)
        {
            if (col < ncols - 1)
                text += ",";
        }
        return text;
    }

    public static void Main(System.String[] args)
    {
        double[][] a = {
            new double[]{0.0, 1.0, 2.0},
            new double[]{4.0, 5.0, 6.0},
            new double[]{8.0, 9.0, 8.0},
            new double[]{6.0, 3.0, 4.0}
        };

        PrintMatrixFormat mf = new PrintMatrixFormatEx2(3);
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();

        //    Print the matrix
        new PrintMatrix().Print(mf, a);
    }
}
```

Output

```
0,  1,  2
4,  5,  6
8,  9,  8
6,  3,  4
```

PrintMatrixFormat.ParsePosition Class

```
public class Imsl.Math.PrintMatrixFormat.ParsePosition
```

Tracks the current position during parsing.

Property

Index

```
public int Index {get; set; }
```

Description

Current parse position.

Property Value

An `int` containing the current index position.

Constructor

ParsePosition

```
public ParsePosition(int index)
```

Description

Creates a `ParsePosition`.

Parameter

`index` – The intial position.

PrintMatrix.MatrixType Enumeration

```
public enumeration Imsl.Math.PrintMatrix.MatrixType
```

MatrixType indicates what part of the matrix is to be printed.

Fields

Full

```
public Imsl.Math.PrintMatrix.MatrixType Full
```

Description

Indicates that the full matrix is to be printed.

LowerTriangular

```
public Imsl.Math.PrintMatrix.MatrixType LowerTriangular
```

Description

Indicates that only the lower triangular elements of the matrix are to be printed. The matrix still must be a rectangular matrix.

StrictLowerTriangular

```
public Imsl.Math.PrintMatrix.MatrixType StrictLowerTriangular
```

Description

Indicates that only the strict lower triangular elements of the matrix are to be printed. The matrix still must be a rectangular matrix.

StrictUpperTriangular

```
public Imsl.Math.PrintMatrix.MatrixType StrictUpperTriangular
```

Description

Indicates that only the strict upper triangular elements of the matrix are to be printed. The matrix still must be a rectangular matrix.

UpperTriangular

```
public Imsl.Math.PrintMatrix.MatrixType UpperTriangular
```

Description

Indicates that only the upper triangular elements of the matrix are to be printed. The matrix still must be a rectangular matrix.

PrintMatrixFormat.FormatType Enumeration

```
public enumeration Imsl.Math.PrintMatrixFormat.FormatType
```

FormatType specifies the argument to format.

Fields

BeginColumnLabel

```
public Imsl.Math.PrintMatrixFormat.FormatType BeginColumnLabel
```

Description

Indicates that the formatting string for ending a column label is to be returned.

BeginColumnLabels

```
public Imsl.Math.PrintMatrixFormat.FormatType BeginColumnLabels
```

Description

Indicates that the formatting string for beginning a column label row is to be returned.

BeginEntry

```
public Imsl.Math.PrintMatrixFormat.FormatType BeginEntry
```

Description

Indicates that the formatted string for beginning an entry is to be returned.

BeginMatrix

```
public Imsl.Math.PrintMatrixFormat.FormatType BeginMatrix
```

Description

Indicates that the formatting string for beginning a matrix is to be returned.

BeginRow

```
public Imsl.Math.PrintMatrixFormat.FormatType BeginRow
```

Description

Indicates that the formatting string for beginning a row is to be returned.

BeginRowLabel

```
public Imsl.Math.PrintMatrixFormat.FormatType BeginRowLabel
```

Description

Indicates that the formatting string for beginning a row label is to be returned.

ColumnLabel

```
public Imsl.Math.PrintMatrixFormat.FormatType ColumnLabel
```

Description

Indicates that the formatted string for a given column label is to be returned.

EndColumnLabel

```
public Imsl.Math.PrintMatrixFormat.FormatType EndColumnLabel
```

Description

Indicates that the formatting string for ending a column label is to be returned.

EndColumnLabels

```
public Imsl.Math.PrintMatrixFormat.FormatType EndColumnLabels
```

Description

Indicates that the formatting string for ending a column label row is to be returned.

EndEntry

```
public Imsl.Math.PrintMatrixFormat.FormatType EndEntry
```

Description

Indicates that the formatted string for ending an entry is to be returned.

EndMatrix

```
public Imsl.Math.PrintMatrixFormat.FormatType EndMatrix
```

Description

Indicates that the formatting string for ending a matrix is to be returned.

EndRow

```
public Imsl.Math.PrintMatrixFormat.FormatType EndRow
```

Description

Indicates that the formatting string for ending a row is to be returned.

EndRowLabel

```
public Imsl.Math.PrintMatrixFormat.FormatType EndRowLabel
```

Description

Indicates that the formatting string for ending a row label is to be returned.

Entry

```
public Imsl.Math.PrintMatrixFormat.FormatType Entry
```

Description

Indicates that the formatted string for a given entry is to be returned.

RowLabel

```
public Imsl.Math.PrintMatrixFormat.FormatType RowLabel
```

Description

Indicates that the formatted string for a given row label is to be returned.

PrintMatrixFormat.ColumnLabelType Enumeration

```
public enumeration Imsl.Math.PrintMatrixFormat.ColumnLabelType
```

Type for column labels.

Fields

LabelNone

```
public Imsl.Math.PrintMatrixFormat.ColumnLabelType LabelNone
```

Description

Specifies no column labels will be displayed.

LabelNumber

```
public Imsl.Math.PrintMatrixFormat.ColumnLabelType LabelNumber
```

Description

Specifies column labels will be an array of `int`s.

LabelString

```
public Imsl.Math.PrintMatrixFormat.ColumnLabelType LabelString
```

Description

Specifies column labels will be an array of `String`s.

PrintMatrixFormat.RowLabelType Enumeration

```
public enumeration Imsl.Math.PrintMatrixFormat.RowLabelType
```

Type for row labels.

Fields

LabelNone

```
public Imsl.Math.PrintMatrixFormat.RowLabelType LabelNone
```

Description

Specifies no row labels will be displayed.

LabelNumber

```
public Imsl.Math.PrintMatrixFormat.RowLabelType LabelNumber
```

Description

Specifies row labels will be an array of `int`s.

Chapter 12: Basic Statistics

Types

Usage Notes

The methods/classes for the computations of basic statistics generally have relatively simple arguments. Most of the methods/classes in this chapter allow for missing values. Missing value codes can be set by using `Double.NaN`.

Several methods/classes in this chapter perform statistical tests. These methods in the classes generally return a "*p*-value" for the test. The *p*-value is between 0 and 1 and is the probability of observing data that would yield a test statistic as extreme or more extreme under the assumption of the null hypothesis. Hence, a small *p*-value is evidence for the rejection of the null hypothesis.

Summary Class

`public class Imsl.Stat.Summary`

Computes basic univariate statistics.

For the data in `x`, `Summary` computes the sample mean, variance, minimum, maximum, and other basic statistics. It also computes confidence intervals for the mean and variance if the sample is assumed to be from a normal population.

Missing values, that is, values equal to NaN (not a number), are excluded from the computations. The sum of the weights is used only in computing the mean (of course, then the weighted mean is used in computing the central moments). The definitions of some of the statistics are given below in terms of a single variable x. The i-th datum is x_i, with corresponding weight w_i. If weights are not specified, the w_i are identically one. The summation in each case is over the set of valid observations, based on the presence of missing values in the data.

Number of nonmissing observations,

$$n = \sum f_i$$

Mean,

$$\bar{x}_w = \frac{\sum f_i w_i x_i}{\sum f_i w_i}$$

Variance,

$$s_w^2 = \frac{\sum f_i w_i (x_i - \bar{x}_w)^2}{n-1}$$

Skewness,

$$\frac{\sum f_i w_i (x_i - \bar{x}_w)^3 / n}{[\sum f_i w_i (x_i - \bar{x}_w)^2 / n]^{3/2}}$$

Excess or Kurtosis,

$$\frac{\sum f_i w_i (x_i - \bar{x}_w)^4 / n}{[\sum f_i w_i (x_i - \bar{x}_w)^2 / n]^2} - 3$$

Minimum,

$$x_{\min} = \min(x_i)$$

Maximum,

$$x_{\max} = \max(x_i)$$

Constructor

Summary

```
public Summary()
```

Description

Constructs a new summary statistics object.

Methods

GetConfidenceMean

```
public double[] GetConfidenceMean(double p)
```

Description

Returns the confidence interval for the mean (assuming normality).

Parameter

`p` – A `double` which specifies the confidence level desired, usually 0.90, 0.95 or 0.99.

Returns

A `double` array of length 2 which contains the lower and upper confidence limits for the mean.

GetConfidenceVariance

```
public double[] GetConfidenceVariance(double p)
```

Description

Returns the confidence interval for the variance (assuming normality).

Parameter

`p` – A `double` which specifies the confidence level desired, usually 0.90, 0.95 or 0.99.

Returns

A `double` array of length 2 which contains the lower and upper confidence limits for the variance.

GetKurtosis

```
public double GetKurtosis()
```

Description

Returns the kurtosis.

Returns

A `double` representing the kurtosis.

GetKurtosis

```
static public double GetKurtosis(double[] x)
```

Description

Returns the kurtosis of the given data set.

Parameter

`x` – A `double` array containing the data set whose kurtosis is to be found.

Returns

A `double` which specifies the kurtosis of the given data set.

GetKurtosis

```
static public double GetKurtosis(double[] x, double[] weight)
```

Description

Returns the kurtosis of the given data set and associated weights.

Parameters

`x` – A `double` array containing the data set whose kurtosis is to be found.

`weight` – A `double` array containing the weights associated with the data points `x`.

Returns

A `double` which specifies the kurtosis of the given data set.

GetMaximum

```
public double GetMaximum()
```

Description

Returns the maximum.

Returns

A `double` representing the maximum.

GetMaximum

```
static public double GetMaximum(double[] x)
```

Description

Returns the maximum of the given data set.

Parameter

`x` – A `double` array containing the data set whose maximum is to be found.

Returns

A `double` which specifies the maximum of the given data set.

GetMaximum

```
static protected internal int GetMaximum(int[] x)
```

Description

Returns the maximum of the given data set.

Parameter

`x` – An `int` array containing the data set whose maximum is to be found.

Returns

An `int` which specifies the maximum of the given data set.

GetMean

```
public double GetMean()
```

Description

Returns the population mean.

Returns

A `double` representing the population mean.

GetMean

```
static public double GetMean(double[] x)
```

Description

Returns the mean of the given data set.

Parameter

`x` – A `double` array containing the data set whose mean is to be found.

Returns

A `double` which specifies the mean of the given data set.

GetMean

```
static public double GetMean(double[] x, double[] weight)
```

Description

Returns the mean of the given data set with associated weights.

Parameters

`x` – A `double` array containing the data set whose mean is to be found.

`weight` – A `double` array containing the weights associated with the data points `x`.

Returns

A `double` which specifies the mean of the given data set.

GetMedian

```
static public double GetMedian(double[] x)
```

Description

Returns the median of the given data set.

Parameter

`x` – A `double` array containing the data set whose median is to be found.

Returns

A `double` which specifies the median of the given data set.

GetMinimum

```
public double GetMinimum()
```

Description

Returns the minimum.

Returns

A `double` representing the minimum.

GetMinimum

```
static public double GetMinimum(double[] x)
```

Description

Returns the minimum of the given data set.

Parameter

`x` – A `double` array containing the data set whose minimum is to be found.

Returns

A `double` which specifies the minimum of the given data set.

GetMinimum

```
static protected internal int GetMinimum(int[] x)
```

Description

Returns the minimum of the given data set.

Parameter

`x` – An `int` array containing the data set whose minimum is to be found.

Returns

An `int` which specifies the minimum of the given data set.

GetMode

```
static public double GetMode(double[] x)
```

Description

Returns the mode of the given data set.

Parameter

`x` – A `double` array containing the data set whose mode is to be found.

Returns

A `double` which specifies the mode of the given data set.

Remarks

Ties are broken at random.

GetSampleStandardDeviation

```
public double GetSampleStandardDeviation()
```

Description

Returns the sample standard deviation.

Returns

A `double` representing the sample standard deviation.

GetSampleStandardDeviation

```
static public double GetSampleStandardDeviation(double[] x)
```

Description

Returns the sample standard deviation of the given data set.

Parameter

`x` – A `double` array containing the data set whose sample standard deviation is to be found.

Returns

A `double` which specifies the sample standard deviation of the given data set.

GetSampleStandardDeviation

```
static public double GetSampleStandardDeviation(double[] x, double[] weight)
```

Description

Returns the sample standard deviation of the given data set and associated weights.

Parameters

`x` – A `double` array containing the data set whose sample standard deviation is to be found.

`weight` – A `double` array containing the weights associated with the data points `x`.

Returns

A `double` which specifies the sample standard deviation of the given data set.

GetSampleVariance

```
public double GetSampleVariance()
```

Description

Returns the sample variance.

Returns

A `double` representing the sample variance.

GetSampleVariance

```
static public double GetSampleVariance(double[] x)
```

Description

Returns the sample variance of the given data set.

Parameter

`x` – A `double` array containing the data set whose sample variance is to be found.

Returns

A `double` which specifies the sample variance of the given data set.

GetSampleVariance

```
static public double GetSampleVariance(double[] x, double[] weight)
```

Description

Returns the sample variance of the given data set and associated weights.

Parameters

`x` – A `double` array containing the data set whose sample variance is to be found.

`weight` – A `double` array containing the weights associated with the data points `x`.

Returns

A `double` which specifies the sample variance of the given data set.

GetSkewness

```
public double GetSkewness()
```

Description

Returns the skewness.

Returns

A `double` representing the skewness.

GetSkewness

```
static public double GetSkewness(double[] x)
```

Description

Returns the skewness of the given data set.

Parameter

`x` – A `double` array containing the data set whose skewness is to be found.

Returns

A `double` which specifies the skewness of the given data set.

GetSkewness

```
static public double GetSkewness(double[] x, double[] weight)
```

Description

Returns the skewness of the given data set and associated weights.

Parameters

`x` – A `double` array containing the data set whose skewness is to be found.

`weight` – A `double` array containing the weights associated with the data points `x`.

Returns

A `double` which specifies the skewness of the given data set.

GetStandardDeviation

```
public double GetStandardDeviation()
```

Description

Returns the population standard deviation.

Returns

A `double` representing the population standard deviation.

GetStandardDeviation

```
static public double GetStandardDeviation(double[] x)
```

Description

Returns the population standard deviation of the given data set.

Parameter

`x` – A `double` array containing the data set whose standard deviation is to be found.

Returns

A `double` which specifies the population standard deviation of the given data set.

GetStandardDeviation

```
static public double GetStandardDeviation(double[] x, double[] weight)
```

Description

Returns the population standard deviation of the given data set and associated weights.

Parameters

`x` – A `double` array containing the data set whose standard deviation is to be found.

`weight` – A `double` array containing the weights associated with the data points `x`.

Returns

A `double` which specifies the population standard deviation of the given data set.

GetVariance

```
public double GetVariance()
```

Description

Returns the population variance.

Returns

A `double` representing the population variance.

GetVariance

```
static public double GetVariance(double[] x)
```

Description

Returns the population variance of the given data set.

Parameter

`x` – A `double` array containing the data set whose population variance is to be found.

Returns

A `double` which specifies the population variance of the given data set.

GetVariance

```
static public double GetVariance(double[] x, double[] weight)
```

Description

Returns the population variance of the given data set and associated weights.

Parameters

`x` – A `double` array containing the data set whose population variance is to be found.

`weight` – A `double` array containing the weights associated with the data points `x`.

Returns

A `double` which specifies the population variance of the given data set.

Update

```
public void Update(double x)
```

Description

Adds an observation to the `Summary` object.

Parameter

x – A double which specifies the data observation to be added.

Update

```
public void Update(double x, double weight)
```

Description

Adds an observation and associated weight to the Summary object.

Parameters

x – A double which specifies the data observation to be added.

weight – A double which specifies the weight associated with the observation.

Update

```
public void Update(double[] x)
```

Description

Adds a set of observations to the Summary object.

Parameter

x – A double array of data observations to be added.

Update

```
public void Update(double[] x, double[] weight)
```

Description

Adds a set of observations and associated weights to the Summary object.

Parameters

x – A double array of data observations to be added.

weight – A double array of weights associated with the observations.

Example: Summary Statistics

Summary statistics for a small data set are computed.

```
using System;
using Imsl.Stat;

public class SummaryEx1
{
    internal static readonly double[] data1 =
        new double[]{     3, 6.4, 2, 1.6, - 8, 12,
                        - 7, 6.4, 22, 1, 0, - 3.2};

    public static void  Main(String[] args)
    {
        Summary summary = new Summary();
        summary.Update(data1);
```

```
        Console.Out.WriteLine
            ("The minimum is " + summary.GetMinimum());
        Console.Out.WriteLine();

        Console.Out.WriteLine
            ("The maximum is " + summary.GetMaximum());
        Console.Out.WriteLine();

        Console.Out.WriteLine("The mean is " + summary.GetMean());
        Console.Out.WriteLine();

        Console.Out.WriteLine
            ("The variance is " + summary.GetVariance());
        Console.Out.WriteLine();

        Console.Out.WriteLine
            ("The sample variance is " + summary.GetSampleVariance());
        Console.Out.WriteLine();

        Console.Out.WriteLine("The standard deviation is " +
            summary.GetStandardDeviation());
        Console.Out.WriteLine();

        Console.Out.WriteLine
            ("The skewness is " + summary.GetSkewness());
        Console.Out.WriteLine();

        Console.Out.WriteLine
            ("The kurtosis is " + summary.GetKurtosis());
        Console.Out.WriteLine();

        double[] confmn = new double[2];
        confmn = summary.GetConfidenceMean(0.95);
        Console.Out.WriteLine("The confidence Mean is {" + confmn[0] +
            ", " + confmn[1] + "}");
        Console.Out.WriteLine();

        double[] confvr = new double[2];
        confvr = summary.GetConfidenceVariance(0.95);
        Console.Out.WriteLine("The confidence Variance is {" +
            confvr[0] + ", " + confvr[1] + "}");
    }
}
```

Output

```
The minimum is -8

The maximum is 22

The mean is 3.01666666666667

The variance is 61.7097222222222
```

```
The sample variance is 67.319696969697

The standard deviation is 7.85555359107315

The skewness is 0.863222413428583

The kurtosis is 0.567706048385121

The confidence Mean is {-2.19645146860124, 8.22978480193457}

The confidence Variance is {33.7826187272066, 194.068533277244}
```

Covariances Class

`public class Imsl.Stat.Covariances`

Computes the sample variance-covariance or correlation matrix.

Class `Covariances` computes estimates of correlations, covariances, or sums of squares and crossproducts for a data matrix *x*. Weights and frequencies are allowed but not required.

The means, (corrected) sums of squares, and (corrected) sums of crossproducts are computed using the method of provisional means. Let x_{ki} denote the mean based on *i* observations for the *k*-th variable, f_i denote the frequency of the *i*-th observation, w_i denote the weight of the *i*-th observations, and c_{jki} denote the sum of crossproducts (or sum of squares if $j = k$) based on *i* observations. Then the method of provisional means finds new means and sums of crossproducts as shown in the example below.

The means and crossproducts are initialized as follows:

$$x_{k0} = 0.0 \quad for\ k = 1, \ldots, p$$

$$c_{jk0} = 0.0 \ \ for\ j, k = 1, \ldots, p$$

where *p* denotes the number of variables. Letting $x_{k,i+1}$ denote the *k*-th variable of observation *i + 1*, each new observation leads to the following updates for x_{ki} and c_{jki} using the update constant r_{i+1}:

$$r_{i+1} = \frac{f_{i+1} w_{i+1}}{\sum_{l=1}^{i+1} f_l w_l}$$

$$\bar{x}_{k,\,i+1} = \bar{x}_{ki} + \left(x_{k,\,i+1} - \bar{x}_{ki}\right) r_{i+1}$$

$$c_{jk,\,i+1} = c_{jki} + f_{i+1} w_{i+1} \left(x_{j,\,i+1} - \bar{x}_{ji}\right) \left(x_{k,\,i+1} - \bar{x}_{ki}\right) (1 - r_{i+1})$$

The default value for weights and frequencies is 1. Means and variances are computed based on the valid data for each variable or, if required, based on all the valid data for each pair of variables.

Properties

MissingValueMethod

```
public int MissingValueMethod {get; set; }
```

Description

Sets the method used to exclude missing values in `x` from the computations.

Property Value

An `int` scalar indicating the method to use.

Remarks

The methods are as follows:

MissingValueMethod	Action
0	The exclusion is listwise, default. (The entire row of `x` is excluded if any of the values of the row is equal to the missing value code.)
1	Raw crossproducts are computed from all valid pairs and means, and variances are computed from all valid data on the individual variables. Corrected crossproducts, covariances, and correlations are computed using these quantities.
2	Raw crossproducts, means, and variances are computed as in the case of `MissingValueMethod` = 1. However, corrected crossproducts and covariances are computed only from the valid pairs of data. Correlations are computed using these covariances and the variances from all valid data.
3	Raw crossproducts, means, variances, and covariances are computed as in the case of `MissingValueMethod` = 2. Correlations are computed using these covariances, but the variances used are computed from the valid pairs of data.

`Double.NaN` is interpreted as the missing value code.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

NumRowMissing

```
public int NumRowMissing {get; }
```

Description

Returns the total number of observations that contain any missing values (`Double.NaN`). Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

An `int` scalar containing the total number of observations that contain any missing values (`Double.NaN`).

Observations

```
public int Observations {get; }
```

Description

Returns the sum of the frequencies. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

An `int` scalar containing the sum of the frequencies.

Remarks

If `MissingValueMethod` = 0, observations with missing values are not included. Otherwise, all observations are included except for observations with missing values for the weight or the frequency.

SumOfWeights

```
public double SumOfWeights {get; }
```

Description

Returns the sum of the weights of all observations. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

A `double` scalar containing the sum of the weights of all observations.

Remarks

If `MissingValueMethod` = 0, observations with missing values are not included. Otherwise, all observations are included except for observations with mssing values for the weight or the frequency.

Constructor

Covariances

```
public Covariances(double[,] x)
```

Description

Constructor for `Covariances`.

Parameter

`x` – A `double` matrix containing the data.

Exception

`System.ArgumentException` is thrown if `x.GetLength(0)`, and `x.GetLength(1)` are equal to 0

Methods

Compute

```
public double[,] Compute(Imsl.Stat.Covariances.MatrixType matrixType)
```

Description

Computes the matrix.

Parameter

`matrixType` – A `Covariances.MatrixType` indicating the type of matrix to compute.

Returns

A `double` matrix containing computed result.

Exceptions

`Imsl.Stat.MoreObsDelThanEnteredException` is thrown if more observations are being deleted from “variance-covariance” matrix than were originally entered.

The corresponding row,column of the incidence matrix is less than zero.

`Imsl.Stat.DiffObsDeletedException` is thrown if different observations are being deleted than were originally entered

GetIncidenceMatrix

```
public int[,] GetIncidenceMatrix()
```

Description

Returns the incidence matrix. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

An `int` matrix containing the incidence matrix.

Remarks

If `MissingValueMethod` is 0, incidence matrix is 1 x 1 and contains the number of valid observations; otherwise, incidence matrix is `x.GetLength(1)` x `x.GetLength(1)` and contains the number of pairs of valid observations used in calculating the crossproducts for covariance.

GetMeans

```
public double[] GetMeans()
```

Description

Returns the means of the variables in `x`.

Returns

A `double` array containing the means of the variables in `x`.

Remarks

The components of the array correspond to the columns of `x`. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

SetFrequencies

```
public void SetFrequencies(double[] frequencies)
```

Description

The frequency for each observation.

Parameter

`frequencies` – A `double` array of size `x.GetLength(0)` containing the frequency for each observation.

Remarks

By default, `frequencies[]` = 1.

SetWeights

```
public void SetWeights(double[] weights)
```

Description

Sets the weight for each observation.

Parameter

`weights` – A `double` array of size `x.GetLength(0)` containing the weight for each observation.

Remarks

By default, `weights[]` = 1.

Example: Covariances

This example illustrates the use of Covariances class for the first 50 observations in the Fisher iris data (Fisher 1936). Note that the first variable is constant over the first 50 observations.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class CovariancesEx1
{
    public static void  Main(String[] args)
    {
        double[,] x = {{1.0, 5.1, 3.5, 1.4, .2},
                       {1.0, 4.9, 3.0, 1.4, .2},
                       {1.0, 4.7, 3.2, 1.3, .2},
                       {1.0, 4.6, 3.1, 1.5, .2},
                       {1.0, 5.0, 3.6, 1.4, .2},
                       {1.0, 5.4, 3.9, 1.7, .4},
                       {1.0, 4.6, 3.4, 1.4, .3},
                       {1.0, 5.0, 3.4, 1.5, .2},
                       {1.0, 4.4, 2.9, 1.4, .2},
                       {1.0, 4.9, 3.1, 1.5, .1},
                       {1.0, 5.4, 3.7, 1.5, .2},
                       {1.0, 4.8, 3.4, 1.6, .2},
                       {1.0, 4.8, 3.0, 1.4, .1},
                       {1.0, 4.3, 3.0, 1.1, .1},
                       {1.0, 5.8, 4.0, 1.2, .2},
                       {1.0, 5.7, 4.4, 1.5, .4},
                       {1.0, 5.4, 3.9, 1.3, .4},
                       {1.0, 5.1, 3.5, 1.4, .3},
                       {1.0, 5.7, 3.8, 1.7, .3},
                       {1.0, 5.1, 3.8, 1.5, .3},
                       {1.0, 5.4, 3.4, 1.7, .2},
                       {1.0, 5.1, 3.7, 1.5, .4},
                       {1.0, 4.6, 3.6, 1.0, .2},
                       {1.0, 5.1, 3.3, 1.7, .5},
                       {1.0, 4.8, 3.4, 1.9, .2},
                       {1.0, 5.0, 3.0, 1.6, .2},
                       {1.0, 5.0, 3.4, 1.6, .4},
                       {1.0, 5.2, 3.5, 1.5, .2},
                       {1.0, 5.2, 3.4, 1.4, .2},
                       {1.0, 4.7, 3.2, 1.6, .2},
                       {1.0, 4.8, 3.1, 1.6, .2},
                       {1.0, 5.4, 3.4, 1.5, .4},
                       {1.0, 5.2, 4.1, 1.5, .1},
                       {1.0, 5.5, 4.2, 1.4, .2},
                       {1.0, 4.9, 3.1, 1.5, .2},
                       {1.0, 5.0, 3.2, 1.2, .2},
                       {1.0, 5.5, 3.5, 1.3, .2},
```

```
                            {1.0, 4.9, 3.6, 1.4, .1},
                            {1.0, 4.4, 3.0, 1.3, .2},
                            {1.0, 5.1, 3.4, 1.5, .2},
                            {1.0, 5.0, 3.5, 1.3, .3},
                            {1.0, 4.5, 2.3, 1.3, .3},
                            {1.0, 4.4, 3.2, 1.3, .2},
                            {1.0, 5.0, 3.5, 1.6, .6},
                            {1.0, 5.1, 3.8, 1.9, .4},
                            {1.0, 4.8, 3.0, 1.4, .3},
                            {1.0, 5.1, 3.8, 1.6, .2},
                            {1.0, 4.6, 3.2, 1.4, .2},
                            {1.0, 5.3, 3.7, 1.5, .2},
                            {1.0, 5.0, 3.3, 1.4, .2}};
        Covariances co = new Covariances(x);

        PrintMatrix pm =
            new PrintMatrix("Sample Variances-covariances Matrix");
        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.0000";
        pm.SetMatrixType(PrintMatrix.MatrixType.UpperTriangular);

        pm.Print(pmf,
            co.Compute(Covariances.MatrixType.VarianceCovariance));
    }
}
```

Output

```
    Sample Variances-covariances Matrix
     0       1       2       3       4
0  0.0000  0.0000  0.0000  0.0000  0.0000
1           0.1242  0.0992  0.0164  0.0103
2                   0.1437  0.0117  0.0093
3                           0.0302  0.0061
4                                   0.0111
```

Covariances.MatrixType Enumeration

`public enumeration Imsl.Stat.Covariances.MatrixType`

Specifies the type of matrix to be computed.

Fields

CorrectedSSCP

```
public Imsl.Stat.Covariances.MatrixType CorrectedSSCP
```

Description

Indicates corrected sums of squares and crossproducts matrix.

Correlation

```
public Imsl.Stat.Covariances.MatrixType Correlation
```

Description

Indicates correlation matrix.

StdevCorrelation

```
public Imsl.Stat.Covariances.MatrixType StdevCorrelation
```

Description

Indicates correlation matrix except for the diagonal elements which are the standard deviations.

VarianceCovariance

```
public Imsl.Stat.Covariances.MatrixType VarianceCovariance
```

Description

Indicates variance-covariance matrix.

PartialCovariances Class

```
public class Imsl.Stat.PartialCovariances
```

Class `PartialCovariances` computes the partial covariances or partial correlations from an input covariance or correlation matrix.

If the “independent” variables (the linear “effect” of the independent variables is removed in computing the partial covariances/correlations) are linearly related to one another, `PartialCovariances` detects the linearity and eliminates one or more of the independent variables from the list of independent variables. The number of variables eliminated, if any, can be determined from the property `PartialDegreesOfFreedom`.

Given a covariance or correlation matrix Σ partitioned as

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

class `PartialCovariances` computes the partial covariances (of the standardized variables if Σ is a correlation matrix) as

$$\Sigma_{22|1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$$

A positive semidefinite solver is used to compute $\Sigma_{11}^{-1}\Sigma_{12}$.

If partial correlations are desired, these are computed as

$$P_{22|1} = \left[\text{diag}(\Sigma_{22|1})\right]^{-1/2} \Sigma_{22|1} \left[\text{diag}(\Sigma_{22|1})\right]^{-1/2}$$

where $\text{diag}(\Sigma)$ denotes the matrix containing the diagonal of its argument along its diagonal with zeros off the diagonal. If Σ_{11} is singular, then as many variables as required are deleted from Σ_{11} (and Σ_{12}) in order to eliminate the linear dependencies. The computations then proceed as above.

The p-value for a partial covariance tests the null hypothesis $H_0 : \sigma_{ij|1} = 0$, where $\sigma_{ij|1}$ is the (i, j) element in matrix $\Sigma_{22|1}$. The p-value for a partial correlation tests the null hypothesis $H_0 : \rho_{ij|1} = 0$, where $\rho_{ij|1}$ is the (i, j) element in matrix $P_{22|1}$. The p-values are returned by `GetPValues`. If the degrees of freedom for `sigma`, `df`, is not known, the resulting p-values may be useful for comparison, but they should not by used as an approximation to the actual probabilities.

Properties

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

PartialDegreesOfFreedom

```
public int PartialDegreesOfFreedom {get; }
```

Description

The degrees of freedom in the test that the partial correlation (covariance) is zero.

Remarks

This will usually be `df` minus the rank of the the independent variables. number of independent variables, but will be greater than this value if the independent variables are computationally linearly related. If this value is not greater than one then there are not enough degrees of freedom for hypothesis testing. A warning is also issued in this case.

Constructors

PartialCovariances

```
public PartialCovariances(int nIndependent, double[,] sigma, int df)
```

Description

Creates a `PartialCovariances` object from a covariance or correleation matrix with a the independent variables in the initial columns and the dependent variables in the final columns.

Parameters

`nIndependent` – is the number of "independent" variables to be used in the partial covariances/correlations. The partial covariances/correlations are the covariances/correlations between the dependent variables after removing the linear effect of the independent variables.

`sigma` – is a correlation or covariance matrix. The rows/columns must be ordered such that the first `nIndependent` rows/columns contain the independent variables, followed by the row/columns containing the dependent variables. The matrix must always be symmetric, positive semidefinite.

`df` – is an `int` indicating the number of degrees of freedom associated with the input matrix. If the number of degrees of freedom in the matrix varies from element to element, then a conservative choice for `df` is the minimum degrees of freedom for all elements in the matrix. The value of `df` must be at least one.

Exceptions

`Imsl.Stat.InvalidMatrixException` is thrown if a computed correlation is greater than one for some pair of variables.

`Imsl.Stat.InvalidPartialCorrelationException` is thrown if a computed partial correlation is greater than one for some pair of variables. The input matrix to the constructor was not positive semidefinite.

PartialCovariances

```
public PartialCovariances(int[] xIndices, double[,] sigma, int df)
```

Description

Creates a `PartialCovariances` object from a covariance or correleation matrix with a mix of dependent and independent variables.

Parameters

`xIndices` – is an array containing values indicating the status of the variable. If the *i*-th entry is 0 then the *i*-th column of the matrix contains a dependent variable. If the *i*-th entry is positive then the *i*-th column of the matrix contains an independent variable. If the *i*-th entry is negative then the *i*-th column of the matrix is not used in the analysis.

`sigma` – is a correlation or covariance matrix. The number of rows and columns in `sigma` must equal the length of the array `xIndices`. The matrix must always be symmetric, positive semidefinite.

`df` – is an `int` indicating the number of degrees of freedom associated with the input matrix. If the number of degrees of freedom in the matrix varies from element to element, then a conservative choice for `df` is the minimum degrees of freedom for all elements in the matrix. The value of `df` must be at least one.

Exceptions

`Imsl.Stat.InvalidMatrixException` is thrown if a computed correlation is greater than one for some pair of variables.

`Imsl.Stat.InvalidPartialCorrelationException` is thrown if a computed partial correlation is greater than one for some pair of variables. The input matrix to the constructor was not positive semidefinite.

Methods

GetPartialCorrelationMatrix

```
public double[,] GetPartialCorrelationMatrix()
```

Description

Returns the partial correlation matrix.

Returns

The partial correlation matrix.

Remarks

This is valid only if the input to the constructor was a correlation matrix.

GetPartialCovarianceMatrix

```
public double[,] GetPartialCovarianceMatrix()
```

Description

Returns the partial covariance matrix.

Returns

The partial covariance matrix.

Remarks

This is valid only if the input to the constructor was a covariance matrix.

GetPValues

```
public double[,] GetPValues()
```

Description

Calculates the *p*-values for testing the null hypothesis that the associated partial covariance/correlation is zero.

Returns

A square array of type `double` containing the *p*-values. The order of the matrix equals the number of dependent variables.

If the partial degrees of freedom is not greater than one then there are not enough degrees of freedom for hypothesis testing. The returned matrix will be set to all NaN values in this case. A warning is also issued in this case.

Remarks

It is assumed that the observations from which `sigma` was computed follows a multivariate normal distribution and that each element in `sigma` has `df` degrees of freedom.

Example 1

This example computes partial covariances, scaled from a nine-variable correlation matrix originally given by Emmett (1949). The first three rows and columns contain the independent variables and the final six rows and columns contain the dependent variables.

```
using System;
using PartialCovariances = Imsl.Stat.PartialCovariances;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class PartialCovariancesEx1
{

   static public void Main(String[] arg)
   {
      double[,] sigma =
      {
         {6.300, 3.050, 1.933, 3.365, 1.317, 2.293, 2.586, 1.242, 4.363},
         {3.050, 5.400, 2.170, 3.346, 1.473, 2.303, 2.274, 0.750, 4.077},
         {1.933, 2.170, 3.800, 1.970, 0.798, 1.062, 1.576, 0.487, 2.673},
         {3.365, 3.346, 1.970, 8.100, 2.983, 4.828, 2.255, 0.925, 3.910},
         {1.317, 1.473, 0.798, 2.983, 2.300, 2.209, 1.039, 0.258, 1.687},
         {2.293, 2.303, 1.062, 4.828, 2.209, 4.600, 1.427, 0.768, 2.754},
         {2.586, 2.274, 1.576, 2.255, 1.039, 1.427, 3.200, 0.785, 3.309},
         {1.242, 0.750, 0.487, 0.925, 0.258, 0.768, 0.785, 1.300, 1.458},
         {4.363, 4.077, 2.673, 3.910, 1.687, 2.754, 3.309, 1.458, 7.400}
      };
      int nIndependent = 3;
      int df = 30;
```

```
      PartialCovariances pcov = new PartialCovariances(nIndependent, sigma, df);

      double[,] covar = pcov.GetPartialCovarianceMatrix();
      PrintMatrixFormat pmf = new PrintMatrixFormat();
      pmf.NumberFormat = "0.0000";
      new PrintMatrix("Partial Covariances").Print(pmf,covar);

      int pdf = pcov.PartialDegreesOfFreedom;
      Console.Out.WriteLine("Partial Degrees of Freedom " + pdf);
      Console.Out.WriteLine();

      double[,] pvalues = pcov.GetPValues();
      new PrintMatrix("p Values").Print(pmf,pvalues);
   }
}
```

Output

```
                Partial Covariances
     0       1       2       3       4       5
0  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
1  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
2  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
3  0.0000  0.0000  0.0000  5.4953  1.8955  3.0836
4  0.0000  0.0000  0.0000  1.8955  1.8407  1.4764
5  0.0000  0.0000  0.0000  3.0836  1.4764  3.4026

Partial Degrees of Freedom 27

                    p Values
     0       1       2       3       4       5
0  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
1  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
2  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
3  0.0000  0.0000  0.0000  0.0000  0.0008  0.0000
4  0.0000  0.0000  0.0000  0.0008  0.0000  0.0009
5  0.0000  0.0000  0.0000  0.0000  0.0009  0.0000
```

Example 2

This example computes partial correlations from a 9 variable correlation matrix originally given by Emmett (1949). The partial correlations between the remaining variables, after adjusting for variables 1, 3 and 9, are computed. Note in the output that the row and column labels are numbers, not variable numbers. The corresponding variable numbers would be 2, 4, 5, 6, 7 and 8, respectively.

```
using System;
using PartialCovariances = Imsl.Stat.PartialCovariances;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class PartialCovariancesEx2
```

```
{

   static public void Main(String[] arg)
   {
      double[,] sigma =
      {
      { 1.000, 0.523, 0.395, 0.471, 0.346, 0.426, 0.576, 0.434, 0.639 },
      { 0.523, 1.000, 0.479, 0.506, 0.418, 0.462, 0.547, 0.283, 0.645 },
      { 0.395, 0.479, 1.000, 0.355, 0.270, 0.254, 0.452, 0.219, 0.504 },
      { 0.471, 0.506, 0.355, 1.000, 0.691, 0.791, 0.443, 0.285, 0.505 },
      { 0.346, 0.418, 0.270, 0.691, 1.000, 0.679, 0.383, 0.149, 0.409 },
      { 0.426, 0.462, 0.254, 0.791, 0.679, 1.000, 0.372, 0.314, 0.472 },
      { 0.576, 0.547, 0.452, 0.443, 0.383, 0.372, 1.000, 0.385, 0.680 },
      { 0.434, 0.283, 0.219, 0.285, 0.149, 0.314, 0.385, 1.000, 0.470 },
      { 0.639, 0.645, 0.504, 0.505, 0.409, 0.472, 0.680, 0.470, 1.000 }
      };

      int[] xIndices = { 1, 0, 1, 0, 0, 0, 0, 0, 1 };
      int df = 30;

      PartialCovariances pcov = new PartialCovariances(xIndices, sigma, df);

      double[,] correl = pcov.GetPartialCorrelationMatrix();
      PrintMatrixFormat pmf = new PrintMatrixFormat();
      pmf.NumberFormat = "0.0000";
      new PrintMatrix("Partial Correlations").Print(pmf, correl);

      double[,] pValues = pcov.GetPValues();
      new PrintMatrix("P-Values").Print(pmf, pValues);
   }
}
```

Output

```
                Partial Correlations
      0       1       2       3       4       5
0   1.0000  0.2235  0.1936  0.2113  0.1253 -0.0610
1   0.2235  1.0000  0.6054  0.7198  0.0919  0.0249
2   0.1936  0.6054  1.0000  0.5977  0.1230 -0.0766
3   0.2113  0.7198  0.5977  1.0000  0.0349  0.0856
4   0.1253  0.0919  0.1230  0.0349  1.0000  0.0622
5  -0.0610  0.0249 -0.0766  0.0856  0.0622  1.0000

                    P-Values
     0       1       2       3       4       5
0  0.0000  0.2525  0.3232  0.2801  0.5249  0.7576
1  0.2525  0.0000  0.0006  0.0000  0.6417  0.9000
2  0.3232  0.0006  0.0000  0.0007  0.5328  0.6982
3  0.2801  0.0000  0.0007  0.0000  0.8602  0.6650
4  0.5249  0.6417  0.5328  0.8602  0.0000  0.7532
5  0.7576  0.9000  0.6982  0.6650  0.7532  0.0000
```

NormOneSample Class

`public class Imsl.Stat.NormOneSample`

Computes statistics for mean and variance inferences using a sample from a normal population.

The statistics for mean and variance inferences are computed by using a sample from a normal population, including methods for the confidence intervals and tests for both mean and variance. The definitions of mean and variance are given below. The summation in each case is over the set of valid observations, based on the presence of missing values in the data.

Property `Mean`, returns value

$$\bar{x} = \frac{\sum x_i}{n}$$

$$\Delta_s^d Z_t$$

Property `StdDev`, returns value

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

The property `TTest` returns the *t* statistic for the two-sided test concerning the population mean which is given by

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where *s* and $\bar{x}$ are given above. This quantity has a *T* distribution with *n - 1* degrees of freedom. The property `TTestDF` returns the degree of freedom.

Property `ChiSquaredTest` returns the chi-squared statistic for the two-sided test concerning the population variance which is given by

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

where *s* is given above. This quantity has a χ^2 distribution with *n - 1* degrees of freedom. Property `ChiSquaredTestDF` returns the degrees of freedom.

Properties

ChiSquaredTest

```
public double ChiSquaredTest {get; }
```

Description

Returns the test statistic associated with the chi-squared test for variances.

Property Value

A `double` containing the test statistic for the chi-squared test.

Remarks

The chi-squared test is a test of the hypothesis $\omega^2 = \omega_0^2$ where ω_0^2 is the null hypothesis value as described in `ChiSquaredTestNull`.

ChiSquaredTestDF

```
public int ChiSquaredTestDF {get; }
```

Description

Returns the degrees of freedom associated with the chi-squared test for variances.

Property Value

A `int` containing the degrees of freedom for the chi-squared test.

Remarks

The chi-squared test is a test of the hypothesis $\omega^2 = \omega_0^2$ where ω_0^2 is the null hypothesis value as described in `ChiSquaredTestNull`.

ChiSquaredTestNull

```
public double ChiSquaredTestNull {get; set; }
```

Description

The null hypothesis value for the chi-squared test.

Property Value

A `double` containing the null hypothesis value for the chi-squared test.

Remarks

The default is 1.0.

ChiSquaredTestP

```
public double ChiSquaredTestP {get; }
```

Description

Returns the probability of a larger chi-squared associated with the chi-squared test for variances.

Property Value

A `double` containing the probability of a larger chi-squared for the chi-squared test for variances.

Remarks

The chi-squared test is a test of the hypothesis $\omega^2 = \omega_0^2$ where ω_0^2 is the null hypothesis value as described in `ChiSquaredTestNull`.

ConfidenceMean

```
public double ConfidenceMean {get; set; }
```

Description

The confidence level (in percent) for a two-sided interval estimate of the mean.

Property Value

A `double` containing the confidence level of the mean.

Remarks

`ConfidenceMean` must be between 0.0 and 1.0 and is often 0.90, 0.95 or 0.99. For a one-sided confidence interval with confidence level `c` less than 50 percent, set `ConfidenceMean` =1.0- 2.0*(`c`/100). This effectively gives the one-sided confidence interval for both `c`% and (100-`c`)%. For example, for a one-sided t-test with confidence level of 40, set `ConfidenceMean`=.2 . This means that 40% of the distribution is lower than `LowerCIMean` and 40% of the distribution is greater than `UpperCIMean`. It also means that 60% of the distribution is greater than `LowerCIMean` and 60% is lower than `UpperCIMean`. If the confidence mean is not specified, a 95-percent confidence interval is computed.

ConfidenceVariance

```
public double ConfidenceVariance {get; set; }
```

Description

The confidence level (in percent) for two-sided interval estimate of the variances.

Property Value

A `double` containing the confidence level of the variance.

Remarks

`ConfidenceVariance` must be between 0.0 and 1.0 and is often 0.90, 0.95 or 0.99. For a one-sided confidence interval with confidence level c (at least 50 percent), set `ConfidenceVariance` = 1.0 - 2.0 * (1.0 - `c`). If the confidence mean is not specified, a 95-percent confidence interval is computed.

LowerCIMean

```
public double LowerCIMean {get; }
```

Description

Returns the lower confidence limit for the mean.

Property Value

A `double` containing the lower confidence limit for the mean.

LowerCIVariance

```
public double LowerCIVariance {get; }
```

Description

Returns the lower confidence limits for the variance.

Property Value

A `double` containing the lower confidence limits for the variance.

Mean

```
public double Mean {get; }
```

Description

Returns the mean of the sample.

Property Value

A `double` containing the mean of `x`.

StdDev

```
public double StdDev {get; }
```

Description

Returns the standard deviation of the sample.

Property Value

A `double` containing the standard deviation of the sample.

TTest

```
public double TTest {get; }
```

Description

Returns the test statistic associated with the *t* test.

Property Value

A `double` containing the test statistic for the *t* test.

Remarks

The *t* test is a test, against a two-sided alternative, of the null hypothesis value described in `TTestNull`.

TTestDF

```
public int TTestDF {get; }
```

Description

Returns the degrees of freedom associated with the *t* test for the mean.

Property Value

A `int` containing the degrees of freedom for the *t* test.

Remarks

The *t* test is a test, against a two-sided alternative, of the null hypothesis value described in `TTestNull`.

TTestNull

```
public double TTestNull {get; set; }
```

Description

Sets the Null hypothesis value for *t* test for the mean.

Property Value

A `double` containing the hypothesis value.

Remarks

`TTestNull` = 0.0 by default.

TTestP

```
public double TTestP {get; }
```

Description

Returns the probability associated with the *t* test of a larger *t* in absolute value.

Property Value

A `double` containing the probability for the *t* test.

Remarks

The *t* test is a test, against a two-sided alternative, of the null hypothesis value described in `TTestNull`.

UpperCIMean

```
public double UpperCIMean {get; }
```

Description

Returns the upper confidence limit for the mean.

Property Value

A `double` containing the upper confidence limit for the mean.

UpperCIVariance

```
public double UpperCIVariance {get; }
```

Description

Returns the upper confidence limits for the variance.

Property Value

A `double` the upper confidence limits for the variance.

Constructor

NormOneSample

```
public NormOneSample(double[] x)
```

Description

Constructor to compute statistics for mean and variance inferences using a sample from a normal population.

Parameter

x – A one-dimension `double` array containing the observations.

Example 1: NormOneSample

This example uses data from Devore (1982, p335), which is based on data published in the *Journal of Materials*. There are 15 observations. The hypothesis $H0 : \mu = 20.0$ is tested. The extremely large t value and the correspondingly small p-value provide strong evidence to reject the null hypothesis.

```
using System;
using Imsl.Stat;

public class NormOneSampleEx1
{
    public static void  Main(String[] args)
    {

        double mean, stdev, lomean, upmean;
        int df;
        double t, pvalue;
        double[] x = new double[]{   26.7, 25.8, 24.0, 24.9, 26.4,
                                     25.9, 24.4, 21.7, 24.1, 25.9,
                                     27.3, 26.9, 27.3, 24.8, 23.6};

        /* Perform Analysis*/

        NormOneSample n1samp = new NormOneSample(x);

        mean = n1samp.Mean;
        stdev = n1samp.StdDev;
        lomean = n1samp.LowerCIMean;
        upmean = n1samp.UpperCIMean;
        n1samp.TTestNull = 20.0;
        df = n1samp.TTestDF;
        t = n1samp.TTest;
        pvalue = n1samp.TTestP;


        /* Print results */

        Console.Out.WriteLine("Sample Mean = " + mean);
        Console.Out.WriteLine("Sample Standard Deviation = " + stdev);
        Console.Out.WriteLine
            ("95% CI for the mean is " + lomean + "   " + upmean);
        Console.Out.WriteLine("T Test results");
        Console.Out.WriteLine("df = " + df);
        Console.Out.WriteLine("t = " + t);
        Console.Out.WriteLine("pvalue = " + pvalue);
        Console.Out.WriteLine("");

        /* CI variance */
        double ciLoVar = n1samp.LowerCIVariance;
        double ciUpVar = n1samp.UpperCIVariance;
        Console.Out.WriteLine
```

```
            ("CI variance is " + ciLoVar + "    " + ciUpVar);
        /*chi-squared test */
        df = n1samp.ChiSquaredTestDF;
        t = n1samp.ChiSquaredTest;
        pvalue = n1samp.ChiSquaredTestP;
        Console.Out.WriteLine("Chi-squared Test results");
        Console.Out.WriteLine("Chi-squared df = " + df);
        Console.Out.WriteLine("Chi-squared t = " + t);
        Console.Out.WriteLine("Chi-squared pvalue = " + pvalue);
    }
}
```

Output

```
Sample Mean = 25.3133333333333
Sample Standard Deviation = 1.57881812336528
95% CI for the mean is 24.4390129997097   26.187653666957
T Test results
df = 14
t = 13.0340861992294
pvalue = 3.2147173398634E-09

CI variance is 1.33609260499922    6.19986346723949
Chi-squared Test results
Chi-squared df = 14
Chi-squared t = 34.8973333333333
Chi-squared pvalue = 0.00152231761418218
```

NormTwoSample Class

`public class Imsl.Stat.NormTwoSample`

Computes statistics for mean and variance inferences using samples from two normal populations.

Class `NormTwoSample` computes statistics for making inferences about the means and variances of two normal populations, using independent samples in `x1` and `x2`. For inferences concerning parameters of a single normal population, see class `NormOneSample`.

Let μ_1 and σ_1^2 be the mean and variance of the first population, and let μ_2 and σ_2^2 be the corresponding quantities of the second population. The function contains test confidence intervals for difference in means, equality of variances, and the pooled variance.

The means and variances for the two samples are as follows:

$$\bar{x_1} = \left(\sum x_{1i}/n_1\right), \qquad \bar{x_2} = \left(\sum x_{2i}\right)/n_2$$

and

$$s_1^2 = \sum (x_{1i} - \bar{x}_1)^2 / (n_1 - 1), \quad s_2^2 = \sum (x_{2i} - \bar{x}_2)^2 / (n_2 - 1)$$

Inferences about the Means

The test that the difference in means equals a certain value, for example, μ_0, depends on whether or not the variances of the two populations can be considered equal. If the variances are equal and `meanHypothesis` equals 0, the test is the two-sample *t*-test, which is equivalent to an analysis-of-variance test. The pooled variance for the difference-in-means test is as follows:

$$s^2 = \frac{(n_1 - 1) s_1 + (n_2 - 1) s_2}{n_1 + n_2 - 2}$$

The *t* statistic is as follows:

$$t = \frac{\bar{x}_1 - \bar{x}_2 - \mu_0}{s\sqrt{(1/n_1) + (1/n_2)}}$$

Also, the confidence interval for the difference in means can be obtained by first assigning the unequal variances flag to false. This can be done by setting the `UnequalVariances` property. The confidence interval can then be obtained by the `LowerCIDiff` and `UpperCIDiff` properties.

If the population variances are not equal, the ordinary *t* statistic does not have a *t* distribution and several approximate tests for the equality of means have been proposed. (See, for example, Anderson and Bancroft 1952, and Kendall and Stuart 1979.) One of the earliest tests devised for this situation is the Fisher-Behrens test, based on Fisher's concept of fiducial probability. A procedure used in the `TTest`, `LowerCIDiff` and `UpperCIDiff` properties assuming unequal variances are specified is the Satterthwaite's procedure, as suggested by H.F. Smith and modified by F.E. Satterthwaite (Anderson and Bancroft 1952, p. 83). Set `UnequalVariances` true to obtain results assuming unequal variances.

The test statistic is

$$t' = (\bar{x}_1 - \bar{x}_2 - \mu_0) / s_d$$

where

$$s_d = \sqrt{(s_1^2/n_1) + (s_2^2/n_2)}$$

Under the null hypothesis of $\mu_1 - \mu_2 = c$, this quantity has an approximate *t* distribution with degrees of freedom `df`, given by the following equation:

$$\mathrm{df} = \frac{s_d^4}{\frac{\left(s_1^2/n_1\right)^2}{n_1-1} + \frac{\left(s_2^2/n_2\right)^2}{n_2-1}}$$

Inferences about Variances

The F statistic for testing the equality of variances is given by $F = s^2_{\max}/s^2_{\min}$, where $s^2_{\max}$ is the larger of s_1^2 and s_2^2. If the variances are equal, this quantity has an F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

It is generally not recommended that the results of the F test be used to decide whether to use the regular t-test or the modified t' on a single set of data. The modified t' (Satterthwaite's procedure) is the more conservative approach to use if there is doubt about the equality of the variances.

Properties

ChiSquaredTest

```
public double ChiSquaredTest {get; }
```

Description

The test statistic associated with the chi-squared test for common, or pooled, variances.

Property Value

A `double` containing the test statistic for the chi-squared test.

Remarks

The chi-squared test is a test of the hypothesis $\omega^2 = \omega_0^2$ where ω_0^2 is the null hypothesis value as described in `ChiSquaredTestNull`.

ChiSquaredTestDF

```
public int ChiSquaredTestDF {get; }
```

Description

The degrees of freedom associated with the chi-squared test for the common, or pooled, variances.

Property Value

A `int` which specifies the degrees of freedom for the chi-squared test.

Remarks

The chi-squared test is a test of the hypothesis $\omega^2 = \omega_0^2$ where ω_0^2 is the null hypothesis value as described in `ChiSquaredTestNull`.

ChiSquaredTestNull

```
public double ChiSquaredTestNull {get; set; }
```

Description

The null hypothesis value for the chi-squared test.

Property Value

A `double` containing the null hypothesis value for the chi-squared test.

Remarks

The default is 1.0.

ChiSquaredTestP

```
public double ChiSquaredTestP {get; }
```

Description

The probability of a larger chi-squared associated with the chi-squared test for common, or pooled, variances.

Property Value

A `double` containing the probability of a larger chi-squared for the chi-squared test for variances.

Remarks

The chi-squared test is a test of the hypothesis $\omega^2 = \omega_0^2$ where ω_0^2 is the null hypothesis value as described in `ChiSquaredTestNull`.

ConfidenceMean

```
public double ConfidenceMean {get; set; }
```

Description

The confidence level (in percent) for a two-sided interval estimate of the mean of `x` - the mean of `y`, in percent.

Property Value

A `double` containing the confidence level of the mean.

Remarks

`ConfidenceMean` must be between 0.0 and 1.0 and is often 0.90, 0.95 or 0.99. For a one-sided confidence interval with confidence level c (at least 50 percent), set ConfidenceMean $= 1.0 - 2.0(1.0 - c)$. If the confidence mean is not specified, a 95-percent confidence interval is computed, `ConfidenceMean` = .95.

ConfidenceVariance

```
public double ConfidenceVariance {get; set; }
```

Description

The confidence level (in percent) for two-sided interval estimate of the variances.

Property Value

A `double` containing the confidence level of the variance.

Remarks

Under the assumption of equal variances, the pooled variance is used to obtain a two-sided `ConfidenceVariance` percent confidence interval for the common variance with Imsl.Stat.NormTwoSample.LowerCICommonVariance (p. 570) or Imsl.Stat.NormTwoSample.UpperCICommonVariance (p. 573). Without making the assumption of equal variances, UnequalVariances (p. 572), the ratio of the variances is of interest. A two-sided `ConfidenceVariance` percent confidence interval for the ratio of the variance of the first sample to that of the second sample is given by the `LowerCIRatioVariance` and `UpperCIRatioVariance`. See UnequalVariances (p. 572) and UpperCIRatioVariance (p. 573). The confidence intervals are symmetric in probability. `ConfidenceVariance` must be between 0.0 and 1.0 and is often 0.90, 0.95 or 0.99. The default is 0.95.

DiffMean

```
public double DiffMean {get; }
```

Description

The difference of means for the two samples.

Property Value

A `double` containing the difference in mean.

Remarks

value = mean of `x` - mean of `y`

FTest

```
public double FTest {get; }
```

Description

The *F* test value of the *F* test for equality of variances.

Property Value

A `double` containing the *F* test value of the *F* test for equality of variances.

FTestDFdenominator

```
public int FTestDFdenominator {get; }
```

Description

The denominator degrees of freedom of the *F* test for equality of variances.

Property Value

A `int` containing the denominator degrees of freedom.

FTestDFnumerator

```
public int FTestDFnumerator {get; }
```

Description

The numerator degrees of freedom of the *F* test for equality of variances.

Property Value

A `int` containing the numerator degrees of freedom.

FTestP

```
public double FTestP {get; }
```

Description

The probability of a larger *F* in absolute value for the *F* test for equality of variances, assuming equal variances.

Property Value

A `double` containing the probability of a larger *F* in absolute value, assuming equal variances.

LowerCICommonVariance

```
public double LowerCICommonVariance {get; }
```

Description

The lower confidence limits for the common, or pooled, variance.

Property Value

A `double` containing the lower confidence limits for the variance.

LowerCIDiff

```
public double LowerCIDiff {get; }
```

Description

The lower confidence limit for the mean of the first population minus the mean of the second for equal or unequal variances.

Property Value

A `double` containing the lower confidence limit for the mean of the first sample minus the mean of the second sample.

Remarks

If UnequalVariances (p. 572) is `true` then the lower confidence limit for unequal variances will be returned.

LowerCIRatioVariance

```
public double LowerCIRatioVariance {get; }
```

Description

The approximate lower confidence limit for the ratio of the variance of the first population to the second.

Property Value

A `double` containing the approximate lower confidence limit variance.

MeanX

```
public double MeanX {get; }
```

Description

The mean of the first sample, `x`.

Property Value

A `double` containing the mean of `x`.

MeanY

```
public double MeanY {get; }
```

Description

The mean of the second sample, `y`.

Property Value

A `double` containing the mean of `y`.

PooledVariance

```
public double PooledVariance {get; }
```

Description

The Pooled variance for the two samples.

Property Value

A `double` containing the Pooled variance for the two samples.

StdDevX

```
public double StdDevX {get; }
```

Description

The standard deviation of the first sample, `x`.

Property Value

A `double` containing the standard deviation of the first sample, `x`.

StdDevY

```
public double StdDevY {get; }
```

Description

The standard deviation of the second sample, `y`.

Property Value

A `double` containing the standard deviation of the second sample, `y`.

TTest

```
public double TTest {get; }
```

Description

The test statistic for the Satterthwaite's approximation for equal or unequal variances.

Property Value

A `double` containing the test statistic for the *t*-test.

Remarks

If UnequalVariances (p. 572) is `true` then the test statistic for unequal variances will be returned.

TTestDF

```
public double TTestDF {get; }
```

Description

The degrees of freedom for the Satterthwaite's approximation for *t*-test for either equal or unequal variances.

Property Value

A `double` containing the degrees of freedom for the *t*-test.

Remarks

If UnequalVariances (p. 572) is `true` then the degrees of freedom for unequal variances will be returned.

TTestNull

```
public double TTestNull {get; set; }
```

Description

The Null hypothesis value for *t*-test for the mean.

Property Value

A `double` containing the hypothesis value.

Remarks

`TTestNull` = 0.0 by default.

TTestP

```
public double TTestP {get; }
```

Description

The approximate probability of a larger `t` for the Satterthwaite's approximation for equal or unequal variances.

Property Value

A `double` containing the probability for the *t*-test.

Remarks

If UnequalVariances (p. 572) is `true` then the approximate probability of a larger `t` for unequal variances will be returned.

UnequalVariances

```
public bool UnequalVariances {get; set; }
```

Description

Specifies whether to return statistics based on equal or unequal variances.

Property Value

A `bool` which specifies whether the sample variances are unequal.

Remarks

A value of `true` will cause statistics for unequal variances to be returned. A value of `false` will cause statistics for equal variances to be returned. The default is to return statistics for equal variances.

UpperCICommonVariance

```
public double UpperCICommonVariance {get; }
```

Description

The upper confidence limits for the common, or pooled, variance.

Property Value

A `double` containing the upper confidence limits for the variance.

UpperCIDiff

```
public double UpperCIDiff {get; }
```

Description

The upper confidence limit for the mean of the first population minus the mean of the second for equal or unequal variances.

Property Value

A `double` containing the upper confidence limit for the mean of the first sample minus the mean of the second sample.

Remarks

If UnequalVariances (p. 572) is `true` then the upper confidence limit for unequal variances will be returned.

UpperCIRatioVariance

```
public double UpperCIRatioVariance {get; }
```

Description

The approximate upper confidence limit for the ratio of the variance of the first population to the second.

Property Value

A `double` containing the approximate upper confidence limit variance.

Constructor

NormTwoSample

```
public NormTwoSample(double[] x, double[] y)
```

Description

Constructor to compute statistics for mean and variance inferences using samples from two normal populations.

Parameters

x – A `double` array containing the first sample.

y – A `double` array containing the second sample.

Methods

DowndateX

```
public void DowndateX(double[] x)
```

Description

Removes the observations in `x` from the first sample.

Parameter

x – A `double` array containing the values to remove from the first sample.

DowndateY

```
public void DowndateY(double[] y)
```

Description

Removes the observations in `y` from the second sample.

Parameter

y – A `double` array containing the values to remove from the second sample.

Update

```
public void Update(double[] x, double[] y)
```

Description

Concatenates samples `x` and `y` to the samples provided in the constructor.

Parameters

x – A `double` array containing updates to the first sample.

y – A `double` array containing updates to the second sample.

UpdateX

```
public void UpdateX(double[] x)
```

Description

Concatenates the values in `x` to the first sample provided in the constructor.

Parameter

x – A `double` array containing updates for the first sample.

UpdateY

```
public void UpdateY(double[] y)
```

Description

Concatenates the values in y to the second sample provided in the constructor.

Parameter

y – A `double` array containing updates for the second sample.

Example 1: NormTwoSample

This example taken from Conover and Iman(1983, p294), involves scores on arithmetic tests of two grade-school classes.

Scores for Standard Group	Scores for Experimental Group
72	111
75	118
77	128
80	138
104	140
110	150
125	163
	164
	169

The question is whether a group taught by an experimental method has a higher mean score. The difference in means and the *t* test are ouput. The variances of the two populations are assumed to be equal. It is seen from the output that there is strong reason to believe that the two means are different (*t* value of -4.804). Since the lower 97.5-percent confidence limit does not include 0, the null hypothesis is that $\mu_1 \leq \mu_2$ would be rejected at the 0.05 significance level. (The closeness of the values of the sample variances provides some qualitative substantiation of the assumption of equal variances.)

```
using System;
using Imsl.Stat;

public class NormTwoSampleEx1
{
    public static void  Main(String[] args)
    {
        double mean;
        double[] x1 =
            new double[]{72.0, 75.0, 77.0, 80.0, 104.0, 110.0, 125.0};
        double[] x2 =
            new double[]{   111.0, 118.0, 128.0, 138.0, 140.0,
                            150.0, 163.0, 164.0, 169.0};

        /* Perform Analysis for one sample x2*/
        NormTwoSample n2samp = new NormTwoSample(x1, x2);
        mean = n2samp.DiffMean;

        Console.Out.WriteLine("x1mean-x2mean  = " + mean);
        Console.Out.WriteLine("X1 mean =" + n2samp.MeanX);
        Console.Out.WriteLine("X2 mean =" + n2samp.MeanY);
```

```
        double pVar = n2samp.PooledVariance;
        Console.Out.WriteLine("pooledVar = " + pVar);

        double loCI = n2samp.LowerCIDiff;
        double upCI = n2samp.UpperCIDiff;
        Console.Out.WriteLine
            ("95% CI for the mean is " + loCI + " " + upCI);

        loCI = n2samp.LowerCIDiff;
        upCI = n2samp.UpperCIDiff;
        Console.Out.WriteLine
            ("95% CI for the ueq mean is " + loCI + " " + upCI);

        Console.Out.WriteLine("T Test Results");
        double tDF = n2samp.TTestDF;
        double tT = n2samp.TTest;
        double tPval = n2samp.TTestP;
        Console.Out.WriteLine("T default = " + tDF);
        Console.Out.WriteLine("t = " + tT);
        Console.Out.WriteLine("p-value = " + tPval);

        double stdevX = n2samp.StdDevX;
        double stdevY = n2samp.StdDevY;
        Console.Out.WriteLine("stdev x1 =" + stdevX);
        Console.Out.WriteLine("stdev x2 =" + stdevY);
    }
}
```

Output

```
x1mean-x2mean  = -50.4761904761905
X1 mean =91.8571428571428
X2 mean =142.333333333333
pooledVar = 434.632653061224
95% CI for the mean is -73.0100196252951 -27.9423613270859
95% CI for the ueq mean is -73.0100196252951 -27.9423613270859
T Test Results
T default = 14
t = -4.80436150471634
p-value = 0.000280258365677279
stdev x1 =20.8760514420118
stdev x2 =20.8266655996585
```

Sort Class

`public class Imsl.Stat.Sort`

A collection of sorting functions.

Class `Sort` contains ascending and descending methods for sorting elements of an array or a matrix.

The QuickSort algorithm is used, except for short sequences which are handled using an insertion sort.

The QuickSort algorithm is a randomized QuickSort with 3-way partitioning. Basic QuickSort is slow if the sequence to be sorted contains many duplicate keys. The 3-way partitioning algorithm elimiates this problem. The pivot is chosen as the middle element of three potential pivots chosen at random.

The matrix ascending method sorts the rows of real matrix `x` using a particular row in `x` as the keys. The sort is algebraic with the first key as the most significant, the second key as the next most significant, etc. When x is sorted in ascending order, the resulting sorted array is such that the following is true:

- For $i = 0, 1, \ldots,$ n_observations $- 2,$ x$[i]$[indices_keys $[0]] \leq$ x$[i+1]$[indices_keys$[0]]$
- For $k = 1, \ldots,$ n_keys $- 1,$ if x[i][indices_keys[j]] $=$ x[i $+ 1$][indices_keys[j]] for $j = 0, 1, \ldots, k-1,$ then x$[i]$[indices_keys$[k]]$ = x$[i+1]$[indices_keys$[k]]$

The observations also can be sorted in descending order. The rows of `x` containing the missing value code NaN in at least one of the specified columns are considered as an additional group. These rows are moved to the end of the sorted `x`.

If all of the sort keys in a pair of rows are equal then the rows keep their original relative order.

The sorting algorithm is based on a quicksort method given by Singleton (1969) with modifications, see Bentley and Sedgewick (1997).

Methods

Ascending

```
static public void Ascending(int[] ra)
```

Description

Function to sort an integer array into ascending order.

Parameter

`ra` – `int` array to be sorted into ascending order

Ascending

```
static public void Ascending(int[] ra, int[] iperm)
```

Description

Sort an integer array into ascending order and returns the permutation vector.

Parameters

`ra` – `int` array to be sorted into ascending order

`iperm` – `int` is on input an array the same length as `ra`. On output, it contains 0, 1, ..., sorted using the same permutations applied to ra.

Ascending

```
static public void Ascending(double[] ra)
```

Description

Sort an array into ascending order.

Parameter

`ra` – `double` array to be sorted into ascending order

Ascending

```
static public void Ascending(double[] ra, int[] iperm)
```

Description

Sort an array into ascending order and returns the permutation vector.

Parameters

`ra` – `double` array to be sorted into ascending order

`iperm` – `int` is on input an array the same length as `ra`. On output, it contains 0, 1, ..., sorted using the same permutations applied to ra.

Ascending

```
static public void Ascending(double[,] ra, int nKeys)
```

Description

Sort a matrix into ascending order by the first `nKeys`.

Parameters

`ra` – `double` matrix to be sorted into ascending order.

`nKeys` – `int` containing the first `nKeys` columns of `ra` to be used as the sorting keys.

Ascending

```
static public void Ascending(double[,] ra, int[] indkeys)
```

Description

Sort a matrix into ascending order by specified keys.

Parameters

`ra` – a `double` matrix to be sorted into ascending order.

`indkeys` – `int` array containing the order the columns of `ra` are to be sorted.

Ascending

```
static public void Ascending(double[,] ra, int nKeys, int[] iperm)
```

Description

Sort an array into ascending order by the first `nKeys` and returns the permutation vector.

Parameters

`ra` – `double` array to be sorted into ascending order.

`nKeys` – `int` containing the first `nKeys` columns of `ra` to be used as the sorting keys.

`iperm` – `int` is on input an array the same length as `ra`. On output, it contains 0, 1, ..., sorted using the same permutations applied to ra.

Ascending

```
static public void Ascending(double[,] ra, int[] indkeys, int[] iperm)
```

Description

Sort a matrix into ascending order by specified keys and returns the permutation vector.

Parameters

`ra` – `double` matrix to be sorted into ascending order.

`indkeys` – `int` array containing the order the columns of `ra` are to be sorted. These values must be between 0 and one less than the number of columns in `ra`.

`iperm` – `int` is on input an array the same length as `ra`. On output, it contains 0, 1, ..., sorted using the same permutations applied to ra.

Descending

```
static public void Descending(int[] ra)
```

Description

Function to sort an integer array into descending order.

Parameter

`ra` – `int` array to be sorted into descending order

Descending

```
static public void Descending(int[] ra, int[] iperm)
```

Description

Sort an integer array into descending order and returns the permutation vector.

Parameters

`ra` – `int` array to be sorted into descending order

`iperm` – `int` is on input an array the same length as `ra`. On output, it contains 0, 1, ..., sorted using the same permutations applied to ra.

Descending

```
static public void Descending(double[] ra)
```

Description

Sort an array into descending order.

Parameter

`ra` – `double` array to be sorted into descending order

Descending

```
static public void Descending(double[] ra, int[] iperm)
```

Description

Sort an array into descending order and returns the permutation vector.

Parameters

`ra` – `double` array to be sorted into descending order

`iperm` – `int` is on input an array the same length as `ra`. On output, it contains 0, 1, ..., sorted using the same permutations applied to ra.

Descending

```
static public void Descending(double[,] ra, int nKeys)
```

Description

Function to sort a matrix into descending order by the first `nKeys`.

Parameters

`ra` – a `double` matrix to be sorted into descending order.

`nKeys` – `int` containing the first `nKeys` columns of `ra` to be used as the sorting keys.

Descending

```
static public void Descending(double[,] ra, int[] indkeys)
```

Description

Function to sort a matrix into descending order by specified keys.

Parameters

`ra` – a `double`“>matrix to be sorted into descending order.

`indkeys` – `int` array containing the order the columns of `ra` are to be sorted.

Descending

```
static public void Descending(double[,] ra, int nKeys, int[] iperm)
```

Description

Function to sort an array into descending order by the first `nKeys` and returns the permutation vector.

Parameters

`ra` – a `double` array to be sorted into descending order.

`nKeys` – `int` containing the first `nKeys` columns of `ra` to be used as the sorting keys.

`iperm` – an `int` array specifying the rearrangement (permutation) of the observations (rows) of `ra`.

Descending

```
static public void Descending(double[,] ra, int[] indkeys, int[] iperm)
```

Description

Function to sort a matrix into descending order by specified keys and return the permutation vector.

Parameters

`ra` – `double` matrix to be sorted into descending order.

`indkeys` – `int` array containing the order the columns of `ra` are to be sorted. These values must be between 0 and one less than the number of columns in `ra`.

`iperm` – an `int` array specifying the rearrangement (permutation) of the observations (rows) of `ra`.

Example 1: Sorting

An array is sorted by increasing value. A permutation array is also computed. Note that the permutation array begins at 0 in this example.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class SortEx1
{
    public static void  Main(String[] args)
    {
        double[] ra = new double[]{  10.0, - 9.0, 8.0, - 7.0, 6.0,
                                     5.0, 4.0, - 3.0, - 2.0, - 1.0};
        int[] iperm = new int[]{0, 1, 2, 3, 4, 5, 6, 7, 8, 9};

        PrintMatrix pm = new PrintMatrix("The Input Array");
        PrintMatrixFormat mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
        //    Print the array
        pm.Print(mf, ra);
        Console.Out.WriteLine();

        //  Sort the array
        Sort.Ascending(ra, iperm);

        pm = new PrintMatrix("The Sorted Array - Lowest to Highest");
        mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();

        //    Print the array
        pm.Print(mf, ra);

        pm = new PrintMatrix("The Resulting Permutation Array");
        mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
        //    Print the array
        pm.Print(mf, iperm);
    }
}
```

Output

```
The Input Array

10
-9
 8
-7
 6
 5
 4
-3
-2
-1


The Sorted Array - Lowest to Highest

-9
-7
-3
-2
-1
 4
 5
 6
 8
10

The Resulting Permutation Array

1
3
7
8
9
6
5
4
2
0
```

Example 2: Sorting

The rows of a 10 x 3 matrix x are sorted in ascending order using Columns 0 and 1 as the keys. There are two missing values (NaNs) in the keys. The observations containing these values are moved to the end of the sorted array.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class SortEx2
{
```

```
    public static void  Main(String[] args)
    {
        int nKeys = 2;
        double[,] x = {
            {1.0, 1.0, 1.0}, {2.0, 1.0, 2.0},
            {1.0, 1.0, 3.0}, {1.0, 1.0, 4.0},
            {2.0, 2.0, 5.0}, {1.0, 2.0, 6.0},
            {1.0, 2.0, 7.0}, {1.0, 1.0, 8.0},
            {2.0, 2.0, 9.0}, {1.0, 1.0, 9.0}
        };

        int[] iperm = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9};
        x[4,1] = Double.NaN;
        x[6,0] = Double.NaN;

        PrintMatrix pm = new PrintMatrix("The Input Array");
        PrintMatrixFormat mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
        //    Print the array
        pm.Print(mf, x);
        Console.Out.WriteLine();

        Sort.Ascending(x, nKeys, iperm);

        pm = new PrintMatrix("The Sorted Array - Lowest to Highest");
        mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();

        //    Print the array
        pm.Print(mf, x);

        pm = new PrintMatrix("The permutation array");
        mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
        pm.Print(mf, iperm);
    }
}
```

Output

```
The Input Array

1    1    1
2    1    2
1    1    3
1    1    4
2    NaN  5
1    2    6
NaN  2    7
1    1    8
2    2    9
1    1    9
```

```
The Sorted Array - Lowest to Highest

1    1    1
1    1    3
1    1    4
1    1    8
1    1    9
1    2    6
2    1    2
2    2    9
2    NaN  5
NaN  2    7

The permutation array

0
2
3
7
9
5
1
8
4
6
```

Ranks Class

```
public class Imsl.Stat.Ranks
```

Compute the ranks, normal scores, or exponential scores for a vector of observations.

The class Ranks can be used to compute the ranks, normal scores, or exponential scores of the data in *X*. Ties in the data can be resolved in four different ways, as specified by property `TieBreaker`. The type of values returned can vary depending on the member function called:

GetRanks: Ordinary Ranks

For this member function, the values output are the ordinary ranks of the data in *X*. If *X[i]* has the smallest value among those in *X* and there is no other element in *X* with this value, then `GetRanks(i)` = 1. If both *X[i]* and *X[j]* have the same smallest value, then

if *TieBreaker* = 0, *Ranks[i]* = `GetRanks([j]` = 1.5

if *TieBreaker* = 1, *Ranks[i]* = *Ranks[j]* = 2.0

if *TieBreaker* = 2, *Ranks[i]* = *Ranks[j]* = 1.0

if *TieBreaker* = 3, *Ranks[i]* = 1.0 and *Ranks[j]* = 2.0

or *Ranks[i]* = 2.0 and *Ranks[j]* = 1.0.

GetBlomScores: Normal Scores, Blom Version

Normal scores are expected values, or approximations to the expected values, of order statistics from a normal distribution. The simplest approximations are obtained by evaluating the inverse cumulative normal distribution function, `Cdf.InverseNormal`, at the ranks scaled into the open interval (0, 1). In the Blom version (see Blom 1958), the scaling transformation for the rank $r_i (1 \leq r_i \leq n$, where n is the sample size is $(r_i - 3/8)/(n + 1/4)$. The Blom normal score corresponding to the observation with rank r_i is

$$\Phi^{-1}\left(\frac{r_i - 3/8}{n + 1/4}\right)$$

where $\Phi(\cdot)$ is the normal cumulative istribution function.

Adjustments for ties are made after the normal score transformation. That is, if *X[i]* equals *X[j]* (within fuzz) and their value is the *k*-th smallest in the data set, the Blom normal scores are determined for ranks of *k* and *k + 1*, and then these normal scores are averaged or selected in the manner specified by *TieBreaker*, which is set by the property `TieBreaker`. (Whether the transformations are made first or ties are resolved first makes no difference except when averaging is done.)

GetTukeyScores: Normal Scores, Tukey Version

In the Tukey version (see Tukey 1962), the scaling transformation for the rank r_i is $(r_i - 1/3)/(n + 1/3)$. The Tukey normal score corresponding to the observation with rank r_i is

$$\Phi^{-1}\left(\frac{r_i - 1/3}{n + 1/3}\right)$$

Ties are handled in the same way as discussed above for the Blom normal scores.

GetVanDerWaerdenScores: Normal Scores, Van der Waerden Version

In the Van der Waerden version (see Lehmann 1975, page 97), the scaling transformation for the rank r_i is $r_i/(n + 1)$. The Van der Waerden normal score corresponding to the observation with rank r_i is

$$\Phi^{-1}\left(\frac{r_i}{n + 1}\right)$$

Ties are handled in the same way as discussed above for the Blom normal scores.

GetNormalScores: Expected Value of Normal Order Statistics

The member function `GetNormalScores` returns the expected values of the normal order statistics. If the value in *X[i]* is the *k*-th smallest, then the value output in `SCORE[i]` is $E(Z_k)$, where $E(\cdot)$ is the expectation operator and Z_k is the *k*-th order statistic in a sample of size `x.length` from a standard normal distribution. Ties are handled in the same way as discussed above for the Blom normal scores.

GetSavageScores: Savage Scores

The member function `GetSavageScores` returns the expected values of the exponential order statistics. These values are called Savage scores because of their use in a test discussed by Savage (1956) (see Lehman 1975). If the value in *X[i]* is the *k*-th smallest, then the *i*-th output value output is $E(Y_k)$, where Y_k is the *k*-th order statistic in a sample of size *n* from a standard exponential distribution. The expected value of the *k*-th order statistic from an exponential sample of size *n* is

$$\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{n-k+1}$$

Ties are handled in the same way as discussed above for the Blom normal scores.

Properties

Fuzz

```
public double Fuzz {get; set; }
```

Description

The fuzz factor used in determining ties.

Property Value

A `double` which represents the fuzz factor.

Random

```
public System.Random Random {get; set; }
```

Description

The `Random` object.

Property Value

A `Random` object used in breaking ties.

TieBreaker

```
public Imsl.Stat.Ranks.Tie TieBreaker {get; set; }
```

Description

The tie breaker for Ranks.

Property Value

A `Tie` which represents the tie breaker.

Constructor

Ranks

```
public Ranks()
```

Description

Constructor for the Ranks class.

Methods

ExpectedNormalOrderStatistic

```
static public double ExpectedNormalOrderStatistic(int i, int n)
```

Description

Returns the expected value of a normal order statistic.

Parameters

`i` – A `int` which specifies the rank of the order statistic.

`n` – A `int` which specifies the sample size.

Returns

A `double`, the expected value of the i-th order statistic in a sample of size n from the standard normal distribution.

GetBlomScores

```
public double[] GetBlomScores(double[] x)
```

Description

Gets the Blom version of normal scores for each observation.

Parameter

`x` – A `double` array which contains the observations to be ranked.

Returns

A `double` array which contains the Blom version of normal scores for each observation in x.

GetNormalScores

```
public double[] GetNormalScores(double[] x)
```

Description

Gets the expected value of normal order statistics (for tied observations, the average of the expected normal scores).

Parameter

`x` – A `double` array which contains the observations.

Returns

A `double` array which contains the expected value of normal order statistics for the observations in x.

Remarks

For tied observations `GetNormalScores` returns an average of the expected normal scores.

GetRanks

```
public double[] GetRanks(double[] x)
```

Description

Gets the rank for each observation.

Parameter

`x` – A `double` array which contains the observations to be ranked.

Returns

A `double` array which contains the rank for each observation in x.

GetSavageScores

```
public double[] GetSavageScores(double[] x)
```

Description

Gets the Savage scores. (the expected value of exponential order statistics)

Parameter

`x` – A `double` array which contains the observations.

Returns

A `double` array which contains the Savage scores for the observations in x. (the expected value of exponential order statistics)

GetTukeyScores

```
public double[] GetTukeyScores(double[] x)
```

Description

Gets the Tukey version of normal scores for each observation.

Parameter

`x` – A `double` array which contains the observations to be ranked.

Returns

A `double` array which contains the Tukey version of normal scores for each observation in x.

GetVanDerWaerdenScores

```
public double[] GetVanDerWaerdenScores(double[] x)
```

Description

Gets the Van der Waerden version of normal scores for each observation.

Parameter

x – A `double` array which contains the observations to be ranked.

Returns

A `double` array which contains the Van der Waerden version of normal scores for each observation in x.

Example: Ranks

In this data from Hinkley (1977) note that the fourth and sixth observations are tied and that the third and twentieth are tied.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class RanksEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{0.77, 1.74, 0.81, 1.20, 1.95, 1.20,
                                  0.47, 1.43, 3.37, 2.20, 3.00,
                                  3.09, 1.51, 2.10, 0.52, 1.62,
                                  1.31, 0.32, 0.59, 0.81, 2.81,
                                  1.87, 1.18, 1.35, 4.75, 2.48,
                                  0.96, 1.89, 0.90, 2.05};

        PrintMatrixFormat mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();

        Ranks ranks = new Ranks();
        double[] score = ranks.GetRanks(x);
        new PrintMatrix("The Ranks of the Observations - " +
                        "Ties Averaged").Print(mf, score);
        Console.Out.WriteLine();

        ranks = new Ranks();
        ranks.TieBreaker = Imsl.Stat.Ranks.Tie.Highest;
        score = ranks.GetBlomScores(x);
        new PrintMatrix("The Blom Scores of the Observations - " +
            "Highest Score used in Ties").Print(mf, score);
        Console.Out.WriteLine();

        ranks = new Ranks();
        ranks.TieBreaker = Imsl.Stat.Ranks.Tie.Lowest;
        score = ranks.GetTukeyScores(x);
        new PrintMatrix("The Tukey Scores of the Observations - " +
                        "Lowest Score used in Ties").Print(mf, score);
        Console.Out.WriteLine();

        ranks = new Ranks();
        ranks.TieBreaker = Imsl.Stat.Ranks.Tie.Random;
        Imsl.Stat.Random random = new Imsl.Stat.Random(123457);
```

```
        random.Multiplier = 16807;
        ranks.Random = random;
        score = ranks.GetVanDerWaerdenScores(x);
        new PrintMatrix("The Van Der Waerden Scores of the " +
            "Observations - Ties untied by Random").Print(mf, score);
    }
}
```

Output

```
The Ranks of the Observations - Ties Averaged

 5
18
 6.5
11.5
21
11.5
 2
15
29
24
27
28
16
23
 3
17
13
 1
 4
 6.5
26
19
10
14
30
25
 9
20
 8
22


The Blom Scores of the Observations - Highest Score used in Ties

-1.02410618374162
 0.208663745751154
-0.775546958322378
-0.294213138930921
 0.472789120992267
-0.294213138930921
-1.60981606718445
-0.0414437330939966
 1.60981606718445
```

```
 0.775546958322378
 1.17581347255003
 1.36087334286719
 0.0414437330939965
 0.668002132269574
-1.36087334286719
 0.124617407947998
-0.208663745751155
-2.04028132201041
-1.17581347255003
-0.775546958322378
 1.02410618374162
 0.294213138930921
-0.472789120992267
-0.124617407947998
 2.04028132201041
 0.892918486444395
-0.56768639112746
 0.381975767696542
-0.668002132269574
 0.56768639112746


The Tukey Scores of the Observations - Lowest Score used in Ties

-1.0200762327862
 0.208082136154993
-0.88970115508476
-0.380874057516038
 0.471389465588488
-0.380874057516038
-1.59868725959458
-0.0413298117447387
 1.59868725959458
 0.772935693128221
 1.17060337087942
 1.35372485367826
 0.0413298117447388
 0.665869518001049
-1.35372485367826
 0.124273282084069
-0.208082136154993
-2.01450973381435
-1.17060337087942
-0.88970115508476
 1.0200762327862
 0.293381232121193
-0.471389465588488
-0.124273282084069
 2.01450973381435
 0.889701155084761
-0.565948821932863
 0.380874057516038
-0.665869518001048
 0.565948821932863
```

```
The Van Der Waerden Scores of the Observations - Ties untied by Random

-0.989168627340635
 0.203544231532486
-0.864894358685283
-0.372289360465191
 0.460494539103116
-0.286893916923039
-1.51792915959428
-0.0404405085656462
 1.51792915959428
 0.75272879425817
 1.13097760824516
 1.30015343336342
 0.0404405085656462
 0.649323913186466
-1.30015343336342
 0.121587382750483
-0.203544231532486
-1.84859628850141
-1.13097760824516
-0.75272879425817
 0.989168627340635
 0.286893916923039
-0.460494539103116
-0.121587382750483
 1.84859628850141
 0.864894358685283
-0.552442584646774
 0.372289360465191
-0.649323913186466
 0.552442584646775
```

Ranks.Tie Enumeration

```
public enumeration Imsl.Stat.Ranks.Tie
```

Determines how to break a tie.

Fields

Average

```
public Imsl.Stat.Ranks.Tie Average
```

Description

Use the average score in the group of ties.

Highest

```
public Imsl.Stat.Ranks.Tie Highest
```

Description

Use the highest score in the group of ties.

Lowest

```
public Imsl.Stat.Ranks.Tie Lowest
```

Description

Use the lowest score in the group of ties.

Random

```
public Imsl.Stat.Ranks.Tie Random
```

Description

Use one of the group of ties chosen at random.

EmpiricalQuantiles Class

```
public class Imsl.Stat.EmpiricalQuantiles
```

Computes empirical quantiles.

The class EmpiricalQuantiles determines the empirical quantiles, as indicated in the array `qProp`, from the data in `x`. The algorithm first checks to see if `x` is sorted; if `x` is not sorted, the algorithm does either a complete or partial sort, depending on how many order statistics are required to compute the quantiles requested. The algorithm returns the empirical quantiles and, for each quantile, the two order statistics from the sample that are at least as large and at least as small as the quantile. For a sample of size n, the quantile corresponding to the proportion p is defined as

$$Q(p) = (1-f)x_{(j)} + fx_{(j+1)}$$

where $j = \lfloor p(n+1) \rfloor$, $f = p(n+1) - j$, and $x_{(j)}$, is the j-th order statistic, if $1 \leq j \leq n$; otherwise, the empirical quantile is the smallest or largest order statistic.

Property

TotalMissing

```
public int TotalMissing {get; }
```

Description

The total number of missing values.

Property Value

An `int` representing the total number of missing values (`NaN`) in input `x`.

Constructor

EmpiricalQuantiles

```
public EmpiricalQuantiles(double[] x, double[] qProp)
```

Description

Computes empirical quantiles.

Parameters

`x` – A `double` array containing the data.

`qProp` – A `double` array containing the quantile proportions.

Methods

GetQ

```
public double[] GetQ()
```

Description

Returns the empirical quantiles.

Returns

A `double` array of length `qProp.Length` containing the empirical quantiles.

Remarks

`Q[i]` corresponds to the empirical quantile at proportion `qProp[i]`. The quantiles are determined by linear interpolation between adjacent ordered sample values.

GetXHi

```
public double[] GetXHi()
```

Description

Returns the smallest element of `x` greater than or equal to the desired quantile.

Returns

A `double` array of length `qProp.Length` containing the smallest element of `x` greater than or equal to the desired quantile.

GetXLo

```
public double[] GetXLo()
```

Description

Returns the largest element of `x` less than or equal to the desired quantile.

Returns

A `double` array of length `qProp.Length` containing the largest element of `x` less than or equal to the desired quantile.

Example: Empirical Quantiles

In this example, five empirical quantiles from a sample of size 30 are obtained. Notice that the 0.5 quantile corresponds to the sample median. The data are from Hinkley (1977) and Velleman and Hoaglin (1981). They are the measurements (in inches) of precipitation in Minneapolis/St. Paul during the month of March for 30 consecutive years.

```
using System;
using Imsl.Stat;

public class EmpiricalQuantilesEx1
{
    public static void  Main(String[] args)
    {
        double[]  x = {0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37,
                       2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59,
                       0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90,
                       2.05};
        double[] qProp = {0.01, 0.5, 0.90, 0.95, 0.99};
        EmpiricalQuantiles eq = new EmpiricalQuantiles(x, qProp);

        double[] Q = eq.GetQ();
        double[] XLo = eq.GetXLo();
        double[] XHi = eq.GetXHi();

        Console.WriteLine("               Smaller        Empirical        Larger");
        Console.WriteLine(" Quantile       Datum          Quantile         Datum");
        for (int i=0; i < qProp.Length; i++)
            Console.WriteLine("  {0}\t\t{1}\t\t{2}\t\t{3}",
                qProp[i], XLo[i], Q[i], XHi[i]);
    }
}
```

Output

```
               Smaller        Empirical        Larger
 Quantile       Datum          Quantile         Datum
```

```
0.01 0.32 0.32 0.32
0.5 1.43 1.47 1.51
0.9 3 3.081 3.09
0.95 3.37 3.991 4.75
0.99 4.75 4.75 4.75
```

TableOneWay Class

`public class Imsl.Stat.TableOneWay`

Tallies observations into a one-way frequency table.

Class `TableOneWay` calculates a frequency table for a data array.

A one-way frequency table can be used to visualize the shape of the data distribution and look for anomalies in the data. There are many approaches to constructing frequency tables. Four approaches are implemented in this class:

1. equal width class intervals based upon the smallest and largest observations,
2. equal width class intervals based upon a user provided minimum and maximum,
3. class intervals defined from user provided class midpoints, and
4. class intervals defined from user provided class boundaries.

The `TableOneWay` class implements the first two approaches by overloading the `GetFrequencyTable` method. If `GetFrequencyTable()` is used without input arguments, `nIntervals` of equal length are formed between the minimum and maximum values in the data. The frequency table returned from this method contains tallies of the number of observations in each interval. The data minimum and maximum can be obtained from the `Minimum` and `Maximum` properties.

Instead of using the minimum and maximum to define the boundaries of the smallest and largest classes, specified boundaries can be used by calling `GetFrequencyTable(lower_bound, upper_bound)`. This method tallies all data less than or equal to the `lower_bound` into the first class, and all data greater than or equal to `upper_bound` into the last class.

The third approach is implemented using the `GetFrequencyTableUsingClassmarks` method. Equally spaced intervals can be defined using class marks. In this approach a double precision array of length nIntervals containing the class midpoints is passed to the `GetFrequencyTableUsingClassmarks(classmarks[])`. The class marks, or midpoints, must be equally spaced.

Finally in those applications where unequal length intervals are preferred, the `GetFrequencyTableUsingCutpoints(cutpoints[])` method can be used. The double precision array `cutpoints` has length `nIntervals-1` and contains the class boundaries listed in ascending order. The first cut point defines the first class which is used to tally all data less than or equal to the first cut

point value. The last cut point defines the last class which is used to tally all data greater than or equal to the last cut point value.

Properties

Maximum

```
public double Maximum {get; }
```

Description

The maximum value of `x`.

Property Value

A `double` containing the maximum data bound.

Minimum

```
public double Minimum {get; }
```

Description

The minimum value of `x`.

Property Value

A `double` containing the minimum data bound.

Constructor

TableOneWay

```
public TableOneWay(double[] x, int nIntervals)
```

Description

Constructor for `TableOneWay`.

Parameters

`x` – A `double` array containing the observations.

`nIntervals` – A `int` scalar containing the number of intervals (bins).

Methods

GetFrequencyTable

```
public double[] GetFrequencyTable()
```

Description

Returns the one-way frequency table.

Returns

A `double` array containing the one-way frequency table.

Remarks

`nIntervals` intervals of equal length are used with the initial interval starting with the minimum value in `x` and the last interval ending with the maximum value in `x`. The initial interval is closed on the left and the right. The remaining intervals are open on the left and the closed on the right. Each interval is of length `(max-min)/nIntervals`, where `max` is the maximum value of `x` and `min` is the minimum value of `x`.

GetFrequencyTable

```
public double[] GetFrequencyTable(double lowerBound, double upperBound)
```

Description

Returns a one-way frequency table using known bounds.

Parameters

`lowerBound` – A `double` specifies the right endpoint.

`upperBound` – A `double` specifies the left endpoint.

Returns

A `double` array containing the one-way frequency table.

Remarks

The one-way frequency table is computed using two semi-infinite intervals as the initial and last intervals. The initial interval is closed on the right and includes `lowerBound` as its right endpoint. The last interval is open on the left and includes all values greater than `upperBound`. The remaining `nIntervals - 2` intervals are each of length `(upperBound - lowerBound) / (nIntervals - 2)` and are open on the left and closed on the right. `nIntervals` must be greater than or equal to 3.

GetFrequencyTableUsingClassmarks

```
public double[] GetFrequencyTableUsingClassmarks(double[] classmarks)
```

Description

Returns the one-way frequency table using class marks.

Parameter

`classmarks` – A `double` array containing either the cutpoints or the class marks.

Returns

A `double` array containing the one-way frequency table.

Remarks

Equally spaced class marks in ascending order must be provided in the array `classmarks` of length `nIntervals`. The class marks are the midpoints of each of the `nIntervals`. Each interval is assumed to have length `classmarks[1] - classmarks[0]`. `nIntervals` must be greater than or equal to 2.

GetFrequencyTableUsingCutpoints

```
public double[] GetFrequencyTableUsingCutpoints(double[] cutpoints)
```

Description

Returns the one-way frequency table using cutpoints.

Parameter

`cutpoints` – A `double` array containing the cutpoints.

Returns

A `double` array containing the one-way frequency table.

Remarks

The cutpoints are boundaries that must be provided in the array `cutpoints` of length `nIntervals`-1. This option allows unequal interval lengths. The initial interval is closed on the right and includes the initial cutpoint as its right endpoint. The last interval is open on the left and includes all values greater than the last cutpoint. The remaining `nIntervals`-2 intervals are open on the left and closed on the right. Argument `nIntervals` must be greater than or equal to 3 for this option.

Example: TableOneWay

The data for this example is from Hinkley (1977) and Belleman and Hoaglin (1981). The measurements (in inches) are for precipitation in Minneapolis/St. Paul during the month of March for 30 consecutive years.

The first test uses the default tally method which may be appropriate when the range of data is unknown. The minimum and maximum data bounds are displayed.

The second test computes the table usings known bounds, where the lower bound is 0.5 and the upper bound is 4.5. The eight interior intervals each have width (4.5 - 0.5)/(10-2) = 0.5. The 10 intervals are $(-\infty, 0.5]$, (0.5,1.0],...,(4.0,4.5], and (4.5,∞].

In the third test, 10 class marks, 0.25, 0.75, 1.25,...,4.75, are input. This defines the class intervals (0.0,0.5],(0.5,1.0],...,(4.0,4.5],(4.5,5.0]. Note that unlike the previous test, the initial and last intervals are the same length as the remaining intervals.

In the fourth test, cutpoints, 0.5,1.0, 1.5, 2.0, ...,4.5, are input to define the same 10 intervals as in the second test. Here again, the initial and last intervals are semi- infinite intervals.

```
using System;
using Imsl.Stat;

public class TableOneWayEx1
{
    public static void  Main(String[] args)
```

```
    {
        int nIntervals = 10;

        double[] x = new double[]{   0.77, 1.74, 0.81, 1.20, 1.95,
                                     1.20, 0.47, 1.43, 3.37, 2.20,
                                     3.00, 3.09, 1.51, 2.10, 0.52,
                                     1.62, 1.31, 0.32, 0.59, 0.81,
                                     2.81, 1.87, 1.18, 1.35, 4.75,
                                     2.48, 0.96, 1.89, 0.9, 2.05};
        double[] cutPoints = new double[]{   0.5, 1.0, 1.5, 2.0, 2.5,
                                             3.0, 3.5, 4.0, 4.5};
        double[] classMarks = new double[]{   0.25, 0.75, 1.25, 1.75,
                                              2.25, 2.75, 3.25, 3.75,
                                              4.25, 4.75};

        TableOneWay fTbl = new TableOneWay(x, nIntervals);

        double[] table = fTbl.GetFrequencyTable();

        Console.Out.WriteLine("Example 1 ");
        for (int i = 0; i < table.Length; i++)
            Console.Out.WriteLine(i + "          " + table[i]);

        Console.Out.WriteLine("--------------------------");
        Console.Out.WriteLine("Lower bounds= " + fTbl.Minimum);
        Console.Out.WriteLine("Upper bounds= " + fTbl.Maximum);
        Console.Out.WriteLine("--------------------------");
        /*  getFrequencyTable using a set of known bounds */
        table = fTbl.GetFrequencyTable(0.5, 4.5);
        for (int i = 0; i < table.Length; i++)
            Console.Out.WriteLine(i + "          " + table[i]);

        Console.Out.WriteLine("----------------------");

        table = fTbl.GetFrequencyTableUsingClassmarks(classMarks);
        for (int i = 0; i < table.Length; i++)
            Console.Out.WriteLine(i + "          " + table[i]);

        Console.Out.WriteLine("--------------------");
        table = fTbl.GetFrequencyTableUsingCutpoints(cutPoints);
        for (int i = 0; i < table.Length; i++)
            Console.Out.WriteLine(i + "          " + table[i]);
    }
}
```

Output

```
Example 1
0          4
1          8
2          5
3          5
4          3
5          1
6          3
```

```
7          0
8          0
9          1
-------------------------
Lower bounds= 0.32
Upper bounds= 4.75
-------------------------
0          2
1          7
2          6
3          6
4          4
5          2
6          2
7          0
8          0
9          1
--------------------
0          2
1          7
2          6
3          6
4          4
5          2
6          2
7          0
8          0
9          1
-------------------
0          2
1          7
2          6
3          6
4          4
5          2
6          2
7          0
8          0
9          1
```

TableTwoWay Class

`public class Imsl.Stat.TableTwoWay`

Tallies observations into a two-way frequency table.

Class `TableTwoWay` calculates a two-dimensional frequency table for a data array based upon two variables.

A two-way frequency table can be used to visualize the shape of the bivariate distribution and look for

anomalies in the data. There are many approaches to constructing two-way frequency tables. Four approaches are implemented in this class:

1. equal width class intervals based upon the smallest and largest observations,
2. equal width class intervals based upon a user provided minimum and maximum,
3. class intervals defined from user provided class midpoints, and
4. class intervals defined from user provided class boundaries.

The `TableTwoWay` class implements the first two approaches by overloading the `GetFrequencyTable` method. If `GetFrequencyTable()` is used without input arguments, `xIntervals` intervals of equal length are formed between the minimum and maximum values in x, and similarly, `yIntervals` intervals are formed for y. The frequency table returned from this method contains tallies of the number of observations in each interval. The data minimum and maximum can be obtained using the `MinimumX`, `MinimumY`, `MaximumX` and `MaximumY` properties.

Instead of using the minimum and maximum to define the boundaries of the smallest and largest classes, specified boundaries can be used by calling `GetFrequencyTable(xLowerBound, xUpperBound, yLowerBound, yUpperBound)`. This method tallies all data less than or equal to the `xLowerBound` and `yLowerBound` into the first class, and all data greater than or equal to `xUpperBound` and `YUpperBround` into the last class

The third approach is implemented using the `GetFrequencyTableUsingClassmarks` method. Equally spaced intervals can be defined using class marks. In this approach two double precision arrays of length `xIntervals` and `yIntervals` containing the class midpoints for x and y respectively are passed to the `GetFrequencyTableUsingClassmarks(cx[], cy[])`. The class marks, or midpoints, must be equally spaced.

Finally in those applications where unequal length intervals are preferred, the `GetFrequencyTableUsingCutpoints(cx[], cy[])` method can be used. The double precision arrays `cx` and `cy` with lengths `xIntervals-1` and `yIntervals-1` respectively contain the class boundaries listed in ascending order. The first cut point defines the first class which is used to tally all data less than or equal to the first cut point value. The last cut point defines the last class which is used to tally all data greater than or equal to the last cut point value.

Properties

MaximumX

```
public double MaximumX {get; }
```

Description

The maximum value of x.

Property Value

A `double` containing the maximum data bound for x.

MaximumY

```
public double MaximumY {get; }
```

Description

The maximum value of y.

Property Value

A `double` containing the maximum data bound for y.

MinimumX

```
public double MinimumX {get; }
```

Description

The minimum value of x.

Property Value

A `double` containing the minimum data bound for x.

MinimumY

```
public double MinimumY {get; }
```

Description

The minimum value of y.

Property Value

A `double` containing the minimum data bound for y.

Constructor

TableTwoWay

```
public TableTwoWay(double[] x, int xIntervals, double[] y, int yIntervals)
```

Description

Constructor for `TableTwoWay`.

Parameters

`x` – A `double` array containing the data for the first variable.

`xIntervals` – A `int` scalar containing the number of intervals (bins) for variable `x`.

`y` – A `double` array containing the data for the second variable.

`yIntervals` – A `int` scalar containing the number of intervals (bins) for variable `y`.

Methods

GetFrequencyTable

```
public double[,] GetFrequencyTable()
```

Description

Returns the two-way frequency table.

Returns

A two-dimensional `double` array containing the two-way frequency table.

Remarks

Intervals of equal length are used. Let `xmin` and `xmax` be the minimum and maximum values in x, respectively, with similiar meanings for `ymin` and `ymax`. Then, the first row of the output table is the tally of observations with the x value less than or equal to `xmin + (xmax - xmin)/xIntervals`, and the y value less than or equal to `ymin + (ymax - ymin)/yIntervals`.

GetFrequencyTable

```
public double[,] GetFrequencyTable(double xLowerBound, double xUpperBound,
double yLowerBound, double yUpperBound)
```

Description

Compute a two-way frequency table using intervals of equal length and user supplied upper and lower bounds, `xLowerBound, xUpperBound, yLowerBound, yUpperBound`.

Parameters

`xLowerBound` – A `double` specifies the right endpoint for `x`.

`xUpperBound` – A `double` specifies the left endpoint for `x`.

`yLowerBound` – A `double` specifies the right endpoint for `y`.

`yUpperBound` – A `double` specifies the left endpoint for `y`.

Returns

A two dimensional `double` array containing the two-way frequency table.

Remarks

The first and last intervals for both variables are semi-infinite in length. `xIntervals` and `yIntervals` must be greater than or equal to 3.

GetFrequencyTableUsingClassmarks

```
public double[,] GetFrequencyTableUsingClassmarks(double[] cx, double[] cy)
```

Description

Returns the two-way frequency table using class marks.

Parameters

`cx` – A `double` array containing the class marks for `x`.

`cy` – A `double` array containing the class marks for `y`.

Returns

A two dimensional `double` array containing the two-way frequency table.

Remarks

Class marks are the midpoints of `xIntervals` and `yIntervals`.

Equally spaced class marks in ascending order must be provided in the arrays `cx` and `cy`. The class marks the midpoints of each interval. Each interval is taken to have length `cx[1] - cx[0]` in the x direction and `cy[1] - cy[0]` in the y direction. The total number of elements in the output table may be less than the number of observations of input data. Arguments `xIntervals` and `yIntervals` must be greater than or equal to 2 for this option.

GetFrequencyTableUsingCutpoints

```
public double[,] GetFrequencyTableUsingCutpoints(double[] cx, double[] cy)
```

Description

Returns the two-way frequency table using cutpoints.

Parameters

`cx` – A `double` array containing the cutpoints for `x`.

`cy` – A `double` array containing the cutpoints for `y`.

Returns

A two dimensional `double` array containing the two-way frequency table.

Remarks

The cutpoints (boundaries) must be provided in the arrays `cx` and `cy`, of length (`xIntervals-1`) and (`yIntervals-1`) respectively. The first row of the output table is the tally of observations for which the x value is less than or equal to `cx[0]`, and the y value is less than or equal to `cy[0]`. This option allows unequal interval lengths. Arguments `cx` and `cy` must be greater than or equal to 2.

Example: TableTwoWay

The data for x in this example is from Hinkley (1977) and Belleman and Hoaglin (1981). The measurements (in inches) are for precipitation in Minneapolis/St. Paul during the month of March for 30 consecutive years. The data for y were created by adding small integers to the data in x.

The first test uses the default tally method which may be appropriate when the range of data is unknown. The minimum and maximum data bounds are displayed.

The second test computes the table using known bounds, where the x lower, x upper, y lower, y upper bounds are chosen so that the intervals will be 0 to 1, 1 to 2, and so on for x and 1 to 2, 2 to 3 and so on for y.

In the third test, the class boundaries are input at the same intervals as in the second test. The first element of cmx and cmy specify the first cutpoint between classes.

The fourth test uses the cutpoints tally option with cutpoints such that the intervals are specified as in the previous tests.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class TableTwoWayEx1
{
    public static void  Main(String[] args)
    {
        int nx = 5;
        int ny = 6;

        double[] x = new double[]{   0.77, 1.74, 0.81, 1.20, 1.95,
                                     1.20, 0.47, 1.43, 3.37, 2.20,
                                     3.00, 3.09, 1.51, 2.10, 0.52,
                                     1.62, 1.31, 0.32, 0.59, 0.81,
                                     2.81, 1.87, 1.18, 1.35, 4.75,
                                     2.48, 0.96, 1.89, 0.9, 2.05};
        double[] y = new double[]{   1.77, 3.74, 3.81, 2.20, 3.95,
                                     4.20, 1.47, 3.43, 6.37, 3.20,
                                     5.00, 6.09, 2.51, 4.10, 3.52,
                                     2.62, 3.31, 3.32, 1.59, 2.81,
                                     5.81, 2.87, 3.18, 4.35, 5.75,
                                     4.48, 3.96, 2.89, 2.9, 5.05};

        TableTwoWay fTbl = new TableTwoWay(x, nx, y, ny);

        double[,] table = fTbl.GetFrequencyTable();

        Console.Out.WriteLine("Example 1 ");
        Console.Out.WriteLine("Use Min and Max for bounds");
        new PrintMatrix("counts").Print(table);

        Console.Out.WriteLine("--------------------------");
        Console.Out.WriteLine("Lower xbounds= " + fTbl.MinimumX);
        Console.Out.WriteLine("Upper xbounds= " + fTbl.MaximumX);
        Console.Out.WriteLine("Lower ybounds= " + fTbl.MinimumY);
        Console.Out.WriteLine("Upper ybounds= " + fTbl.MaximumY);
        Console.Out.WriteLine("--------------------------");

        double xlo = 1.0;
        double xhi = 4.0;
        double ylo = 2.0;
        double yhi = 6.0;
        Console.Out.WriteLine("");
        Console.Out.WriteLine("Use Known bounds");
        table = fTbl.GetFrequencyTable(xlo, xhi, ylo, yhi);
        new PrintMatrix("counts").Print(table);

        double[] cmx = new double[]{0.5, 1.5, 2.5, 3.5, 4.5};
        double[] cmy = new double[]{1.5, 2.5, 3.5, 4.5, 5.5, 6.5};
        table = fTbl.GetFrequencyTableUsingClassmarks(cmx, cmy);
        Console.Out.WriteLine("");
        Console.Out.WriteLine("Use Class Marks");
        new PrintMatrix("counts").Print(table);

        double[] cpx = new double[]{1, 2, 3, 4};
```

```
        double[] cpy = new double[]{2, 3, 4, 5, 6};
        table = fTbl.GetFrequencyTableUsingCutpoints(cpx, cpy);
        Console.Out.WriteLine("");
        Console.Out.WriteLine("Use Cutpoints");
        new PrintMatrix("counts").Print(table);
    }
}
```

Output

```
Example 1
Use Min and Max for bounds
       counts
   0  1  2  3  4  5
0  4  2  4  2  0  0
1  0  4  3  2  1  0
2  0  0  1  2  0  1
3  0  0  0  0  1  2
4  0  0  0  0  0  1

--------------------------
Lower xbounds= 0.32
Upper xbounds= 4.75
Lower ybounds= 1.47
Upper ybounds= 6.37
--------------------------

Use Known bounds
       counts
   0  1  2  3  4  5
0  3  2  4  0  0  0
1  0  5  5  2  0  0
2  0  0  1  3  2  0
3  0  0  0  0  0  2
4  0  0  0  0  1  0


Use Class Marks
       counts
   0  1  2  3  4  5
0  3  2  4  0  0  0
1  0  5  5  2  0  0
2  0  0  1  3  2  0
3  0  0  0  0  0  2
4  0  0  0  0  1  0


Use Cutpoints
       counts
   0  1  2  3  4  5
0  3  2  4  0  0  0
1  0  5  5  2  0  0
2  0  0  1  3  2  0
3  0  0  0  0  0  2
4  0  0  0  0  1  0
```

TableMultiWay Class

```
public class Imsl.Stat.TableMultiWay
```

Tallies observations into a multi-way frequency table.

The `TableMultiWay` class determines the distinct values in multivariate data and computes frequencies for the data. This class accepts the data in the matrix `x`, but performs computations only for the variables (columns) in the first `nkeys` columns of `x` or by the variables specified in `indkeys`. In general, the variables for which frequencies should be computed are discrete; they should take on a relatively small number of different values. Variables that are continuous can be grouped first. `TableMultiWay` can be used to group variables and determine the frequencies of groups.

The read-only property `BalancedTable` returns a `TableBalanced` object. Its `GetValues` method returns an array with the unique values in the vector of the variables and tallies the number of unique values of each variable table. Each combination of one value from each variable forms a cell in a multi-way table. The frequencies of these cells are entered in a table so that the first variable cycles through its values exactly once, and the last variable cycles through its values most rapidly. Some cells cannot correspond to any observations in the data; in other words, "missing cells" are included in table and have a value of 0.

The read-only property `UnbalancedTable` returns a `TableUnbalanced` object. The frequency of each cell is entered in the unbalanced table so that the first variable cycles through its values exactly once and the last variable cycles through its values most rapidly. `table` is returned by `UnbalancedTable` property. All cells have a frequency of at least 1, i.e., there is no "missing cell." The array `listCells`, returned by method `GetListCells` can be considered "parallel" to `table` because row i of `listCells` is the set of `nkeys` values that describes the cell for which row *i* of `table`contains the corresponding frequency.

Properties

BalancedTable

```
public Imsl.Stat.TableMultiWay.TableBalanced BalancedTable {get; }
```

Description

An object containing the balanced table.

Property Value

A `TableMultiWay.TableBalanced` object.

UnbalancedTable

```
public Imsl.Stat.TableMultiWay.TableUnbalanced UnbalancedTable {get; }
```

Description

An object containing the unbalanced table.

Property Value

A `TableMultiWay.TableUnBalanced` object.

Constructors

TableMultiWay

```
public TableMultiWay(double[,] x, int nKeys)
```

Description

Constructor for `TableMultiWay`.

Parameters

`x` – A `double` matrix containing the observations and variables.

`nKeys` – A `int` array containing the variables(columns) for which computations are to be performed.

TableMultiWay

```
public TableMultiWay(double[,] x, int[] indkeys)
```

Description

Constructor for `TableMultiWay`.

Parameters

`x` – A `double` matrix containing the observations and variables.

`indkeys` – A `int` array containing the variables(columns) for which computations are to be performed.

Methods

GetGroups

```
public int[] GetGroups()
```

Description

Returns the number of observations (rows) in each group.

Returns

A `int` array containing the number of observations (row) in each group.

Remarks

The number of groups is the length of the returned array. A group contains observations in `x` that are equal with respect to the method of comparison. If `n` contains the returned integer array, then the first `n[0]` rows of the sorted `x` are group number 1. The next `n[1]` rows of the sorted `x` are group number 2, etc. The last `n[n.length - 1]` rows of the sorted `x` are group number `n.length`.

SetFrequencies

```
public void SetFrequencies(double[] frequencies)
```

Description

Sets the frequencies for each observation in x.

Parameter

`frequencies` – A `double` array containing the frequency for each observation in x.

Remarks

Length of input must be the same as the number of observations or number of rows in `x`.

By default, `frequencies`[] = 1.

Example 1: TableMultiWay

The same data used in SortEx2 is used in this example. It is a 10 x 3 matrix using Columns 0 and 1 as keys. There are two missing values (NaNs) in the keys. NaN is displayed as a ?. Table MultiWay determines the number of groups of different observations.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class TableMultiWayEx1
{
    public static void  Main(String[] args)
    {
        int nKeys = 2;
        double[,] x = {
            {1.0, 1.0, 1.0}, {2.0, 1.0, 2.0},
            {1.0, 1.0, 3.0}, {1.0, 1.0, 4.0},
            {2.0, 2.0, 5.0}, {1.0, 2.0, 6.0},
            {1.0, 2.0, 7.0}, {1.0, 1.0, 8.0},
            {2.0, 2.0, 9.0}, {1.0, 1.0, 9.0}};

        x[4,1] = Double.NaN;
        x[6,0] = Double.NaN;

        PrintMatrix pm = new PrintMatrix("The Input Array");
        PrintMatrixFormat mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
        //    Print the array
        pm.Print(mf, x);
        Console.Out.WriteLine();
```

```
        TableMultiWay tbl = new TableMultiWay(x, nKeys);
        int[] ngroups = tbl.GetGroups();
        Console.Out.WriteLine(" ngroups");
        for (int i = 0; i < ngroups.Length; i++)
            Console.Out.Write(ngroups[i] + "  ");
    }
}
```

Output

```
The Input Array

1    1    1
2    1    2
1    1    3
1    1    4
2    NaN  5
1    2    6
NaN  2    7
1    1    8
2    2    9
1    1    9


 ngroups
5  1  1  1
```

Example 2: TableMultiWay

The table of frequencies for a data matrix of size 30 x 2 is output.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class TableMultiWayEx2
{
    public static void  Main(String[] args)
    {
        int[] indkeys = new int[]{0, 1};
        double[,] x = {
                {0.5, 1.5}, {1.5, 3.5},
                {0.5, 3.5}, {1.5, 2.5},
                {1.5, 3.5}, {1.5, 4.5},
                {0.5, 1.5}, {1.5, 3.5},
                {3.5, 6.5}, {2.5, 3.5},
                {2.5, 4.5}, {3.5, 6.5},
                {1.5, 2.5}, {2.5, 4.5},
                {0.5, 3.5}, {1.5, 2.5},
                {1.5, 3.5}, {0.5, 3.5},
                {0.5, 1.5}, {0.5, 2.5},
                {2.5, 5.5}, {1.5, 2.5},
                {1.5, 3.5}, {1.5, 4.5},
```

```
                {4.5, 5.5}, {2.5, 4.5},
                {0.5, 3.5}, {1.5, 2.5},
                {0.5, 2.5}, {2.5, 5.5}};

            TableMultiWay tbl = new TableMultiWay(x, indkeys);

            int[] nvalues = tbl.BalancedTable.GetNvalues();

            double[] values = tbl.BalancedTable.GetValues();

            Console.Out.WriteLine("          row values");
            for (int i = 0; i < nvalues[0]; i++)
                Console.Out.Write(values[i] + "  ");
            Console.Out.WriteLine("");
            Console.Out.WriteLine("");
            Console.Out.WriteLine("           column values");
            for (int i = 0; i < nvalues[1]; i++)
                Console.Out.Write(values[i + nvalues[0]] + "   ");

            double[] table = tbl.BalancedTable.GetTable();

            Console.Out.WriteLine("");
            Console.Out.WriteLine("");
            Console.Out.WriteLine("        Table");



            Console.Out.Write("      ");
            for (int i = 0; i < nvalues[1]; i++)
                Console.Out.Write(values[i + nvalues[0]] + "   ");
            Console.Out.WriteLine("");
            for (int i = 0; i < nvalues[0]; i++)
            {
                Console.Out.Write(values[i] + "   ");
                for (int j = 0; j < nvalues[1]; j++)
                    Console.Out.Write(table[j + (nvalues[1] * i)] + "   ");

                Console.Out.WriteLine(" ");
            }
        }
    }
```

Output

```
          row values
0.5  1.5  2.5  3.5  4.5

           column values
1.5   2.5   3.5   4.5   5.5   6.5

        Table
      1.5   2.5   3.5   4.5   5.5   6.5
0.5   3   2   4   0   0   0
1.5   0   5   5   2   0   0
2.5   0   0   1   3   2   0
```

```
3.5   0   0   0   0   0   2
4.5   0   0   0   0   1   0
```

Example 3: TableMultiWay

The unbalanced table of frequencies for a data matrix of size 4 x 3 is output.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class TableMultiWayEx3
{
    public static void  Main(String[] args)
    {
        int[] indkeys = new int[]{0, 1};
        double[,] x = {
            {2.0, 5.0, 1.0}, {1.0, 5.0, 2.0},
            {1.0, 6.0, 3.0}, {2.0, 6.0, 4.0}};
        double[] frq = new double[]{1.0, 2.0, 3.0, 4.0};

        TableMultiWay tbl = new TableMultiWay(x, indkeys);
        tbl.SetFrequencies(frq);

        int ncells = tbl.UnbalancedTable.NCells;
        double[] listCells = tbl.UnbalancedTable.GetListCells();
        double[] table = tbl.UnbalancedTable.GetTable();

        PrintMatrix pm = new PrintMatrix("List Cells");
        PrintMatrixFormat mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
        //    Print the array
        pm.Print(mf, listCells);
        Console.Out.WriteLine();

        pm = new PrintMatrix("Unbalanced Table");
        mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
        //    Print the array
        pm.Print(mf, table);
        Console.Out.WriteLine();
    }
}
```

Output

```
List Cells

1
5
1
6
```

```
2
5
2
6


Unbalanced Table

2
3
1
4
```

TableMultiWay.TableBalanced Class

`public class Imsl.Stat.TableMultiWay.TableBalanced`

Tallies the number of unique values of each variable.

Methods

GetNvalues

`public int[] GetNvalues()`

Description

Returns an array of length `nkeys` containing in its *i*-th element (*i*=0,1,...nkeys-1), the number of levels or categories of the *i*-th classification variable (column).

Returns

A `int` array containing the number of levels or for each variable (column) in `x`.

GetTable

`public double[] GetTable()`

Description

Returns an array containing the frequencies for each variable.

Returns

A `double` array containing the frequencies for each variable in `x`.

Remarks

The array is of length `nValues[0] x nValues[1] x ... x nValues[nkeys]` containing the frequencies in the cells of the table to be fit, where `nValues` contains the result from `getNValues`.

Empty cells are included in table, and each element of table is nonnegative. The cells of table are sequenced so that the first variable cycles through its `nValues[0]` categories one time, the second variable cycles through its `nValues[1]` categories `nValues[0]` times, the third variable cycles through its `nValues[2]` categories `nValues[0] * nValues[1]` times, etc., up to the `nkeys`-th variable, which cycles through its `nValues[nkeys - 1]` categories `nValues[0] * nValues[1] * ... * nValues[nkeys - 2]` times.

GetValues

```
public double[] GetValues()
```

Description

Returns the values of the classification variables.

Returns

A `double` array containing the values of the classification variables.

Remarks

`GetValues` returns an array of length `nValues[0] + nValues[1] + ... + nValues[nkeys - 1]`. The first `nValues[0]` elements contain the values for the first classification variable. The next `nValues[1]` contain the values for the second variable. The last `nValues[nkeys - 1]` positions contain the values for the last classification variable, where `nValues` contains the result from `getNValues`.

TableMultiWay.TableUnbalanced Class

```
public class Imsl.Stat.TableMultiWay.TableUnbalanced
```

Tallies the frequency of each cell in `x`.

Property

NCells

```
public int NCells {get; }
```

Description

Returns the number of non-empty cells.

Property Value

A `int` containing the number of non-empty cells.

Methods

GetListCells

```
public double[] GetListCells()
```

Description

Returns for each row, a list of the levels of `nkeys` coorresponding classification variables that describe a cell.

Returns

A `double` array containing the list of levels of `nkeys` corresponding classification variables that describe a cell.

GetTable

```
public double[] GetTable()
```

Description

Returns the frequency for each cell.

Returns

A `double` array containing the frequency for each cell.

Chapter 13: Regression

Types

Usage Notes

The regression models in this chapter include the simple and multiple linear regression models, the multivariate general linear model, and the nonlinear regression model. Functions for fitting regression models, computing summary statistics from a fitted regression, computing diagnostics, and computing confidence intervals for individual cases are provided. This chapter also provides methods for building a model from a set of candidate variables.

Simple and Multiple Linear Regression

The simple linear regression model is

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_1 \; i = 1, 2, \ldots, n$$

where the observed values of the y_i's constitute the responses or values of the dependent variable, the x_i's

are the settings of the independent (explanatory) variable, β_0 and β_1 are the intercept and slope parameters (respectively) and the ε_1's are independently distributed normal errors, each with mean 0 and variance σ^2.

The multiple linear regression model is

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + \varepsilon_1 \; i = 1, 2, \ldots, n$$

where the observed values of the y_i's constitute the responses or values of the dependent variable; the x_{i1}'s,x_{i2}'s,...,x_{ik}'s are the settings of the k independent (explanatory) variables; $\beta_0, \beta_1, \ldots, \beta_k$ are the regression coefficients; and the ε_1's are independently distributed normal errors, each with mean 0 and variance σ^2.

The class `LinearRegression` fits both the simple and multiple linear regression models using a fast Given's transformation and includes an option for excluding the intercept β_0. The responses are input in array `y`, and the independent variables are input in array `x`, where the individual cases correspond to the rows and the variables correspond to the columns.

After the model has been fitted using the `LinearRegression` class, properties such as `CoefficientTTests` can be used to retrieve summary statistics. Predicted values, confidence intervals, and case statistics for the fitted model can be obtained from inner class `LinearRegression.CaseStatistics`.

No Intercept Model

Several functions provide the option for excluding the intercept from a model. In most practical applications, the intercept should be included in the model. For functions that use the sums of squares and crossproducts matrix as input, the no-intercept case can be handled by using the raw sums of squares and crossproducts matrix as input in place of the corrected sums of squares and crossproducts. The raw sums of squares and crossproducts matrix can be computed as $(x_1, x_2, \ldots, x_k, y)^T (x_1, x_2, \ldots, x_k, y)$.

Specification of X for the General Linear Model

Variables used in the general linear model are either continuous or classification variables. Typically, multiple regression models use continuous variables, whereas analysis of variance models use classification variables. Although the notation used to specify analysis of variance models and multiple regression models may look quite different, the models are essentially the same. The term *general linear model* emphasizes that a common notational scheme is used for specifying a model that may contain both continuous and classification variables.

A general linear model is specified by its effects (sources of variation). An effect is referred to in this text as a single variable or a product of variables. (The term *effect* is often used in a narrower sense, referring only to a single regression coefficient.) In particular, an *effect* is composed of one of the following:

1. a single continuous variable
2. a single classification variable
3. several different classification variables
4. several continuous variables, some of which may be the same

5. continuous variables, some of which may be the same, and classification variables, which must be distinct

Effects of the first type are common in multiple regression models. Effects of the second type appear as main effects in analysis of variance models. Effects of the third type appear as interactions in analysis of variance models. Effects of the fourth type appear in polynomial models and response surface models as powers and crossproducts of some basic variables. Effects of the fifth type appear in one-way analysis of covariance models as regression coefficients that indicate lack of parallelism of a regression function across the groups.

The analysis of a general linear model occurs in two stages. The first stage calls class `RegressorsForGLM` to specify all regressors except the intercept. The second stage uses `LinearRegression`, at which point the model will be specified as either having or not having an intercept.

Suppose the data matrix has as its first four columns two continuous variables in Columns 0 and 1 and two classification variables in Columns 2 and 3. The data might appear as follows:

Column 0	Column 1	Column 2	Column 3
11.23	1.23	1.0	5.0
12.12	2.34	1.0	4.0
12.34	1.23	1.0	4.0
4.34	2.21	1.0	5.0
5.67	4.31	2.0	4.0
4.12	5.34	2.0	1.0
4.89	9.31	2.0	1.0
9.12	3.71	2.0	1.0

Each distinct value of a classification variable determines a level. The classification variable in Column 2 has two levels. The classification variable in Column 3 has three levels. (Integer values are recommended, but not required, for values of the classification variables. The values of the classification variables corresponding to the same level must be identical.) Some examples of regression functions and their specifications are as follows:

	Intercept	Class Columns
$\beta_0 + \beta_1 x_1$	`true`	$\{\}$
$\beta_0 + \beta_1 x_1 + \beta_2 x_1^2$	`true`	$\{\}$
$\mu + \alpha_i$	`true`	$\{2\}$
$\mu + \alpha_i + \beta_j + \gamma_{ij}$	`true`	$\{2, 3\}$
μ_{ij}	`false`	$\{2, 3\}$
$\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$	`true`	$\{\}$

	Effects
$\beta_0 + \beta_1 x_1$	{ {0} }
$\beta_0 + \beta_1 x_1 + \beta_2 x_1^2$	{{0}, {0,0}}
$\mu + \alpha_i$	{ {2} }
$\mu + \alpha_i + \beta_j + \gamma_{ij}$	{{2}, {3}, {2, 3}}
μ_{ij}	{ {2, 3} }
$\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$	{{0}, {1}, {0, 1}}
$\mu + \alpha_i + \beta_j + \gamma_{ij}$	{{2}, {0}, {0, 2}}

Variable Selection

Variable selection can be performed by `SelectionRegression`, which computes all best-subset regressions, or by `StepwiseRegression` , which computes stepwise regression. The method used by `SelectionRegression` is generally preferred over that used by `StepwiseRegression` because `SelectionRegression` implicitly examines all possible models in the search for a model that optimizes some criterion while stepwise does not examine all possible models. However, the computer time and memory requirements for `SelectionRegression` can be much greater than that for `StepwiseRegression` when the number of candidate variables is large.

Nonlinear Regression Model

The nonlinear regression model is

$$y_i = f(x_i; \theta) + \varepsilon_i \; i = 1, 2, \ldots, n$$

where the observed values of the y_i's constitute the responses or values of the dependent variable, the x_i's are the known vectors of values of the independent (explanatory) variables, f is a known function of an unknown regression parameter vector θ, and the ε_i's are independently distributed normal errors each with mean 0 and variance σ^2.

Class `NonlinearRegression` performs the least-squares fit to the data for this model.

Weighted Least Squares

Classes throughout the chapter generally allow weights to be assigned to the observations. A `weight` argument is used throughout to specify the weighting for particular rows of X.

Computations that relate to statistical inference-e.g., t tests, F tests, and confidence intervals-are based on the multiple regression model except that the variance of ε_i is assumed to equal σ^2 times the reciprocal of the corresponding weight.

If a single row of the data matrix corresponds to n_i observations, the vector `frequencies` can be used to specify the frequency for each row of X. Degrees of freedom for error are affected by frequencies but are unaffected by weights.

Summary Statistics

Property and methods `LinearRegression.ANOVA`, `LinearRegression.CoefficientTTests`, `NonlinearRegression.GetR()` and `StepwiseRegression.CoefficientVIF` can be used to compute statistics related to a regression for each of the dependent variables fitted by the indicated

regression. The summary statistics include the model analysis of variance table, sequential sums of squares and *F*-statistics, coefficient estimates, estimated standard errors, t-statistics, variance inflation factors and estimated variance-covariance matrix of the estimated regression coefficients.

The summary statistics are computed under the model $y = X\beta + \varepsilon$, where y is the $n \times 1$ vector of responses, X is the $n \times p$ matrix of regressors with rank (X) = r, is the $p \times 1$ vector of regression coefficients, and $\varepsilon$is the $n \times 1$ vector of errors whose elements are independently normally distributed with mean 0 and variance σ^2/w_i.

Given the results of a weighted least-squares fit of this model (with the w_i's as the weights), most of the computed summary statistics are output in the following variables:

`ANOVA` Class

The `ANOVA` property in several of the regression classes returns an `ANOVA` object. Summary statistics can be retrieved via specific "get" methods or the `ANOVA.GetArray()` method. This returns a one-dimensional array. In `StepwiseRegression`, `ANOVA.GetArray()` returns `Double.NaN` for the last two elements of the array because they cannot be computed from the input. The array contains statistics related to the analysis of variance. The sources of variation examined are the regression, error, and total. The first 10 elements of the `ANOVA.GetArray()` and the notation frequently used for these is described in the following table (here, `AOV = ANOVA.GetArray()`):

Model Analysis of Variance Table

Variation Src.	**Deg. of Freedom**	**Sum of Squares**	**Mean Square**	F	p-**value**
Regression	DFR = AOV[0]	SSR = AOV[3]	MSR = AOV[6]	`AOV[8]`	`AOV[9]`
Error	DFE = AOV[1]	SSE = AOV[4]	s^2 = AOV[7]		
Total	DFT = AOV[2]	SST = AOV[5]			

If the model has an intercept (default), the total sum of squares is the sum of squares of the deviations of y_i from its (weighted) mean $\bar{y}$–the so-called *corrected total sum of squares*, denoted by the following:

$$\text{SST} = \sum_{i=1}^{n} w_i (y_i - \bar{y})^2$$

If the model does not have an intercept (`hasIntercept = false`), the total sum of squares is the sum of squares of y_i-the so-called *uncorrected total sum of squares*, denoted by the following:

$$\text{SST} = \sum_{i=1}^{n} w_i y_i^2$$

The error sum of squares is given as follows:

$$\text{SSE} = \sum_{i=1}^{n} w_i (y_i - \hat{y}_i)^2$$

The error degrees of freedom is defined by $\text{DFE} = n - r$.

The estimate of σ^2 is given by $s^2 = \text{SSE}/\text{DFE}$, which is the error mean square.

The computed F statistic for the null hypothesis, $H_0 : \beta_1 = \beta_2 = \ldots \beta_k = 0$, versus the alternative that at least one coefficient is nonzero is given by $F = s^2 = \text{MSR}/s^2$. The p-value associated with the test is the probability of an F larger than that computed under the assumption of the model and the null hypothesis. A small p-value (less than 0.05) is customarily used to indicate there is sufficient evidence from the data to reject the null hypothesis.

The remaining five elements in `AOV` frequently are displayed together with the actual analysis of variance table. The quantities R-squared ($R^2 = \text{AOV}[10]$) and adjusted R-squared

$$R_a^2 = (\text{AOV}[11])$$

are expressed as a percentage and are defined as follows:

$$R^2 = 100\,(\text{SSR}/\text{SST}) = 100\,(1 - \text{SSE}/\text{SST})$$

$$R_a^2 = 100 \max\left\{0, 1 - \frac{s^2}{\text{SST}/\text{DFT}}\right\}$$

The square root of s^2 ($s = \text{AOV}[12]$) is frequently referred to as the estimated standard deviation of the model error.

The overall mean of the responses $\bar{y}$ is output in `AOV[13]`.

The coefficient of variation ($CV = \text{AOV}[14]$) is expressed as a percentage and defined by $\text{CV} = 100s/\bar{y}$.

`LinearRegression.CoefficientTTests`

A nested class within the `LinearRegression` and `StepwiseRegression` classes. The statistics (estimated standard error, t statistic and p-value) associated with each coefficient can be retrieved via associated “Get” methods.

`GetR()`

Estimated variance-covariance matrix of the estimated regression coefficients.

Diagnostics for Individual Cases

Diagnostics for individual cases (observations) are computed by the `LinearRegression.CaseStatistics` class for linear regression.

Statistics computed include predicted values, confidence intervals, and diagnostics for detecting outliers and cases that greatly influence the fitted regression.

The diagnostics are computed under the model $y = X\beta + \varepsilon$, where y is the $n \times 1$ vector of responses, X is the $n \times p$ matrix of regressors with rank(X) = r, β is the $p \times 1$ vector of regression coefficients, and ε is the $n \times 1$ vector of errors whose elements are independently normally distributed with mean 0 and variance ϕ^2/w_i.

Given the results of a weighted least-squares fit of this model (with the w_i’s as the weights), the following five diagnostics are computed:

1. leverage
2. standardized residual
3. jackknife residual
4. Cook's distance
5. DFFITS

The definition of these terms is given in the discussion that follows:Let x_i be a column vector containing the elements of the i-th row of X. A case can be unusual either because of x_i or because of the response y_i. The leverage h_i is a measure of uniqueness of the x_i. The leverage is defined by

$$h_i = [x_i^T \left(X^T W X\right)^- x_i] w_i$$

where $W = \text{diag}\,(w_1, w_2 \ldots, w_n)$ and $\left(X^T W T\right)^-$ denotes a generalized inverse of $X^T W T$. The average value of the h_i's is r/n. Regression functions declare x_i unusual if $h_i > 2r/n$. Hoaglin and Welsch (1978) call a data point highly influential (i.e., a leverage point) when this occurs.

Let e_i denote the residual

$$y_i - \hat{y}_i$$

for the *i*-th case. The estimated variance of e_i is $(1-h_i)s^2 w_i$, where s^2 is the residual mean square from the fitted regression. The *i*-th *standardized residual* (also called the internally studentized residual) is by definition

$$r_i = e_i \sqrt{\frac{w_i}{s^2\,(1-h_i)}}$$

and r_i follows an approximate standard normal distribution in large samples.

The *i*-th *jackknife residual or deleted residual* involves the difference between y_i and its predicted value, based on the data set in which the *i*-th case is deleted. This difference equals $e_i/\,(1-h_i)$. The jackknife residual is obtained by standardizing this difference. The residual mean square for the regression in which the *i*-th case is deleted is as follows:

$$s_i^2 = \frac{(n-r)\,s^2 - w_i e_i^2/\,(1-h_i)}{n-r-1}$$

The jackknife residual is defined as

$$t_i = e_i \sqrt{\frac{w_i}{s_i^2\,(1-h_i)}}$$

and t_i follows a t_i distribution with $n-r \times 1$ degrees of freedom.

Cook's distance for the *i*-th case is a measure of how much an individual case affects the estimated regression coefficients. It is given as follows:

$$D_i = \frac{w_i h_i e_i^2}{rs^2 (1-h_i)^2}$$

Weisberg (1985) states that if D_i exceeds the 50-th percentile of the *F(r, n - r)* distribution, it should be considered large. (This value is about 1. This statistic does not have an *F* distribution.)

DFFITS, like Cook's distance, is also a measure of influence. For the *i*-th case, DFFITS is computed by the formula below.

$$\text{DFFITS}_i = e_i \sqrt{\frac{w_i h_i}{s_i^2 (1-h_i)^2}}$$

Hoaglin and Welsch (1978) suggest that DFFITS greater than

$$2\sqrt{r/n}$$

is large.

Transformations

Transformations of the independent variables are sometimes useful in order to satisfy the regression model. The inclusion of squares and crossproducts of the variables

$$\left(x_1, x_2, x_1^2, x_2^2, x_1 x_2\right)$$

is often needed. Logarithms of the independent variables are used also. (See Draper and Smith 1981, pp. 218-222; Box and Tidwell 1962; Atkinson 1985, pp. 177-180; Cook and Weisberg 1982, pp. 78-86.)

When the responses are described by a nonlinear function of the parameters, a transformation of the model equation often can be selected so that the transformed model is linear in the regression parameters. For example, by taking natural logarithms on both sides of the equation, the exponential model

$$y = e^{\beta_0 + \beta_1 x_1} \varepsilon$$

can be transformed to a model that satisfies the linear regression model provided the ε_i's have a log-normal distribution (Draper and Smith, pp. 222-225).

When the responses are nonnormal and their distribution is known, a transformation of the responses can often be selected so that the transformed responses closely satisfy the regression model, assumptions. The square-root transformation for counts with a Poisson distribution and the arc-sine transformation for binomial proportions are common examples (Snedecor and Cochran 1967, pp. 325-330; Draper and Smith, pp. 237-239).

Missing Values

NaN (Not a Number) is the missing value code used by the regression functions. Use field `Double.NaN` to retrieve NaN. Any element of the data matrix that is missing must be set to `Double.NaN`. In fitting regression models, any observation containing `NaN` for the independent, dependent, weight, or frequency variables is omitted from the computation of the regression parameters.

RegressorsForGLM Class

`public class Imsl.Stat.RegressorsForGLM`

Generates regressors for a general linear model.

Class `RegressorsForGLM` generates regressors for a general linear model from a data matrix. The data matrix can contain classification variables as well as continuous variables. Regressors for effects composed solely of continuous variables are generated as powers and crossproducts. Consider a data matrix containing continuous variables as Columns 3 and 4. The effect indices (3, 3) generate a regressor whose *i*-th value is the square of the *i*-th value in Column 3. The effect indices (3, 4) generates a regressor whose *i*-th value is the product of the *i*-th value in Column 3 with the *i*-th value in Column 4.

Regressors for an effect (source of variation) composed of a single classification variable are generated using indicator variables. Let the classification variable A take on values $a_1, a_2, \ldots, a_n$. From this classification variable, `RegressorsForGLM` creates n indicator variables. For $k = 1, 2, \ldots, n$, we have

$$I_k = \begin{cases} 1 & \text{if } A = a_k \\ 0 & \text{otherwise} \end{cases}$$

For each classification variable, another set of variables is created from the indicator variables. These new variables are called *dummy variables*. Dummy variables are generated from the indicator variables in one of three manners:

1. The dummies are the n indicator variables.
2. The dummies are the first $n-1$ indicator variables.
3. The $n-1$ dummies are defined in terms of the indicator variables so that for balanced data, the usual summation restrictions are imposed on the regression coefficients.

In particular, for dummy method `All`, the dummy variables are $A_k = I_k \, (k = 1, 2, \ldots, n)$. For dummy method `LeaveOutLast`, the dummy variables are $A_k = I_k \, (k = 1, 2, ..., n-1)$. For dummy method `SumToZero`, the dummy variables are $A_k = I_k - I_n \, (k = 1, 2, \ldots, n-1)$. The regressors generated for an effect composed of a single-classification variable are the associated dummy variables.

Let m_j be the number of dummies generated for the j-th classification variable. Suppose there are two classification variables A and B with dummies

$$A_1, A_2, \ldots, A_{m_1}$$

and

$$B_1, B_2, \ldots, B_{m_2}$$

The regressors generated for an effect composed of two classification variables A and B are

$$\begin{aligned} A \otimes B = & \quad (A_1, A_2, \ldots, A_{m_1}) \otimes (B_1, B_2, \ldots, B_{m_2}) \\ = & \quad (A_1B_1, A_1B_2, \ldots, A_1B_{m_2}, A_2, B_1, A_2B_2, \ldots, \\ = & \quad A_2B_{m_2}, \ldots, A_{m_1}, B_1, A_{m_1}, B_2, \ldots, A_{m_1}B_{m_2}) \end{aligned}$$

More generally, the regressors generated for an effect composed of several classification variables and several continuous variables are given by the Kronecker products of variables, where the order of the variables is specified in `SetEffects`. Consider a data matrix containing classification variables in Columns 0 and 1 and continuous variables in Columns 2 and 3. Label these four columns A, B, X_1, and X_2. The regressors generated by the effect indices $(0,1,2,2,3)$ are $A \otimes B \otimes X_1X_1X_2$.

Remarks

Let the data matrix $\mathtt{x} = (A, B, X_1)$, where A and B are classification variables and X_1 is a continuous variable. The model containing the effects A, B, AB, X_1, AX_1, BX_1, and ABX_1 is specified by setting `nClassVariables=2` in the constructor and calling `SetEffects(effects)`, with `int effects[][] = { {0}, {1}, {0, 1}, {2}, {0, 2}, {1, 2}, {0, 1, 2} };`

For this model, suppose that variable A has two levels, A_1 and A_2, and that variable B has three levels, B_1, B_2, and B_3. For each `DummyMethod` option, the regressors in their order of appearance in regressors are given below.

DummyMethod	Regressors
`All`	A_1, A_2, B_1, B_2, B_3, A_1B_1, A_1B_2, A_1B_3, A_2B_1, A_2B_2, $A_2B_3, X1, A_1X_1, A_2X_1$, B_1X_1, B_2X_1, B_3X_1, $A_1B_1X_1$, $A_1B_2X_1$, $A_1B_3X_1$, $A_2B_1X_1$, $A_2B_2X_1$, $A_2B_3X_1$
`LeaveOutLast`	A_1, B_1, B_2, A_1B_1, A_1B_2, X_1, A_1X_1, B_1X_1, B_2X_1, $A_1B_1X_1$, $A_1B_2X_1$
`SumToZero`	$A_1 - A_2$, $B_1 - B_3$, $B_2 - B_3$, $(A_1 - A_2)(B_1 - B_2), (A_1 - A_2)(B_2 - B_3)$, X_1, $(A_1 - A_2)X_1$, $(B_1 - B_3)X_1$, $(B_2 - B_3)X_1$, $(A_1 - A_2)(B_1 - B_2)X_1$, $(A_1 - A_2)(B_2 - B_3)X_1$

Within a group of regressors corresponding to an interaction effect, the indicator variables composing the regressors vary most rapidly for the last classification variable, next most rapidly for the next to last classification variable, etc.

By default, `RegressorsForGLM` internally generates values for effects which correspond to a first order model with `nEffects` = `nContinuousVariables` + `nClassVariables`, where `nContinuousVariables` is the number of continuous variables and `nClassVariables` is the number of classification variables. The variables then are used to create the regressor variables. The effects are ordered such that the first effect corresponds to the first column of `x`, the second effect corresponds to the second column of `x`, etc. A second order model corresponding to the columns (variables) of `x` is generated if `ModelOrder = 2` is used.

The effects array for a first or second order model can be obtained by first using `ModelOrder` followed by `GetEffects`. This array can then be modified and used as the argument to `SetEffects`. This may be an easier way of setting the effects for an almost linear or quadratic model than creating the effects array from scratch.

There are

$$\texttt{nEffects} = \texttt{nClassVariables} + \texttt{nContinuousVariables} + \frac{\texttt{nVar}(\texttt{nVar}-1)}{2}$$

effects, where `nVar` = `nClassVariables`+`nContinuousVariables`. The first `nVar` effects correspond to the columns of `x`, such that the first effect corresponds to the first column of `x`, the second effect corresponds to the second column of `x`, ..., the `nVar`-th effect corresponds to the `nVar`-th column of `x` (i.e. `x[nVar-1]`). The next `nContinuousVariables` effects correspond to squares of the continuous variables. The last $\texttt{nVar}(\texttt{nVar}-1)/2$ effects correspond to the two-variable interactions.

- Let the data matrix $\texttt{x} = (A, B, X_1)$, where A and B are classification variables and X_1 is a continuous variable. The effects generated and order of appearance is

$$A,\ B,\ X_1,\ X_1^2,\ AB,\ AX_1,\ BX_1$$

- Let the data matrix $\texttt{x} = (A, X_1, X_2)$, where A is a classification variable and X_1 and X_2 are continuous variables. The effects generated and order of appearance is

$$A,\ X_1,\ X_2,\ X_1^2,\ X_2^2,\ AX_1,\ AX_2,\ X_1X_2$$

- Let the data matrix $\texttt{x} = (X_1, A, X_2)$, where A is a classification variable and X_1 and X_2 are continuous variables. The effects generated and order of appearance is

$$X_1,\ A,\ X_2,\ X_1^2,\ X_2^2,\ X_1A,\ X_1X_2,\ AX_2$$

Higher-order and more complicated models can be specified using `SetEffects`.

Properties

DummyMethod

```
public Imsl.Stat.RegressorsForGLM.DummyType DummyMethod {get; set; }
```

Description

The dummy method.

Remarks

One of `All` (the default), `LeaveOutLast` or `SumToZero`.

ModelOrder

```
public int ModelOrder {set; }
```

Description

The order of the model.

Remarks

Model order can be specified as 1 or 2. Use `SetEffects` to specify more complicated models. This overrides previously set effects.

NumberOfMissingRows

```
public int NumberOfMissingRows {get; }
```

Description

Returns the number of rows in the regressors matrix containing NaN (not a number).

Remarks

A row of the regressors matrix contains NaN for a regressor when any of the variables involved in generation of the regressor equals NaN.

NumberOfRegressors

```
public int NumberOfRegressors {get; }
```

Description

Returns the number of regressors.

Constructors

RegressorsForGLM

```
public RegressorsForGLM(double[,] x, int nClassVariables)
```

Description

Constructor where the class columns are the first columns.

Parameters

`x` – is an `nObservations` by `nClassVariables`+`nContinuousVariables` array containing the data, where `nObservations` is the number of observations. The columns must be ordered such that the first `nClassVariables` columns contain the class variables and the next `nContinuousVariables` columns contain the continuous variables.

`nClassVariables` – is number of class variables. The number of continuous variables is assumed to be the number of columns in `x`-`nClassVariables`.

RegressorsForGLM

```
public RegressorsForGLM(double[,] x, int[] classColumns)
```

Description

Constructor with an explicit set of class column indices.

Parameters

`x` – is an `nObservations` by `nClassVariables+nContinuousVariables` array containing the data. The columns containing the class variables are specified by `classColumns`.

`classColumns` – is an array containing the column indices, in `x`, of the class variables.

Methods

GetEffects

```
public int[][] GetEffects()
```

Description

Returns the effects.

Returns

a jagged array containing the effects. The number of rows in the matrix is the number of effects. For each row, the values are the 0-based column numbers of `x`.

GetEffectsColumns

```
public int[][] GetEffectsColumns()
```

Description

Returns a mapping of effects to regressor columns.

Returns

A jagged `int` array. The number of rows is equal to the number of effects. Each row contains the column numbers of the regressor matrix into which the corresponding effect is mapped.

GetRegressors

```
public double[,] GetRegressors()
```

Description

Returns the regressor array.

Returns

An array of size number of observations by number of regresssors.

SetEffects

```
public void SetEffects(int[][] effects)
```

Description

Set the effects.

Parameter

`effects` – A jagged array containing the effects. The number of rows in the matrix is the number of effects. For each row, the values are the 0-based column numbers of `x`.

Remarks

This overrides any previously set model order. Effects is a jagged array. The number of rows in the matrix is the number of effects. For each row, the values are the 0-based column numbers of `x`.

Example 1

In the following example, there are two classification variables, *A* and *B*, with two and three values, respectively. Regressors for a one-way model (the default model order) are generated using the the default dummy method. The five regressors generated are A_1, A_2, B_1, B_2 and B_3.

```
using System;
using Imsl.Stat;
using Imsl.Math;


public class RegressorsForGLMEx1
{
    public static void Main(String[] args)
    {
        double[,] x = {
                {10.0,  5.0},
                {20.0, 15.0},
                {20.0, 10.0},
                {10.0, 10.0},
                {10.0, 15.0},
                {20.0,  5.0}
        };

        RegressorsForGLM r = new RegressorsForGLM(x, 2);
        double[,] regressors = r.GetRegressors();
        int n = r.NumberOfRegressors;
        Console.WriteLine("Number of regressors = "+n);
        Console.WriteLine();
        new PrintMatrix("Regressors").Print(regressors);
    }
}
```

Output

```
Number of regressors = 5

    Regressors
   0  1  2  3  4
0  1  0  1  0  0
1  0  1  0  0  1
2  0  1  0  1  0
3  1  0  0  1  0
4  1  0  0  0  1
5  0  1  1  0  0
```

Example 2

In this example, a two-way analysis of covariance model containing all the interaction terms is fit. First, `RegressorsForGLM` is used to produce a matrix of regressors, `regressors`, from the data `x`. Then, `regressors` is used as the input matrix into `Regression` to produce the final fit. The regressors, generated using the dummy method `LeaveOutLast`, are the model whose mean function is

$$\mu + \alpha_i + \beta_j + \gamma_{ij} + \delta x_{ij} + \zeta_i x_{ij} + \eta x_{ij} + \theta_i x_{ij} \quad i = 1,2;\ j = 1,2,3$$

where $\alpha_2 = \beta_3 = \gamma_{21} + \gamma_{22} + \gamma 23 + \zeta_2 + \eta_3 + \theta_{21} + \theta_{22} + \theta_{23} = 0$.

```
using System;
using Imsl.Stat;
using Imsl.Math;


public class RegressorsForEx2
{
    public static void  Main(String[] args)
    {
        double[,] x = {
                {1.0, 1.0, 1.11},
                {1.0, 1.0, 2.22},
                {1.0, 1.0, 3.33},
                {1.0, 2.0, 1.11},
                {1.0, 2.0, 2.22},
                {1.0, 2.0, 3.33},
                {1.0, 3.0, 1.11},
                {1.0, 3.0, 2.22},
                {1.0, 3.0, 3.33},
                {2.0, 1.0, 1.11},
                {2.0, 1.0, 2.22},
                {2.0, 1.0, 3.33},
                {2.0, 2.0, 1.11},
                {2.0, 2.0, 2.22},
                {2.0, 2.0, 3.33},
                {2.0, 3.0, 1.11},
                {2.0, 3.0, 2.22},
                {2.0, 3.0, 3.33}
                      };
        double[] y = {
                        1.0, 2.0, 2.0, 4.0, 4.0, 6.0,
                        3.0, 3.5, 4.0, 4.5, 5.0, 5.5,
                        2.0, 3.0, 4.0, 5.0, 6.0, 7.0
                    };
        int[][] effects = {
                             new int[]{0},
                             new int[]{1},
                             new int[]{0, 1},
                             new int[]{2},
                             new int[]{0, 2},
                             new int[]{1, 2},
                             new int[]{0, 1, 2}
                          };
```

```
int nClassVariables = 2;

RegressorsForGLM r = new RegressorsForGLM(x, nClassVariables);
r.DummyMethod = RegressorsForGLM.DummyType.LeaveOutLast;
r.SetEffects(effects);

int nRegressors = r.NumberOfRegressors;

double[,] regressors = r.GetRegressors();

Imsl.Math.PrintMatrixFormat pmf = new Imsl.Math.PrintMatrixFormat();
pmf.SetColumnLabels(new String[]{"Alpha1", "Beta1", "Beta2",
                                 "Gamma11", "Gamma12", "Delta", "Zeta1", "Eta1", "Eta2",
                                 "Theta11", "Theta12"});
new Imsl.Math.PrintMatrix("Regressors").Print(pmf,regressors);

LinearRegression regression = new LinearRegression(nRegressors, true);
regression.Update(regressors, y);

Console.WriteLine("        * * * Analysis of Variance * * *");
ANOVA anova = regression.ANOVA;
Object[,] table = new Object[15,2];
table[0,0] = "Degrees of freedom for the model       ";
table[0,1] = anova.DegreesOfFreedomForModel;
table[1,0] = "Degrees of freedom for the error       ";
table[1,1] = anova.DegreesOfFreedomForError;
table[2,0] = "Total degrees of freedom               ";
table[2,1] = anova.TotalDegreesOfFreedom;
table[3,0] = "Sum of squares for the model           ";
table[3,1] = anova.SumOfSquaresForModel;
table[4,0] = "Sum of squares for error               ";
table[4,1] = anova.SumOfSquaresForError;
table[5,0] = "Total sum of squares                   ";
table[5,1] = anova.TotalSumOfSquares;
table[6,0] = "Model mean square                      ";
table[6,1] = anova.ModelMeanSquare;
table[7,0] = "Error mean square                      ";
table[7,1] = anova.ErrorMeanSquare;
table[8,0] = "F-statistic                            ";
table[8,1] = anova.F;
table[9,0] = "p-value                                ";
table[9,1] = anova.P;
table[10,0] = "R-squared                             ";
table[10,1] = anova.RSquared;
table[11,0] = "Adjusted R-squared                    ";
table[11,1] = anova.AdjustedRSquared;
table[12,0] = "Standard deviation for the model error";
table[12,1] = anova.ModelErrorStdev;
table[13,0] = "Overall mean of y                     ";
table[13,1] = anova.MeanOfY;
table[14,0] = "Coefficient of variation              ";
table[14,1] = anova.CoefficientOfVariation;
pmf = new Imsl.Math.PrintMatrixFormat();
pmf.SetNoColumnLabels();
pmf.SetNoRowLabels();
pmf.NumberFormat = "0.0000";
```

```
        new Imsl.Math.PrintMatrix().Print(pmf, table);
    }
}
```

Output

```
                                    Regressors
    Alpha1  Beta1  Beta2  Gamma11  Gamma12  Delta  Zeta1  Eta1  Eta2  Theta11  Theta12
 0    1       1      0       1        0     1.11   1.11   1.11  0      1.11     0
 1    1       1      0       1        0     2.22   2.22   2.22  0      2.22     0
 2    1       1      0       1        0     3.33   3.33   3.33  0      3.33     0
 3    1       0      1       0        1     1.11   1.11   0     1.11   0        1.11
 4    1       0      1       0        1     2.22   2.22   0     2.22   0        2.22
 5    1       0      1       0        1     3.33   3.33   0     3.33   0        3.33
 6    1       0      0       0        0     1.11   1.11   0     0      0        0
 7    1       0      0       0        0     2.22   2.22   0     0      0        0
 8    1       0      0       0        0     3.33   3.33   0     0      0        0
 9    0       1      0       0        0     1.11   0      1.11  0      0        0
10    0       1      0       0        0     2.22   0      2.22  0      0        0
11    0       1      0       0        0     3.33   0      3.33  0      0        0
12    0       0      1       0        0     1.11   0      0     1.11   0        0
13    0       0      1       0        0     2.22   0      0     2.22   0        0
14    0       0      1       0        0     3.33   0      0     3.33   0        0
15    0       0      0       0        0     1.11   0      0     0      0        0
16    0       0      0       0        0     2.22   0      0     0      0        0
17    0       0      0       0        0     3.33   0      0     0      0        0

        * * * Analysis of Variance * * *

Degrees of freedom for the model         11.0000
Degrees of freedom for the error         6.0000
Total degrees of freedom                 17.0000
Sum of squares for the model             43.9028
Sum of squares for error                 0.8333
Total sum of squares                     44.7361
Model mean square                        3.9912
Error mean square                        0.1389
F-statistic                              28.7364
p-value                                  0.0003
R-squared                                98.1372
Adjusted R-squared                       94.7221
Standard deviation for the model error   0.3727
Overall mean of y                        3.9722
Coefficient of variation                 9.3821
```

RegressorsForGLM.DummyType Enumeration

```
public enumeration Imsl.Stat.RegressorsForGLM.DummyType
```

Dummy variable types.

Fields

All

```
public Imsl.Stat.RegressorsForGLM.DummyType All
```

Description

The *n* indicator variables are the dummy variables.

LeaveOutLast

```
public Imsl.Stat.RegressorsForGLM.DummyType LeaveOutLast
```

Description

The dummies are the first *n-1* indicator variables.

SumToZero

```
public Imsl.Stat.RegressorsForGLM.DummyType SumToZero
```

Description

The $n-1$ dummies are defined in terms of the indicator variables so that for balanced data, the usual summation restrictions are imposed on the regression coefficients.

LinearRegression Class

```
public class Imsl.Stat.LinearRegression
```

Fits a multiple linear regression model with or without an intercept.

Fits a multiple linear regression model with or without an intercept. If the constructor argument `hasIntercept` is true, the multiple linear regression model is

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + \varepsilon_i \quad i = 1, 2, \ldots, n$$

where the observed values of the y_i's constitute the responses or values of the dependent variable, the x_{i1}'s, x_{i2}'s, $\ldots, x_{ik}$'s are the settings of the independent variables, $\beta_0, \beta_1, \ldots, \beta_k$ are the regression coefficients, and the e_i's are independently distributed normal errors each with mean zero and variance σ^2/w_i. If `hasIntercept` is `false`, β_0 is not included in the model.

`LinearRegression` computes estimates of the regression coefficients by minimizing the sum of squares of the deviations of the observed response y_i from the fitted response

$$\hat{y}_i$$

for the observations. This minimum sum of squares (the error sum of squares) is in the ANOVA output and denoted by

$$\text{SSE} = \sum_{i=1}^{n} w_i (y_i - \hat{y}_i)^2$$

In addition, the total sum of squares is output in the ANOVA table. For the case, `hasIntercept` is true; the total sum of squares is the sum of squares of the deviations of y_i from its mean

$$\bar{y}$$

–the so-called *corrected total sum of squares*; it is denoted by

$$\text{SST} = \sum_{i=1}^{n} w_i (y_i - \bar{y})^2$$

For the case `hasIntercept` is `false`, the total sum of squares is the sum of squares of y_i –the so-called *uncorrected total sum of squares*; it is denoted by

$$\text{SST} = \sum_{i=1}^{n} y_i^2$$

See Draper and Smith (1981) for a good general treatment of the multiple linear regression model, its analysis, and many examples.

In order to compute a least-squares solution, `LinearRegression` performs an orthogonal reduction of the matrix of regressors to upper triangular form. Givens rotations are used to reduce the matrix. This method has the advantage that the loss of accuracy resulting from forming the crossproduct matrix used in the normal equations is avoided, while not requiring the storage of the full matrix of regressors. The method is described by Lawson and Hanson, pages 207-212.

From a general linear model fitted using the w_i's as the weights, inner class Imsl.Stat.LinearRegression.CaseStatistics (p. 643) can also compute predicted values, confidence intervals, and diagnostics for detecting outliers and cases that greatly influence the fitted regression. Let x_i be a column vector containing elements of the i-th row of X. Let $W = diag(w_1, w_2, ..., w_n)$. The leverage is defined as

$$h_i = [x_i^T (X^T W X)^- x_i] w_i$$

(In the case of linear equality restrictions on β, the leverage is defined in terms of the reduced model.) Put $D = diag(d_1, d_2, ..., d_k)$ with $d_j = 1$ if the j-th diagonal element of R is positive and 0 otherwise. The leverage is computed as $h_i = (a^T D a) w_i$ where a is a solution to $R^T a = x_i$. The estimated variance of

$$\hat{y}_i = x_i^T \hat{\beta}$$

is given by $h_i s^2 / w_i$, where $s^2 = SSE/DFE$. The computation of the remainder of the case statistics follows easily from their definitions.

Let e_i denote the residual

$$y_i - \hat{y}_i$$

for the ith case. The estimated variance of e_i is $(1 - h_i) s^2 / w_i$ where s^2 is the residual mean square from the fitted regression. The ith standardized residual (also called the internally studentized residual) is by definition

$$r_i = e_i \sqrt{\frac{w_i}{s^2 (1 - h_i)}}$$

and r_i follows an approximate standard normal distribution in large samples.

The *i*th jackknife residual or deleted residual involves the difference between y_i and its predicted value based on the data set in which the *i*th case is deleted. This difference equals $e_i/(1-h_i)$. The jackknife residual is obtained by standardizing this difference. The residual mean square for the regression in which the *i*th case is deleted is

$$s_i^2 = \frac{(n-r)s^2 - w_i e_i^2/(1-h_i)}{n-r-1}$$

The jackknife residual is defined to be

$$t_i = e_i \sqrt{\frac{w_i}{s_i^2(1-h_i)}}$$

and t_i follows a t distribution with $n-r-1$ degrees of freedom.

Cook's distance for the *i*th case is a measure of how much an individual case affects the estimated regression coefficients. It is given by

$$D_i = \frac{w_i h_i e_i^2}{r s^2 (1-h_i)^2}$$

Weisberg (1985) states that if D_i exceeds the 50-th percentile of the $F(r, n-r)$ distribution, it should be considered large. (This value is about 1. This statistic does not have an F distribution.)

DFFITS, like Cook's distance, is also a measure of influence. For the *i*th case, DFFITS is computed by the formula

$$DFFITS_i = e_i \sqrt{\frac{w_i h_i}{s_i^2 (1-h_i)^2}}$$

Hoaglin and Welsch (1978) suggest that $DFFITS_i$ greater than

$$2\sqrt{r/n}$$

is large.

Often predicted values and confidence intervals are desired for combinations of settings of the effect variables not used in computing the regression fit. This can be accomplished using a single data matrix by including these settings of the variables as part of the data matrix and by setting the response equal to `Double.NaN`. `LinearRegression` will omit the case when performing the fit and a predicted value and confidence interval for the missing response will be computed from the given settings of the effect variables.

Properties

ANOVA

```
public Imsl.Stat.ANOVA ANOVA {get; }
```

Description

Returns an analysis of variance table and related statistics.

Property Value

An `ANOVA` table and related statistics.

CoefficientTTests

```
public Imsl.Stat.LinearRegression.CoefficientTTestsValue CoefficientTTests
{get; }
```

Description

Returns statistics relating to the regression coefficients.

Property Value

A `LinearRegression.CoefficientTTestsValue` object which possesses regression statistics related to the coefficients.

HasIntercept

```
public bool HasIntercept {get; }
```

Description

A `bool` which indicates whether or not an intercept is in this regression model.

Property Value

A `bool` indicating whether or not the model includes an intercept.

Rank

```
public int Rank {get; }
```

Description

Returns the rank of the matrix.

Property Value

An `int` which specifies the matrix rank.

Constructor

LinearRegression

```
public LinearRegression(int nVariables, bool hasIntercept)
```

Description

Constructs a new linear regression object.

Parameters

`nVariables` – An `int` which specifies the number of regression variables.

`hasIntercept` – A `bool` which indicates whether or not an intercept is in this regression model.

Methods

GetCaseStatistics

```
virtual public Imsl.Stat.LinearRegression.CaseStatistics
GetCaseStatistics(double[] x, double y)
```

Description

Returns the case statistics for an observation.

Parameters

`x` – A `double` array containing the independent (explanatory) variables. Its length must be equal to the number of variables set in the `LinearRegression` constructor.

`y` – A `double` representing the dependent (response) variable.

Returns

The `CaseStatistics` for the observation.

GetCaseStatistics

```
virtual public Imsl.Stat.LinearRegression.CaseStatistics
GetCaseStatistics(double[] x, double y, double w)
```

Description

Returns the case statistics for an observation and a weight.

Parameters

`x` – A `double` array containing the independent (explanatory) variables. Its length must be equal to the number of variables set in the constructor.

`y` – A `double` representing the dependent (response) variable.

`w` – A `double` representing the weight.

Returns

The `CaseStatistics` for the observation.

GetCaseStatistics

```
virtual public Imsl.Stat.LinearRegression.CaseStatistics
GetCaseStatistics(double[] x, double y, int pred)
```

Description

Returns the case statistics for an observation and future response count for the desired prediction interval.

Parameters

`x` – A `double` array containing the independent (explanatory) variables. Its length must be equal to the number of variables set in the constructor.

`y` – A `double` representing the dependent (response) variable.

`pred` – An `int` representing the number of future responses for which the prediction interval is desired on the average of the future responses.

Returns

The `CaseStatistics` for the observation.

GetCaseStatistics

```
virtual public Imsl.Stat.LinearRegression.CaseStatistics
GetCaseStatistics(double[] x, double y, double w, int pred)
```

Description

Returns the case statistics for an observation, weight, and future response count for the desired prediction interval.

Parameters

`x` – A `double` array containing the independent (explanatory) variables. Its length must be equal to the number of variables set in the constructor.

`y` – A `double` representing the dependent (response) variable.

`w` – A `double` representing the weight.

`pred` – An `int` representing the number of future responses for which the prediction interval is desired on the average of the future responses.

Returns

The `CaseStatistics` for the observation.

GetCoefficients

```
public double[] GetCoefficients()
```

Description

Returns the regression coefficients.

Returns

A `double` array containing the regression coefficients.

Remarks

If `HasIntercept` is `false` its length is equal to the number of variables. If `HasIntercept` is true then its length is the number of variables plus one and the 0-th entry is the value of the intercept.

If the model is not full rank, the regression coefficients are not uniquely determined. In this case, a warning is issued and a solution with all linearly dependent regressors set to zero is returned.

GetR

```
public double[,] GetR()
```

Description

Returns a copy of the *R* matrix.

Returns

A `double` matrix containing a copy of the *R* matrix.

Remarks

R is the upper triangular matrix containing the *R* matrix from a QR decomposition of the matrix of regressors.

GetRank

```
public int GetRank()
```

Description

Returns the rank of the matrix.

Returns

An `int` containing the rank of the matrix.

Update

```
public void Update(double[] x, double y)
```

Description

Updates the regression object with a new observation.

Parameters

`x` – A `double` array containing the independent (explanatory) variables.

`y` – A `double` representing the dependent (response) variable.

Remarks

`x.Length` must be equal to the number of variables set in the constructor.

Update

```
public void Update(double[] x, double y, double w)
```

Description

Updates the regression object with a new observation and weight.

Parameters

`x` – A `double` array containing the independent (explanatory) variables.

`y` – A `double` representing the dependent (response) variable.

`w` – A `double` representing the weight.

Remarks

`x.Length` must be equal to the number of variables set in the constructor.

Update

```
public void Update(double[,] x, double[] y)
```

Description

Updates the regression object with a new set of observations.

Parameters

x – A `double` matrix containing the independent (explanatory) variables.

y – A `double` array containing the dependent (response) variables.

Remarks

The number of rows in `x` must equal `y.Length` and the number of columns must be equal to the number of variables set in the constructor.

Update

```
public void Update(double[,] x, double[] y, double[] w)
```

Description

Updates the regression object with a new set of observations and weights.

Parameters

x – A `double` matrix containing the independent (explanatory) variables.

y – A `double` array containing the dependent (response) variables.

w – A `double` array representing the weights.

Remarks

The number of rows in `x` must equal `y.Length` and the number of columns must be equal to the number of variables set in the constructor.

Example: Linear Regression

The coefficients of a simple linear regression model, without an intercept, are computed.

```
using System;
using Imsl.Stat;

public class LinearRegressionEx1
{
    public static void  Main(String[] args)
    {
        // y = 4*x0 + 3*x1
        LinearRegression r = new LinearRegression(2, false);
        double[] c = new double[]{4, 3};
        double[] x0 = {1, 5};
        double[] x1 = {0, 2};
        double[] x2 = {-1, 4};

        r.Update(x0, 1 * c[0] + 5 * c[1]);
        r.Update(x1, 0 * c[0] + 2 * c[1]);
        r.Update(x2, - 1 * c[0] + 4 * c[1]);
        double[] coef = r.GetCoefficients();
        Console.Out.WriteLine
            ("The computed regression coefficients are {" +
            coef[0] + ", " + coef[1] + "}");
    }
}
```

Output

```
The computed regression coefficients are {4, 3}
```

Example: Linear Regression Case Statistics

Selected case statistics of a simple linear regression model, with an intercept, are computed.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class LinearRegressionEx2
{
    public static void  Main(String[] args)
    {
        LinearRegression r = new LinearRegression(2, true);
        double[] y = {3, 4, 5, 7, 7, 8, 9};
        double[,] x = {{1, 1},{1, 2},{1, 3},{1,4},{1,5},{0,6},{1,7}};
        double[,] results = new double[7,5];
        double[] confint = new double[2];
        r.Update(x, y);
        double[] xTmp = new double[2];
        for (int k=0; k<7; k++){
            xTmp[0] = x[k,0];
            xTmp[1] = x[k,1];
            LinearRegression.CaseStatistics cs = r.GetCaseStatistics(xTmp,y[k]);
            results[k,0] = cs.JackknifeResidual;
            results[k,1] = cs.CooksDistance;
            results[k,2] = cs.DFFITS;
            confint = cs.ConfidenceInterval;
            results[k,3] = confint[0];
            results[k,4] = confint[1];
            }
        PrintMatrix p = new PrintMatrix("Selected Case Statistics");
        PrintMatrixFormat mf = new PrintMatrixFormat();
        String[] labels = {"Jackknife Residual.","Cook's D","DFFITS",
         "[Conf. Interval", "on the Mean]"};
      mf.SetColumnLabels(labels);
      mf.NumberFormat = "0.#########";
        p.Print(mf, results);
    }
}
```

Output

```
                          Selected Case Statistics
   Jackknife Residual.   Cook's D       DFFITS     [Conf. Interval  on the Mean]
0     -0.343038693      0.044885519  -0.323965838    2.260946521    3.996196336
1     -0.327326835      0.018390805  -0.207019668    3.467412069    4.818302217
2     -0.337597012      0.011129861  -0.16122517     4.612581629    5.701704085
3      Infinity         0.275862069   Infinity       5.648231067    6.694626076
4     -0.417763902      0.023512274  -0.236601469    6.562984697    7.808443875
5      NaN              NaN           NaN            6.736357974    9.263642026
```

```
6     -0.742307489      0.372413793  -0.995910003    8.201118103    10.227453326
```

LinearRegression.CaseStatistics Class

```
public class Imsl.Stat.LinearRegression.CaseStatistics
```

Inner Class `CaseStatistics` allows for the computation of predicted values, confidence intervals, and diagnostics for detecting outliers and cases that greatly influence the fitted regression.

Properties

ConfidenceInterval

```
virtual public double[] ConfidenceInterval {get; }
```

Description

Returns the Confidence Interval on the mean for an observation.

Property Value

A `double`[2] array containing the Confidence Interval for the observation.

ConLevelMean

```
virtual public double ConLevelMean {set; }
```

Description

Sets the confidence level for two-sided interval estimates on the mean, in percent.

Property Value

A `double` containing the confidence level on the mean.

Remarks

By default, `ConLevelMean` = 0.95.

ConLevelPred

```
virtual public double ConLevelPred {set; }
```

Description

Sets the confidence level for two-sided prediction intervals, in percent.

Property Value

A `double` containing the confidence level.

Remarks

By default, ConLevelPred = 0.95.

CooksDistance

```
virtual public double CooksDistance {get; }
```

Description

Returns Cook's Distance for an observation.

Property Value

A double containing Cook's Distance for an observation.

DFFITS

```
virtual public double DFFITS {get; }
```

Description

Returns DFFITS for an observation.

Property Value

A double containing the DFFITS value for an observation.

JackknifeResidual

```
virtual public double JackknifeResidual {get; }
```

Description

Returns the Jackknife Residual for an observation.

Property Value

A double containing the Jackknife Residual for an observation.

Leverage

```
virtual public double Leverage {get; }
```

Description

Returns the Leverage for an observation.

Property Value

A double containing the Leverage for an observation.

ObservedResponse

```
virtual public double ObservedResponse {get; }
```

Description

Returns the observed response for an observation.

Property Value

A double containing the observed response for an observation.

PredictedResponse

```
virtual public double PredictedResponse {get; }
```

Description

Returns the predicted response for an observation.

Property Value

A `double` containing the predicted response for an observation.

PredictionInterval

```
virtual public double[] PredictionInterval {get; }
```

Description

Returns the Prediction Interval for an observation.

Property Value

A `double[2]` array containing the Prediction Interval for the observation.

Residual

```
virtual public double Residual {get; }
```

Description

Returns the `Residual` for an observation.

Property Value

A `double` containing the residual for an observation.

StandardizedResidual

```
virtual public double StandardizedResidual {get; }
```

Description

Returns the Standardized Residual for an observation.

Property Value

A `double` containing the Standardized Residual for an observation.

Statistics

```
virtual public double[] Statistics {get; }
```

Description

Returns the case statistics for an observation.

Property Value

A `double[12]` array containing the case statistics.

Remarks

Elements 0 through 11 contain the following:

Index	*Description*
0	Observed response
1	Predicted response
2	Residual
3	Leverage
4	Standardized residual
5	Jackknife residual
6	Cook's distance
7	DFFITS
8,9	Confidence interval on the mean
10,11	Prediction interval

Example: Linear Regression Case Statistics

Selected case statistics of a simple linear regression model, with an intercept, are computed.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class LinearRegressionEx2
{
    public static void  Main(String[] args)
    {
        LinearRegression r = new LinearRegression(2, true);
        double[] y = {3, 4, 5, 7, 7, 8, 9};
        double[,] x = {{1, 1},{1, 2},{1, 3},{1,4},{1,5},{0,6},{1,7}};
        double[,] results = new double[7,5];
        double[] confint = new double[2];
        r.Update(x, y);
        double[] xTmp = new double[2];
        for (int k=0; k<7; k++){
            xTmp[0] = x[k,0];
            xTmp[1] = x[k,1];
            LinearRegression.CaseStatistics cs = r.GetCaseStatistics(xTmp,y[k]);
            results[k,0] = cs.JackknifeResidual;
            results[k,1] = cs.CooksDistance;
            results[k,2] = cs.DFFITS;
            confint = cs.ConfidenceInterval;
            results[k,3] = confint[0];
            results[k,4] = confint[1];
            }
        PrintMatrix p = new PrintMatrix("Selected Case Statistics");
        PrintMatrixFormat mf = new PrintMatrixFormat();
        String[] labels = {"Jackknife Residual.","Cook's D","DFFITS",
         "[Conf. Interval", "on the Mean]"};
      mf.SetColumnLabels(labels);
      mf.NumberFormat = "0.#########";
        p.Print(mf, results);
    }
}
```

Output

```
                          Selected Case Statistics
   Jackknife Residual.   Cook's D       DFFITS     [Conf. Interval  on the Mean]
0     -0.343038693      0.044885519  -0.323965838    2.260946521    3.996196336
1     -0.327326835      0.018390805  -0.207019668    3.467412069    4.818302217
2     -0.337597012      0.011129861  -0.16122517     4.612581629    5.701704085
3      Infinity         0.275862069   Infinity       5.648231067    6.694626076
4     -0.417763902      0.023512274  -0.236601469    6.562984697    7.808443875
5      NaN              NaN           NaN            6.736357974    9.263642026
6     -0.742307489      0.372413793  -0.995910003    8.201118103    10.227453326
```

LinearRegression.CoefficientTTestsValue Class

`public class Imsl.Stat.LinearRegression.CoefficientTTestsValue`

`CoefficientTTestsValue` contains statistics related to the regression coefficients.

Constructor

CoefficientTTestsValue

`public CoefficientTTestsValue(Imsl.Stat.LinearRegression lr)`

Description

`CoefficientTTestsValue` contains statistics related to the regression coefficients.

Parameter

`lr` – A `LinearRegression` object used to calculate the regression statistics.

Methods

GetCoefficient

`public double GetCoefficient(int i)`

Description

Returns the estimate for a coefficient.

Parameter

`i` – An `int` which specifies the index of the coefficient whose estimate is to be returned.

Returns

A `double` which specifies the estimate for the *i*-th coefficient.

GetPValue

```
public double GetPValue(int i)
```

Description

Returns the p-value for the two-sided test.

Parameter

`i` – An `int` which specifies the index of the coefficient whose *p*-value estimate is to be returned.

Returns

A `double` which specifies the estimated *p*-value for the *i*-th coefficient estimate.

GetStandardError

```
public double GetStandardError(int i)
```

Description

Returns the estimated standard error for a coefficient estimate.

Parameter

`i` – An `int` which specifies the index of the coefficient whose standard error estimate is to be returned.

Returns

A `double` which specifies the estimated standard error for the *i*-th coefficient estimate.

GetTStatistic

```
public double GetTStatistic(int i)
```

Description

Returns the t-statistic for the test that the *i*-th coefficient is zero.

Parameter

`i` – An `int` which specifies the index of the coefficient whose standard error estimate is to be returned.

Returns

A `double` which specifies the estimated standard error for the *i*-th coefficient estimate.

NonlinearRegression Class

```
public class Imsl.Stat.NonlinearRegression
```

Fits a multivariate nonlinear regression model using least squares.

The nonlinear regression model is

$$y_i = f(x_i;\theta) + \varepsilon_i \quad i = 1, 2, \ldots, n$$

where the observed values of the y_i constitute the responses or values of the dependent variable, the known x_i are vectors of values of the independent (explanatory) variables, θ is the vector of p regression parameters, and the ε_i are independently distributed normal errors each with mean zero and variance σ^2. For this model, a least squares estimate of θ is also a maximum likelihood estimate of θ.

The residuals for the model are

$$e_i(\theta) = y_i - f(x_i;\theta) \quad i = 1, 2, \ldots, n$$

A value of θ that minimizes

$$\sum_{i=1}^{n} [e_i(\theta)]^2$$

is the least-squares estimate of θ calculated by this class. `NonlinearRegression` accepts these residuals one at a time as input from a user-supplied function. This allows `NonlinearRegression` to handle cases where n is so large that data cannot reside in an array but must reside in a secondary storage device.

`NonlinearRegression` is based on MINPACK routines `LMDIF` and `LMDER` by More' et al. (1980). `NonlinearRegression` uses a modified Levenberg-Marquardt method to generate a sequence of approximations to the solution. Let $\hat{\theta}_c$ be the current estimate of θ. A new estimate is given by

$$\hat{\theta}_c + s_c$$

where s_c is a solution to

$$(J(\hat{\theta}_c)^T J(\hat{\theta}_c) + \mu_c I)s_c = J(\hat{\theta}_c)^T e(\hat{\theta}_c)$$

Here, $J(\hat{\theta}_c)$ is the Jacobian evaluated at $\hat{\theta}_c$.

The algorithm uses a "trust region" approach with a step bound of $\hat{\delta}_c$. A solution of the equations is first obtained for $\mu_c = 0$. If $||s_c||_2 < \delta_c$, this update is accepted; otherwise, μ_c is set to a positive value and another solution is obtained. The method is discussed by Levenberg (1944), Marquardt (1963), and Dennis and Schnabel (1983, pages 129 - 147, 218 - 338).

Forward finite differences are used to estimate the Jacobian numerically unless the user supplied function computes the derivatives. In this case the Jacobian is computed analytically via the user-supplied function.

`NonlinearRegression` does not actually store the Jacobian but uses fast Givens transformations to construct an orthogonal reduction of the Jacobian to upper triangular form. The reduction is based on fast Givens transformations (see Golub and Van Loan 1983, pages 156-162, Gentleman 1974). This method has two main advantages:

1. The loss of accuracy resulting from forming the crossproduct matrix used in the equations for s_c is avoided.

2. The n x p Jacobian need not be stored saving space when $n > p$.

A weighted least squares fit can also be performed. This is appropriate when the variance of ε_i in the nonlinear regression model is not constant but instead is σ^2/w_i. Here, w_i are weights input via the user supplied function. For the weighted case, `NonlinearRegression` finds the estimate by minimizing a weighted sum of squares error.

Programming Notes

Nonlinear regression allows users to specify the model's functional form. This added flexibility can cause unexpected convergence problems for users who are unaware of the limitations of the algorithm. Also, in many cases, there are possible remedies that may not be immediately obvious. The following is a list of possible convergence problems and some remedies. No one-to-one correspondence exists between the problems and the remedies. Remedies for some problems may also be relevant for the other problems.

1. A local minimum is found. Try a different starting value. Good starting values can often be obtained by fitting simpler models. For example, for a nonlinear function

   $$f(x;\theta) = \theta_1 e^{\theta_2 x}$$

   good starting values can be obtained from the estimated linear regression coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ from a simple linear regression of *ln y* on *ln x*. The starting values for the nonlinear regression in this case would be

   $$\theta_1 = e^{\hat{\beta}_0} \text{ and } \theta_2 = \hat{\beta}_1$$

   If an approximate linear model is unclear, then simplify the model by reducing the number of nonlinear regression parameters. For example, some nonlinear parameters for which good starting values are known could be set to these values. This simplifies the approach to computing starting values for the remaining parameters.

2. The estimate of θ is incorrectly returned as the same or very close to the initial estimate.

   - The scale of the problem may be orders of magnitude smaller than the assumed default of 1 causing premature stopping. For example, if the sums of squares for error is less than approximately $(2.22e^{-16})^2$, the routine stops. See Example 3, which shows how to shut down some of the stopping criteria that may not be relevant for your particular problem and which also shows how to improve the speed of convergence by the input of the scale of the model parameters.
   - The scale of the problem may be orders of magnitude larger than the assumed default causing premature stopping. The information with regard to the input of the scale of the model parameters in Example 3 is also relevant here. In addition, the maximum allowable step size Imsl.Stat.NonlinearRegression.MaxStepsize (p. 654) in Example 3 may need to be increased.
   - The residuals are input with accuracy much less than machine accuracy, causing premature stopping because a local minimum is found. Again see Example 3 to see how to change some default tolerances. If you cannot improve the precision of the computations of the residual, you need to use method Imsl.Stat.NonlinearRegression.Digits (p. 652) to indicate the actual number of good digits in the residuals.

3. The model is discontinuous as a function of θ. There may be a mistake in the user-supplied function. Note that the function $f(x;\theta)$ can be a discontinuous function of x.
4. The `R` matrix value given by Imsl.Stat.NonlinearRegression.R (p. 655) is inaccurate. If only a function is supplied try providing the Imsl.Stat.NonlinearRegression.IDerivative (p. 662). If the derivative is supplied try providing only Imsl.Stat.NonlinearRegression.IFunction (p. 663).
5. Overflow occurs during the computations. Make sure the user-supplied functions do not overflow at some value of θ.
6. The estimate of θ is going to infinity. A parameterization of the problem in terms of reciprocals may help.
7. Some components of θ are outside known bounds. This can sometimes be handled by making a function that produces artificially large residuals outside of the bounds (even though this introduces a discontinuity in the model function).

Note that the Imsl.Stat.NonlinearRegression.Solve(Imsl.Stat.NonlinearRegression.IFunction) (p. 657) method must be called before using any property as a right operand, otherwise the value is `null`.

Properties

AbsoluteTolerance

```
virtual public double AbsoluteTolerance {set; }
```

Description

The absolute function tolerance.

Property Value

A `double` scalar value specifying the absolute function tolerance.

Remarks

The tolerance must be greater than or equal to zero. By default, `AbsoluteTolerance` = 4.93e-32.

Exception

`System.ArgumentException` is thrown if `AbsoluteTolerance` is set less than 0

Coefficients

```
virtual public double[] Coefficients {get; }
```

Description

The regression coefficients.

Property Value

A `double` array containing the regression coefficients or `null` if Imsl.Stat.NonlinearRegression.Solve(Imsl.Stat.NonlinearRegression.IFunction) (p. 657) has not been called.

DFError

```
virtual public double DFError {get; }
```

Description

The degrees of freedom for error.

Property Value

A `double` which specifies the degrees of freedom for error or `null` if Imsl.Stat.NonlinearRegression.Solve(Imsl.Stat.NonlinearRegression.IFunction) (p. 657) has not been called.

Digits

```
virtual public int Digits {set; }
```

Description

The number of good digits in the residuals.

Property Value

An `int` specifying the number of good digits in the residuals.

Remarks

The number of digits must be greater than zero. By default, `Digits` = 15.

Exception

`System.ArgumentException` is thrown if `Digits` is set less than or equal to 0

ErrorStatus

```
virtual public int ErrorStatus {get; }
```

Description

Characterizes the performance of `NonlinearRegression`.

Property Value

An `int` specifying information about convergence.

Remarks

Value	Description
0	All convergence tests were met.
1	Scaled step tolerance was satisfied. The current point may be an approximate local solution, or the algorithm is making very slow progress and is not near a solution, or `StepTolerance` is too big.
2	Scaled actual and predicted reductions in the function are less than or equal to the relative function convergence tolerance `RelativeTolerance`.
3	Iterates appear to be converging to a noncritical point. Incorrect gradient information, a discontinuous function, or stopping tolerances being too tight may be the cause.
4	Five consecutive steps with the maximum stepsize have been taken. Either the function is unbounded below, or has a finite asymptote in some direction, or the `MaxStepsize` is too small.

See Also: RelativeTolerance (p. 655), StepTolerance (p. 656), MaxStepsize (p. 654)

FalseConvergenceTolerance

```
virtual public double FalseConvergenceTolerance {set; }
```

Description

The false convergence tolerance.

Property Value

A `double` scalar value specifying the false convergence tolerance.

Remarks

The tolerance must be greater than or equal to zero. By default, `FalseConvergenceTolerance` = 2.22e-14.

Exception

`System.ArgumentException` is thrown if `FalseConvergenceTolerance` is set less than 0

GradientTolerance

```
virtual public double GradientTolerance {set; }
```

Description

The gradient tolerance.

Property Value

A `double` specifying the gradient tolerance used to compute the gradient.

Remarks

The tolerance must be greater than or equal to zero. By default, `GradientTolerance`= 6.055e-6.

Exception

`System.ArgumentException` is thrown if `GradientTolerance` is set less than 0

Guess

```
virtual public double[] Guess {set; }
```

Description

The initial guess of the parameter values.

Property Value

A `double` array of initial values for the parameters.

Remarks

By default, `Guess` is an array of zeroes.

InitialTrustRegion

```
virtual public double InitialTrustRegion {set; }
```

Description

The initial trust region radius.

Property Value

A `double` scalar value specifying the initial trust region radius.

Remarks

The initial trust radius must be greater than zero. By default, `InitialTrustRegion` is set based on the initial scaled Cauchy step.

Exception

`System.ArgumentException` is thrown if `InitialTrustRegion` is less than or equal to 0

MaxIterations

```
virtual public int MaxIterations {set; }
```

Description

The maximum number of iterations allowed during optimization

Property Value

An `int` specifying the maximum number of iterations allowed during optimization.

Remarks

The value must be greater than 0. By default, `MaxIterations` = 100.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is set less than or equal to 0

MaxStepsize

```
virtual public double MaxStepsize {set; }
```

Description

The maximum allowable stepsize.

Property Value

A nonnegative `double` value specifying the maximum allowable stepsize.

Remarks

The maximum allowable stepsize must be greater than zero.

If this property is not set then the maximum stepsize is set to a default value based on a scaled *theta*.

Exception

`System.ArgumentException` is thrown if `MaxStepsize` is less than or equal to 0

R

```
virtual public double[,] R {get; }
```

Description

A copy of the *R* matrix.

Property Value

A two dimensional `double` array containing a copy of the *R* matrix or `null` if Imsl.Stat.NonlinearRegression.Solve(Imsl.Stat.NonlinearRegression.IFunction) (p. 657) has not been called.

Remarks

The upper triangular matrix containing the *R* matrix from a QR decomposition of the matrix of regressors.

Rank

```
virtual public int Rank {get; }
```

Description

The rank of the matrix.

Property Value

An `int` which specifies the rank of the matrix or `null` if Imsl.Stat.NonlinearRegression.Solve(Imsl.Stat.NonlinearRegression.IFunction) (p. 657) has not been called.

RelativeTolerance

```
virtual public double RelativeTolerance {set; }
```

Description

The relative function tolerance

Property Value

A `double` scalar value specifying the relative function tolerance.

Remarks

The relative function tolerance must be greater than or equal to zero. By default, `RelativeTolerance` = 1.0e-20.

Exception

`System.ArgumentException` is thrown if `RelativeTolerance` is set less than 0

Scale

```
virtual public double[] Scale {set; }
```

Description

The scaling array for *theta*.

Property Value

A `double` array containing the scaling values for the parameters (*theta*).

Remarks

The elements of the scaling array must be greater than zero. `Scale` is used mainly in scaling the gradient and the distance between points. If good starting values of *theta* are known and are nonzero, then a good choice is `Scale[i]=1.0/theta[i]`. Otherwise, if *theta* is known to be in the interval (-10.e5, 10.e5), set `Scale[i]=10.e-5`. By default, the elements of the scaling array are set to 1.0.

By default, `Scale` is an array of ones.

Exception

`System.ArgumentException` is thrown thrown if any of the elements of `Scale` is set less than or equal to 0

StepTolerance

```
virtual public double StepTolerance {set; }
```

Description

The step tolerance.

Property Value

A `double` scalar value specifying the step tolerance used to step between two points.

Remarks

The step tolerance must be greater than or equal to zero. bY default `StepTolerance` = 3.667e-11.

Exception

`System.ArgumentException` is thrown if `StepTolerance` is set less than 0

Constructor

NonlinearRegression

```
public NonlinearRegression(int nparm)
```

Description

Constructs a new nonlinear regression object.

Parameter

`nparm` – An `int` which specifies the number of unknown parameters in the regression.

Methods

GetCoefficient

```
virtual public double GetCoefficient(int i)
```

Description

Returns the estimate for a coefficient.

Parameter

`i` – An `int` which specifies the index of a coefficient whose estimate is to be returned.

Returns

A `double` which contains the estimate for the *i*-th coefficient or `null` if Imsl.Stat.NonlinearRegression.Solve(Imsl.Stat.NonlinearRegression.IFunction) (p. 657) has not been called.

GetSSE

```
virtual public double GetSSE()
```

Description

Returns the sums of squares for error.

Returns

A `double` which contains the sum of squares for error or `null` if Imsl.Stat.NonlinearRegression.Solve(Imsl.Stat.NonlinearRegression.IFunction) (p. 657) has not been called.

Solve

```
virtual public double[] Solve(Imsl.Stat.NonlinearRegression.IFunction F)
```

Description

Solves the least squares problem and returns the regression coefficients.

Parameter

`F` – An Imsl.Stat.NonlinearRegression.IFunction (p. 663) whose coefficients are to be computed.

Returns

A `double` array containing the regression coefficients.

Exceptions

`Imsl.Stat.TooManyIterationsException` is thrown when the number of allowed iterations is exceeded

`Imsl.Stat.NegativeFreqException` is thrown when the specified frequency is negative

`Imsl.Stat.NegativeWeightException` is thrown when the weight is negative

`Imsl.Stat.NoProgressException` is thrown if the algorithm is not making any progress.

Example 1: Nonlinear Regression using Finite Differences

In this example a nonlinear model is fitted. The derivatives are obtained by finite differences.

```
using System;
using Imsl.Math;
using Imsl.Stat;
public class NonlinearRegressionEx1 : NonlinearRegression.IFunction
{
    public bool f(double[] theta, int iobs, double[] frq, double[] wt, double[] e)
    {
            double[] ydata =
                new double[]{   54.0, 50.0, 45.0, 37.0, 35.0,
                                25.0, 20.0, 16.0, 18.0, 13.0,
                                 8.0, 11.0,  8.0,  4.0,  6.0};
            double[] xdata =
                new double[]{    2.0,  5.0, 7.0, 10.0, 14.0,
                                19.0, 26.0, 31.0, 34.0, 38.0,
                                45.0, 52.0, 53.0, 60.0, 65.0};
            bool iend;
            int nobs = 15;

            if (iobs < nobs)
            {
                wt[0] = 1.0;
                frq[0] = 1.0;
                iend = true;
                e[0] = ydata[iobs] - theta[0] * Math.Exp(theta[1] * xdata[iobs]);
            }
            else
            {
                iend = false;
            }
            return iend;
    }
    public static void Main(String[] args)
    {
        int nparm = 2;
        double[] theta = new double[]{60.0, - 0.03};
        NonlinearRegression regression = new NonlinearRegression(nparm);
        regression.Guess = theta;
        NonlinearRegression.IFunction fcn = new NonlinearRegressionEx1();
        double[] coef = regression.Solve(fcn);

        Console.Out.WriteLine
```

```
            ("The computed regression coefficients are {" + coef[0] + ", "
            + coef[1] + "}");
        Console.Out.WriteLine("The computed rank is " + regression.Rank);
        Console.Out.WriteLine("The degrees of freedom for error are " +
            regression.DFError);
        Console.Out.WriteLine("The sums of squares for error is "
            + regression.GetSSE());
        new PrintMatrix("R from the QR decomposition ").Print(regression.R);
    }
}
```

Output

```
The computed regression coefficients are {58.6065629318535, -0.039586447289031}
The computed rank is 2
The degrees of freedom for error are 13
The sums of squares for error is 49.459299862472
     R from the QR decomposition
          0                1
0  1.87385998175942  1139.92836180938
1  0                 1139.79754530433
```

Example 2: Nonlinear Regression with User-supplied Derivatives

In this example a nonlinear model is fitted. The derivatives are supplied by the user.

```
using System;
using Imsl.Math;
using Imsl.Stat;
public class NonlinearRegressionEx2 : NonlinearRegression.IDerivative
{


        double[] ydata = new double[]{
            54.0, 50.0, 45.0, 37.0, 35.0, 25.0, 20.0,
            16.0, 18.0, 13.0, 8.0, 11.0, 8.0, 4.0, 6.0};
        double[] xdata = new double[]{
            2.0, 5.0, 7.0, 10.0, 14.0, 19.0, 26.0, 31.0,
            34.0, 38.0, 45.0, 52.0, 53.0, 60.0, 65.0};
        bool iend;
        int nobs = 15;

        public bool f(double[] theta, int iobs, double[] frq, double[] wt, double[] e)
        {

            if (iobs < nobs)
            {
                wt[0] = 1.0;
                frq[0] = 1.0;
```

```
            iend = true;
            e[0] = ydata[iobs] - theta[0] * Math.Exp(theta[1] * xdata[iobs]);
        }
        else
        {
            iend = false;
        }
        return iend;
    }

    public bool derivative(double[] theta, int iobs, double[] frq,
        double[] wt, double[] de)
    {
        if (iobs < nobs)
        {
            wt[0] = 1.0;
            frq[0] = 1.0;
            iend = true;
            de[0] = - Math.Exp(theta[1] * xdata[iobs]);
            de[1] = (- theta[0]) * xdata[iobs] *
                Math.Exp(theta[1] * xdata[iobs]);
        }
        else
        {
            iend = false;
        }
        return iend;
    }
public static void Main(String[] args)
{
    int nparm = 2;
    double[] theta = new double[]{60.0, - 0.03};
    NonlinearRegression regression = new NonlinearRegression(nparm);
    regression.Guess = theta;
    double[] coef = regression.Solve(new NonlinearRegressionEx2());

    Console.Out.WriteLine("The computed regression coefficients are {" +
        coef[0] + ", " + coef[1] + "}");
    Console.Out.WriteLine("The computed rank is " + regression.Rank);
    Console.Out.WriteLine("The degrees of freedom for error are " +
        regression.DFError);
    Console.Out.WriteLine("The sums of squares for error is " +
        regression.GetSSE());
    new PrintMatrix("R from the QR decomposition ").Print(regression.R);
}
}
```

Output

```
The computed regression coefficients are {58.6065629254192, -0.0395864472775247}
The computed rank is 2
The degrees of freedom for error are 13
The sums of squares for error is 49.4592998624722
```

```
     R from the QR decomposition
          0                 1
0  1.87385998422826  1139.92837730064
1  0                 1139.79757620697
```

Example 3: NonlinearRegression using Set Methods

In this example, some nondefault tolerances and scales are used to fit a nonlinear model. The data is 1.e-10 times the data of Example 1. In order to fit this model without rescaling the data, we first set the absolute function tolerance to 0.0. The default value would cause the program to terminate after one iteration because the residual sum of squares is roughly 1.e-19. We also set the relative function tolerance to 0.0. The gradient tolerance is properly scaled for this problem so we leave it at its default value. Finally, we set the elements of scale to the absolute value of the recipricol of the starting value. The derivatives are obtained by finite differences.

```
using System;
using Imsl.Math;
using Imsl.Stat;
public class NonlinearRegressionEx3 : NonlinearRegression.IFunction
{
    public bool f(double[] theta, int iobs, double[] frq, double[] wt, double[] e)
    {
            double[] ydata = new double[]{
                54e-10, 50e-10, 45e-10, 37e-10, 35e-10, 25e-10, 20e-10,
                16e-10, 18e-10, 13e-10, 8e-10, 11e-10, 8e-10, 4e-10, 6e-10};
            double[] xdata = new double[]{
                2.0, 5.0, 7.0, 10.0, 14.0, 19.0, 26.0, 31.0, 34.0, 38.0,
                45.0, 52.0, 53.0, 60.0, 65.0};
            bool iend;
            int nobs = 15;
            if (iobs < nobs)
            {
                wt[0] = 1.0;
                frq[0] = 1.0;
                iend = true;
                e[0] = ydata[iobs] - theta[0] * Math.Exp(theta[1] * xdata[iobs]);
            }
            else
            {
                iend = false;
            }
            return iend;
    }
    public static void Main(String[] args)
    {
        int nparm = 2;
        double[] theta = new double[]{6e-9, - 0.03};
        double[] scale = new double[nparm];
        NonlinearRegression regression = new NonlinearRegression(nparm);
        regression.Guess = theta;
        regression.AbsoluteTolerance = 0.0;
        regression.RelativeTolerance = 0.0;
        scale[0] = 1.0 / Math.Abs(theta[0]);
```

```
        scale[1] = 1.0 / Math.Abs(theta[1]);
        regression.Scale = scale;
        NonlinearRegression.IFunction fcn = new NonlinearRegressionEx3();
        double[] coef = regression.Solve(fcn);
        Console.Out.WriteLine("The computed regression coefficients are {" +
                    coef[0] + ", " + coef[1] + "}");
        Console.Out.WriteLine("The computed rank is " + regression.Rank);
        Console.Out.WriteLine("The degrees of freedom for error are " +
         regression.DFError);
        Console.Out.WriteLine("The sums of squares for error is " +
         regression.GetSSE());
        new PrintMatrix("R from the QR decomposition ").Print(regression.R);
    }
}
```

Output

```
The computed regression coefficients are {5.78378362108798E-09, -0.0396252538296399}
The computed rank is 2
The degrees of freedom for error are 13
The sums of squares for error is 5.16637661043416E-19
       R from the QR decomposition
          0                   1
0  1.87310563212442  5.74734586541055E-09
1  0                 5.8371399105394E-11

Imsl.Stat.NonlinearRegression: Scaled step tolerance was satisfied. The current point may be an approximate local s
```

NonlinearRegression.IDerivative Interface

```
public interface Imsl.Stat.NonlinearRegression.IDerivative :
Imsl.Stat.NonlinearRegression.IFunction
```

Public interface for the user supplied function to compute the derivative for `NonlinearRegression`.

Method

derivative

```
abstract public bool derivative(double[] theta, int iobs, double[] frq,
double[] wt, double[] de)
```

Description

Computes the weight, frequency, and partial derivatives of the residual given the parameter vector *theta*

for a single observation.

Parameters

`theta` – An input `double` array which contains the parameter values of the regression function.

`iobs` – An input `int` value indicating the observation index.

`frq` – An output `double` array of length 1 containing the frequency for observation `y[iobs]`.

`wt` – An output `double` array of length 1 containing the weight for the observation `y[iobs]`.

`de` – An output `double` array containing the partial derivatives of the error (residual) for observation `y[iobs]`.

Returns

A `bool` value representing the completion indicator. `true` indicates `iobs` is less than the number of observations. `false` indicates `iobs` is greater than or equal to the number of observations and `wt`, `freq`, and `de` are not output.

Remarks

The length of *theta* corresponds to the number of unknown parameters in the regression function.

The function is evaluated at observation `y[iobs]`.

Use `wt` = 1.0 for equal weighting (unweighted least squares).

The length of `de` corresponds to the number of unknown parameters in the regression function.

NonlinearRegression.IFunction Interface

`public interface Imsl.Stat.NonlinearRegression.IFunction`

Public interface for the user supplied function for `NonlinearRegression`.

Method

f

```
abstract public bool f(double[] theta, int iobs, double[] frq, double[] wt,
double[] e)
```

Description

Computes the weight, frequency, and residual given the parameter vector `theta` for a single observation.

Parameters

`theta` – An input `double` array containing the parameter values of the model.

`iobs` – An input `int` value indicating the observation index.

`frq` – An output `double` array of length 1 containing the frequency for observation `y[iobs]`.

`wt` – An output `double` array of length 1 containing the weight for observation `y[iobs]`.

`e` – An output `double` array of length 1 which contains the error (residual) for observation `y[iobs]`.

Returns

A `bool` value representing the completion indicator. `true` indicates `iobs` is less than the number of observations. `false` indicates `iobs` is greater than or equal to the number of observations and `wt`, `freq`, and `e` are not output.

Remarks

The length of `theta` corresponds to the number of unknown parameters in the model.

The function is evaluated at observation `y[iobs]`.

Use `wt` = 1.0 for equal weighting (unweighted least squares).

SelectionRegression Class

```
public class Imsl.Stat.SelectionRegression
```

Selects the best multiple linear regression models.

Class `SelectionRegression` finds the best subset regressions for a regression problem with three or more independent variables. Typically, the intercept is forced into all models and is not a candidate variable. In this case, a sum-of-squares and crossproducts matrix for the independent and dependent variables corrected for the mean is computed internally. Optionally, `SelectionRegression` supports user-calculated sum-of-squares and crossproducts matrices; see the description of the Imsl.Stat.SelectionRegression.Compute(System.Double[0:,0:],System.Double[]) (p. 668) method.

"Best" is defined by using one of the following three criteria:

- R^2 (in percent)

$$R^2 = 100(1 - \frac{\text{SSE}_p}{\text{SST}})$$

- R_a^2 (adjusted R^2)

$$R_a^2 = 100[1 - (\frac{n-1}{n-p})\frac{\text{SSE}_p}{\text{SST}}]$$

Note that maximizing the R_a^2 is equivalent to minimizing the residual mean squared error:

$$\frac{\text{SSE}_p}{(n-p)}$$

- Mallow's C_p statistic

$$C_p = \frac{\text{SSE}_p}{s^2_{\text{k}}} + 2p - n$$

Here, *n* is equal to the sum of the frequencies (or the number of rows in `x` if frequencies are not specified in the `Compute` method), and SST is the total sum-of-squares. *k* is the number of candidate or independent variables, represented as the `nCandidate` argument in the `SelectionRegression` constructor. SSE_p is the error sum-of-squares in a model containing *p* regression parameters including β_0 (or *p* - 1 of the *k* candidate variables). Variable

$$S^2_{\text{k}}$$

is the error mean square from the model with all *k* variables in the model. Hocking (1972) and Draper and Smith (1981, pp. 296-302) discuss these criteria.

Class `SelectionRegression` is based on the algorithm of Furnival and Wilson (1974). This algorithm finds the maximum number of good saved candidate regressions for each possible subset size. For more details, see method MaximumGoodSaved. These regressions are used to identify a set of best regressions. In large problems, many regressions are not computed. They may be rejected without computation based on results for other subsets; this yields an efficient technique for considering all possible regressions.

There are cases when the user may want to input the variance-covariance matrix rather than allow it to be calculated. This can be accomplished using the appropriate `Compute` method. Three situations in which the user may want to do this are as follows:

1. The intercept is not in the model. A raw (uncorrected) sum of squares and crossproducts matrix for the independent and dependent variables is required. Argument `nObservations` must be set to 1 greater than the number of observations. Form A^TA, where *A = [A, Y]*, to compute the raw sum-of-squares and crossproducts matrix.

2. An intercept is a candidate variable. A raw (uncorrected) sum of squares and crossproducts matrix for the constant regressor (= 1.0), independent, and dependent variables is required for `cov`. In this case, `cov` contains one additional row and column corresponding to the constant regressor. This row and column contain the sum-of-squares and crossproducts of the constant regressor with the independent and dependent variables. The remaining elements in `cov` are the same as in the previous case. Argument `nObservations` must be set to 1 greater than the number of observations.

3. There are *m* variables that must be forced into the models. A sum-of-squares and crossproducts matrix adjusted for the *m* variables is required (calculated by regressing the candidate variables on the variables to be forced into the model). Argument `nObservations` must be set to *m* less than the number of observations.

Programming Notes

`SelectionRegression` can save considerable CPU time over explicitly computing all possible regressions. However, the function has some limitations that can cause unexpected results for users who are unaware of the limitations of the software.

1. For $k+1 > -\log_2(\varepsilon)$, where ε is the largest relative spacing for double precision, some results can be incorrect. This limitation arises because the possible models indicated (the model numbers 1, 2, ..., 2k) are stored as floating-point values; for sufficiently large k, the model numbers cannot be stored exactly. On many computers, this means `SelectionRegression` (for $k > 49$) can produce incorrect results.

2. `SelectionRegression` eliminates some subsets of candidate variables by obtaining lower bounds on the error sum-of-squares from fitting larger models. First, the full model containing all independent variables is fit sequentially using a forward stepwise procedure in which one variable enters the model at a time, and criterion values and model numbers for all the candidate variables that can enter at each step are stored. If linearly dependent variables are removed from the full model, a "VariablesDeleted" warning is issued. In this case, some submodels that contain variables removed from the full model because of linear dependency can be overlooked if they have not already been identified during the initial forward stepwise procedure. If this warning is issued and you want the variables that were removed from the full model to be considered in smaller models, you can rerun the program with a set of linearly independent variables.

Properties

CriterionOption

```
virtual public Imsl.Stat.SelectionRegression.Criterion CriterionOption {get;
set; }
```

Description

The criterion option used to calculate the regression estimates.

Property Value

An `int` containing the criterion option.

Remarks

By default for all criteria, subset size 1,2, ..., k = `nCandidate` are considered. However, for R^2 the maximum number of subsets can be restricted using property Imsl.Stat.SelectionRegression.MaximumSubsetSize (p. 667).

Criterion Option	Description
RSquared	For R^2, subset sizes 1, 2, ..., `MaximumSubsetSize` are examined. This is the default with `MaximumSubsetSize` = `nCandidate`.
AdjustedRSquared	For Adjusted R^2, subset sizes 1, 2, ..., `nCandidate` are examined.
MallowsCP	For Mallow's C_p Subset sizes 1, 2, ..., `nCandidate` are examined.

See Also: RSquared (p. 675), AdjustedRSquared (p. 675), MallowsCP (p. 675)

MaximumBestFound

```
virtual public int MaximumBestFound {set; }
```

Description

The maximum number of best regressions to be found.

Property Value

An `int` containing the maximum number of best regressions to be reported.

Remarks

If the R^2 criterion option is selected, the `MaximumBestFound` best regressions for each subset size examined are reported. If the adjusted R^2 or Mallow's C_p criteria are selected, the `MaximumBestFound` among all possible regressions are found.

By default, `MaximumBestFound` = 1.

See Also: RSquared (p. 675), AdjustedRSquared (p. 675), MallowsCP (p. 675)

MaximumGoodSaved

```
virtual public int MaximumGoodSaved {set; }
```

Description

The maximum number of good regressions for each subset size saved.

Property Value

An `int` containing the maximum number of good regressions saved for each subset size.

Remarks

`MaximumGoodSaved` must be greater than or equal to Imsl.Stat.SelectionRegression.MaximumBestFound (p. 667). Normally, `MaximumGoodSaved` should be less than or equal to 10. It should never need be larger than `MaximumSubsetSize`, the maximum number of subsets for any subset size. Computing time required is inversely related to `MaximumGoodSaved`.

The default value is maximum(10,Imsl.Stat.SelectionRegression.MaximumSubsetSize (p. 667)).

MaximumSubsetSize

```
virtual public int MaximumSubsetSize {set; }
```

Description

The maximum subset size if R^2 criterion is used.

Property Value

An `int` containing the maximum subset size when R^2 criterion is used.

Remarks

By default, `MaximumSubsetSize` = `nCandidate`.

See Also: RSquared (p. 675), AdjustedRSquared (p. 675), MallowsCP (p. 675)

Statistics

```
virtual public Imsl.Stat.SelectionRegression.SummaryStatistics Statistics {get;
}
```

Description

A `SummaryStatistics` object.

Property Value

A `SummaryStatistics` object containing the Coefficient statistics.

Constructor

SelectionRegression

```
public SelectionRegression(int nCandidate)
```

Description

Constructs a new `SelectionRegression` object.

Parameter

`nCandidate` – An `int` containing the number of candidate variables (independent variables).

Remarks

`nCandidate` must be greater than 2.

Methods

Compute

```
virtual public void Compute(double[,] x, double[] y)
```

Description

Computes the best multiple linear regression models.

Parameters

`x` – A `double` matrix containing the observations of the candidate (independent) variables.

`y` – A `double` array containing the observations of the dependent variable.

Remarks

The number of columns in `x` must be equal to the number of variables set in the constructor.

Exceptions

`Imsl.Stat.NoVariablesException` is thrown if no variables can enter any model

`Imsl.Stat.TooManyObsDeletedException` is thrown if more observations have been deleted than were originally entered

`Imsl.Stat.MoreObsDelThanEnteredException` is thrown if more observations are being deleted from the output covariance matrix than were originally entered

`Imsl.Stat.DiffObsDeletedException` is thrown if different observations are being deleted from the return matrix than were originally entered

Compute

```
virtual public void Compute(double[,] x, double[] y, double[] weights)
```

Description

Computes the best weighted multiple linear regression models.

Parameters

`x` – A `double` matrix containing the observations of the candidate (independent) variables.

`y` – A `double` array containing the observations of the dependent variable.

`weights` – A `double` array containing the weight for each of the observations.

Remarks

The number of columns in `x` must be equal to the number of variables set in the constructor.

Exceptions

`Imsl.Stat.NoVariablesException` is thrown if no variables can enter any model

`Imsl.Stat.NegativeWeightException` is thrown if a weight is less than zero.

`Imsl.Stat.TooManyObsDeletedException` is thrown if more observations have been deleted than were originally entered

`Imsl.Stat.MoreObsDelThanEnteredException` is thrown if more observations are being deleted from the output covariance matrix than were originally entered

`Imsl.Stat.DiffObsDeletedException` is thrown if different observations are being deleted from the return matrix than were originally entered

Compute

```
virtual public void Compute(double[,] x, double[] y, double[] weights, double[]
frequencies)
```

Description

Computes the best weighted multiple linear regression models using frequencies for each observation.

Parameters

`x` – A `double` matrix containing the observations of the candidate (independent) variables.

`y` – A `double` array containing the observations of the dependent variable.

`weights` – A `double` array containing the weight for each of the observations.

`frequencies` – A `double` array containing the frequency for each of the observations of `x`.

Remarks

The number of columns in `x` must be equal to the number of variables set in the constructor.

Exceptions

`Imsl.Stat.NoVariablesException` is thrown if no variables can enter any model

`Imsl.Stat.NegativeFreqException` is thrown if a frequency is less than zero.

`Imsl.Stat.NegativeWeightException` is thrown if a weight is less than zero.

`Imsl.Stat.TooManyObsDeletedException` is thrown if more observations have been deleted than were originally entered

`Imsl.Stat.MoreObsDelThanEnteredException` is thrown if more observations are being deleted from the output covariance matrix than were originally entered

`Imsl.Stat.DiffObsDeletedException` is thrown if different observations are being deleted from the return matrix than were originally entered

Compute

```
virtual public void Compute(double[,] cov, int nObservations)
```

Description

Computes the best multiple linear regression models using a user-supplied covariance matrix.

Parameters

`cov` – A `double` matrix containing a variance-covariance or sum-of- squares and crossproducts matrix, in which the last column must correspond to the dependent variable.

`nObservations` – An `int` containing the number of observations used to compute `cov`.

Remarks

`cov` can be computed using the Imsl.Stat.Covariances (p. 545) class.

Exception

`Imsl.Stat.NoVariablesException` is thrown if no variables can enter any model

Example 1: SelectionRegression

This example uses a data set from Draper and Smith (1981, pp. 629-630). Class `SelectionRegression` is invoked to find the best regression for each subset size using the R^2 criterion.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class SelectionRegressionEx1
{
    public static void  Main(String[] args)
    {
           double[,] x =  {   {7.0, 26.0, 6.0, 60.0},
                              {1.0, 29.0, 15.0, 52.0},
                              {11.0, 56.0, 8.0, 20.0},
                              {11.0, 31.0, 8.0, 47.0},
```

```
                            {7.0, 52.0, 6.0, 33.0},
                            {11.0, 55.0, 9.0, 22.0},
                            {3.0, 71.0, 17.0, 6.0},
                            {1.0, 31.0, 22.0, 44.0},
                            {2.0, 54.0, 18.0, 22.0},
                            {21.0, 47.0, 4.0, 26},
                            {1.0, 40.0, 23.0, 34.0},
                            {11.0, 66.0, 9.0, 12.0},
                            {10.0, 68.0, 8.0, 12.0}
                        };

           double[] y = new double[]{   78.5, 74.3, 104.3,
                                        87.6, 95.9, 109.2,
                                        102.7, 72.5, 93.1,
                                        115.9, 83.8, 113.3,
                                        109.4};
        SelectionRegression sr = new SelectionRegression(4);
        sr.Compute(x, y);
        SelectionRegression.SummaryStatistics stats = sr.Statistics;

        for (int i = 1; i <= 4; i++)
        {
            double[] tmpCrit = stats.GetCriterionValues(i);
            int[,] indvar = stats.GetIndependentVariables(i);
            Console.Out.WriteLine("Regressions with "+i+" variable(s)  (R-squared)");
            for (int j = 0; j < tmpCrit.GetLength(0); j++)
            {
                Console.Out.Write("     " + tmpCrit[j] + "        ");
                for (int k = 0; k < indvar.GetLength(1); k++)
                    Console.Out.Write(indvar[j,k] + "   ");
                Console.Out.WriteLine("");
            }
            Console.Out.WriteLine("");
        }

        // Setup a PrintMatrix object for use in the loop below.
        PrintMatrix pm = new PrintMatrix();
        pm.SetColumnSpacing(8);
        PrintMatrixFormat tst = new PrintMatrixFormat();
        tst.SetNoColumnLabels();
        tst.SetNoRowLabels();
                tst.NumberFormat = "0.000";
        for (int i = 0; i < 4; i++)
        {
            double[,] tmpCoef = stats.GetCoefficientStatistics(i);
            Console.Out.WriteLine("\n\nRegressions with "+(i+1)+" variable(s)  " +
            "(R-squared)");
            Console.Out.WriteLine("Variable   Coefficient   Standard Error  " +
            "t-statistic   p-value");
            pm.Print(tst, tmpCoef);
        }
    }
}
```

Output

```
Regressions with 1 variable(s)  (R-squared)
     67.4541964131609        4
     66.6268257633294        2
     53.3948023835033        1
     28.5872731229812        3

Regressions with 2 variable(s)  (R-squared)
     97.8678374535631        1   2
     97.2471047716931        1   4
     93.5289640615807        3   4
     68.006040795005        2   4
     54.8166748844857        1   3

Regressions with 3 variable(s)  (R-squared)
     98.2335451200426        1   2   4
     98.2284679219086        1   2   3
     98.1281092587343        1   3   4
     97.2819959386273        2   3   4

Regressions with 4 variable(s)  (R-squared)
     98.237562040768        1   2   3   4



Regressions with 1 variable(s)  (R-squared)
Variable   Coefficient   Standard Error  t-statistic   p-value

4.000        -0.738        0.155        -4.775        0.001



Regressions with 2 variable(s)  (R-squared)
Variable   Coefficient   Standard Error  t-statistic   p-value

1.000         1.468        0.121        12.105        0.000
2.000         0.662        0.046        14.442        0.000



Regressions with 3 variable(s)  (R-squared)
Variable   Coefficient   Standard Error  t-statistic   p-value

1.000          1.452        0.117         12.410        0.000
2.000          0.416        0.186          2.242        0.052
4.000         -0.237        0.173         -1.365        0.205



Regressions with 4 variable(s)  (R-squared)
Variable   Coefficient   Standard Error  t-statistic   p-value

1.000          1.551        0.745          2.083        0.071
2.000          0.510        0.724          0.705        0.501
3.000          0.102        0.755          0.135        0.896
```

```
4.000        -0.144        0.709        -0.203        0.844
```

Example 2: SelectionRegression

This example uses the same data set as the first example, but Mallow's C_p statistic is used as the criterion rather than R^2. Note that when Mallow's C_p statistic (or adjusted R^2) is specified, `MaximumBestFound` is used to indicate the total number of "best" regressions (rather than indicating the number of best regressions per subset size, as in the case of the R^2 criterion). In this example, the three best regressions are found to be (1, 2), (1, 2, 4), and (1, 2, 3).

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class SelectionRegressionEx2
{
    public static void  Main(String[] args)
    {
        double[,] x =  {   {7.0, 26.0, 6.0, 60.0},
                           {1.0, 29.0, 15.0, 52.0},
                           {11.0, 56.0, 8.0, 20.0},
                           {11.0, 31.0, 8.0, 47.0},
                           {7.0, 52.0, 6.0, 33.0},
                           {11.0, 55.0, 9.0, 22.0},
                           {3.0, 71.0, 17.0, 6.0},
                           {1.0, 31.0, 22.0, 44.0},
                           {2.0, 54.0, 18.0, 22.0},
                           {21.0, 47.0, 4.0, 26},
                           {1.0, 40.0, 23.0, 34.0},
                           {11.0, 66.0, 9.0, 12.0},
                           {10.0, 68.0, 8.0, 12.0}
                       };

        double[] y = new double[]{   78.5, 74.3, 104.3, 87.6,
                                     95.9, 109.2, 102.7, 72.5,
                                     93.1, 115.9, 83.8, 113.3,
                                     109.4};

        SelectionRegression sr = new SelectionRegression(4);
        sr.CriterionOption = Imsl.Stat.SelectionRegression.Criterion.MallowsCP;
        sr.MaximumBestFound = 3;
        sr.Compute(x, y);
        SelectionRegression.SummaryStatistics stats = sr.Statistics;

        for (int i = 1; i <= 4; i++)
        {
            double[] tmpCrit = stats.GetCriterionValues(i);
            int[,] indvar = stats.GetIndependentVariables(i);
            Console.Out.WriteLine("Regressions with "+i+" variable(s)  (MallowsCP)");
            for (int j = 0; j < tmpCrit.GetLength(0); j++)
            {
                Console.Out.Write("     " + tmpCrit[j] + "        ");
                for (int k = 0; k < indvar.GetLength(1); k++)
```

```
                    Console.Out.Write(indvar[j,k] + "   ");
                Console.Out.WriteLine("");
            }
            Console.Out.WriteLine("");
        }

        // Setup a PrintMatrix object for use in the loop below.
        PrintMatrix pm = new PrintMatrix();
        pm.SetColumnSpacing(9);
        PrintMatrixFormat tst = new PrintMatrixFormat();
        tst.SetNoColumnLabels();
        tst.SetNoRowLabels();
                tst.NumberFormat = "0.000";
        for (int i = 0; i < 3; i++)
        {
            double[,] tmpCoef = stats.GetCoefficientStatistics(i);
            Console.Out.WriteLine("\n\nRegressions with "+(i+1)+" variable(s)  (MallowsCP)");
            Console.Out.WriteLine("Variable   Coefficient   Standard Error  t-statistic   p-value");
            pm.Print(tst, tmpCoef);
        }
    }
}
```

Output

```
Regressions with 1 variable(s)  (MallowsCP)
     138.730833491674        4
     142.486406936958        2
     202.548769123445        1
     315.154284140073        3

Regressions with 2 variable(s)  (MallowsCP)
     2.6782415983184        1   2
     5.4958508247584        1   4
     22.3731119646967        3   4
     138.225919754638        2   4
     198.094652569569        1   3

Regressions with 3 variable(s)  (MallowsCP)
     3.01823347348731        1   2   4
     3.04127972306423        1   2   3
     3.49682444234832        1   3   4
     7.33747399565576        2   3   4

Regressions with 4 variable(s)  (MallowsCP)
     5        1   2   3   4



Regressions with 1 variable(s)  (MallowsCP)
Variable   Coefficient   Standard Error  t-statistic   p-value

1.000         1.468         0.121         12.105         0.000
2.000         0.662         0.046         14.442         0.000
```

```
Regressions with 2 variable(s)  (MallowsCP)
Variable   Coefficient   Standard Error  t-statistic   p-value

1.000          1.452         0.117          12.410         0.000
2.000          0.416         0.186          2.242          0.052
4.000         -0.237         0.173         -1.365          0.205


Regressions with 3 variable(s)  (MallowsCP)
Variable   Coefficient   Standard Error  t-statistic   p-value

1.000         1.696         0.205         8.290          0.000
2.000         0.657         0.044         14.851         0.000
3.000         0.250         0.185         1.354          0.209
```

SelectionRegression.Criterion Enumeration

`public enumeration Imsl.Stat.SelectionRegression.Criterion`

Criterion Methods.

Fields

AdjustedRSquared

`public Imsl.Stat.SelectionRegression.Criterion AdjustedRSquared`

Description

Indicates R_a^2 (adjusted R^2) criterion regression.

MallowsCP

`public Imsl.Stat.SelectionRegression.Criterion MallowsCP`

Description

Indicates Mallow's C_p criterion regression.

RSquared

`public Imsl.Stat.SelectionRegression.Criterion RSquared`

Description

Indicates R^2 criterion regression.

SelectionRegression.SummaryStatistics Class

```
public class Imsl.Stat.SelectionRegression.SummaryStatistics
```

`SummaryStatistics` contains statistics related to the regression coefficients.

Methods

GetCoefficientStatistics

```
virtual public double[,] GetCoefficientStatistics(int regressionIndex)
```

Description

Returns the coefficients statistics for each of the best regressions found for each subset considered.

Parameter

`regressionIndex` – An `int` which specifies the index of the best regression statistics to return.

Returns

A two-dimensional `double` array containing the regression statistics.

Remarks

The value set using Imsl.Stat.SelectionRegression.MaximumBestFound (p. 667) determines the total number of best regressions to find. The number of best regression is equal to (Imsl.Stat.SelectionRegression.MaximumSubsetSize (p. 667) x `MaximumBestFound`), if criterion `RSquared` is specified or it is equal to `MaximumBestFound` if either `MallowsCP` or `AdjustedRSquared` is specified.

Each row contains statistics related to the regression coefficients of the best models. The regressions are ordered so that the better regressions appear first. The statistic in the columns are as follows (inferences are conditional on the selected model):

Column	Description
0	variable number
1	coefficient estimate
2	estimated standard error of the estimate
3	*t*-statistic for the test that the coefficient is 0
4	*p*-value for the two-sided *t* test

There will be 0 to (`MaximumSubsetSize` x `MaximumBestFound` - 1) best regressions if `RSquared` is specified or 0 to (`MaximumBestFound` - 1) if either `MallowsCP` or `AdjustedRSquared` is specified.

See Also: RSquared (p. 675), AdjustedRSquared (p. 675), MallowsCP (p. 675)

GetCriterionValues

```
virtual public double[] GetCriterionValues(int numVariables)
```

Description

Returns an array containing the values of the best criterion for the number of variables considered.

Parameter

`numVariables` – An `int` which specifies the number of variables considered.

Returns

A `double` array with Imsl.Stat.SelectionRegression.MaximumSubsetSize (p. 667) rows and `nCandidate` columns containing the criterion values.

GetIndependentVariables

```
virtual public int[,] GetIndependentVariables(int numVariables)
```

Description

Returns the identification numbers for the independent variables for the number of variables considered and in the same order as the criteria returned by Imsl.Stat.SelectionRegression.SummaryStatistics.GetCriterionValues(System.Int32) (p. 676).

Parameter

`numVariables` – An `int` which specifies the number of variables considered.

Returns

An `int` array containing the identification numbers for the independent variables considered.

StepwiseRegression Class

```
public class Imsl.Stat.StepwiseRegression
```

Builds multiple linear regression models using forward selection, backward selection, or stepwise selection.

Class `StepwiseRegression` builds a multiple linear regression model using forward selection, backward selection, or forward stepwise (with a backward glance) selection.

Levels of priority can be assigned to the candidate independent variables using Imsl.Stat.StepwiseRegression.Levels (p. 680). All variables with a priority level of 1 must enter the model before variables with a priority level of 2. Similarly, variables with a level of 2 must enter before variables with a level of 3, etc. Variables also can be forced into the model using Imsl.Stat.StepwiseRegression.Force (p. 679). Note that specifying “force” without also specifying levels of priority will result in all variables being forced into the model.

Typically, the intercept is forced into all models and is not a candidate variable. In this case, a sum-of-squares and crossproducts matrix for the independent and dependent variables corrected for the mean is required. Other possibilities are as follows:

1. The intercept is not in the model. A raw (uncorrected) sum-of-squares and crossproducts matrix for the independent and dependent variables is required as input in `cov`. Argument `nObservations` must be set to one greater than the number of observations.

2. An intercept is a candidate variable. A raw (uncorrected) sum-of-squares and crossproducts matrix for the constant regressor (=1), independent and dependent variables are required for `cov`. In this case, `cov` contains one additional row and column corresponding to the constant regressor. This row/column contains the sum-of-squares and crossproducts of the constant regressor with the independent and dependent variables. The remaining elements in `cov` are the same as in the previous case. Argument `nObservations` must be set to one greater than the number of observations.

The stepwise regression algorithm is due to Efroymson (1960). `StepwiseRegression` uses sweeps of the covariance matrix (input in `cov`, if the covariance matrix is specified, or generated internally) to move variables in and out of the model (Hemmerle 1967, Chapter 3). The SWEEP operator discussed in Goodnight (1979) is used. A description of the stepwise algorithm is also given by Kennedy and Gentle (1980, pp. 335-340). The advantage of stepwise model building over all possible regression (`SelectionRegression`) is that it is less demanding computationally when the number of candidate independent variables is very large. However, there is no guarantee that the model selected will be the best model (highest R^2) for any subset size of independent variables.

Properties

ANOVA

```
virtual public Imsl.Stat.ANOVA ANOVA {get; }
```

Description

An analysis of variance table and related statistics.

Property Value

An `ANOVA` table and related statistics.

CoefficientTTests

```
virtual public Imsl.Stat.StepwiseRegression.CoefficientTTestsValue
CoefficientTTests {get; }
```

Description

The student-*t* test statistics for the regression coefficients.

Property Value

A `StepwiseRegression.CoefficientTTestsValue` object containing statistics relating to the regression coefficients.

CoefficientVIF

```
virtual public double[] CoefficientVIF {get; }
```

Description

The variance inflation factors for the final model in this invocation.

Property Value

A `double` array containing the variance inflation factors for the final model in this invocation.

Remarks

The elements are in the same order as the independent variables in `x` (or, if the covariance matrix is specified, the elements are in the same order as the variables in `cov`). Each element corresponding to a variable not in the model contains statistics for a model which includes the variables of the final model and the variables corresponding to the element in question.

The square of the multiple correlation coefficient for the *i*-th regressor after all others can be obtained from the *i*-th element for the returned array by the following formula:

$$1.0 - \frac{1.0}{VIF}$$

CovariancesSwept

```
virtual public double[,] CovariancesSwept {get; }
```

Description

Results after `cov` has been swept for the columns corresponding to the variables in the model.

Property Value

A `double` matrix containing the results after `cov` has been swept on the columns corresponding to the variables in the model.

Remarks

The estimated variance-covariance matrix of the estimated regression coefficients in the final model can be obtained by extracting the rows and columns corresponding to the independent variables in the final model and multiplying the elements of this matrix by the error mean square.

Force

```
virtual public int Force {set; }
```

Description

Forces independent variables into the model based on their level assigned from `Levels`.

Property Value

An `int` specifying the upper bound on the variables forced into the model.

Remarks

Variables with levels 1, 2, ..., `Force` are forced into the model as independent variables.

See Also: Levels (p. 680)

History

```
virtual public double[] History {get; }
```

Description

The stepwise regression history for the independent variables.

Property Value

A `double` array containing the recent history of the independent variables.

Remarks

The last element corresponds to the dependent variable.

History[i]	Status of i-th Variable
0.0	This variable has never been added to the model.
0.5	This variable was added into the model during initialization.
$k > 0.0$	This variable was added to the model during the k-th step.
$k < 0.0$	This variable was deleted from model during the k-th step

See Also: Levels (p. 680)

Intercept

```
virtual public double Intercept {get; }
```

Description

Returns the intercept.

Remarks

The intercept is computed as follows:

$$\beta_0 = \bar{y} - \sum_{i=1}^{n} \beta_i \bar{x}_{i-1}$$

where $\bar{y}$ is the mean of the dependent variable `y`, β_i are the coefficients, and $\bar{x}_i$ are the mean values for each independent variable x_i in the final model. If the covariance matrix is used for input, use method `SetMean` to specify the means of the variables. If `x` and `y` are used for input, the means are computed internally and do not need to be specified.

Levels

```
virtual public int[] Levels {set; }
```

Description

The levels of priority for variables entering and leaving the regression.

Property Value

An `int` array containing the levels of entry into the model for each variable.

Remarks

Each variable is assigned a positive value which indicates its level of entry into the model. A variable can enter the model only after all variables with smaller nonzero levels of entry have entered. Similarly, a variable can only leave the model after all variables with higher levels of entry have left. Variables with the same level of entry compete for entry (deletion) at each step. A value `Levels[i]=0` means the i-th variable never enters the model. A value `Levels[i]=-1` means the i-th variable is the dependent

variable. The last element in `Levels` must correspond to the dependent variable, except when the variance-covariance or sum-of-squares and crossproducts matrix is supplied.

By default, `Levels` = {1, 1, ..., 1, -1}, where -1 corresponds to the dependent variable.

See Also: Force (p. 679)

Method

```
virtual public Imsl.Stat.StepwiseRegression.Direction Method {set; }
```

Description

Specifies the stepwise selection method, forward, backward, or stepwise Regression.

Property Value

An `int` value between -1 and 1 specifying the stepwise selection method.

Remarks

Fields `Forward`, `Backward`, and `Stepwise` should be used.

By default, `Direction.Stepwise`.

See Also: Forward (p. 688), Backward (p. 688), Stepwise (p. 688)

PValueIn

```
virtual public double PValueIn {set; }
```

Description

Defines the largest *p*-value for variables entering the model.

Property Value

A `double` containing the largest *p*-value for variables entering the model.

Remarks

Variables with *p*-value less than `PValueIn` may enter the model. Backward regression does not use this value.

By default, `PValueIn` = 0.05.

PValueOut

```
virtual public double PValueOut {set; }
```

Description

Defines the smallest *p*-value for removing variables.

Property Value

A `double` containing the smallest *p*-value for removing variables from the model.

Remarks

Variables with *p*-values greater than `PValueOut` may leave the model. `PValueOut` must be greater than or equal to `PValueIn`. A common choice for `PValueOut` is 2*`PValueIn`. Forward regression does not use this value.

By default, `PValueOut` = 0.10.

Swept

```
virtual public double[] Swept {get; }
```

Description

An array containing information indicating whether or not a particular variable is in the model.

Property Value

A `double` array with information to indicate the independent variables in the model.

Remarks

The last element corresponds to the dependent variable. A +1 in the *i*-th position indicates that the variable is in the selected model. A -1 indicates that the variable is not in the selected model.

See Also: Levels (p. 680)

Tolerance

```
virtual public double Tolerance {set; }
```

Description

The tolerance used to detect linear dependence among the independent variables.

Property Value

A `double` containing the tolerance used for detecting linear dependence.

Remarks

By default, `Tolerance` = 2.2204460492503e-16.

Constructors

StepwiseRegression

```
public StepwiseRegression(double[,] x, double[] y)
```

Description

Creates a new instance of `StepwiseRegression`.

Parameters

`x` – A `double` matrix of *nObs* by *nVars*, where *nObs* is the number of observations and *nVars* is the number of independent variables.

`y` – A `double` array containing the observations of the dependent variable.

Exceptions

`Imsl.Stat.TooManyObsDeletedException` is thrown if more observations have been deleted than were originally entered

`Imsl.Stat.MoreObsDelThanEnteredException` is thrown if more observations are being deleted from the output covariance matrix than were originally entered

`Imsl.Stat.DiffObsDeletedException` is thrown if different observations are being deleted from the return matrix than were originally entered

StepwiseRegression

```
public StepwiseRegression(double[,] x, double[] y, double[] weights)
```

Description

Creates a new instance of weighted `StepwiseRegression`.

Parameters

`x` – A `double` matrix of *nObs* by *nVars*, where *nObs* is the number of observations and *nVars* is the number of independent variables.

`y` – A `double` array containing the observations of the dependent variable.

`weights` – A `double` array containing the weight for each observation of `x`.

Exceptions

`Imsl.Stat.TooManyObsDeletedException` is thrown if more observations have been deleted than were originally entered

`Imsl.Stat.MoreObsDelThanEnteredException` is thrown if more observations are being deleted from the output covariance matrix than were originally entered

`Imsl.Stat.DiffObsDeletedException` is thrown if different observations are being deleted from the return matrix than were originally entered

`Imsl.Stat.NegativeWeightException` is thrown if a weight is less than zero.

StepwiseRegression

```
public StepwiseRegression(double[,] x, double[] y, double[] weights, double[]
frequencies)
```

Description

Creates a new instance of weighted `StepwiseRegression` using observation frequencies.

Parameters

`x` – A `double` matrix of *nObs* by *nVars*, where *nObs* is the number of observations and *nVars* is the number of independent variables.

`y` – A `double` array containing the observations of the dependent variable.

`weights` – A `double` array containing the weight for each observation of `x`.

`frequencies` – A `double` array containing the frequency for each row of `x`.

Exceptions

`Imsl.Stat.TooManyObsDeletedException` is thrown if more observations have been deleted than were originally entered

`Imsl.Stat.MoreObsDelThanEnteredException` is thrown if more observations are being deleted from the output covariance matrix than were originally entered

`Imsl.Stat.DiffObsDeletedException` is thrown if different observations are being deleted from the return matrix than were originally entered

`Imsl.Stat.NegativeWeightException` is thrown if a weight is less than zero.

`Imsl.Stat.NegativeFreqException` is thrown if a frequency is less than zero.

StepwiseRegression

```
public StepwiseRegression(double[,] cov, int nObservations)
```

Description

Creates a new instance of `StepwiseRegression` from a user-supplied variance-covariance matrix.

Parameters

`cov` – A `double` matrix containing a variance-covariance or sum-of- squares and crossproducts matrix, in which the last column must correspond to the dependent variable.

`nObservations` – An `int` containing the number of observations associated with `cov`.

Remarks

`cov` can be computed using the Imsl.Stat.Covariances (p. 545) class.

Methods

Compute

```
virtual public void Compute()
```

Description

Builds the multiple linear regression models using forward selection, backward selection, or stepwise selection.

Exceptions

`Imsl.Stat.NoVariablesEnteredException` is thrown if no variables entered the model. All elements of the Imsl.Stat.StepwiseRegression.ANOVA (p. 678) table are set to `NaN`

`Imsl.Stat.CyclingIsOccurringException` is thrown if cycling occurs

SetMeans

```
virtual public void SetMeans(double[] means)
```

Description

Sets the means of the variables.

Parameter

`means` – A `double` array of length *nVars*+1, where *nVars* is the number of independent variables. `means[0]` through `means[`*nVars*`-1]` are the means of the independent variables and `means[`*nVars*`]` is the mean of the dependent variable.

Remarks

This is required when the covariance array is input and the intercept Imsl.Stat.StepwiseRegression.Intercept (p. 680) is requested. Otherwise, it is not used.

Example: StepwiseRegression

This example uses a data set from Draper and Smith (1981, pp. 629-630). Method `compute` is invoked to find the best regression for each subset size using the R^2 criterion. By default, stepwise regression is performed.

```
using System;
using Imsl;
using Imsl.Math;
using Imsl.Stat;

public class StepwiseRegressionEx1
{

    public static void  Main(System.String[] args)
    {
        double[,] x = {{7.0, 26.0, 6.0, 60.0},
                       {1.0, 29.0, 15.0, 52.0},
                       {11.0, 56.0, 8.0, 20.0},
                       {11.0, 31.0, 8.0, 47.0},
                       {7.0, 52.0, 6.0, 33.0},
                       {11.0, 55.0, 9.0, 22.0},
                       {3.0, 71.0, 17.0, 6.0},
                       {1.0, 31.0, 22.0, 44.0},
                       {2.0, 54.0, 18.0, 22.0},
                       {21.0, 47.0, 4.0, 26},
                       {1.0, 40.0, 23.0, 34.0},
                       {11.0, 66.0, 9.0, 12.0},
                       {10.0, 68.0, 8.0, 12.0}};

        double[] y = new double[]{78.5, 74.3, 104.3, 87.6,
                                  95.9, 109.2, 102.7, 72.5,
                                  93.1, 115.9, 83.8, 113.3, 109.4};

        StepwiseRegression sr = new StepwiseRegression(x, y);
        sr.Compute();

        PrintMatrix pm = new PrintMatrix();
        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.000";
        pm.SetTitle("*** ANOVA *** "); pm.Print(sr.ANOVA.GetArray());

        StepwiseRegression.CoefficientTTestsValue coefT = sr.CoefficientTTests;
        double[,] coef = new double[4, 4];
        for (int i = 0; i < 4; i++)
        {
            coef[i,0] = coefT.GetCoefficient(i);
            coef[i,1] = coefT.GetStandardError(i);
            coef[i,2] = coefT.GetTStatistic(i);
            coef[i,3] = coefT.GetPValue(i);
```

```
        }
        pm.SetTitle("*** Coef *** "); pm.Print(pmf, coef);
        pm.SetTitle("*** Swept *** "); pm.Print(sr.Swept);
        pm.SetTitle("*** History *** "); pm.Print(sr.History);
        pm.SetTitle("*** VIF *** "); pm.Print(sr.CoefficientVIF);
        pm.SetTitle("*** CovS *** "); pm.Print(pmf, sr.CovariancesSwept);
        Console.WriteLine("*** Intercept ***   " + sr.Intercept);
    }
}
```

Output

```
       *** ANOVA ***
              0
 0     2
 1    10
 2    12
 3  2641.00096476634
 4    74.7621121567356
 5  2715.76307692308
 6  1320.50048238317
 7     7.47621121567356
 8   176.626963081888
 9     1.58106023181439E-08
10    97.2471047716931
11    96.6965257260317
12     2.73426612012685
13     NaN
14     NaN

          *** Coef ***
     0       1       2       3
0   1.440  0.138   10.403  0.000
1   0.416  0.186   2.242   0.052
2  -0.410  0.199  -2.058   0.070
3  -0.614  0.049  -12.621  0.000

*** Swept ***
   0
0   1
1  -1
2  -1
3   1
4  -1

*** History ***
   0
0  2
1  0
2  0
3  1
4  0

     *** VIF ***
           0
```

```
0   1.0641052101769
1  18.7803086409578
2   3.45960147891528
3   1.0641052101769

                *** CovS ***
     0        1         2         3         4
0   0.003  -0.029    -0.946     0.000   1.440
1  -0.029   154.720  -142.800   0.907   64.381
2  -0.946  -142.800   142.302   0.070  -58.350
3   0.000   0.907     0.070     0.000  -0.614
4   1.440   64.381   -58.350   -0.614   74.762

*** Intercept ***   103.097381636675
```

StepwiseRegression.CoefficientTTestsValue Class

`public class Imsl.Stat.StepwiseRegression.CoefficientTTestsValue`

`CoefficientTTestsValue` contains statistics related to the student-*t* test, for each regression coefficient.

Methods

GetCoefficient

`virtual public double GetCoefficient(int index)`

Description

Returns the estimate for a coefficient of the independent variable.

Parameter

`index` – An `int` which specifies the index of the coefficient whose estimate is to be returned.

Returns

A `double` which contains the estimate for the coefficient.

Remarks

`index` must be between 1 and the number of independent variables.

GetPValue

`virtual public double GetPValue(int index)`

Description

Returns the *p*-value for the two-sided test $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$.

Parameter

`index` – An `int` which specifies the index of the coefficient whose *p*-value is to be returned.

Returns

A `double` which contains the estimated *p*-value for the coefficient.

Remarks

`index` must be between 1 and the number of independent variables.

GetStandardError

```
virtual public double GetStandardError(int index)
```

Description

Returns the estimated standard error for a coefficient estimate.

Parameter

`index` – An `int` which specifies the index of the coefficient whose standard error estimate is to be returned.

Returns

A `double` which contains the estimated standard error for the coefficient.

Remarks

`index` must be between 1 and the number of independent variables.

GetTStatistic

```
virtual public double GetTStatistic(int index)
```

Description

Returns the student-*t* test statistic for testing the *i*-th coefficient equal to zero ($\beta_{index} = 0$).

Parameter

`index` – An `int` which specifies the index of the coefficient whose *t*-test statistic is to be returned.

Returns

A `double` which contains the estimated *t*-test statistic for the coefficient.

Remarks

`index` must be between 1 and the number of independent variables.

StepwiseRegression.Direction Enumeration

```
public enumeration Imsl.Stat.StepwiseRegression.Direction
```

Direction indicator.

Fields

Backward

```
public Imsl.Stat.StepwiseRegression.Direction Backward
```

Description

Indicates backward regression. An attempt is made to remove a variable from the model. A variable is removed if its *p*-value exceeds `PValueOut`. During initialization, all candidate independent variables enter the model.

Forward

```
public Imsl.Stat.StepwiseRegression.Direction Forward
```

Description

Indicates forward regression. An attempt is made to add a variable to the model. A variable is added if its *p*-value is less than `PValueIn`. During intitialization, only forced variables enter the model.

Stepwise

```
public Imsl.Stat.StepwiseRegression.Direction Stepwise
```

Description

Indicates stepwise regression. A backward step is attempted. After the backward step, a forward step is attempted. This is a stepwise step. Any forced variables enter the model during initialization.

UserBasisRegression Class

```
public class Imsl.Stat.UserBasisRegression
```

Generates summary statistics using user-supplied functions in a nonlinear regression model.

Fits a linear function of the form

$$y = c_0 + c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) + \varepsilon$$

, where $f_1(x), f_2(x), \cdots, f_k(x)$ are the user basis functions $f_i(x)$ evaluated at index values $i = 1, 2, \ldots, k, c_0$ is the intercept, $c_1, c_2, \cdots, c_k$ are the coefficients associated with the basis functions, and is the random error associated with y. The coefficients $c_0, c_1, \cdots, c_k$ are determined by least squares.

Description

`UserBasisRegression` generalizes the concept of linear regression to user defined basis functions. The linear regression model is

$$y = c_0 + c_1 x_1 + \cdots + c_k x_k + \varepsilon$$

, where are the k independent variables. UserBasisRegression generalizes this concept by setting $x_i = f_i(x)$, where $f_i(x)$ is any user defined function of x.

This makes it easier for users to fit complex univariate models. For example, the `LinearRegression` class can be used to fit polynomials such as

$$y = c_0 + c_1 x + c_2 x^2 \cdots + c_k x^k + \varepsilon$$

, but this requires an input matrix where the ith column of that array contains the values of x^i.

With `UserBasisRegression`, these columns can be automatically generated. For this polynomial model, the user would define a user basis function $f_i(x) = x^{i+1}$. The `UserBasisRegression` class automatically inserts the necessary values into the regression equation and then calculates the coefficients and analysis of variance statistics.

Since the user provides a method for calculating the basis function, other more complex user basis functions are possible such as

$$y = c_1 Sin(x) + c_2 Cos(x) + \varepsilon$$

. In this case, `nBasis`=2, $f_0(x) = Sin(x)$, and $f_1(x) = Cos(x)$.

Property

ANOVA

```
public Imsl.Stat.ANOVA ANOVA {get; }
```

Description

An analysis of variance table and related statistics.

Property Value

An `ANOVA` table and related statistics.

Constructor

UserBasisRegression

```
public UserBasisRegression(Imsl.Stat.IRegressionBasis basis, int nBasis, bool
hasIntercept)
```

Description

Constructs a `UserBasisRegression` object.

Parameters

`basis` – A `IRegressionBasis` basis function supplied by the user.

`nBasis` – A `int` which specifies the number of basis functions.

`hasIntercept` – A `bool` which specifies whether or not the model has an intercept.

Methods

GetCoefficients

```
public double[] GetCoefficients()
```

Description

Returns the regression coefficients.

Returns

A `double` array containing the regression coefficients.

Remarks

If hasIntercept is `false` its length is equal to the number of variables. If hasIntercept is `true` then its length is the number of variables plus one and the 0-th entry is the value of the intercept.

Exception

`Imsl.Math.SingularMatrixException` is thrown when the regression matrix is singular

Update

```
public void Update(double x, double y, double w)
```

Description

Adds a new observation and associated weight to the `IRegressionBasis` object.

Parameters

`x` – A `double` containing the independent (explanatory) variable.

`y` – A `double` containing the dependent (response) variable.

`w` – A `double` representing the weight.

Example: Regression with User-supplied Basis Functions

In this example, we fit the function 1 + sin(x) + 7 * sin(3x) with no error introduced. The function is evaluated at 90 equally spaced points on the interval [0, 6]. Four basis functions are used, sin(kx) for k = 1,...,4 with no intercept.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class UserBasisRegressionEx1 : IRegressionBasis
{
    public double Basis(int index, double x)
    {
        return System.Math.Sin((index + 1) * x);
    }

    public static void Main(string[] args)
    {
        double[] coef = new double[4];
```

```
        IRegressionBasis basis = new UserBasisRegressionEx1();
        UserBasisRegression ubr =
            new UserBasisRegression(basis, 4, false);

        for (int k = 0; k < 90; k++)
        {
            double x = 6.0 * k / 89.0;
            double y = 1.0 + Math.Sin(x) + 7.0 * Math.Sin(3.0 * x);
            ubr.Update(x, y, 1.0);
        }
        coef = ubr.GetCoefficients();
        PrintMatrix pm = new PrintMatrix("The regression coefficients are:");
        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.000";
        pm.Print(pmf, coef);
    }
}
```

Output

```
The regression coefficients are:
     0
0  1.010
1  0.020
2  7.029
3  0.037
```

Example: Regression with User-supplied Basis Functions

In this example, we fit the polynomial $y = c_0 + c_1 x + c_2 x^2 \cdots + c_4 x^4 + \varepsilon$. For this model, the user basis function is $f_i(x) = x^{i+1}$ with $i = 0, 1, \cdots, nBasis = 4$ and `hasIntercept=true`.

Data are generated using the model $y = 10 + 2x + 5x^3$ with $x = 0, 1, \cdots 9$.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class UserBasisRegressionEx2 : IRegressionBasis
{
    public double Basis(int index, double x)
    {
        // Notice zero-based indexing requires index be incremented
        return System.Math.Pow(x, index + 1);
    }

    public static void Main(string[] args)
    {
        double y = 0;
        double[] coef ;
        IRegressionBasis basis = new UserBasisRegressionEx2();
        UserBasisRegression ubr =
            new UserBasisRegression(basis, 4, true);
```

```
        for (double x = 0; x < 10; x++)
        {
            y = 10.0 + x + 3 * System.Math.Pow(x, 3);
            ubr.Update(x, y, 1.0);
        }
        coef = ubr.GetCoefficients();
        PrintMatrix pm = new PrintMatrix("The regression coefficients are:");
        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.0";
        pm.Print(pmf, coef);

        // Print the Anova Table
        ANOVA anovaTable = ubr.ANOVA;

        System.Console.Out.WriteLine("Degrees Of Freedom For Model:          "
         + anovaTable.DegreesOfFreedomForModel);
        System.Console.Out.WriteLine("Degrees Of Freedom For Error:          "
         + anovaTable.DegreesOfFreedomForError);
        System.Console.Out.WriteLine("Total (Corrected) Degrees Of Freedom:  "
         + anovaTable.TotalDegreesOfFreedom);
        System.Console.Out.WriteLine("Sum Of Squares For Model:              "
         + anovaTable.SumOfSquaresForModel);
        System.Console.Out.WriteLine("Sum Of Squares For Error:              "
         + anovaTable.SumOfSquaresForError);
        System.Console.Out.WriteLine("Total (Corrected) Sum Of Squares:      "
         + anovaTable.TotalSumOfSquares);
        System.Console.Out.WriteLine("Model Mean Square:                     "
         + anovaTable.ModelMeanSquare);
        System.Console.Out.WriteLine("Error Mean Square:                     "
         + anovaTable.ErrorMeanSquare);
        System.Console.Out.WriteLine("F statistic:                           "
         + anovaTable.F);
        System.Console.Out.WriteLine("P value:                               "
         + anovaTable.P);
        System.Console.Out.WriteLine("R Squared (in percent):                "
         + anovaTable.RSquared);
        System.Console.Out.WriteLine("Adjusted R Squared (in percent):       "
         + anovaTable.AdjustedRSquared);
        System.Console.Out.WriteLine("Model Error Standard deviation:        "
         + anovaTable.ModelErrorStdev);
        System.Console.Out.WriteLine("Mean Of Y:                             "
         + anovaTable.MeanOfY);
        System.Console.Out.WriteLine("Coefficient Of Variation (in percent): "
         + anovaTable.CoefficientOfVariation);
    }
}
```

Output

```
The regression coefficients are:
    0
0  10.0
1  1.0
2  0.0
3  3.0
```

```
4  0.0

Degrees Of Freedom For Model:          4
Degrees Of Freedom For Error:          5
Total (Corrected) Degrees Of Freedom:  9
Sum Of Squares For Model:              5152488
Sum Of Squares For Error:              1.86264514923096E-09
Total (Corrected) Sum Of Squares:      5152488
Model Mean Square:                     1288122
Error Mean Square:                     3.72529029846191E-10
F statistic:                           3.45777616453632E+15
P value:                               8.69662785564134E-39
R Squared (in percent):                100
Adjusted R Squared (in percent):       99.9999999999999
Model Error Standard deviation:        1.93010111094261E-05
Mean Of Y:                             622
Coefficient Of Variation (in percent): 3.10305644846079E-06
```

IRegressionBasis Interface

`public interface Imsl.Stat.IRegressionBasis`

Interface for user supplied function to `UserBasisRegression` object.

Method

Basis

`abstract public double Basis(int index, double x)`

Description

Basis function for the nonlinear least-squares function.

Parameters

`index` – A `int` which specifies the index of the basis function to be evaluated at x.

`x` – A `double` which specifies the point at which the function is to be evaluated.

Returns

A `double` which specifies the returned value of the function at x.

Chapter 14: Analysis of Variance

Types

Usage Notes

The classes described in this chapter are for commonly-used experimental designs. Typically, responses are stored in the input vector *y* in a pattern that takes advantage of the balanced design structure. Consequently, the full set of model subscripts is not needed to identify each response. The classes assume the usual pattern, which requires that the last model subscript change most rapidly, followed by the model subscript next in line, and so forth, with the first subscript changing at the slowest rate. This pattern is referred to as *lexicographical ordering*.

`ANOVA` class allows missing responses if confidence interval information is not requested. Double.NaN (Not a Number) is the missing value code used by these classes. Any element of *y* that is missing must be set to NaN. Other classes described in this chapter do not allow missing responses because the classes generally deal with balanced designs.

As a diagnostic tool for determination of the validity of a model, classes in this chapter typically perform a test for lack of fit when $n(n > 1)$ responses are available in each cell of the experimental design.

ANCOVA Class

`public class Imsl.Stat.ANCOVA`

Analyzes a one-way classification model with covariates.

Class ANCOVA performs analyses for models that combine the features of a one-way analysis of variance model with that of a multiple linear regression model. The basic one-way analysis of covariance model is

$$y_{ij} = \beta_{0i} + \beta_1 x_{ij1} + \beta_2 x_{ij2} + \ldots + \beta_m x_{ijm} + \varepsilon_{ij}$$

$$i = 1, 2, \ldots, ngroup$$

$$j = 1, 2, \ldots, n_i$$

where, *ngroup* is the number of treatment groups, the observed value of y_{ij} constitutes the *j*-th response in the *i*-th group, β_{0i} denotes the y intercept for the regression function for the *i*-th group, β_1, β_2, ..., β_m are the regression coefficients for the covariates, and the ε_{ij}'s are independently distributed normal errors with mean zero and variance σ^2. This model allows the regression function for each group to have different intercepts. However, the remaining *m* regression coefficients are the same for each group, i.e., the regression functions are parallel.

In practice, sometimes the regression functions are not parallel. In addition to estimates for the model assuming parallelism (parallel regression planes), ANCOVA computes estimates and summary statistics for the separate regressions of each group. These estimates can be examined using the methods GetCoefficientTables and GetANOVATables.

Estimates for the β_{0i}'s and β_1, β_2, ..., β_m in the model assuming parallelism are returned using the method GetModelCoefficients. Summary statistics are also computed for this model and returned by the Compute method.

The adjusted group means, stored in the last column of *xymean*, are computed using the formula:

$$\hat{\beta}_{0i} + \hat{\beta}_1 \bar{x}_1 + \hat{\beta}_2 \bar{x}_2 + \ldots + \hat{\beta}_m \bar{x}_m$$

where *xymean* is the matrix returned by GetMeans and *ncov* is the number of covariates.

The estimated covariance between the i_1-th and i_2-th adjusted group mean is given by

$$v_{i_1 i_2} + \sum_{r=1}^{m}\sum_{s=1}^{m} \bar{x}_r v_{k+r,k+s} \bar{x}_s + \sum_{r=1}^{m} \bar{x}_r v_{i_1,k+r} + \sum_{r=1}^{m} \bar{x}_r v_{i_2,k+r}$$

where v_{pq} is the entry in covb[p-1][q-1], where covb is returned by GetVarCovCoefficients and is the estimated covariance between the *p*-th and *q*-th estimated coefficients in the regression function.

Property

NumberOfMissing

```
virtual public int NumberOfMissing {get; }
```

Description

The number of cases with missing values in covariates or responses. Cases with any missing values are not used in the analysis.

Property Value

An `int` scalar value indicating the number of cases with missing values in `covariates` or `responses`.

Constructor

ANCOVA

```
public ANCOVA(double[][] responses, double[][][] covariates)
```

Description

Constructs a one-way classification model with covariates.

Parameters

`responses` – A `double` matrix containing the responses. Each row in `responses` corresponds to a treatment group. Each row of `responses` can contain a different number of observations. There must be at least two groups (responses.Length $>$ 1).

`covariates` – A three-dimensional `double` array containing the covariates. The first dimension corresponds to the number of covariates (consider each element an individual covariate matrix or `covariates.Length` = number of covariates). Each row in `covariates[i]` corresponds to a treatment group. There must be the same number of rows, for each covariate, as there are in the responses matrix (`covariates[i].Length = responses.Length`). There must be at least one covariate (covariate.Length $>$ 1). Each row of `covariates[i]` must contain the same number of elements as the corresponding row in `responses`
(`covariates[i][j].Length = responses[j].Length`).

Methods

Compute

```
public double[] Compute()
```

Description

Performs one-way analysis of covariance assuming parallelism and returns an array containing the parallelism tests for the one-way analysis of covariance.

Returns

A `double` array containing the parallelism tests for the one-way analysis of covariance organized as follows:

index	Description
0	Extra degrees of freedom for model not assuming parallelism.
1	Degrees of freedom for error for model not assuming parallelism.
2	Degrees of freedom for error for model assuming parallelism.
3	Extra sum of squares for model not assuming parallelism.
4	Sum of squares for error for model not assuming parallelism.
5	Sum of squares for error for model assuming parallelism.
6	Mean square for *index* = 0.
7	Mean square for *index* = 1.
8	*F* statistic
9	*p*-value

GetAdjustedANOVA

```
public double[] GetAdjustedANOVA()
```

Description

Returns the partial sum of squares for the one-way analysis of covariance.

Returns

A `double` array containing the partial sum of squares for the one-way analysis of covariance organized as follows:

index	Description
0	Degrees of freedom for groups after covariates.
1	Degrees of freedom for covariates after groups.
2	Sum of squares for groups after covariates.
3	Sum of squares for model (groups and covariates combined).
4	*F*-statistic for groups.
5	*F*-statistic for covariates.
6	*p*-value for groups.
7	*p*-value for covariates.

GetANCOVA

```
public double[] GetANCOVA()
```

Description

Returns an array containing the one-way analysis of covariance assuming parallelism.

Returns

A `double` array of length 15 containing the following values:

index	ANCOVA Table Value
0	Degrees of freedom for model
1	Degrees of freedom for error
2	Total degrees of freedom
3	Sum of squares for model
4	Sum of squares for error
5	Total sum of squares
6	Model mean square
7	Error mean square
8	*F* statistic
9	*p*-value
10	*R*-squared (in percent)
11	Adjusted *R*-squared (in percent)
12	Estimated standard deviation of the model error
13	Mean of the response (dependent variable)
14	Coefficient of variation (in percent)

GetANOVATables

```
public double[][] GetANOVATables()
```

Description

Returns a matrix of size *ngroup* by 15 containing the analysis of variance tables for each linear regression model fitted separately to each treatment group.

Returns

A `double` matrix containing the analysis of variance tables for each linear regression model fitted separately to each treatment group. The 15 values in the *i*-th row are for treatment group *i* organized as follows:

index	Description
0	Degrees of freedom for model
1	Degrees of freedom for error
2	Total degrees of freedom
3	Sum of squares for model
4	Sum of squares for error
5	Total sum of squares
6	Model mean square
7	Error mean square
8	*F* statistic
9	*p*-value
10	*R*-squared (in percent)
11	Adjusted *R*-squared (in percent)
12	Estimated standard deviation of the model error
13	Mean of the response (dependent variable)
14	Coefficient of variation (in percent)

GetCoefficientTable

```
public double[][] GetCoefficientTable(int group)
```

Description

Returns a matrix of size *ncov* + 1 by 4 containing statistics for a linear regression model fitted separately for each of the *ngroup* treatment groups.

Parameter

`group` – An `int` specifying the group for which the regression statistics will be retrieved.

Returns

A `double` matrix containing statistics for a `group`. Each row corresponds to the model coefficients. For row = 0, the statistics relate to the intercept in the regression model. For row = 1, 2, ..., *ncov*, the statistics relate to the slopes for the covariates. The column dimension corresponds to the row described for `GetModelCoefficients` as follows:

Column	Description
0	Coefficient estimate.
1	Estimated standard error of the estimate.
2	*t*-statistic.
3	*p*-value.

GetCoefficientTables

```
public double[][][] GetCoefficientTables()
```

Description

Returns an array containing statistics for a linear regression model fitted separately for all *ngroup* treatments.

Returns

A `double[][][]` array containing statistics for a linear regression model fitted separately for each of the *ngroup* treatment groups. The 3 dimensional array organized with *ngroup* rows, *ncov* + 1 columns, and depth of 4. Each row corresponds to one of the *ngroup* treatment groups. Each column corresponds to the model coefficients. For column = 0, the statistics relate to the intercept in the regression model. For column = 1, 2, ..., *ncov*, the statistics relate to the slopes for the covariates. The depth dimension corresponds to the columns described for `GetModelCoefficients` as follows:

Column	Description
0	Coefficient estimate.
1	Estimated standard error of the estimate.
2	*t*-statistic.
3	*p*-value.

GetMeans

```
public double[][] GetMeans()
```

Description

Returns a matrix containing the unadjusted means for the covariates and the response variate and the means for the response variate adjusted for the covariates.

Returns

A `double` matrix of size *ngroup* + 1 by *ncov* + 3 containing the unadjusted means for the covariates and the response variate and the means for the response variate adjusted for the covariates. Each row for i = 0, 1, ..., *ngroup* - 1 corresponds to a group. Row *ngroup* contains overall statistics. The means are organized in columns as follows:

Column	Description
0	Number of non-missing cases
1 thru *ncov*	Covariate means.
ncov + 1	Response mean.
ncov + 2	Response mean adjusted assuming parallelism.

GetModelCoefficients

```
public double[][] GetModelCoefficients()
```

Description

Returns a matrix containing statistics for the regression coefficients for the model assuming parallelism.

Returns

A `double` matrix of size *ngroup* + *ncov* by 4 containing statistics for the regression coefficients for the model assuming parallelism. Each row corresponds to a coefficient in the model. For *i* = 0, 1, ..., *ngroup* - 1, row `i` is for the y intercept for the *i*-th group. The remaining *ncov* rows are for the covariate coefficients. The statistics in the columns are organized as follows:

Column	Description
0	Coefficient estimate.
1	Estimated standard error of the estimate.
2	*t*-statistic.
3	*p*-value.

GetR

```
public double[][] GetR()
```

Description

Returns the *R* matrix from the *QR* decomposition.

Returns

A `double` matrix of size *ngroup* + *ncov* by *ngroup* + *ncov* which contains the *R* from the *QR* decomposition.

Remarks

The *R* matrix is from the regression assuming parallelism.

GetVarCovAdjustedMeans

```
public double[][] GetVarCovAdjustedMeans()
```

Description

Returns a matrix containing the estimated variances and covariances for the adjusted means assuming parallelism.

Returns

A `double` matrix of size *ngroup* by *ngroup* containing the estimated variances and covariances for the adjusted means assuming parallelism.

GetVarCovCoefficients

```
public double[][] GetVarCovCoefficients()
```

Description

Returns a matrix containing the estimated variances and covariances for the coefficients returned using `GetModelCoefficients`.

Returns

A `matrix` of size *ngroup* + *ncov* by *ngroup* + *ncov* containing the estimated variances and covariances for the coefficients returned using `GetModelCoefficients`.

Example 1: One-way analysis of covariance model

This example fits a one-way analysis of covariance model assuming parallelism using data discussed by Snedecor and Cochran (Table 14.6.1, pages 432-436). The responses are concentrations of cholesterol (in mg/100 ml) in the blood of two groups of women: women from Iowa and women from Nebraska. The age of a woman is the single covariate. The cholesterol concentrations and ages of the women according to state are shown in the following table. (There are 11 Iowa women and 19 Nebraska women in the study. Only the first 5 women from each state are shown here.)

Iowa		Nebraska	
Age	Cholesterol	Age	Cholesterol
46	181	18	137
52	228	44	173
39	182	33	177
65	249	78	241
54	259	51	225

There is no evidence from the data to indicate that the regression lines for cholesterol concentration as a function of age are not parallel for Iowa and Nebraska women (p-value is 0.5425). The parallel line model suggests that Nebraska women may have higher cholesterol concentrations than Iowa women. The cholesterol concentrations (adjusted for age) are 195.5 for Iowa women versus 224.2 for Nebraska women. The difference is 28.7 with an estimated standard error of

$$\sqrt{170.4 + 97.4 - 2(2.9)} = 16.1$$

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class ANCOVAEx1
{
    public static void  Main(String[] args)
    {

        double[][] y = new double[][]{
            new double[]{181.0, 228.0, 182.0, 249.0, 259.0, 201.0, 121.0,
                         339.0, 224.0, 112.0, 189.0},
            new double[]{137.0, 173.0, 177.0, 241.0, 225.0, 223.0, 190.0,
                         257.0, 337.0, 189.0, 214.0, 140.0, 196.0, 262.0,
                         261.0, 356.0, 159.0, 191.0, 197.0}};
        double[][][] x = new double[1][][];
        for (int i = 0; i < 1; i++)
        {
            x[i] = new double[2][];
        }
        x[0][0] = new double[]{46.0, 52.0, 39.0, 65.0, 54.0, 33.0, 49.0, 76.0,
                               71.0, 41.0, 58.0};
        x[0][1] = new double[]{18.0, 44.0, 33.0, 78.0, 51.0, 43.0, 44.0, 58.0,
                               63.0, 19.0, 42.0, 30.0, 47.0, 58.0, 70.0,
                               67.0, 31.0, 21.0, 56.0};
        ANCOVA awc = new ANCOVA(y, x);

        double[] testpl = awc.Compute();
        double[] aov = awc.GetANCOVA();

        Console.WriteLine("             * * * Analysis of Variance * * *\n");
        Console.WriteLine("                  Sum of         Mean            " +
            "    Prob of");
        Console.WriteLine("Source   DF      Squares        Square    " +
            "Overall F  Larger F");
        Console.WriteLine("Model  {0,3:0.#}     {1,10:0.##}      {2,9:0.##}" +
            "  {3,2:0.##}      {4,8:0.######}", aov[0], aov[3], aov[6], aov[8],
            aov[9]);
        Console.WriteLine("Error   {0,3:0.#}    {1,10:0.##}     {2,9:0.##} ",
            aov[1],    aov[4], aov[7]);
        Console.WriteLine("Total   {0,3:0.#}    {1,10:0.##} \n\n", aov[2],
            aov[5]);
        Console.WriteLine("           * * * Test for Parallelism  * * *\n");
        Console.WriteLine("                        Sum of     Mean     F     " +
            " Prob of");
        Console.WriteLine("Source           DF    Squares   Square   Test   " +
            " Larger F");
        Console.WriteLine("Extra due to");
        Console.WriteLine("Nonparallelism {0,3:0.#} {1,10:0.##}   {2,7:0.##}" +
            "   {3,7:0.#####} {4,5:0.####}", testpl[0], testpl[3], testpl[6],
            testpl[8], testpl[9]);
        Console.WriteLine("Extra Assuming");
        Console.WriteLine("Nonparallelism  {0,3:0.#} {1,10:0.##}  {2,7:0.##} ",
            testpl[1], testpl[4], testpl[7]);
        Console.WriteLine("Error Assuming");
        Console.WriteLine("Parallelism     {0,3:0.#} {1,10:0.##} \n\n",
```

```
            testpl[2], testpl[5]);
        PrintMatrixFormat pmf = new Imsl.Math.PrintMatrixFormat();
        pmf.NumberFormat = "0.###";
        new PrintMatrix("XY Mean Matrix\n").Print(pmf, awc.GetMeans());
        new PrintMatrix("\n\nVar./Covar. Matrix of Adjusted Group Means" +
            "\n").Print(pmf, awc.GetVarCovAdjustedMeans());
    }
}
```

Output

```
             * * * Analysis of Variance * * *

                   Sum of         Mean                Prob of
Source   DF       Squares        Square    Overall F  Larger F
Model    2       54432.75       27216.38  14.97      0.000042
Error    27      49103.91       1818.66
Total    29     103536.67


           * * * Test for Parallelism  * * *

                         Sum of     Mean      F      Prob of
Source            DF     Squares    Square    Test   Larger F
Extra due to
Nonparallelism    1      709.05     709.05    0.38093 0.5425
Extra Assuming
Nonparallelism    26    48394.86   1861.34
Error Assuming
Parallelism       27    49103.91


          XY Mean Matrix

   0     1        2        3
0  11  53.091  207.727  195.521
1  19  45.947  217.105  224.172
2  30  48.567  213.667  213.667



Var./Covar. Matrix of Adjusted Group Means

      0         1
0   170.368  -2.915
1  -2.915     97.407
```

Example 2: One-way analysis of covariance model

This example fits a one-way analysis of covariance model and performs a test for parallelism using data discussed by Snedecor and Cochran (1967, Table 14.8.1, pages 438-443). The responses are weight gains (in pounds per day) of 40 pigs for four groups of pigs under varying treatments. Two covariates-initial age (in days) and initial weight (in pounds) are used. For each treatment, there are 10

pigs. Only the first 5 pigs from each treatment are shown here.

Treatment 1			Treatment 2			Treatment 3			Treatment 4		
Age	Wt.	Gain	Age	Wt.	Gain	Age	Wt.	Gain	Age	Wt.	Gain
78	61	1.4	78	74	1.61	78	80	1.67	77	62	1.4
90	59	1.79	99	75	1.31	83	61	1.41	71	55	1.47
94	76	1.72	80	64	1.12	79	62	1.73	78	62	1.37
71	50	1.47	75	48	1.35	70	47	1.23	70	43	1.15
99	61	1.26	94	62	1.29	85	59	1.49	95	57	1.22

For these data, the test for non-parallelism is not statistically significant (p = 0.901). The one-way analysis of covariance test for the treatment means adjusted for the covariates, assuming parallel slopes, is statistically significant at a stated significance level of $\alpha = 0.05$, (p = 0.04931). Multiple comparisons can be done using the least significant difference approach of comparing each pair of treatment groups with the two-sample student-t test. Since the adjusted means in the one-way analysis of covariance are correlated, the standard error for these comparisons must be computed using the variances and covariances in covm. The standard errors for these comparisons are fairly similar ranging from 0.0630 to 0.0638. The Student's t comparisons identify differences between groups 1 and 2, and 1 and 4 as being statistically significant with p-values of 0.01225 and 0.03854 respectively.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class ANCOVAEx2
{
    public static void  Main(String[] args)
    {
        int j;
        int ngroup = 4;
        double[][] x1 = new double[][]{
            new double[]{78.0, 90.0, 94.0, 71.0, 99.0, 80.0, 83.0, 75.0,
                         62.0, 67.0},
            new double[]{78.0, 99.0, 80.0, 75.0, 94.0, 91.0, 75.0, 63.0,
                         62.0, 67.0},
            new double[]{78.0, 83.0, 79.0, 70.0, 85.0, 83.0, 71.0, 66.0,
                         67.0, 67.0},
            new double[]{77.0, 71.0, 78.0, 70.0, 95.0, 96.0, 71.0, 63.0,
                         62.0, 67.0}};
        double[][] x2 = new double[][]{
            new double[]{61.0, 59.0, 76.0, 50.0, 61.0, 54.0, 57.0, 45.0,
                         41.0, 40.0},
            new double[]{74.0, 75.0, 64.0, 48.0, 62.0, 42.0, 52.0, 43.0,
                         50.0, 40.0},
            new double[]{80.0, 61.0, 62.0, 47.0, 59.0, 42.0, 47.0, 42.0,
                         40.0, 40.0},
            new double[]{62.0, 55.0, 62.0, 43.0, 57.0, 51.0, 41.0, 40.0,
                         45.0, 39.0}};
        double[][] y = new double[][]{
            new double[]{1.40, 1.79, 1.72, 1.47, 1.26, 1.28, 1.34, 1.55,
                         1.57, 1.26},
            new double[]{1.61, 1.31, 1.12, 1.35, 1.29, 1.24, 1.29, 1.43,
```

```
                    1.29, 1.26},
    new double[]{1.67, 1.41, 1.73, 1.23, 1.49, 1.22, 1.39, 1.39,
                    1.56, 1.36},
    new double[]{1.40, 1.47, 1.37, 1.15, 1.22, 1.48, 1.31, 1.27,
                    1.22, 1.36}};
double[][][] x = new double[][][]{x1, x2};
/* setup covariate input matrix */

ANCOVA awc = new ANCOVA(y, x);

double[] testpl = awc.Compute();
double[] aov = awc.GetANCOVA();
double[] adjAov = awc.GetAdjustedANOVA();
double[][] xymean = awc.GetMeans();
double[][] covm = awc.GetVarCovAdjustedMeans();

Console.WriteLine("\n           * * * Test for Parallelism  * * *\n");
Console.WriteLine("                        Sum of     Mean         " + "F     Prob of");
Console.WriteLine("Source           DF    Squares   Square       " + "Test   Larger F");
Console.WriteLine("Extra due to");
Console.WriteLine("Nonparallelism {0,3:0.#} {1,10:0.##}    " +
    "{2,7:0.##}    {3,7:0.#####}   {4,5:0.###}", testpl[0], testpl[3],
    testpl[6], testpl[8], testpl[9]);
Console.WriteLine("Extra Assuming");
Console.WriteLine("Nonparallelism {0,3:0.#} {1,10:0.##}    {2,7:0.##}",
    testpl[1], testpl[4], testpl[7]);
Console.WriteLine("Error Assuming");
Console.WriteLine("Parallelism    {0,3:0.#} {1,10:0.##}   \n\n",
    testpl[2], testpl[5]);
Console.WriteLine("             * * * Analysis of Variance * * *\n");
Console.WriteLine("                  Sum of         " + "Mean       " +
    "         Prob of");
Console.WriteLine("Source   DF      Squares        Square    " +
    "Overall F  Larger F");
Console.WriteLine("Model   {0,3:0.#}      {1,7:0.######}     {2,7:0.######}    {3,7:0.######}    " +
    "{4,5:0.####}", aov[0], aov[3], aov[6], aov[8], aov[9]);
Console.WriteLine("Error   {0,3:0.#}      {1,7:0.######}     {2,7:0.######}      ", aov[1],
    aov[4], aov[7]);
Console.WriteLine("Total   {0,3:0.#}      {1,7:0.######}  \n\n", aov[2], aov[5]);
Console.WriteLine("         * * * Adjusted Analysis of Variance  * " +
    "* *\n");
Console.WriteLine("                                  Sum of        " +
    "F     Prob of");
Console.WriteLine("Source                     DF    Squares      " +
    "Test   Larger F");
Console.WriteLine("Groups after Covariates {0,3:0.#}   {1,10:0.##}" +
    "    {2,5:0.##}   " + " {3,7:0.#####}", adjAov[0], adjAov[2],
    adjAov[4], adjAov[6]);
Console.WriteLine("Covariates after Groups {0,3:0.#}   {1,10:0.##}" +
    "    {2,5:0.##}   " + " {3,7:0.#####}\n\n", adjAov[1], adjAov[3],
    adjAov[5], adjAov[7]);
Console.WriteLine("           * * * Group Means * * *\n");
Console.WriteLine("GROUP  | Unadjusted   |  Adjusted |  Std. Error");
for (int i = 0; i < ngroup; i++)
{
    double se = Math.Sqrt(covm[i][i]);
```

```
            Console.WriteLine("  {0}    |   {1,5:0.####}     |   {2,5:0.####}  " +
                "|   {3,7:0.####}", i + 1, xymean[i][ngroup - 1],
                xymean[i][ngroup], se);
        }
        Console.WriteLine("\n\n      * * * Student-t Multiple Comparisons * " +
            "* *\n");
        Console.WriteLine(" Groups  |    Diff   | Std. Error | Student-t | " +
            "P-Value");
        for (int i = 0; i < ngroup - 1; i++)
        {
            for (j = i + 1; j < ngroup; j++)
            {
                double delta = xymean[i][ngroup] - xymean[j][ngroup];
                double se = Math.Sqrt(covm[i][i] + covm[j][j] - 2.0 * covm[i][j]);
                double t = delta / se;
                double df = xymean[i][0] + xymean[j][0] - 2;
                double pvalue = 1.0 - Cdf.StudentsT(t, df);
                Console.WriteLine(" {0} vs {1}  |  {2,7:0.####}  |  " +
                    "{3,7:0.####}   | {4,7:0.####}   " + "| {5,7:0.#####}",
                    i + 1, j + 1, delta, se, t, pvalue);
            }
        }
    }
}
```

Output

```
          * * * Test for Parallelism  * * *

                       Sum of     Mean         F     Prob of
Source           DF    Squares   Square       Test   Larger F
Extra due to
Nonparallelism   6       0.05       0.01    0.35534   0.901
Extra Assuming
Nonparallelism  28       0.62       0.02
Error Assuming
Parallelism     34       0.67


             * * * Analysis of Variance * * *

                  Sum of         Mean               Prob of
Source   DF      Squares        Square    Overall F  Larger F
Model     5      0.352517     0.070503    3.57639    0.0105
Error    34      0.670261     0.019714
Total    39      1.022778


         * * * Adjusted Analysis of Variance  * * *

                               Sum of        F     Prob of
Source                    DF   Squares      Test   Larger F
Groups after Covariates   3        0.17     2.9    0.04931
Covariates after Groups   2        0.17    4.44    0.01939
```

```
          * * * Group Means * * *

GROUP  | Unadjusted   |  Adjusted |  Std. Error
  1    |   1.464      |   1.4614  |    0.0448
  2    |   1.319      |   1.3068  |    0.0446
  3    |   1.445      |   1.4429  |    0.0447
  4    |   1.325      |   1.3418  |    0.0449


     * * * Student-t Multiple Comparisons * * *

 Groups  |    Diff   | Std. Error | Student-t | P-Value
 1 vs 2  |   0.1546  |    0.063   |  2.4548   | 0.01225
 1 vs 3  |   0.0185  |   0.0637   |  0.2902   |  0.3875
 1 vs 4  |   0.1196  |   0.0638   |  1.8752   | 0.03854
 2 vs 3  |  -0.1362  |   0.0632   | -2.1528   | 0.97743
 2 vs 4  |   -0.035  |   0.0638   | -0.5495   | 0.70528
 3 vs 4  |   0.1011  |   0.0631   |  1.6018   |  0.0633
```

ANOVA Class

`public class Imsl.Stat.ANOVA`

Analysis of Variance table and related statistics.

Properties

AdjustedRSquared

`public double AdjustedRSquared {get; }`

Description

Returns the adjusted R-squared (in percent).

Property Value

A `double` representing the adjusted R-squared (in percent).

CoefficientOfVariation

`public double CoefficientOfVariation {get; }`

Description

Returns the coefficient of variation (in percent).

Property Value

A `double` representing the coefficient of variation (in percent).

DegreesOfFreedomForError

```
public double DegreesOfFreedomForError {get; }
```

Description

Returns the degrees of freedom for error.

Property Value

A `double` representing the degrees of freedom for the error.

DegreesOfFreedomForModel

```
public double DegreesOfFreedomForModel {get; }
```

Description

Returns the degrees of freedom for the model.

Property Value

A `double` representing the degrees of freedom for the model.

ErrorMeanSquare

```
public double ErrorMeanSquare {get; }
```

Description

Returns the error mean square.

Property Value

A `double` representing the error mean square.

F

```
public double F {get; }
```

Description

Returns the F statistic.

Property Value

A `double` representing the F statistic

MeanOfY

```
public double MeanOfY {get; }
```

Description

Returns the mean of the response (dependent variable).

Property Value

A `double` representing the mean of the response (dependent variable).

ModelErrorStdev

```
public double ModelErrorStdev {get; }
```

Description

Returns the estimated standard deviation of the model error.

Property Value

A `double` representing the estimated standard deviation of the model error.

ModelMeanSquare

```
public double ModelMeanSquare {get; }
```

Description

Returns the model mean square.

Property Value

A `double` representing the model mean square.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

P

```
public double P {get; }
```

Description

Returns the *p*-value.

Property Value

A `double` representing the *p*-value.

RSquared

```
public double RSquared {get; }
```

Description

Returns the *R*-squared (in percent).

Property Value

A `double` representing the *R*-squared (in percent).

SumOfSquaresForError

```
public double SumOfSquaresForError {get; }
```

Description

Returns the sum of squares for error.

Property Value

A `double` representing the sum of squares for the error.

SumOfSquaresForModel

```
public double SumOfSquaresForModel {get; }
```

Description

Returns the sum of squares for model.

Property Value

A `double` representing the sum of squares for the model.

TotalDegreesOfFreedom

```
public double TotalDegreesOfFreedom {get; }
```

Description

Returns the total degrees of freedom.

Property Value

A `double` representing the total degrees of freedom.

TotalMissing

```
public int TotalMissing {get; }
```

Description

Returns the total number of missing values.

Property Value

An `int` representing the total number of missing values (`NaN`) in input Y.

Remarks

Elements of Y containing `NaN` (not a number) are omitted from the computations.

TotalSumOfSquares

```
public double TotalSumOfSquares {get; }
```

Description

Returns the total sum of squares.

Property Value

A `double` representing the total sum of squares.

Constructors

ANOVA

```
public ANOVA(double[][] y)
```

Description

Analyzes a one-way classification model.

Parameter

`y` – Two-dimension `double` array containing the responses.

Remarks

The rows in y correspond to observation groups. Each row of y can contain a different number of observations.

ANOVA

```
public ANOVA(double dfr, double ssr, double dfe, double sse, double gmean)
```

Description

Construct an analysis of variance table and related statistics. Intended for use by the LinearRegression class.

Parameters

`dfr` – A `double` representing the degrees of freedom for the model.

`ssr` – A `double` representing the sum of squares for the model.

`dfe` – A `double` representing the degrees of freedom for the error.

`sse` – A `double` representing the sum of squares for the error.

`gmean` – A `double` representing the grand mean.

Remarks

If the grand mean is not known it may be set to not-a-number.

Methods

GetArray

```
public double[] GetArray()
```

Description

Returns the ANOVA values as an array.

Returns

A `double`[15] array containing the following values.

index	Value
0	Degrees of freedom for model
1	Degrees of freedom for error
2	Total degrees of freedom
3	Sum of squares for model
4	Sum of squares for error
5	Total sum of squares
6	Model mean square
7	Error mean square
8	F statistic
9	*p*-value
10	R-squared (in percent)
11	Adjusted R-squared (in percent)
12	Estimated standard deviation of the model error
13	Mean of the response (dependent variable)
14	Coefficient of variation (in percent)

GetConfidenceInterval

```
virtual public double[] GetConfidenceInterval(double conLevel, int i, int j,
Imsl.Stat.ANOVA.ComputeOption compMethod)
```

Description

Computes the confidence interval associated with the difference of means between two groups using a specified method.

Parameters

`conLevel` – A `double` specifying the confidence level for simultaneous interval estimation. If the Tukey method for computing the confidence intervals on the pairwise difference of means is to be used, `conLevel` must be in the range [90.0, 99.0]. Otherwise, `conLevel` must be in the range [0.0, 100.0). One normally sets this value to 95.0.

`i` – An `int` indicating the *i*-th member of the pair difference, $\mu_i - \mu_j$. `i` must be a valid group index.

`j` – An `int` indicating the *j*-th member of the pair difference, $\mu_i - \mu_j$. `j` must be a valid group index.

`compMethod` – An `ANOVA.ComputeOption`. `compMethod` must be one of the following:

compMethod	Description
Tukey	Uses the Tukey method. This method is valid for balanced one-way designs.
TukeyKramer	Uses the Tukey-Kramer method. This method simplifies to the Tukey method for the balanced case.
DunnSidak	Uses the Dunn-Sidak method.
Bonferroni	Uses the Bonferroni method.
Scheffe	Uses the Scheffe method.
OneAtATime	Uses the One-at-a-Time (Fisher's LSD) method.

Returns

A `double` array containing the group numbers, difference of means, and lower and upper confidence limits.

Array Element	Description
0	Group number for the *i*-th mean.
1	Group number for the *j*-th mean.
2	Difference of means (*i*-th mean) - (*j*-th mean).
3	Lower confidence limit for the difference.
4	Upper confidence limit for the difference.

Remarks

`GetConfidenceInterval` computes the simultaneous confidence interval on the pairwise comparison of means μ_i and μ_j in the one-way analysis of variance model. Any of several methods can be chosen. A good review of these methods is given by Stoline (1981). Also the methods are discussed in many elementary statistics texts, e.g., Kirk (1982, pages 114-127). Let s^2 be the estimated variance of a single observation. Let ν be the degrees of freedom associated with s^2. Let

$$\alpha = 1 - \frac{conLevel}{100.0}$$

The methods are summarized as follows:

Tukey method: The Tukey method gives the narrowest simultaneous confidence intervals for the pairwise differences of means $\mu_i - \mu_j$ in balanced $(n_1 = n_2 = \ldots = n_k = n)$ one-way designs. The method is exact and uses the Studentized range distribution. The formula for the difference $\mu_i - \mu_j$ is given by

$$\bar{y_i} - \bar{y_j} \pm q_{1-\alpha;k,\nu}\sqrt{\frac{s^2}{n}}$$

where $q_{1-a,k,\nu}$ is the $(1-\alpha)100$ percentage point of the Studentized range distribution with parameters k and ν. If the group sizes are unequal, the Tukey-Kramer method is used instead.

Tukey-Kramer method: The Tukey-Kramer method is an approximate extension of the Tukey method for the unbalanced case. (The method simplifies to the Tukey method for the balanced case.) The method always produces confidence intervals narrower than the Dunn-Sidak and Bonferroni methods. Hayter (1984) proved that the method is conservative, i.e., the method guarantees a confidence coverage of at least $(1-\alpha)100\%$. Hayter's proof gave further support to earlier recommendations for its use

(Stoline 1981). (Methods that are currently better are restricted to special cases and only offer improvement in severely unbalanced cases, see, e.g., Spurrier and Isham 1985). The formula for the difference $\mu_i - \mu_j$ is given by the following:

$$\bar{y_i} - \bar{y_j} \pm q_{1-\alpha;\nu,k}\sqrt{\frac{s^2}{2n_i} + \frac{s^2}{2n_j}}$$

Dunn-Sidak method: The Dunn-Sidak method is a conservative method. The method gives wider intervals than the Tukey-Kramer method. (For large ν and small α and k, the difference is only slight.) The method is slightly better than the Bonferroni method and is based on an improved Bonferroni (multiplicative) inequality (Miller, pages 101, 254-255). The method uses the *t* distribution. The formula for the difference $\mu_i - \mu_j$ is given by

$$\bar{y_i} - \bar{y_j} \pm t_{\frac{1}{2}+\frac{1}{2}(1-\alpha)^{1/k^*};\nu}\sqrt{\frac{s^2}{n_i} + \frac{s^2}{n_j}}$$

where $t_{f;\nu}$ is the $100f$ percentage point of the *t* distribution with ν degrees of freedom.

Bonferroni method: The Bonferroni method is a conservative method based on the Bonferroni (additive) inequality (Miller, page 8). The method uses the *t* distribution. The formula for the difference $\mu_i - \mu_j$ is given by

$$\bar{y_i} - \bar{y_j} \pm t_{1-\frac{\alpha}{2k^*};\nu}\sqrt{\frac{s^2}{n_i} + \frac{s^2}{n_j}}$$

Scheffè method: The Scheffè; method is an overly conservative method for simultaneous confidence intervals on pairwise difference of means. The method is applicable for simultaneous confidence intervals on all contrasts, i.e., all linear combinations

$$\sum_{i=1}^{k} c_i \mu_i$$

where the following is true:

$$\sum_{i=1}^{k} c_i = 0$$

The method can be recommended here only if a large number of confidence intervals on contrasts in addition to the pairwise differences of means are to be constructed. The method uses the *F* distribution. The formula for the difference $\mu_i - \mu_j$ is given by

$$\bar{y_i} - \bar{y_j} \pm \sqrt{(k-1)F_{1-\alpha;k-1,\nu}\left(\frac{s^2}{n_i} + \frac{s^2}{n_j}\right)}$$

where $F_{1-a;(k-1),\nu}$ is the $(1-\alpha)100$ percentage point of the *F* distribution with $k-1$ and ν degrees of freedom.

One-at-a-time *t* method (Fisher's LSD): The one-at-a-time *t* method is the method appropriate for constructing a single confidence interval. The confidence percentage input is appropriate for one interval

at a time. The method has been used widely in conjunction with the overall test of the null hypothesis $\mu_1 = \mu_2 = \ldots = \mu_k$ by the use of the F statistic. Fisher's LSD (least significant difference) test is a two-stage test that proceeds to make pairwise comparisons of means only if the overall *F* test is significant. Milliken and Johnson (1984, page 31) recommend LSD comparisons after a significant *F* only if the number of comparisons is small and the comparisons were planned prior to the analysis. If many unplanned comparisons are made, they recommend Scheffe's method. If the *F* test is insignificant, a few planned comparisons for differences in means can still be performed by using either Tukey, Tukey-Kramer, Dunn-Sidak or Bonferroni methods. Because the *F* test is insignificant, Scheffe's method will not yield any significant differences. The formula for the difference $\mu_i - \mu_j$ is given by

$$\bar{y_i} - \bar{y_j} \pm t_{1-\frac{\alpha}{2};\nu}\sqrt{\frac{s^2}{n_i} + \frac{s^2}{n_j}}$$

GetGroupInformation

```
public double[][] GetGroupInformation()
```

Description

Returns information concerning the groups.

Returns

A two-dimensional `double` array containing information concerning the groups.

Remarks

Row *i* contains information pertaining to the *i*-th group. The information in the columns is as follows:

Column	Information
0	Group Number
1	Number of nonmissing observations
2	Group Mean
3	Group Standard Deviation

Example: ANOVA

This example computes a one-way analysis of variance for data discussed by Searle (1971, Table 5.1, pages 165-179). The responses are plant weights for 6 plants of 3 different types - 3 normal, 2 off-types, and 1 aberrant. The 3 normal plant weights are 101, 105, and 94. The 2 off-type plant weights are 84 and 88. The 1 aberrant plant weight is 32. Note in the results that for the group with only one response, the standard deviation is undefined and is set to NaN (not a number).

```
using System;
using Imsl.Stat;
using Imsl.Math;


public class ANOVAEx1
{
    public static void  Main(String[] args)
```

```
    {
        double[][] y = {   new double[]{101, 105, 94},
                           new double[]{84, 88},
                           new double[]{32}};
        ANOVA anova = new ANOVA(y);
        double[] aov = anova.GetArray();

        Console.Out.WriteLine
            ("Degrees Of Freedom For Model = " + aov[0]);
        Console.Out.WriteLine
            ("Degrees Of Freedom For Error = " + aov[1]);
        Console.Out.WriteLine
            ("Total (Corrected) Degrees Of Freedom = " + aov[2]);
        Console.Out.WriteLine("Sum Of Squares For Model = " + aov[3]);
        Console.Out.WriteLine("Sum Of Squares For Error = " + aov[4]);
        Console.Out.WriteLine
            ("Total (Corrected) Sum Of Squares = " + aov[5]);
        Console.Out.WriteLine("Model Mean Square = " + aov[6]);
        Console.Out.WriteLine("Error Mean Square = " + aov[7]);
        Console.Out.WriteLine("F statistic = " + aov[8]);
        Console.Out.WriteLine("P value= " + aov[9]);
        Console.Out.WriteLine("R Squared (in percent) = " + aov[10]);
        Console.Out.WriteLine
            ("Adjusted R Squared (in percent) = " + aov[11]);
        Console.Out.WriteLine
            ("Model Error Standard deviation = " + aov[12]);
        Console.Out.WriteLine("Mean Of Y = " + aov[13]);
        Console.Out.WriteLine
            ("Coefficient Of Variation (in percent) = " + aov[14]);
        Console.Out.WriteLine
            ("Total number of missing values = " + anova.TotalMissing);

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        String[] labels =
            new String[]{"Group", "N", "Mean", "Std. Deviation"};
        pmf.SetColumnLabels(labels);
        pmf.NumberFormat = null;
        new PrintMatrix("Group Information").Print(pmf,
            (Object)anova.GetGroupInformation());
    }
}
```

Output

```
Degrees Of Freedom For Model = 2
Degrees Of Freedom For Error = 3
Total (Corrected) Degrees Of Freedom = 5
Sum Of Squares For Model = 3480
Sum Of Squares For Error = 70
Total (Corrected) Sum Of Squares = 3550
Model Mean Square = 1740
Error Mean Square = 23.3333333333333
F statistic = 74.5714285714286
P value= 0.00276888252534978
R Squared (in percent) = 98.0281690140845
```

```
Adjusted R Squared (in percent) = 96.7136150234742
Model Error Standard deviation = 4.83045891539648
Mean Of Y = 84
Coefficient Of Variation (in percent) = 5.75054632785295
Total number of missing values = 0
          Group Information
   Group  N  Mean   Std. Deviation
0    0    3  100   5.56776436283002
1    1    2   86   2.82842712474619
2    2    1   32   NaN
```

ANOVA.ComputeOption Enumeration

```
public enumeration Imsl.Stat.ANOVA.ComputeOption
```

Compute option.

Fields

Bonferroni

```
public Imsl.Stat.ANOVA.ComputeOption Bonferroni
```

Description

The Bonferroni method.

DunnSidak

```
public Imsl.Stat.ANOVA.ComputeOption DunnSidak
```

Description

The Dunn-Sidak method.

OneAtATime

```
public Imsl.Stat.ANOVA.ComputeOption OneAtATime
```

Description

The One-at-a-Time (Fisher’s LSD) method.

Scheffe

```
public Imsl.Stat.ANOVA.ComputeOption Scheffe
```

Description

The Scheffe method.

Tukey

```
public Imsl.Stat.ANOVA.ComputeOption Tukey
```

Description

The Tukey method.

TukeyKramer

```
public Imsl.Stat.ANOVA.ComputeOption TukeyKramer
```

Description

The Tukey-Kramer method.

ANOVAFactorial Class

```
public class Imsl.Stat.ANOVAFactorial
```

Analyzes a balanced factorial design with fixed effects.

Class `ANOVAFactorial` performs an analysis for an *n*-way classification design with balanced data. For balanced data, here must be an equal number of responses in each cell of the *n*-way layout. The effects are assumed to be fixed effects. The model is an extension of the two-way model to include *n* factors. The interactions (two-way, three-way, up to *n*-way) can be included in the model, or some of the higher-way interactions can be pooled into error. The `ModelOrder` property specifies the number of factors to be included in the highest-way interaction. For example, if three-way and higher-way interactions are to be pooled into error, set `ModelOrder` = 2. (By default, `ModelOrder` = `nSubscripts` - 1 with the last subscript being the error subscript.) `Pure` indicates there are repeated responses within the *n*-way cell; `Pooled` indicates otherwise.

Class `ANOVAFactorial` requires the responses as input into a single vector y in lexicographical order, so that the response subscript associated with the first factor varies least rapidly, followed by the subscript associated with the second factor, and so forth. Hemmerle (1967, Chapter 5) discusses the computational method.

Properties

ErrorIncludeType

```
public Imsl.Stat.ANOVAFactorial.ErrorCalculation ErrorIncludeType {get; set; }
```

Description

The error included type.

Property Value

An `int` scalar specifying the included error type.

Remarks

`ANOVAFactorial.ErrorCalculation.Pure`, the default option, indicates factor `nSubscripts` is error. Its main effect and all its interaction effects are pooled into the error with the other (`ModelOrder` + 1)-way and higher-way interactions.

`ANOVAFactorial.ErrorCalculation.Pooled` indicates factor `nSubscripts` is not error. Only (`ModelOrder` + 1)-way and higher-way interactions are included in the error.

ModelOrder

```
public int ModelOrder {get; set; }
```

Description

The number of factors to be included in the highest-way interaction in the model.

Property Value

An `int` scalar containing the number of factors to be included in the highest-way interaction in the model.

Remarks

`ModelOrder` must be in the interval [1, `nSubscripts`-1]. For example:

`ModelOrder` of 1 indicates that a main effect model will be analyzed.

`ModelOrder` of 2 indicates that two-way interactions will be included in the model.

By default, `ModelOrder` = `nSubscripts - 1`.

Constructor

ANOVAFactorial

```
public ANOVAFactorial(int nSubscripts, int[] nLevels, double[] y)
```

Description

Constructor for `ANOVAFactorial`.

Parameters

`nSubscripts` – An `int` scalar containing the number of subscripts. Number of factors in the model + 1 (for the error term).

`nLevels` – An `int` array of length `nSubscripts` containing the number of levels for each of the factors for the first `nSubscripts`-1 elements. `nLevels[nSubscripts-1]` is the number of observations per cell.

`y` – A `double` array of length `nLevels[0] * nLevels[1] * ... * nLevels[nSubscripts-1]` containing the responses.

Remarks

y must not contain `NaN` for any of its elements; i.e., missing values are not allowed.

Exception

`System.ArgumentException` is thrown if `nLevels.length`, and `y.length` are not consistent

Methods

Compute

```
public double Compute()
```

Description

Analyzes a balanced factorial design with fixed effects.

Returns

A `double` scalar containing the *p*-value for the overall *F* test.

GetANOVATable

```
public double[] GetANOVATable()
```

Description

Returns the analysis of variance table. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` array containing the analysis of variance table.

Remarks

The analysis of variance statistics are given as follows:

Element	Analysis of Variance Statistics
0	Degrees of freedom for the model
1	Degrees of freedom for error
2	Total (corrected) degrees of freedom
3	Sum of squares for the model
4	Sum of squares for error
5	Total (corrected) sum of squares
6	Model mean square
7	Error mean square
8	Overall *F*-statistic
9	*p*-value
10	R^2 (in percent)
11	Adjusted R^2 (in percent)
12	Estimate of the standard deviation
13	Overall mean of y
14	Coefficient of variation (in percent)

GetMeans

```
public double[] GetMeans()
```

Description

Returns the subgroup means. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` array containing the subgroup means.

GetTestEffects

```
public double[,] GetTestEffects()
```

Description

Returns statistics relating to the sums of squares for the effects in the model. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` matrix containing statistics relating to the sums of squares for the effects in the model.

Remarks

Here,

$$\text{NEF} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{\min(n, |\text{model\_order}|)}$$

where `n` is given by `nSubscripts` if `ANOVAFactorial.ErrorCalculation.Pooled` is specified; otherwise, `nSubscripts`-1. Suppose the factors are A, B, C, and error. With `ModelOrder` = 3, rows 0 through NEF-1 would correspond to A, B, C, AB, AC, BC, and ABC, respectively.

The columns of the output matrix are as follows:

Column	Description
0	Degrees of freedom
1	Sum of squares
2	*F*-statistic
3	*p*-value

Example 1: Two-way Analysis of Variance

A two-way analysis of variance is performed with balanced data discussed by Snedecor and Cochran (1967, Table 12.5.1, p. 347). The responses are the weight gains (in grams) of rats that were fed diets varying in the source (A) and level (B) of protein. The model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk} \quad i = 1,\ 2;\ j = 1,\ 2,\ 3;\ k = 1,\ 2,\ \ldots,\ 10$$

where

$$\sum_{i=1}^{2} \alpha_i = 0; \sum_{j=1}^{3} \beta_j = 0; \sum_{i=1}^{2} \gamma_{ij} = 0 \text{ for } j = 1, 2, 3;$$

and

$$\sum_{j=1}^{3} \gamma_{ij} = 0 \text{ for } j = 1, 2$$

The first responses in each cell in the two-way layout are given in the following table:

	Protein Source (A)		
Protein Level (B)	**Beef**	**Cereal**	**Pork**
High	73, 102, 118, 104, 81, 107, 100, 87, 117, 111	98, 74, 56, 111, 95, 88, 82, 77, 86, 92	94, 79, 96, 98, 102, 102, 108, 91, 120, 105
Low	90, 76, 90, 64, 86, 51, 72, 90, 95, 78	107, 95, 97, 80, 98, 74, 74, 67, 89, 58	49, 82, 73, 86, 81, 97, 106, 70, 61, 82

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class ANOVAFactorialEx1
{
    public static void  Main(String[] args)
    {
        int nSubscripts = 3;
        int[] nLevels = new int[]{3, 2, 10};
        double[] y = new double[]{   73.0, 102.0, 118.0,
                                     104.0, 81.0, 107.0,
                                     100.0, 87.0, 117.0,
                                     111.0, 90.0, 76.0,
                                     90.0, 64.0, 86.0,
                                     51.0, 72.0, 90.0,
                                     95.0, 78.0, 98.0,
                                     74.0, 56.0, 111.0,
                                     95.0, 88.0, 82.0,
                                     77.0, 86.0, 92.0,
                                     107.0, 95.0, 97.0,
                                     80.0, 98.0, 74.0,
                                     74.0, 67.0, 89.0,
                                     58.0, 94.0, 79.0,
                                     96.0, 98.0, 102.0,
                                     102.0, 108.0, 91.0,
                                     120.0, 105.0, 49.0,
                                     82.0, 73.0, 86.0,
                                     81.0, 97.0, 106.0,
                                     70.0, 61.0, 82.0};
```

```
        ANOVAFactorial af =
            new ANOVAFactorial(nSubscripts, nLevels, y);
        Console.Out.WriteLine
            ("P-value = " + af.Compute().ToString("0.000000"));
    }
}
```

Output

```
P-value = 0.002299
```

Example 2: Two-way Analysis of Variance

In this example, the same model and data is fit as in the example 1, but additional information is printed.

```
using System;
using Imsl.Stat;

public class ANOVAFactorialEx2
{
    public static void  Main(String[] args)
    {
        int nSubscripts = 3, i;
        int[] nLevels = new int[]{3, 2, 10};
        double[] y = new double[]{   73.0, 102.0, 118.0,
                                     104.0, 81.0, 107.0,
                                     100.0, 87.0, 117.0,
                                     111.0, 90.0, 76.0,
                                     90.0, 64.0, 86.0,
                                     51.0, 72.0, 90.0,
                                     95.0, 78.0, 98.0,
                                     74.0, 56.0, 111.0,
                                     95.0, 88.0, 82.0,
                                     77.0, 86.0, 92.0,
                                     107.0, 95.0, 97.0,
                                     80.0, 98.0, 74.0,
                                     74.0, 67.0, 89.0,
                                     58.0, 94.0, 79.0,
                                     96.0, 98.0, 102.0,
                                     102.0, 108.0, 91.0,
                                     120.0, 105.0, 49.0,
                                     82.0, 73.0, 86.0,
                                     81.0, 97.0, 106.0,
                                     70.0, 61.0, 82.0};
        String[] labels =
            new String[]{"degrees of freedom for the model" +
            "                 ", "degrees of freedom for error" +
            "                    ",
            "total (corrected) degrees of freedom            ",
            "sum of squares for the model                 ",
            "sum of squares for error                    ",
            "total (corrected) sum of squares            ",
            "model mean square                              ",
```

```
            "error mean square                                ",
            "F-statistic                                       ",
            "p-value                                           ",
            "R-squared (in percent)                           ",
            "Adjusted R-squared (in percent)                  ",
            "est. standard deviation of the model error       ",
            "overall mean of y                                ",
            "coefficient of variation (in percent)            "};
        String[] rlabels = new String[]{"A", "B", "A*B"};
        String[] mlabels = new String[]{"grand mean     ",
            "A1              ", "A2              ", "A3              ",
            "B1              ", "B2              ", "A1*B1          ",
            "A1*B2           ", "A2*B1           ", "A2*B2           ",
            "A3*B1           ", "A3*B2           "};

        ANOVAFactorial af =
            new ANOVAFactorial(nSubscripts, nLevels, y);

        Console.Out.WriteLine
            ("P-value = " + af.Compute().ToString("0.000000"));

        Console.Out.WriteLine
            ("\n          * * * Analysis of Variance * * *");
        double[] anova = af.GetANOVATable();
        for (i = 0; i < anova.Length; i++)
        {
            Console.Out.WriteLine
                (labels[i] + " " + anova[i].ToString("0.0000"));
        }

        Console.Out.WriteLine
            ("\n          * * * Variation Due to the " + "Model * * *");
        Console.Out.WriteLine
            ("Source\tDF\tSum of Squares\tMean Square" +
             "\tProb. of Larger F");
        double[,] te = af.GetTestEffects();
        for (i = 0; i < te.GetLength(0); i++)
        {
            Console.Out.WriteLine(
                rlabels[i] + "\t" +
                te[i,0].ToString("0.0000") + "\t" +
                te[i,1].ToString("0.0000") + "\t" +
                te[i,2].ToString("0.0000") + "\t\t" +
                te[i,3].ToString("0.0000"));
        }

        Console.Out.WriteLine("\n* * * Subgroup Means * * *");
        double[] means = af.GetMeans();
        for (i = 0; i < means.Length; i++)
        {
            Console.Out.WriteLine
                (mlabels[i] + " " + means[i].ToString("0.0000"));
        }
    }
}
```

Output

```
P-value = 0.002299

          * * * Analysis of Variance * * *
degrees of freedom for the model                  5.0000
degrees of freedom for error                     54.0000
total (corrected) degrees of freedom             59.0000
sum of squares for the model                   4612.9333
sum of squares for error                      11586.0000
total (corrected) sum of squares              16198.9333
model mean square                               922.5867
error mean square                               214.5556
F-statistic                                       4.3000
p-value                                           0.0023
R-squared (in percent)                           28.4768
Adjusted R-squared (in percent)                  21.8543
est. standard deviation of the model error       14.6477
overall mean of y                                87.8667
coefficient of variation (in percent)            16.6704

          * * * Variation Due to the Model * * *
Source DF Sum of Squares Mean Square Prob. of Larger F
A 2.0000 266.5333 0.6211 0.5411
B 1.0000 3168.2667 14.7666 0.0003
A*B 2.0000 1178.1333 2.7455 0.0732

* * * Subgroup Means * * *
grand mean      87.8667
A1              89.6000
A2              84.9000
A3              89.1000
B1              95.1333
B2              80.6000
A1*B1          100.0000
A1*B2           79.2000
A2*B1           85.9000
A2*B2           83.9000
A3*B1           99.5000
A3*B2           78.7000
```

Example 3: Three-way Analysis of Variance

This example performs a three-way analysis of variance using data discussed by John (1971, pp. 91 92). The responses are weights (in grams) of roots of carrots grown with varying amounts of applied nitrogen (A), potassium (B), and phosphorus (C). Each cell of the three-way layout has one response. Note that the ABC interactions sum of squares, which is 186, is given incorrectly by John (1971, Table 5.2.) The three-way layout is given in the following table:

	A_0		
	B_0	B_1	B_2
C_0	88.76	91.41	97.85
C_1	87.45	98.27	95.85
C_2	86.01	104.20	90.09

	A_1		
	B_0	B_1	B_2
C_0	94.83	100.49	99.75
C_1	84.57	97.20	112.30
C_2	81.06	120.80	108.77

	A_2		
	B_0	B_1	B_2
C_0	99.90	100.23	104.50
C_1	92.98	107.77	110.94
C_2	94.72	118.39	102.87

```
using System;
using Imsl.Stat;

public class ANOVAFactorialEx3
{
    public static void  Main(String[] args)
    {
        int nSubscripts = 3, i;
        int[] nLevels = new int[]{3, 3, 3};
        double[] y = new double[]{   88.76, 87.45, 86.01,
                                     91.41, 98.27, 104.2,
                                     97.85, 95.85, 90.09,
                                     94.83, 84.57, 81.06,
                                     100.49, 97.2, 120.8,
                                     99.75, 112.3, 108.77,
                                     99.9, 92.98, 94.72,
                                     100.23, 107.77, 118.39,
                                     104.51, 110.94, 102.87};
        String[] labels =
            new String[]{"degrees of freedom for the model" +
            "                  ", "degrees of freedom for error" +
            "                    ",
            "total (corrected) degrees of freedom             ",
            "sum of squares for the model                   ",
            "sum of squares for error                         ",
            "total (corrected) sum of squares               ",
            "model mean square                                  ",
            "error mean square                                   ",
            "F-statistic                                          ",
           "p-value                                              ",
           "R-squared (in percent)                             ",
           "Adjusted R-squared (in percent)                    ",
           "est. standard deviation of the model error          ",
           "overall mean of y                                  ",
           "coefficient of variation (in percent)               "};
        String[] rlabels =
            new String[]{"A", "B", "C", "A*B", "A*C", "B*C"};

        ANOVAFactorial af =
            new ANOVAFactorial(nSubscripts, nLevels, y);
```

```
        af.ErrorIncludeType = ANOVAFactorial.ErrorCalculation.Pooled;
        Console.Out.WriteLine
            ("P-value = " + af.Compute().ToString("0.000000"));

        Console.Out.WriteLine
            ("\n          * * * Analysis of Variance * * *");
        double[] anova = af.GetANOVATable();
        for (i = 0; i < anova.Length; i++)
        {
            Console.Out.WriteLine
                (labels[i] + " " + anova[i].ToString("0.0000"));
        }

        Console.Out.WriteLine
            ("\n          * * * Variation Due to the " + "Model * * *");
        Console.Out.WriteLine
            ("Source\tDF\tSum of Squares\tMean Square" +
             "\tProb. of Larger F");
        double[,] te = af.GetTestEffects();
        for (i = 0; i < te.GetLength(0); i++)
        {
            System.Text.StringBuilder sb =
                new System.Text.StringBuilder(rlabels[i]);

            int len = sb.Length;
            for (int j = 0; j < (8 - len); j++)
                sb.Append(' ');
            sb.Append(te[i,0].ToString("0.0000"));

            len = sb.Length;
            for (int j = 0; j < (16 - len); j++)
                sb.Append(' ');
            sb.Append(te[i,1].ToString("0.0000"));

            len = sb.Length;
            for (int j = 0; j < (32 - len); j++)
                sb.Append(' ');
            sb.Append(te[i,2].ToString("0.0000"));

            len = sb.Length;
            for (int j = 0; j < (48 - len); j++)
                sb.Append(' ');
            sb.Append(te[i,3].ToString("0.0000"));

            Console.Out.WriteLine(sb.ToString());
        }
    }
}
```

Output

```
P-value = 0.008299

          * * * Analysis of Variance * * *
degrees of freedom for the model                 18.0000
```

```
degrees of freedom for error                       8.0000
total (corrected) degrees of freedom              26.0000
sum of squares for the model                    2395.7290
sum of squares for error                         185.7763
total (corrected) sum of squares                2581.5052
model mean square                                133.0961
error mean square                                 23.2220
F-statistic                                        5.7315
p-value                                            0.0083
R-squared (in percent)                            92.8036
Adjusted R-squared (in percent)                   76.6116
est. standard deviation of the model error         4.8189
overall mean of y                                 98.9619
coefficient of variation (in percent)              4.8695

          * * * Variation Due to the Model * * *
Source DF Sum of Squares Mean Square Prob. of Larger F
A       2.0000  488.3675        10.5152         0.0058
B       2.0000  1090.6564       23.4832         0.0004
C       2.0000  49.1485         1.0582          0.3911
A*B     4.0000  142.5853        1.5350          0.2804
A*C     4.0000  32.3474         0.3482          0.8383
B*C     4.0000  592.6238        6.3800          0.0131
```

ANOVAFactorial.ErrorCalculation Enumeration

`public enumeration Imsl.Stat.ANOVAFactorial.ErrorCalculation`

ErrorCalculation members indicate whether interaction effects are pooled into the error or not.

Fields

Pooled

`public Imsl.Stat.ANOVAFactorial.ErrorCalculation Pooled`

Description

Indicates factor `nSubscripts` is not error.

Pure

`public Imsl.Stat.ANOVAFactorial.ErrorCalculation Pure`

Description

Indicates factor `nSubscripts` is error. This is the default.

MultipleComparisons Class

`public class Imsl.Stat.MultipleComparisons`

Performs Student-Newman-Keuls multiple comparisons test.

Class `MultipleComparisons` performs a multiple comparison analysis of means using the Student-Newman-Keuls method. The null hypothesis is equality of all possible ordered subsets of a set of means. This null hypothesis is tested using the Studentized range of each of the corresponding subsets of sample means. The method is discussed in many elementary statistics texts, e.g., Kirk (1982, pp. 123-125).

Property

Alpha

`public double Alpha {get; set; }`

Description

The significance level of the test

Property Value

A `double` scalar containing the significance level of test.

Remarks

`Alpha` must be in the interval [0.01, 0.10]. By default, `Alpha` = 0.01.

Constructor

MultipleComparisons

`public MultipleComparisons(double[] means, int df, double stdError)`

Description

Constructor for `MultipleComparisons`.

Parameters

`means` – A `double` array containing the means.

`df` – A `int` scalar containing the degrees of freedom associated with `stdError`.

`stdError` – A `double` scalar containing the effective estimated standard error of a mean.

Remarks

In fixed effects models, `stdError` equals the estimated standard error of a mean. For example, in a one-way model stdError $= \sqrt{s^2/n}$ where s^2 is the estimate of σ^2 and n is the number of responses in a sample mean. In models with random components, use stdError $=$ sedif$/\sqrt{2}$ where `sedif` is the estimated standard error of the difference of two means.

Method

Compute

```
public int[] Compute()
```

Description

Performs Student-Newman-Keuls multiple comparisons test.

Returns

A `int` array , call it `equalMeans`, indicating the size of the groups of means declared to be equal.

Remarks

Value `equalMeans[I] = J` indicates the *I*-th smallest mean and the next *J*-1 larger means are declared equal. Value `equalMeans[I] = 0` indicates no group of means starts with the *I*-th smallest mean.

Example: Multiple Comparisons Test

A multiple-comparisons analysis is performed using data discussed by Kirk (1982, pp. 123-125). The results show that there are three groups of means with three separate sets of values: (36.7, 40.3, 43.4), (40.3, 43.4, 47.2), and (43.4, 47.2, 48.7).

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class MultipleComparisonsEx1
{
    public static void  Main(String[] args)
    {
        double[] means = new double[]{36.7, 48.7, 43.4, 47.2, 40.3};

        /* Perform multiple comparisons tests */
        MultipleComparisons mc =
            new MultipleComparisons(means, 45, 1.6970563);

        new PrintMatrix("Size of Groups of Means").Print(mc.Compute());
    }
}
```

Output

```
Size of Groups of Means
   0
0  3
1  3
2  3
3  0
```

Chapter 15: Categorical and Discrete Data Analysis

Types

Usage Notes

The `ContingencyTable` class computes many statistics of interest in a two-way table. Statistics computed by this routine include the usual chi-squared statistics, measures of association, Kappa, and many others.

ContingencyTable Class

```
public class Imsl.Stat.ContingencyTable
```

Performs a chi-squared analysis of a two-way contingency table.

Class `ContingencyTable` computes statistics associated with an $r \times c$ contingency table. The function computes the chi-squared test of independence, expected values, contributions to chi-squared, row and column marginal totals, some measures of association, correlation, prediction, uncertainty, the McNemar test for symmetry, a test for linear trend, the odds and the log odds ratio, and the kappa statistic (if the appropriate optional arguments are selected).

Notation

Let x_{ij} denote the observed cell frequency in the ij cell of the table and n denote the total count in the table. Let $p_{ij} = p_{i\bullet}p_{j\bullet}$ denote the predicted cell probabilities under the null hypothesis of independence, where $p_{i\bullet}$ and $p_{j\bullet}$ are the row and column marginal relative frequencies. Next, compute the expected

cell counts as $e_{ij} = np_{ij}$.

Also required in the following are a_{uv} and b_{uv} for $u, v = 1, \ldots, n$. Let (r_s, c_s) denote the row and column response of observation s. Then, $a_{uv} = 1, 0$, or -1, depending on whether $r_u < r_v, r_u = r_v$, or $r_u > r_v$, respectively. The b_{uv} are similarly defined in terms of the c_s variables.

Chi-squared Statistic

For each cell in the table, the contribution to χ^2 is given as $(x_{ij} - e_{ij})^2 / e_{ij}$. The Pearson chi-squared statistic (denoted χ^2) is computed as the sum of the cell contributions to chi-squared. It has *(r - 1) (c - 1)* degrees of freedom and tests the null hypothesis of independence, i.e., $H_0 : p_{ij} = p_{i\bullet} p_{j\bullet}$. The null hypothesis is rejected if the computed value of χ^2 is too large.

The maximum likelihood equivalent of χ^2, G^2 is computed as follows:

$$G^2 = -2 \sum_{i,j} x_{ij} \ln (x_{ij} / np_{ij})$$

G^2 is asymptotically equivalent to χ^2 and tests the same hypothesis with the same degrees of freedom.

Measures Related to Chi-squared (Phi, Contingency Coefficient, and Cramer's V)

There are three measures related to chi-squared that do not depend on sample size:

$$\text{phi, } \phi = \sqrt{\chi^2 / n}$$

$$\text{contingency coefficient, } P = \sqrt{\chi^2 / (n + \chi^2)}$$

$$\text{Cramer's } V, V = \sqrt{\chi^2 / (n \min (r, c))}$$

Since these statistics do not depend on sample size and are large when the hypothesis of independence is rejected, they can be thought of as measures of association and can be compared across tables with different sized samples. While both P and V have a range between 0.0 and 1.0, the upper bound of P is actually somewhat less than 1.0 for any given table (see Kendall and Stuart 1979, p. 587). The significance of all three statistics is the same as that of the χ^2 statistic, which is contained in the `ChiSquared` property.

The distribution of the χ^2 statistic in finite samples approximates a chi-squared distribution. To compute the exact mean and standard deviation of the χ^2 statistic, Haldane (1939) uses the multinomial distribution with fixed table marginals. The exact mean and standard deviation generally differ little from the mean and standard deviation of the associated chi-squared distribution.

Standard Errors and p-values for Some Measures of Association

In Columns 1 through 4 of statistics, estimated standard errors and asymptotic *p*-values are reported. Estimates of the standard errors are computed in two ways. The first estimate, in Column 1 of the return matrix from the `Statistics` property, is asymptotically valid for any value of the statistic. The second estimate, in Column 2 of the array, is only correct under the null hypothesis of no association. The *z*-scores in Column 3 of statistics are computed using this second estimate of the standard errors. The *p*-values in Column 4 are computed from this *z*-score. See Brown and Benedetti (1977) for a discussion and formulas for the standard errors in Column 2.

Measures of Association for Ranked Rows and Columns

The measures of association, ϕ, *P*, and *V*, do not require any ordering of the row and column categories. Class `ContingencyTable` also computes several measures of association for tables in which the rows and column categories correspond to ranked observations. Two of these tests, the product-moment correlation and the Spearman correlation, are correlation coefficients computed using assigned scores for the row and column categories. The cell indices are used for the product-moment correlation, while the average of the tied ranks of the row and column marginals is used for the Spearman rank correlation. Other scores are possible.

Gamma, Kendall's τ_b, Stuart's τ_c, and Somers' *D* are measures of association that are computed like a correlation coefficient in the numerator. In all these measures, the numerator is computed as the "covariance" between the a_{uv} variables and b_{uv} variables defined above, i.e., as follows:

$$\sum_u \sum_v a_{uv} b_{uv}$$

Recall that a_{uv} and b_{uv} can take values -1, 0, or 1. Since the product $a_{uv}b_{uv} = 1$ only if a_{uv} and b_{uv} are both 1 or are both -1, it is easy to show that this "covariance" is twice the total number of agreements minus the number of disagreements, where a disagreement occurs when $a_{uv}b_{uv} = -1$.

Kendall's τ_b is computed as the correlation between the a_{uv} variables and the b_{uv} variables (see Kendall and Stuart 1979, p. 593). In a rectangular table $(r \neq c)$, Kendall's τ_b cannot be 1.0 (if all marginal totals are positive). For this reason, Stuart suggested a modification to the denominator of τ in which the denominator becomes the largest possible value of the "covariance." This maximizing value is approximately $n^2 m/(m-1)$, where *m = min (r, c)*. Stuart's τ_c uses this approximate value in its denominator. For large $n, \tau_c \approx m\tau_b/(m-1)$.

Gamma can be motivated in a slightly different manner. Because the "covariance" of the a_{uv} variables and the b_{uv} variables can be thought of as twice the number of agreements minus the disagreements, *2(A - D)*, where *A* is the number of agreements and *D* is the number of disagreements, Gamma is motivated as the probability of agreement minus the probability of disagreement, given that either agreement or disagreement occurred. This is shown as $\gamma = (A-D)/(A+D)$.

Two definitions of Somers' *D* are possible, one for rows and a second for columns. Somers' *D* for rows can be thought of as the regression coefficient for predicting a_{uv} from b_{uv}. Moreover, Somer's *D* for rows is the probability of agreement minus the probability of disagreement, given that the column variable, b_{uv}, is not 0. Somers' *D* for columns is defined in a similar manner.

A discussion of all of the measures of association in this section can be found in Kendall and Stuart (1979, p. 592).

Measures of Prediction and Uncertainty

Optimal Prediction Coefficients: The measures in this section do not require any ordering of the row or column variables. They are based entirely upon probabilities. Most are discussed in Bishop et al. (1975, p. 385).

Consider predicting (or classifying) the column for a given row in the table. Under the null hypothesis of independence, choose the column with the highest column marginal probability for all rows. In this case, the probability of misclassification for any row is 1 minus this marginal probability. If independence is not assumed within each row, choose the column with the highest row conditional probability. The probability of misclassification for the row becomes 1 minus this conditional probability.

Define the optimal prediction coefficient $\lambda_{c|r}$ for predicting columns from rows as the proportion of the probability of misclassification that is eliminated because the random variables are not independent. It is estimated by

$$\lambda_{c|r} = \frac{(1 - p_{\bullet m}) - (1 - \sum_i p_{im})}{1 - p_{\bullet m}}$$

where m is the index of the maximum estimated probability in the row (p_{im}) or row margin $(p_{\bullet m})$. A similar coefficient is defined for predicting the rows from the columns. The symmetric version of the optimal prediction λ is obtained by summing the numerators and denominators of $\lambda_{r|c}$ and $\lambda_{c|r}$ then dividing. Standard errors for these coefficients are given in Bishop et al. (1975, p. 388).

A problem with the optimal prediction coefficients λ is that they vary with the marginal probabilities. One way to correct this is to use row conditional probabilities. The optimal prediction $\lambda*$ coefficients are defined as the corresponding λ coefficients in which first the row (or column) marginals are adjusted to the same number of observations. This yields

$$\lambda^*_{c|r} = \frac{\sum_i \max_j p_{j|i} - \max_j(\sum_i p_{j|i})}{R - \max_j(\sum_i p_{j|i})}$$

where i indexes the rows, j indexes the columns, and $p_{j|i}$ is the (estimated) probability of column j given row i.

$$\lambda^*_{r|c}$$

is similarly defined.

Goodman and Kruskal τ: A second kind of prediction measure attempts to explain the proportion of the explained variation of the row (column) measure given the column (row) measure. Define the total variation in the rows as follows:

$$n/2 - (\sum_i x^2_{i\bullet})/(2n)$$

Note that this is *1/(2n)* times the sums of squares of the a_{uv} variables.

With this definition of variation, the Goodman and Kruskal τ coefficient for rows is computed as the reduction of the total variation for rows accounted for by the columns, divided by the total variation for the rows. To compute the reduction in the total variation of the rows accounted for by the columns, note that the total variation for the rows within column j is defined as follows:

$$q_j = x_{\bullet j}/2 - (\sum_i x_{ij}^2)/(2x_{i\bullet})$$

The total variation for rows within columns is the sum of the q_j variables. Consistent with the usual methods in the analysis of variance, the reduction in the total variation is given as the difference between the total variation for rows and the total variation for rows within the columns.

Goodman and Kruskal's τ for columns is similarly defined. See Bishop et al. (1975, p. 391) for the standard errors.

Uncertainty Coefficients: The uncertainty coefficient for rows is the increase in the log-likelihood that is achieved by the most general model over the independence model, divided by the marginal log-likelihood for the rows. This is given by the following equation:

$$U_{r|c} = \frac{\sum_{i,j} x_{ij} \log(x_{i\bullet} x_{\bullet j}/n x_{ij})}{\sum_i x_{i\bullet} \log(x_{i\bullet}/n)}$$

The uncertainty coefficient for columns is similarly defined. The symmetric uncertainty coefficient contains the same numerator as $U_{r|c}$ and $U_{c|r}$but averages the denominators of these two statistics. Standard errors for U are given in Brown (1983).

Kruskal-Wallis: The Kruskal-Wallis statistic for rows is a one-way analysis-of-variance-type test that assumes the column variable is monotonically ordered. It tests the null hypothesis that no row populations are identical, using average ranks for the column variable. The Kruskal-Wallis statistic for columns is similarly defined. Conover (1980) discusses the Kruskal-Wallis test.

Test for Linear Trend: When there are two rows, it is possible to test for a linear trend in the row probabilities if it is assumed that the column variable is monotonically ordered. In this test, the probabilities for row 1 are predicted by the column index using weighted simple linear regression. This slope is given by

$$\hat{\beta} = \frac{\sum_j x_{\bullet j}\left(x_{1j}/x_{\bullet j} - x_{1\bullet}/n\right)(j - \bar{j})}{\sum_j x_{\bullet j}(j - \bar{j})^2}$$

where

$$\bar{j} = \sum_j x_{\bullet j} j / n$$

is the average column index. An asymptotic test that the slope is 0 may then be obtained (in large samples) as the usual regression test of zero slope.

In two-column data, a similar test for a linear trend in the column probabilities is computed. This test assumes that the rows are monotonically ordered.

Kappa: Kappa is a measure of agreement computed on square tables only. In the kappa statistic, the rows and columns correspond to the responses of two judges. The judges agree along the diagonal and disagree off the diagonal. Let

$$p_0 = \sum_i x_{ii} / n$$

denote the probability that the two judges agree, and let

$$p_c = \sum_i e_{ii} / n$$

denote the expected probability of agreement under the independence model. Kappa is then given by $(p_0 - p_c)/(1 - p_c)$.

McNemar Tests: The McNemar test is a test of symmetry in a square contingency table. In other words, it is a test of the null hypothesis $H_0 : \theta_{ij} = \theta_{ji}$. The multiple degrees-of-freedom version of the McNemar test with *r (r - 1)/2* degrees of freedom is computed as follows:

$$\sum_{i<j} \frac{(x_{ij} - x_{ji})^2}{(x_{ij} + x_{ji})}$$

The single degree-of-freedom test assumes that the differences, $x_{ij} - x_{ji}$, are all in one direction. The single degree-of-freedom test will be more powerful than the multiple degrees-of-freedom test when this is the case. The test statistic is given as follows:

$$\frac{\left(\sum_{i<j} (x_{ij} - x_{ji})\right)^2}{\sum_{i<j} (x_{ij} + x_{ji})}$$

The exact probability can be computed by the binomial distribution.

Properties

ChiSquared

```
public double ChiSquared {get; }
```

Description

Returns the Pearson chi-squared test statistic.

Property Value

A `double` scalar containing the Pearson chi-squared test statistic.

ContingencyCoef

```
public double ContingencyCoef {get; }
```

Description

Returns contingency coefficient.

Property Value

A `double` scalar containing the contingency coefficient based on Pearson chi-squared statistic.

CramersV

```
public double CramersV {get; }
```

Description

Returns Cramer's V.

Property Value

A `double` scalar containing the Cramer's V based on Pearson chi-squared statistic.

DegreesOfFreedom

```
public int DegreesOfFreedom {get; }
```

Description

Returns the degrees of freedom for the chi-squared tests associated with the table.

Property Value

An `int` scalar containing the degrees of freedom for the chi-squared tests associated with the table.

ExactMean

```
public double ExactMean {get; }
```

Description

Returns the exact mean.

Property Value

A `double` scalar containing the exact mean based on Pearson's chi-square statistic.

ExactStdev

```
public double ExactStdev {get; }
```

Description

Returns the exact standard deviation.

Property Value

A `double` scalar containing the exact standard deviation based on Pearson's chi-square statistic.

GSquared

```
public double GSquared {get; }
```

Description

Returns the likelihood ratio G^2 (chi-squared).

Property Value

A `double` scalar containing the likelihood ratio G^2 (chi-squared).

GSquaredP

```
public double GSquaredP {get; }
```

Description

Returns the probability of a larger G^2 (chi-squared).

Property Value

A `double` scalar containing the probability of a larger G^2 (chi-squared).

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

P

```
public double P {get; }
```

Description

Returns the Pearson chi-squared *p*-value for independence of rows and columns.

Property Value

A `double` scalar containing the Pearson chi-squared *p*-value for independence of rows and columns.

Phi

```
public double Phi {get; }
```

Description

Returns phi.

Property Value

A `double` scalar containing the phi based on Pearson chi-squared statistic.

Constructor

ContingencyTable

```
public ContingencyTable(double[,] table)
```

Description

Constructs and performs a chi-squared analysis of a two-way contingency table.

Parameter

`table` – A `double` matrix containing the observed counts in the contingency table.

Methods

GetContributions

```
public double[,] GetContributions()
```

Description

Returns the contributions to chi-squared for each cell in the table.

Returns

A `double` matrix of size `(table.GetLength(0)+1) * (table.GetLength(1)+1)`

containing the contributions to chi-squared for each cell in the table.

Remarks

The last row and column contain the total contribution to chi-squared for that row or column.

GetExpectedValues

```
public double[,] GetExpectedValues()
```

Description

Returns the expected values of each cell in the table.

Returns

A `double` matrix of size `(table.GetLength(0)+1) * (table.GetLength(1)+1)` containing the expected values of each cell in the table, under the null hypothesis.

Remarks

The marginal totals are in the last row and column.

GetStatistics

```
public double[,] GetStatistics()
```

Description

Returns the statistics associated with this table.

Returns

A `double` matrix of size 23 * 5 containing statistics associated with this table.

Remarks

Each row corresponds to a statistic.

Row	Statistics
0	gamma
1	Kendall's τ_b
2	Stuart's τ_c
3	Somers' D for rows (given columns)
4	Somers' D for columns (given rows)
5	product moment correlation
6	Spearman rank correlation
7	Goodman and Kruskal τ for rows (given columns)
8	Goodman and Kruskal τ for columns (given rows)
9	uncertainty coefficient U (symmetric)
10	uncertainty $U_{r\|c}$ (rows)
11	uncertainty $U_{c\|r}$ (columns)
12	optimal prediction λ (symmetric)
13	optimal prediction $\lambda_{r\|c}$ (rows)
14	optimal prediction $\lambda_{c\|r}$ (columns)
15	optimal prediction $\lambda^*_{r\|c}$ (rows)
16	optimal prediction $\lambda^*_{c\|r}$ (columns)
17	Test for linear trend in row probabilities if `table.GetLength(0)` = 2. Test for linear trend in column probabilities if `table.GetLength(1)` = 2 and `table.GetLength(0)` is not 2
18	Kruskal-Wallis test for no row effect
19	Kruskal-Wallis test for no column effect
20	kappa (square tables only)
21	McNemar test of symmetry (square tables only)
22	McNemar one degree of freedom test of symmetry (square tables only)

The columns are as follows:

Column	Value
0	estimated statistic
1	standard error for any parameter value
2	standard error under the null hypothesis
3	*t* value for testing the null hypothesis
4	*p*-value of the test in column 3

If a statistic cannot be computed, or if some value is not relevant for the computed statistic, the entry is NaN (Not a Number).

In the McNemar tests, column 0 contains the statistic, column 1 contains the chi-squared degrees of freedom, column 3 contains the exact *p*-value (1 degree of freedom only), and column 4 contains the chi-squared asymptotic *p*-value. The Kruskal-Wallis test is the same except no exact *p*-value is computed.

Example 1: Contingency Table

The following example is taken from Kendall and Stuart (1979) and involves the distance vision in the right and left eyes.

```
using System;
using Imsl.Stat;

public class ContingencyTableEx1
{
    public static void  Main(String[] args)
    {
        double[,] table = {{821, 112, 85, 35},
                           {116, 494, 145, 27},
                           {72, 151, 583, 87},
                           {43, 34, 106, 331}};
        ContingencyTable ct = new ContingencyTable(table);
        Console.Out.WriteLine("P-value = " + ct.P);
    }
}
```

Output

```
P-value = 0
```

Example 2: Contingency Table

The following example, which illustrates the use of Kappa and McNemar tests, uses the same distance vision data as in Example 1.

```
using System;
using Imsl.Stat;
```

```
using Imsl.Math;

public class ContingencyTableEx2
{
    public static void  Main(String[] args)
    {
        double[,] table = {{821.0, 112.0, 85.0, 35.0},
                           {116.0, 494.0, 145.0, 27.0},
                           {72.0, 151.0, 583.0, 87.0},
                           {43.0, 34.0, 106.0, 331.0}};
        String[] rlabels = new String[]{"Gamma", "Tau B"
                                        , "Tau C", "D-Row"
                                        , "D-Column", "Correlation"
                                        , "Spearman", "GK tau rows"
                                        , "GK tau cols.", "U - sym."
                                        , "U - rows", "U - cols."
                                        , "Lambda-sym.", "Lambda-row"
                                        , "Lambda-col."
                                        , "l-star-rows"
                                        , "l-star-col."
                                        , "Lin. trend"
                                        , "Kruskal row"
                                        , "Kruskal col.", "Kappa"
                                        , "McNemar"
                                        , "McNemar df=1"};
        ContingencyTable ct = new ContingencyTable(table);

        Console.Out.WriteLine("Pearson chi-squared statistic = " +
            ct.ChiSquared.ToString("0.0000"));
        Console.Out.WriteLine("p-value for Pearson chi-squared = " +
            ct.P.ToString("0.0000"));
        Console.Out.WriteLine("degrees of freedom = " +
            ct.DegreesOfFreedom);
        Console.Out.WriteLine("G-squared statistic = " +
            ct.GSquared.ToString("0.0000"));
        Console.Out.WriteLine("p-value for G-squared = " +
            ct.GSquaredP.ToString("0.0000"));
        Console.Out.WriteLine("degrees of freedom = " +
            ct.DegreesOfFreedom);


        PrintMatrix pm = new PrintMatrix("\n* * * Table Values * * *");
        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.00";
        pm.Print(pmf, table);

        pm.SetTitle("* * * Expected Values * * *");
        pm.Print(pmf, ct.GetExpectedValues());

        pmf.NumberFormat = "0.0000";
        pm.SetTitle("* * * Contributions to Chi-squared* * *");
        pm.Print(pmf, ct.GetContributions());

        Console.Out.WriteLine("* * * Chi-square Statistics * * *");
        Console.Out.WriteLine
            ("Exact mean = " + ct.ExactMean.ToString("0.0000"));
```

```
        Console.Out.WriteLine("Exact standard deviation = " +
            ct.ExactStdev.ToString("0.0000"));
        Console.Out.WriteLine("Phi = " + ct.Phi.ToString("0.0000"));
        Console.Out.WriteLine
            ("P = " + ct.ContingencyCoef.ToString("0.0000"));
        Console.Out.WriteLine
            ("Cramer's V = " + ct.CramersV.ToString("0.0000"));

        Console.Out.WriteLine("\n             stat.      std. err.   "
            + "std. err.(Ho) t-value(Ho)  p-value");
        double[,] stat = ct.GetStatistics();
        for (int i = 0; i < stat.GetLength(0); i++)
        {
            System.Text.StringBuilder sb =
                new System.Text.StringBuilder(rlabels[i]);

            int len = sb.Length;
            for (int j = 0; j < (13 - len); j++)
                sb.Append(' ');
            sb.Append(stat[i,0].ToString("0.0000"));

            len = sb.Length;
            for (int j = 0; j < (24 - len); j++)
                sb.Append(' ');
            sb.Append(stat[i,1].ToString("0.0000"));

            len = sb.Length;
            for (int j = 0; j < (36 - len); j++)
                sb.Append(' ');
            sb.Append(stat[i,2].ToString("0.0000"));

            len = sb.Length;
            for (int j = 0; j < (50 - len); j++)
                sb.Append(' ');
            sb.Append(stat[i,3].ToString("0.0000"));

            len = sb.Length;
            for (int j = 0; j < (63 - len); j++)
                sb.Append(' ');
            sb.Append(stat[i,4].ToString("0.0000"));

            Console.Out.WriteLine(sb.ToString());
        }
    }
}
```

Output

```
Pearson chi-squared statistic = 3304.3684
p-value for Pearson chi-squared = 0.0000
degrees of freedom = 9
G-squared statistic = 2781.0190
p-value for G-squared = 0.0000
degrees of freedom = 9
```

```
* * * Table Values * * *
     0       1       2       3
0  821.00  112.00  85.00   35.00
1  116.00  494.00  145.00  27.00
2  72.00   151.00  583.00  87.00
3  43.00   34.00   106.00  331.00

         * * * Expected Values * * *
      0       1       2       3        4
0  341.69   256.92  298.49  155.90  1053.00
1  253.75   190.80  221.67  115.78  782.00
2  289.77   217.88  253.14  132.21  893.00
3  166.79   125.41  145.70  76.10   514.00
4  1052.00  791.00  919.00  480.00  3242.00

        * * * Contributions to Chi-squared* * *
       0         1         2         3          4
0  672.3626   81.7416   152.6959  93.7612    1000.5613
1  74.7802    481.8351  26.5189   68.0768    651.2109
2  163.6605   20.5287   429.8489  15.4625    629.5006
3  91.8743    66.6263   10.8183   853.7768   1023.0957
4  1002.6776  650.7317  619.8819  1031.0772  3304.3684

* * * Chi-square Statistics * * *
Exact mean = 9.0028
Exact standard deviation = 4.2402
Phi = 1.0096
P = 0.7105
Cramer's V = 0.5829

              stat.      std. err.   std. err.(Ho) t-value(Ho)  p-value
Gamma         0.7757     0.0123      0.0149        52.1897      0.0000
Tau B         0.6429     0.0122      0.0123        52.1897      0.0000
Tau C         0.6293     0.0121      NaN           52.1897      0.0000
D-Row         0.6418     0.0122      0.0123        52.1897      0.0000
D-Column      0.6439     0.0122      0.0123        52.1897      0.0000
Correlation   0.6926     0.0128      0.0172        40.2669      0.0000
Spearman      0.6939     0.0127      0.0127        54.6614      0.0000
GK tau rows   0.3420     0.0123      NaN           NaN          NaN
GK tau cols.  0.3430     0.0122      NaN           NaN          NaN
U - sym.      0.3171     0.0110      NaN           NaN          NaN
U - rows      0.3178     0.0110      NaN           NaN          NaN
U - cols.     0.3164     0.0110      NaN           NaN          NaN
Lambda-sym.   0.5373     0.0124      NaN           NaN          NaN
Lambda-row    0.5374     0.0126      NaN           NaN          NaN
Lambda-col.   0.5372     0.0126      NaN           NaN          NaN
l-star-rows   0.5506     0.0136      NaN           NaN          NaN
l-star-col.   0.5636     0.0127      NaN           NaN          NaN
Lin. trend    NaN        NaN         NaN           NaN          NaN
Kruskal row   1561.4859  3.0000      NaN           NaN          0.0000
Kruskal col.  1563.0303  3.0000      NaN           NaN          0.0000
Kappa         0.5744     0.0111      0.0106        54.3583      0.0000
McNemar       4.7625     6.0000      NaN           NaN          0.5746
McNemar df=1  0.9487     1.0000      NaN           0.3459       0.3301
```

CategoricalGenLinModel Class

`public class Imsl.Stat.CategoricalGenLinModel`

Analyzes categorical data using logistic, probit, Poisson, and other linear models.

Reweighted least squares is used to compute (extended) maximum likelihood estimates in some generalized linear models involving categorized data. One of several models, including probit, logistic, Poisson, logarithmic, and negative binomial models, may be fit for input point or interval observations. (In the usual case, only point observations are observed.)

Let

$$\gamma_i = w_i + x_i^T \beta = w_i + \eta_i$$

be the linear response where x_i is a design column vector obtained from a row of x, β is the column vector of coefficients to be estimated, and w_i is a fixed parameter that may be input in `x`. When some of the γ_i are infinite at the supremum of the likelihood, then extended *maximum likelihood estimates* are computed. Extended maximum likelihood is computed as the finite (but nonunique) estimates $\hat{\beta}$ that optimize the likelihood containing only the observations with finite $\hat{\gamma}_i$. These estimates, when combined with the set of indices of the observations such that $\hat{\gamma}_i$ is infinite at the supremum of the likelihood, are called extended maximum estimates. When none of the optimal $\hat{\gamma}_i$ are infinite, extended maximum likelihood estimates are identical to maximum likelihood estimates. Extended maximum likelihood estimation is discussed in more detail by Clarkson and Jennrich (1991). In `CategoricalGenLinModel`, observations with potentially infinite

$$\hat{\eta}_i = x_i^T \hat{\beta}$$

are detected and removed from the likelihood if Imsl.Stat.CategoricalGenLinModel.InfiniteEstimateMethod (p. 756) = 0. See below.

The models available in CategoricalGenLinModel are:

Model Name	Parameterization	Response PDF
Model0 (Poisson)	$\lambda = N \times e^{w+\eta}$	$f(y) = \lambda^y e^{-\lambda}/y!$
Model1 (Negative Binomial)	$\theta = \frac{e^{w+\eta}}{1+e^{w+\eta}}$	$f(y) = \begin{pmatrix} S+y-1 \\ y-1 \end{pmatrix} \theta^S (1-\theta)^y$
Model2 (Logarithmic)	$\theta = \frac{e^{w+\eta}}{1+e^{w+\eta}}$	$f(y) = (1-\theta)^y/(y \ln \theta)$
Model3 (Logistic)	$\theta = \frac{e^{w+\eta}}{1+e^{w+\eta}}$	$f(y) = \begin{pmatrix} N \\ y \end{pmatrix} \theta^y (1-\theta)^{N-y}$
Model4 (Probit)	$\theta = \Phi(w+\eta)$	$f(y) = \begin{pmatrix} N \\ y \end{pmatrix} \theta^y (1-\theta)^{N-y}$
Model5 (Log-log)	$\theta = 1 - e^{-e^{w+\eta}}$	$f(y) = \begin{pmatrix} N \\ y \end{pmatrix} \theta^y (1-\theta)^{N-y}$

Here Φ denotes the cumulative normal distribution, N and S are known parameters specified for each observation via column Imsl.Stat.CategoricalGenLinModel.OptionalDistributionParameterColumn (p. 759) of `x`, and w is an optional fixed parameter specified for each observation via column

Imsl.Stat.CategoricalGenLinModel.FixedParameterColumn (p. 755) of x. (By default N is taken to be 1 for *model* = 0, 3, 4 and 5 and S is taken to be 1 for *model* = 1. By default w is taken to be 0.) Since the log-log model (*model* = 5) probabilities are not symmetric with respect to 0.5, quantitatively, as well as qualitatively, different models result when the definitions of "success" and "failure" are interchanged in this distribution. In this model and all other models involving θ, θ is taken to be the probability of a "success."

Note that each row vector in the data matrix can represent a single observation; or, through the use of column Imsl.Stat.CategoricalGenLinModel.FrequencyColumn (p. 755) of the matrix x, each vector can represent several observations.

Computational Details

For interval observations, the probability of the observation is computed by summing the probability distribution function over the range of values in the observation interval. For right-interval observations, $\Pr(Y \geq y)$ is computed as a sum based upon the equality $\Pr(Y \geq y) = 1 - \Pr(Y < y)$. Derivatives are similarly computed. `CategoricalGenLinModel` allows three types of interval observations. In full interval observations, both the lower and the upper endpoints of the interval must be specified. For right-interval observations, only the lower endpoint need be given while for left-interval observations, only the upper endpoint is given.

The computations proceed as follows:

1. The input parameters are checked for consistency and validity.
2. Estimates of the means of the "independent" or design variables are computed. The frequency of the observation in all but the binomial distribution model is taken from column `FrequencyColumn` of the data matrix `x`. In binomial distribution models, the frequency is taken as the product of n = `x[i,OptionalDistributionParameterColumn]` and `x[i,FrequencyColumn]`. In all cases these values default to 1. Means are computed as

$$\bar{x} = \frac{\Sigma_i f_i x_i}{\Sigma_i f_i}$$

3. If `init` = 0, initial estimates of the coefficients are obtained (based upon the observation intervals) as multiple regression estimates relating transformed observation probabilities to the observation design vector. For example, in the binomial distribution models, θ for point observations may be estimated as

$$\hat{\theta} = x[i, LowerEndpointColumn] / x[i, OptionalDistributionParameterColumn]$$

and, when *model* = 3, the linear relationship is given by

$$\left(\ln(\hat{\theta}/(1-\hat{\theta})) \approx x\beta\right)$$

while if `model` = 4,

$$\left(\Phi^{-1}(\hat{\theta}) = x\beta\right)$$

For bounded interval observations, the midpoint of the interval is used for `x[i,Imsl.Stat.CategoricalGenLinModel.LowerEndpointColumn (p. 757)]`.

Right-interval observations are not used in obtaining initial estimates when the distribution has unbounded support (since the midpoint of the interval is not defined). When computing initial estimates, standard modifications are made to prevent illegal operations such as division by zero.

Regression estimates are obtained at this point, as well as later, by use of linear regression.

4. Newton-Raphson iteration for the maximum likelihood estimates is implemented via iteratively reweighted least squares. Let
$$\Psi(x_i^T \beta)$$
denote the log of the probability of the i-th observation for coefficients β. In the least-squares model, the weight of the i-th observation is taken as the absolute value of the second derivative of
$$\Psi(x_i^T \beta)$$
with respect to
$$\gamma_i = x_i^T \beta$$
(times the frequency of the observation), and the dependent variable is taken as the first derivative Ψ with respect to γ_i, divided by the square root of the weight times the frequency. The Newton step is given by
$$\Delta\beta = \left(\sum_i |\Psi''(\gamma_i)| x_i x_i^T \right)^{-1} \sum_i \Psi'(\gamma_i) x_i$$
where all derivatives are evaluated at the current estimate of γ, and $\beta_{n+1} = \beta_n - \Delta\beta$. This step is computed as the estimated regression coefficients in the least-squares model. Step halving is used when necessary to ensure a decrease in the criterion.

5. Convergence is assumed when the maximum relative change in any coefficient update from one iteration to the next is less than Imsl.Stat.CategoricalGenLinModel.ConvergenceTolerance (p. 753) or when the relative change in the log-likelihood from one iteration to the next is less than `ConvergenceTolerance`/100. Convergence is also assumed after Imsl.Stat.CategoricalGenLinModel.MaxIterations (p. 758) or when step halving leads to a step size of less than .0001 with no increase in the log-likelihood.

6. For interval observations, the contribution to the log-likelihood is the log of the sum of the probabilities of each possible outcome in the interval. Because the distributions are discrete, the sum may involve many terms. The user should be aware that data with wide intervals can lead to expensive (in terms of computer time) computations.

7. If `InfiniteEstimateMethod` is set to 0, then the methods of Clarkson and Jennrich (1991) are used to check for the existence of infinite estimates in
$$\eta_i = x_i^T \beta$$
As an example of a situation in which infinite estimates can occur, suppose that observation j is right censored with $t_j > 15$ in a logistic model. If design matrix `x` is such that $x_{jm} = 1$ and $x_{im} = 0$ for all $i \neq j$, then the optimal estimate of β_m occurs at
$$\hat{\beta}_m = \infty$$

leading to an infinite estimate of both β_m and η_j. In `CategoricalGenLinModel`, such estimates may be "computed." In all models fit by `CategoricalGenLinModel`, infinite estimates can only occur when the optimal estimated probability associated with the left- or right-censored observation is 1. If `InfiniteEstimateMethod` is set to 0, left- or right- censored observations that have estimated probability greater than 0.995 at some point during the iterations are excluded from the log-likelihood, and the iterations proceed with a log-likelihood based upon the remaining observations. This allows convergence of the algorithm when the maximum relative change in the estimated coefficients is small and also allows for the determination of observations with infinite

$$\eta_i = x_i^T \beta$$

At convergence, linear programming is used to ensure that the eliminated observations have infinite η_i. If some (or all) of the removed observations should not have been removed (because their estimated $\eta_{i's}$ must be finite), then the iterations are restarted with a log-likelihood based upon the finite η_i observations. See Clarkson and Jennrich (1991) for more details.

When `InfiniteEstimateMethod` is set to 1, no observations are eliminated during the iterations. In this case, when infinite estimates occur, some (or all) of the coefficient estimates $\hat{\beta}$ will become large, and it is likely that the Hessian will become (numerically) singular prior to convergence.

When infinite estimates for the $\hat{\eta}_i$ are detected, linear regression (see Chapter 2, Regression;) is used at the convergence of the algorithm to obtain unique estimates $\hat{\beta}$. This is accomplished by regressing the optimal $\hat{\eta}_i$ or the observations with finite η against $x\beta$, yielding a unique $\hat{\beta}$ (by setting coefficients $\hat{\beta}$ that are linearly related to previous coefficients in the model to zero). All of the final statistics relating to $\hat{\beta}$ are based upon these estimates.

8. Residuals are computed according to methods discussed by Pregibon (1981). Let $\ell_i(\gamma_i)$ denote the log-likelihood of the *i*-th observation evaluated at γ_i. Then, the standardized residual is computed as

$$r_i = \frac{\ell_i'(\hat{\gamma}_i)}{\sqrt{\ell_i''(\hat{\gamma}_i)}}$$

where $\hat{\gamma}_i$ is the value of γ_i when evaluated at the optimal $\hat{\beta}$ and the derivatives here (and only here) are with respect to γ rather than with respect to β. The denominator of this expression is used as the "standard error of the residual" while the numerator is the "raw" residual.

Following Cook and Weisberg (1982), we take the influence of the *i*-th observation to be

$$\ell_i'(\hat{\gamma}_i)^T \ell''(\hat{\gamma})^{-1} \ell'(\hat{\gamma}_i)$$

This quantity is a one-step approximation to the change in the estimates when the *i*-th observation is deleted. Here, the partial derivatives are with respect to β.

Programming Notes

1. Classification variables are specified via Imsl.Stat.CategoricalGenLinModel.ClassificationVariableColumn (p. 752). Indicator or dummy variables are created for the classification variables.

2. To enhance precision “centering” of covariates is performed if Imsl.Stat.CategoricalGenLinModel.ModelIntercept (p. 758) is set to 1 and (number of observations) - (number of rows in `x` missing one or more values) `>’` 1. In doing so, the sample means of the design variables are subtracted from each observation prior to its inclusion in the model. On convergence the intercept, its variance and its covariance with the remaining estimates are transformed to the uncentered estimate values.

3. Two methods for specifying a binomial distribution model are possible. In the first method, `x[i,FrequencyColumn]` contains the frequency of the observation while `x[i,LowerEndpointColumn]` is 0 or 1 depending upon whether the observation is a success or failure. In this case, N = `x[i,OptionalDistributionParameterColumn]` is always 1. The model is treated as repeated Bernoulli trials, and interval observations are not possible.

A second method for specifying binomial models is to use `x[i,LowerEndpointColumn]` to represent the number of successes in the `x[i,OptionalDistributionParameterColumn]` trials. In this case, `x[i,FrequencyColumn]` will usually be 1, but it may be greater than 1, in which case interval observations are possible.

Note that the Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) method must be called before using any property as a right operand, otherwise the value is `null`.

Properties

CaseAnalysis

```
virtual public double[,] CaseAnalysis {get; }
```

Description

The case analysis.

Property Value

A `double` matrix containing the case analysis or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

Remarks

The matrix is *nobs* × 5 where *nobs* is the number of observations. The matrix contains:

Column	Statistic
0	Prediction.
1	The residual.
2	The estimated standard error of the residual.
3	The estimated influence of the observation.
4	The standardized residual.

Case studies are computed for all observations except where missing values prevent their computation. The prediction in column 0 depends upon the model used as follows:

Model	Prediction
0	The predicted mean for the observation.
1-4	The probability of a success on a single trial.

CensorColumn

```
virtual public int CensorColumn {set; }
```

Description

The column number in `x` which contains the interval type for each observation.

Property Value

An `int` scalar which indicates the column number `x` which contains the interval type code for each observation.

Remarks

The valid codes are interpreted as:

`x[i,CensorColumn]`	**Censoring**
0	Point observation. The response is unique and is given by `x[i,LowerEndpointColumn]`.
1	Right interval. The response is greater than or equal to `x[i,LowerEndpointColumn]` and less than or equal to the upper bound, if any, of the distribution.
2	Left interval. The response is less than or equal to `x[i,UpperEndpointColumn]` and greater than or equal to the lower bound of the distribution.
3	Full interval. The response is greater than or equal to `x[i,LowerEndpointColumn]` but less than or equal to `x[i,UpperEndpointColumn]`.

By default, `CensorColumn` = 0.

Exception

`System.ArgumentException` is thrown when `CensorColumn` is less than 0 or greater than or equal to the number of columns of `x`

ClassificationVariableColumn

```
virtual public int[] ClassificationVariableColumn {set; }
```

Description

An index vector to contain the column numbers in `x` that are classification variables.

Property Value

An `int` vector which contains the column numbers in `x` that are classification variables.

Remarks

By default, `ClassificationVariableColumn` is not referenced.

Exception

`System.ArgumentException` is thrown when an element of `ClassificationVariableColumn` is less than 0 or greater than or equal to the number of columns of `x`

ClassificationVariableCounts

```
virtual public int[] ClassificationVariableCounts {get; }
```

Description

The number of values taken by each classification variable.

Property Value

An `int` array of length *nclvar* containing the number of values taken by each classification variable where *nclvar* is the number of classification variables or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

Exception

`Imsl.Stat.ClassificationVariableException` is thrown when the number of values taken by each classification variable has been set by the user to be less than or equal to 1

ClassificationVariableValues

```
virtual public double[] ClassificationVariableValues {get; }
```

Description

The distinct values of the classification variables in ascending order.

Property Value

A `double` array of length $\sum_{k=0}^{\text{nclvar}}$ nclval[k] containing the distinct values of the classification variables in ascending order where *nclvar* is the number of classification variables and *nclval[i]* is the number of values taken by the *i*-th classification variable.

Remarks

A `null` is returned if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called prior to calling this method.

Exception

`Imsl.Stat.ClassificationVariableException` is thrown when the number of values taken by each classification variable has been set by the user to be less than or equal to 1

ConvergenceTolerance

```
virtual public double ConvergenceTolerance {set; }
```

Description

The convergence criterion.

Property Value

A `double` scalar specifying the convergence criterion.

Remarks

Convergence is assumed when the maximum relative change in any coefficient estimate is less than `ConvergenceTolerance` from one iteration to the next or when the relative change in the log-likelihood, Imsl.Stat.CategoricalGenLinModel.OptimizedCriterion (p. 759), from one iteration to the next is less than `ConvergenceTolerance`/100. `ConvergenceTolerance` must be greater than 0.

By default, `ConvergenceTolerance` = .001.

Exception

`System.ArgumentException` is thrown if `ConvergenceTolerance` is or equal to 0

CovarianceMatrix

```
virtual public double[,] CovarianceMatrix {get; }
```

Description

The estimated asymptotic covariance matrix of the coefficients.

Property Value

A `double` matrix containing the estimated asymptotic covariance matrix of the coefficients or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

Remarks

The covariance matrix is *nCoef* by *nCoef* where *nCoef* is the number of coefficients in the model.

DesignVariableMeans

```
virtual public double[] DesignVariableMeans {get; }
```

Description

The means of the design variables.

Property Value

A `double` array of length *nCoef* containing the means of the design variables where *nCoef* is the number of coefficients in the model or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

ExtendedLikelihoodObservations

```
virtual public int[] ExtendedLikelihoodObservations {get; set; }
```

Description

A vector indicating which observations are included in the extended likelihood.

Property Value

An `int` array of length *nobs* indicating which observations are included in the extended likelihood where *nobs* is the number of observations.

Remarks

`ExtendedLikelihoodObservations` is an `int` array of length *nobs* indicating which observations are included in the extended likelihood where *nobs* is the number of observations. The values within the array are interpreted as:

Value	Status of observation
0	Observation *i* is in the likelihood.
1	Observation *i* cannot be in the likelihood because it contains at least one missing value in `x`.
2	Observation *i* is not in the likelihood. Its estimated parameter is infinite.

A `null` is returned if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called prior to calling this method.

By default, all elements are zero.

Exception

`System.ArgumentException` is thrown when an element of `ExtendedLikelihoodObservations` is not in the range [0,2]

FixedParameterColumn

```
virtual public int FixedParameterColumn {set; }
```

Description

The column number in `x` that contains a fixed parameter for each observation that is added to the linear response prior to computing the model parameter.

Property Value

An `int` scalar which indicates the column number in `x` that contains a fixed parameter for each observation that is added to the linear response prior to computing the model parameter.

Remarks

The "fixed" parameter allows one to test hypothesis about the parameters via the log-likelihoods. By default the fixed parameter is assumed to be zero.

Exception

`System.ArgumentException` is thrown when `FixedParameterColumn` is less than 0 or greater than or equal to the number of columns of `x`

FrequencyColumn

```
virtual public int FrequencyColumn {set; }
```

Description

The column number in `x` that contains the frequency of response for each observation.

Property Value

An `int` scalar which indicates the column number in `x` that contains the frequency of response for each observation.

Remarks

By default a frequency of 1 for each observation is assumed.

Exception

`System.ArgumentException` is thrown when `FrequencyColumn` is less than 0 or greater than or equal to the number of columns of `x`

Hessian

```
virtual public double[,] Hessian {get; }
```

Description

The Hessian computed at the initial parameter estimates.

Property Value

A `double` matrix containing the Hessian computed at the input parameter estimates.

Remarks

The Hessian matrix is *nCoef* by *nCoef* where *nCoef* is the number of coefficients in the model. This member function will call Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) to get the Hessian if the Hessian has not already been computed.

Exceptions

`Imsl.Stat.ClassificationVariableException` is thrown when the number of values taken by each classification variable has been set by the user to be less than or equal to 1

`Imsl.Stat.ClassificationVariableLimitException` is thrown when the sum of the number of distinct values taken on by each classification variable exceeds the maximum allowed, Imsl.Stat.CategoricalGenLinModel.UpperBound (p. 761)

`Imsl.Stat.DeleteObservationsException` is thrown if the number of observations to be deleted has grown too large

`Imsl.Stat.RankDeficientException` is thrown if the model has been determined to be rank deficient

InfiniteEstimateMethod

```
virtual public int InfiniteEstimateMethod {set; }
```

Description

Specifies the method used for handling infinite estimates.

Property Value

An `int` scalar indicating which method to use for handling infinite estimates.

Remarks

The value of `InfiniteEstimateMethod` is interpreted as follows:

InfiniteEstimateMethod	**Method**
0	Remove a right or left-censored observation from the log-likelihood whenever the probability of the observation exceeds 0.995. At convergence, use linear programming to check that all removed observations actually have an estimated linear response that is infinite. Set `ExtendedLikelihoodObservations[i]` for observation *i* to 2 if the linear response is infinite. If not all removed observations have infinite linear response, recompute the estimates based upon the observations with estimated linear response that is finite. This option is valid only for censoring codes 1 and 2.
1	Iterate without checking for infinite estimates.

By default `InfiniteEstimateMethod` = 1.

Exception

`System.ArgumentException` is thrown when `InfiniteEstimateMethod` is less than 0 or greater than 1

LastParameterUpdates

```
virtual public double[] LastParameterUpdates {get; }
```

Description

The last parameter updates (excluding step halvings).

Property Value

A `double` array of length *nCoef* containing the last parameter updates (excluding step halvings) or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

LowerEndpointColumn

```
virtual public int LowerEndpointColumn {set; }
```

Description

The column number in `x` that contains the lower endpoint of the observation interval for full interval and right interval observations.

Property Value

An `int` scalar which indicates the column number in `x` that contains the lower endpoint of the observation interval for full interval and right interval observations.

Remarks

By default all observations are treated as "point" observations and `x[i,LowerEndpointColumn]` contains the observation point. If this member function is not called, the last column of `x` is assumed to contain the "point" observations.

Exception

`System.ArgumentException` is thrown when `LowerEndpointColumn` is less than 0 or greater than or equal to the number of columns of `x`

MaxIterations

```
virtual public int MaxIterations {set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` specifying the maximum number of iterations allowed.

Remarks

By default, `MaxIterations` = 30.

Exception

`System.ArgumentException` is thrown if `MaxIterations` is less than or equal to 0

ModelIntercept

```
virtual public int ModelIntercept {set; }
```

Description

The intercept option.

Property Value

An `int` scalar which indicates whether or not the model has an intercept.

Remarks

Input `ModelIntercept` is interpreted as follows:

Value	Action
0	No intercept is in the model (unless otherwise provided for by the user).
1	Intercept is automatically included in the model.

By default `ModelIntercept` = 1.

Exception

`System.ArgumentException` is thrown when `ModelIntercept` is less than 0 or greater than 1

NRowsMissing

```
virtual public int NRowsMissing {get; }
```

Description

The number of rows of data in `x` that contain missing values in one or more specific columns of `x`.

Property Value

An `int` scalar representing the number of rows of data in `x` that contain missing values in one or more specific columns of `x` or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

Remarks

The columns of `x` included in the count are the columns containing the upper or lower endpoints of full interval, left interval, or right interval observations. Also included are the columns containing the frequency responses, fixed parameters, optional distribution parameters, and interval type for each observation. Columns containing classification variables and columns associated with each effect in the model are also included.

ObservationMax

```
virtual public int ObservationMax {set; }
```

Description

The maximum number of observations that can be handled in the linear programming.

Property Value

An `int` scalar which sets the maximum number of observations that can be handled in the linear programming.

Remarks

By default, `ObservationMax` is set to the number of observations.

Exception

`System.ArgumentException` is thrown if `ObservationMax` is less than 0.

OptimizedCriterion

```
virtual public double OptimizedCriterion {get; }
```

Description

The optimized criterion.

Property Value

A `double` scalar representing the optimized criterion or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

Remarks

The criterion to be maximized is a constant plus the log-likelihood.

OptionalDistributionParameterColumn

```
virtual public int OptionalDistributionParameterColumn {set; }
```

Description

The column number in `x` that contains an optional distribution parameter for each observation.

Property Value

An `int` scalar which indicates the column number in `x` that contains an optional distribution parameter for each observation.

Remarks

The distribution parameter values are interpreted as follows depending on the model chosen:

Model	Meaning of `x[i,OptionalDistributionParameterColumn]`
0	The Poisson parameter is given by $x[i, OptionalDistributionParameterColumn] \times e^{\rho}$.
1	The number of successes required in the negative binomial is given by `x[i,OptionalDistributionParameterColumn]`.
2	`x[i,OptionalDistributionParameterColumn]` is not used.
3-5	The number of trials in the binomial distribution is given by `x[i,OptionalDistributionParameterColumn]`.

By default, the distribution parameter is assumed to be 1.

Exception

`System.ArgumentException` is thrown when `OptionalDistributionParameterColumn` is less than 0 or greater than or equal to the number of columns of `x`

Parameters

```
virtual public double[,] Parameters {get; }
```

Description

Parameter estimates and associated statistics.

Property Value

An *nCoef* row by 4 column `double` matrix containing the parameter estimates and associated statistics or `null` if Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) has not been called.

Remarks

Here, *nCoef* is the number of coefficients in the model. The statistics returned are as follows:

Column	Statistic
0	Coefficient estimate.
1	Estimated standard deviation of the estimated coefficient.
2	Asymptotic normal score for testing that the coefficient is zero.
3	ρ - value associated with the normal score in column 2.

Product

```
virtual public double[] Product {get; }
```

Description

The inverse of the Hessian times the gradient vector computed at the input parameter estimates.

Property Value

A `double` array of length *nCoef* containing the inverse of the Hessian times the gradient vector computed at the input parameter estimates.

Remarks

nCoef is the number of coefficients in the model. This member function will call Imsl.Stat.CategoricalGenLinModel.Solve (p. 764) to get the product if the product has not already been computed.

Exceptions

`Imsl.Stat.ClassificationVariableException` is thrown when the number of values taken by each classification variable has been set by the user to be less than or equal to 1

`Imsl.Stat.ClassificationVariableLimitException` is thrown when the sum of the number of distinct values taken on by each classification variable exceeds the maximum allowed, Imsl.Stat.CategoricalGenLinModel.UpperBound (p. 761)

`Imsl.Stat.DeleteObservationsException` is thrown if the number of observations to be deleted has grown too large

`Imsl.Stat.RankDeficientException` is thrown if the model has been determined to be rank deficient

Tolerance

```
virtual public double Tolerance {get; set; }
```

Description

The tolerance used in determining linear dependence.

Property Value

A `double` value used in determining linear dependence.

Default: `Tolerance` = 0.22204460492503130808e-14.

Remarks

When linear dependence is detected, a `RankDeficientException` is thrown and no results are computed. Computations for a rank deficient model can be forced to continue by specifying a negative tolerance. If `Tolerance` is negative, the absolute value of `Tolerance` will be used to determine linear dependence, but computations will proceed with a rank deficient warning. In this case the results should be carefully inspected and used with caution.

UpperBound

```
virtual public int UpperBound {set; }
```

Description

Defines the upper bound on the sum of the number of distinct values taken on by each classification variable.

Property Value

An `int` scalar specifying the upper bound on the sum of the number of distinct values taken on by each classification variable.

Remarks

By default, if property `ClassificationVariableColumn` has not been referenced `UpperBound` = 1. If property `ClassificationVariableColumn` has been referenced `UpperBound` = `x.GetLength(0) * nclvar` where `nclvar` is the number of classification variables.

Exception

`System.ArgumentException` is thrown thrown when `UpperBound` is less than 1 and the number of classification variables is greater than 0

UpperEndpointColumn

```
virtual public int UpperEndpointColumn {set; }
```

Description

The column number in `x` that contains the upper endpoint of the observation interval for full interval and left interval observations.

Property Value

An `int` scalar which indicates the column number in `x` that contains the upper endpoint of the observation interval for full interval and left interval observations.

Remarks

By default all observations are treated as “point” observations.

Exception

`System.ArgumentException` is thrown when `UpperEndpointColumn` is less than 0 or greater than or equal to the number of columns of `x`

Constructor

CategoricalGenLinModel

```
public CategoricalGenLinModel(double[,] x,
Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel model)
```

Description

Constructs a new `CategoricalGenLinModel`.

Parameters

`x` – A `double` input matrix containing the data where the number of rows in the matrix is equal to the number of observations.

`model` – An `int` scalar which specifies the distribution of the response variable and the function used to model the distribution parameter.

Remarks

Use one of the class members from the following table. The lower bound given in the table is the minimum possible value of the response variable:

Model	Distribution	Function	Lower-bound
0	Poisson	Exponential	0
1	Negative Binomial	Logistic	0
2	Logarithmic	Logistic	1
3	Binomial	Logistic	0
4	Binomial	Probit	0
5	Binomial	Log-log	0

Let γ be the dot product of a row in the design matrix with the parameters (plus the fixed parameter, if used). Then, the functions used to model the distribution parameter are given by:

Name	Function
Exponential	e^{γ}
Logistic	$e^{\gamma}/(1+e^{\gamma})$
Probit	$\Phi(\gamma)$ (where Φ is the normal cdf)
Log-log	$1-e^{-\gamma}$

Methods

SetEffects

```
virtual public void SetEffects(int[] indef, int[] nvef)
```

Description

Initializes an index vector to contain the column numbers in `x` associated with each effect.

Parameters

`indef` – An `int` vector of length $\sum_{k=0}^{\text{nef}-1}$ nvef[k] where *nef* is the number of effects in the model.

`nvef` – An `int` vector of length *nef* where *nef* is the number of effects in the model.

Remarks

`indef` contains the column numbers in `x` that are associated with each effect. Member function `SetEffects(int [], nvef [])` sets the number of variables associated with each effect in the model. The first `nvef[0]` elements of `indef` give the column numbers of the variables in the first effect. The next `nvef[0]` elements give the column numbers of the variables in the second effect, etc. By default this vector is not referenced.

`nvef` contains the number of variables associated with each effect in the model. By default this vector is not referenced.

Exception

`System.ArgumentException` is thrown when an element of `indef` is less than 0 or greater than or equal to the number of columns of `x` or if an element of `nvef` is less than or equal to 0

SetInitialEstimates

```
virtual public void SetInitialEstimates(int init, double[] estimates)
```

Description

Sets the initial parameter estimates option.

Parameters

`init` – An input `int` indicating the desired initialization method for the initial estimates of the parameters.

`estimates` – An input `double` array of length *nCoef* containing the initial estimates of the parameters where *nCoef* is the number of estimated coefficients in the model.

Remarks

If this method is not called, `init` is set to 0.

`init`	**Action**
0	Unweighted linear regression is used to obtain initial estimates.
1	The *nCoef*, number of coefficients, elements of `estimates` contain initial estimates of the parameters. Use of this option requires that the user know *nCoef* beforehand.

`estimates` is used if `init = 1`. If this member function is not called, unweighted linear regression is used to obtain the initial estimates.

Exception

`System.ArgumentException` is thrown when `init` is not in the range [0,1]

Solve

```
virtual public double[,] Solve()
```

Description

Returns the parameter estimates and associated statistics for a `CategoricalGenLinModel` object.

Returns

An *nCoef* row by 4 column `double` matrix containing the parameter estimates and associated statistics.

Remarks

Here, *nCoef* is the number of coefficients in the model. The statistics returned are as follows:

Column	**Statistic**
0	Coefficient estimate.
1	Estimated standard deviation of the estimated coefficient.
2	Asymptotic normal score for testing that the coefficient is zero.
3	ρ - value associated with the normal score in column 2.

Exceptions

`Imsl.Stat.ClassificationVariableException` is thrown when the number of values taken by each classification variable has been set by the user to be less than or equal to 1

`Imsl.Stat.ClassificationVariableLimitException` is thrown when the sum of the number of distinct values taken on by each classification variable exceeds the maximum allowed, Imsl.Stat.CategoricalGenLinModel.UpperBound (p. 761)

`Imsl.Stat.DeleteObservationsException` is thrown if the number of observations to delete has grown too large

`Imsl.Stat.RankDeficientException` is thrown if the model has been determined to be rank deficient

Example 1: Example: Mortality of beetles.

The first example is from Prentice (1976) and involves the mortality of beetles after exposure to various concentrations of carbon disulphide. Both a logit and a probit fit are produced for linear model $\mu + \beta x$. The data is given as

Covariate(x)	N	y
1.690	59	6
1.724	60	13
1.755	62	18
1.784	56	28
1.811	63	52
1.836	59	53
1.861	62	61
1.883	60	60

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class CategoricalGenLinModelEx1
{
    public static void  Main(String[] args)
    {
        // Set up a PrintMatrix object for later use.
        PrintMatrixFormat mf;
        PrintMatrix p;
        p = new PrintMatrix();
        mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
                mf.NumberFormat = "0.0000";

        double[,] x = {
```

```
                {1.69, 59.0, 6.0},
                {1.724, 60.0, 13.0},
                {1.755, 62.0, 18.0},
                {1.784, 56.0, 28.0},
                {1.811, 63.0, 52.0},
                {1.836, 59.0, 53.0},
                {1.861, 62.0, 61.0},
                {1.883, 60.0, 60.0}
               };

CategoricalGenLinModel CATGLM3, CATGLM4;
// MODEL3
CATGLM3 = new CategoricalGenLinModel(x,
 CategoricalGenLinModel.DistributionParameterModel.Model3);
CATGLM3.LowerEndpointColumn = 2;
CATGLM3.OptionalDistributionParameterColumn = 1;
CATGLM3.InfiniteEstimateMethod = 1;
CATGLM3.ModelIntercept = 1;
int[] nvef = new int[]{1};
int[] indef = new int[]{0};
CATGLM3.SetEffects(indef, nvef);
CATGLM3.UpperBound = 1;

Console.Out.WriteLine("MODEL3");
p.SetTitle("Coefficient Statistics");
p.Print(mf, CATGLM3.Solve());
Console.Out.WriteLine("Log likelihood " + CATGLM3.OptimizedCriterion);
p.SetTitle("Asymptotic Coefficient Covariance");
p.SetMatrixType(PrintMatrix.MatrixType.UpperTriangular);
p.Print(mf, CATGLM3.CovarianceMatrix);
p.SetMatrixType(PrintMatrix.MatrixType.Full);
p.SetTitle("Case Analysis");
p.Print(mf, CATGLM3.CaseAnalysis);
p.SetTitle("Last Coefficient Update");
p.Print(CATGLM3.LastParameterUpdates);
p.SetTitle("Covariate Means");
p.Print(CATGLM3.DesignVariableMeans);
p.SetTitle("Observation Codes");
p.Print(CATGLM3.ExtendedLikelihoodObservations);
Console.Out.WriteLine("Number of Missing Values " + CATGLM3.NRowsMissing);

// MODEL4
CATGLM4 = new CategoricalGenLinModel(x,
 CategoricalGenLinModel.DistributionParameterModel.Model4);
CATGLM4.LowerEndpointColumn = 2;
CATGLM4.OptionalDistributionParameterColumn = 1;
CATGLM4.InfiniteEstimateMethod = 1;
CATGLM4.ModelIntercept = 1;
CATGLM4.SetEffects(indef, nvef);
CATGLM4.UpperBound = 1;
CATGLM4.Solve();

Console.Out.WriteLine("\nMODEL4");
Console.Out.WriteLine("Log likelihood " + CATGLM4.OptimizedCriterion);
p.SetTitle("Coefficient Statistics");
p.Print(mf, CATGLM4.Parameters);
```

```
    }
}
```

Output

```
MODEL3
        Coefficient Statistics

-60.7568  5.1876  -11.7118  0.0000
 34.2985  2.9164   11.7607  0.0000

Log likelihood -18.7781790423339
Asymptotic Coefficient Covariance

 26.9117  -15.1243
           8.5052

               Case Analysis

0.0577   2.5934  1.7916  0.2674   1.4475
0.1644   3.1390  2.8706  0.3470   1.0935
0.3629  -4.4976  3.7860  0.3108  -1.1879
0.6063  -5.9517  3.6562  0.2322  -1.6279
0.7954   1.8901  3.2020  0.2688   0.5903
0.9016  -0.1949  2.2878  0.2380  -0.0852
0.9558   1.7434  1.6193  0.1976   1.0767
0.9787   1.2783  1.1185  0.1382   1.1429

 Last Coefficient Update
             0
0  1.85192237936898E-07
1  1.33163785440457E-05

Covariate Means
     0
0  1.793
1  0

Observation Codes
   0
0  0
1  0
2  0
3  0
4  0
5  0
6  0
7  0

Number of Missing Values 0

MODEL4
Log likelihood -18.2323545743846
        Coefficient Statistics
```

```
-34.9441  2.6412  -13.2305  0.0000
 19.7367  1.4852   13.2888  0.0000
```

Example 2: Example: Poisson Model.

In this example, the following data illustrate the Poisson model when all types of interval data are present. The example also illustrates the use of classification variables and the detection of potentially infinite estimates (which turn out here to be finite). These potential estimates lead to the two iteration summaries. The input data is

ilt	irt	icen	Class 1	Class 2
0	5	0	1	0
9	4	3	0	0
0	4	1	0	0
9	0	2	1	1
0	1	0	0	1

A linear model $\mu + \beta_1 x_1 + \beta_2 x_2$ is fit where $x_1 = 1$ if the Class 1 variable is 0, $x_1 = 1$, otherwise, and the x_2 variable is similarly defined.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class CategoricalGenLinModelEx2
{
    public static void  Main(String[] args)
    {
        // Set up a PrintMatrix object for later use.
        PrintMatrixFormat mf;
        PrintMatrix p;
        p = new PrintMatrix();
        mf = new PrintMatrixFormat();
        mf.SetNoRowLabels();
        mf.SetNoColumnLabels();
                mf.NumberFormat = "0.0000";


        double[,] x = {
                                {0.0, 5.0, 0.0, 1.0, 0.0},
                                {9.0, 4.0, 3.0, 0.0, 0.0},
                                {0.0, 4.0, 1.0, 0.0, 0.0},
                                {9.0, 0.0, 2.0, 1.0, 1.0},
                                {0.0, 1.0, 0.0, 0.0, 1.0}};
        CategoricalGenLinModel CATGLM;
        CATGLM = new CategoricalGenLinModel(x,
         CategoricalGenLinModel.DistributionParameterModel.Model0);
        CATGLM.UpperEndpointColumn = 0;
        CATGLM.LowerEndpointColumn = 1;
        CATGLM.OptionalDistributionParameterColumn = 1;
```

```
        CATGLM.CensorColumn = 2;
        CATGLM.InfiniteEstimateMethod = 0;
        CATGLM.ModelIntercept = 1;
        int[] indcl = new int[]{3, 4};
        CATGLM.ClassificationVariableColumn = indcl;
        int[] nvef = new int[]{1, 1};
        int[] indef = new int[]{3, 4};
        CATGLM.SetEffects(indef, nvef);
        CATGLM.UpperBound = 4;


        p.SetTitle("Coefficient Statistics");
        p.Print(mf, CATGLM.Solve());
        Console.Out.WriteLine("Log likelihood " + CATGLM.OptimizedCriterion);
        p.SetTitle("Asymptotic Coefficient Covariance");
        p.SetMatrixType(PrintMatrix.MatrixType.UpperTriangular);
        p.Print(mf, CATGLM.CovarianceMatrix);
        p.SetMatrixType(PrintMatrix.MatrixType.Full);
        p.SetTitle("Case Analysis");
        p.Print(mf, CATGLM.CaseAnalysis);
        p.SetTitle("Last Coefficient Update");
        p.Print(CATGLM.LastParameterUpdates);
        p.SetTitle("Covariate Means");
        p.Print(CATGLM.DesignVariableMeans);
        p.SetTitle("Distinct Values For Each Class Variable");
        p.Print(CATGLM.ClassificationVariableValues);
        Console.Out.WriteLine("Number of Missing Values " + CATGLM.NRowsMissing);
    }
}
```

Output

```
       Coefficient Statistics

-0.5488  1.1713  -0.4685  0.6395
 0.5488  0.6098   0.8999  0.3684
 0.5488  1.0825   0.5069  0.6123

Log likelihood -3.11463849257844
Asymptotic Coefficient Covariance

 1.3719  -0.3719  -1.1719
          0.3719   0.1719
                   1.1719

               Case Analysis

5.0000   0.0000  2.2361  1.0000   0.0000
6.9246  -0.4122  2.1078  0.7636  -0.1955
6.9246   0.4122  1.1727  0.2364   0.3515
0.0000   0.0000  0.0000  0.0000   NaN
1.0000   0.0000  1.0000  1.0000   0.0000

 Last Coefficient Update
             0
```

```
0  -2.84092901774647E-07
1   3.53822065313335E-10
2   7.09878431984082E-07

Covariate Means
    0
0  0.6
1  0.6
2  0

Distinct Values For Each Class Variable
   0
0  0
1  1
2  0
3  1

Number of Missing Values 0
```

CategoricalGenLinModel.DistributionParameterModel Enumeration

```
public enumeration Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel
```

Indicates the function used to model the distribution parameter.

Fields

Model0

```
public Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel Model0
```

Description

Indicates an exponential function is used to model the distribution parameter. The distribution of the response variable is Poisson. The lower bound of the response variable is 0.

Model1

```
public Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel Model1
```

Description

Indicates a logistic function is used to model the distribution parameter. The distribution of the response variable is negative Binomial. The lower bound of the response variable is 0.

Model2

```
public Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel Model2
```

Description

Indicates a logistic function is used to model the distribution parameter. The distribution of the response variable is Logarithmic. The lower bound of the response variable is 1.

Model3

```
public Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel Model3
```

Description

Indicates a logistic function is used to model the distribution parameter. The distribution of the response variable is Binomial. The lower bound of the response variable is 0.

Model4

```
public Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel Model4
```

Description

Indicates a probit function is used to model the distribution parameter. The distribution of the response variable is Binomial. The lower bound of the response variable is 0.

Model5

```
public Imsl.Stat.CategoricalGenLinModel.DistributionParameterModel Model5
```

Description

Indicates a log-log function is used to model the distribution parameter. The distribution of the response variable is Binomial. The lower bound of the response variable is 0.

Chapter 16: Nonparametric Statistics

Types

Usage Notes

Much of what is considered nonparametric statistics is included in other chapters. Topics of possible interest in other chapters are: nonparametric measures of location and scale (see “Basic Statistics”), nonparametric measures in a contingency table (see “Categorical and Discrete Data Analysis”), measures of correlation in a contingency table (see “Correlation and Covariance”), and tests of goodness of fit and randomness (see “Tests of Goodness of Fit and Randomness”).

Missing Values

Most classes described in this chapter automatically handle missing values (NaN, “Not a Number”; see `Double.NaN`).

Tied Observations

The `WilcoxonRankSum` class described in this chapter contains a set method, `setFuzz`. Observations that are within fuzz of each other in absolute value are said to be tied. If `fuzz` = 0.0, observations must be identically equal before they are considered to be tied. Other positive values of fuzz allow for numerical imprecision or roundoff error.

SignTest Class

```
public class Imsl.Stat.SignTest
```

Performs a sign test.

Class `SignTest` tests hypotheses about the proportion p of a population that lies below a value q, where p and q corresponds to the `Percentage` and `Percentile` properties, respectively. In continuous distributions, this can be a test that q is the 100 p-th percentile of the population from which `x` was obtained. To carry out testing, `SignTest` tallies the number of values above q in the number of positive differences $x[j-1]$ – Percentile for $j = 1, 2, \ldots,$ x.length. The binomial probability of the number of values above q in the number of positive differences $x[j-1]$ – Percentile for $j = 1, 2, \ldots, \ldots,$ x.length or more values above q is then computed using the proportion p and the sample size in `x` (adjusted for the missing observations and ties).

Hypothesis testing is performed as follows for the usual null and alternative hypotheses:

- $H_0 : Pr(x \leq q) \geq p$ (the p-th quantile is at least q)
  $H_1 : Pr(x \leq q) < p$
  Reject H_0 if *probability* is less than or equal to the significance level.
- $H_0 : Pr(x \leq q) \leq p$ (the p-th quantile is at least q)
  $H_1 : Pr(x \leq q) > p$
  Reject H_0 if *probability* is greater than or equal to 1 minus the significance level.
- $H_0 : Pr(x = q) = p$(the p-th quantile is q)
  $H_1 : Pr((x \leq q) < p)$ or $Pr((x \leq q) > p)$
  Reject H_0 if *probability* is less than or equal to half the significance level or greater than or equal to 1 minus half the significance level.

The assumptions are as follows:

1. They are independent and identically distributed.
2. Measurement scale is at least ordinal; i.e., an ordering less than, greater than, and equal to exists in the observations.

Many uses for the sign test are possible with various values of p and q. For example, to perform a matched sample test that the difference of the medians of y and z is 0.0, let *p = 0.5*, *q = 0.0*, and $x_i = y_i - z_i$ in matched observations y and z. To test that the median difference is c, let *q = c*.

Properties

NumPositiveDev

```
public int NumPositiveDev {get; }
```

Description

Returns the number of positive differences. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

A scalar `int` containing the number of positive differences `x[j-1]-Percentile` for j = 1, 2, ..., `x.Length`.

NumZeroDev

```
public int NumZeroDev {get; }
```

Description

Returns the number of zero differences. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

A scalar `int` containing the number of zero differences (ties) `x[j-1]-Percentile` for j = 1, 2, ..., `x.Length`.

Percentage

```
public double Percentage {get; set; }
```

Description

The percentage percentile of the population.

Property Value

A `double` scalar containing the value in the range (0, 1).

Remarks

`Percentile` is the 100 * `percentage` percentile of the population.

By default, `Percentage` = 0.5.

Percentile

```
public double Percentile {get; set; }
```

Description

The hypothesized percentile of the population.

Property Value

A `double` scalar containing the hypothesized percentile of the population from which `x` was drawn.

Remarks

By default, `Percentile` = 0.0.

Constructor

SignTest

```
public SignTest(double[] x)
```

Description

Constructor for `SignTest`.

Parameter

`x` – A `double` array containing the data.

Method

Compute

```
public double Compute()
```

Description

Performs a sign test.

Returns

A `double` scalar containing the Binomial probability of `NumPositiveDev` or more positive differences in `x.length` - number of zero differences trials.

Remarks

Call this value probability. If using default values, the null hypothesis is that the median equals 0.0.

Example 1: Sign Test

This example tests the hypothesis that at least 50 percent of a population is negative. Because $0.18 < 0.95$, the null hypothesis at the 5-percent level of significance is not rejected.

```
using System;
using Imsl.Stat;

public class SignTestEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{   92.0, 139.0, - 6.0,
                                     10.0, 81.0, - 11.0,
                                     45.0, - 25.0, - 4.0,
                                     22.0, 2.0, 41.0,
                                     13.0, 8.0, 33.0,
                                     45.0, - 33.0, - 45.0,
                                     - 12.0};
        SignTest st = new SignTest(x);

        Console.Out.WriteLine
            ("Probability = " + st.Compute().ToString("0.000000"));
    }
}
```

Output

```
Probability = 0.179642
```

Example 2: Sign Test

This example tests the null hypothesis that at least 75 percent of a population is negative. Because $0.923 < 0.95$, the null hypothesis at the 5-percent level of significance is rejected.

```
using System;
using Imsl.Stat;

public class SignTestEx2
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{   92.0, 139.0, - 6.0,
                                     10.0, 81.0, - 11.0,
                                     45.0, - 25.0, - 4.0,
                                     22.0, 2.0, 41.0,
                                     13.0, 8.0, 33.0,
                                     45.0, - 33.0, - 45.0,
                                     - 12.0};
        SignTest st = new SignTest(x);

        st.Percentage = 0.75;
        st.Percentile = 0.0;
        Console.Out.WriteLine
            ("Probability = " + st.Compute().ToString("0.000000"));
        Console.Out.WriteLine
            ("Number of positive deviations = " + st.NumPositiveDev);
        Console.Out.WriteLine("Number of ties = " + st.NumZeroDev);
    }
}
```

Output

```
Probability = 0.922543
Number of positive deviations = 12
Number of ties = 0
```

WilcoxonRankSum Class

`public class Imsl.Stat.WilcoxonRankSum`

Performs a Wilcoxon rank sum test.

Class `WilcoxonRankSum` performs the Wilcoxon rank sum test for identical population distribution functions. The Wilcoxon test is a linear transformation of the Mann-Whitney *U* test. If the difference

between the two populations can be attributed solely to a difference in location, then the Wilcoxon test becomes a test of equality of the population means (or medians) and is the nonparametric equivalent of the two-sample *t*-test. Class `WilcoxonRankSum` obtains ranks in the combined sample after first eliminating missing values from the data. The rank sum statistic is then computed as the sum of the ranks in the `x` sample. Three methods for handling ties are used. (A tie is counted when two observations are within `fuzz` of each other.) Method 1 uses the largest possible rank for tied observations in the smallest sample, while Method 2 uses the smallest possible rank for these observations. Thus, the range of possible rank sums is obtained.

Method 3 for handling tied observations between samples uses the average rank of the tied observations. Asymptotic standard normal scores are computed for the *W* score (based on a variance that has been adjusted for ties) when average ranks are used (see Conover 1980, p. 217), and the probability associated with the two-sided alternative is computed.

Hypothesis Tests

In each of the following tests, the first line gives the hypothesis (and its alternative) under the assumptions 1 to 3 below, while the second line gives the hypothesis when assumption 4 is also true. The rejection region is the same for both hypotheses and is given in terms of Method 3 for handling ties. If another method for handling ties is desired, another output statistic, `stat[0]` or `stat[3]`, should be used, where `stat` is the array containing the statistics returned from the `getStatistics` method.

Test	Null Hypothesis	Alternative Hypothesis	Action
1	$H_0 : \Pr(x1 < x2) = 0.5$ $H_0 : E(x1) = E(x2)$	$H_1 : \Pr(x1 < x2) \neq 0.5$ $H_1 : E(x1) \neq E(x2)$	Reject if `stat[9]` is less than the significance level of the test. Alternatively, reject the null hypothesis if `stat[6]` is too large or too small.
2	$H_0 : \Pr(x1 < x2) \leq 0.5$ $H_0 : E(x1) \geq E(x2)$	$H_1 : \Pr(x1 < x2) \neq 0.5$ $H_1 : E(x1) < E(x2)$	Reject if `stat[6]` is too small
3	$H_0 : \Pr(x1 < x2) \geq 0.5$ $H_0 : E(x1) \leq E(x2)$	$H_1 : \Pr(x1 < x2) < 0.5$ $H_1 : E(x1) > E(x2)$	Reject if `stat[6]` is too large

Assumptions

1. *x* and *y* contain random samples from their respective populations.
2. All observations are mutually independent.
3. The measurement scale is at least ordinal (i.e., an ordering less than, greater than, or equal to exists among the observations).
4. If *f(x)* and *g(y)* are the distribution functions of *x* and *y*, then *g(y) = f(x + c)* for some constant *c*(i.e., the distribution of *y* is, at worst, a translation of the distribution of *x*).

Tables of critical values of the *W* statistic are given in the references for small samples.

Constructor

WilcoxonRankSum

```
public WilcoxonRankSum(double[] x, double[] y)
```

Description

Constructor for `WilcoxonRankSum`.

Parameters

`x` – A `double` array containing the first sample.

`y` – A `double` array containing the second sample.

Methods

Compute

```
public double Compute()
```

Description

Performs a Wilcoxon rank sum test.

Returns

A `double` scalar containing the two-sided p-value for the Wilcoxon rank sum statistic that is computed with average ranks used in the case of ties.

GetStatistics

```
public double[] GetStatistics()
```

Description

Returns the statistics. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` array of length 10 containing statistics.

Remarks

The statistics are as follows:

Row	Statistics
0	Wilcoxon *W* statistic (the sum of the ranks of the *x* observations) adjusted for ties in such a manner that *W*is as small as possible
1	2 x E*(*W) - W, where E*(*W) is the expected value of *W*
2	probability of obtaining a statistic less than or equal to min{W, 2 x E*(*W) - W}
3	*W* statistic adjusted for ties in such a manner that *W*is as large as possible
4	2 x E*(*W) - W, where E*(*W) is the expected value of W, adjusted for ties in such a manner that *W* is as large as possible
5	probability of obtaining a statistic less than or equal to min{W, 2 x E*(*W) - W}, adjusted for ties in such a manner that *W* is as large as possible
6	*W*statistic with average ranks used in case of ties
7	estimated standard error of Row 6 under the null hypothesis of no difference
8	standard normal score associated with Row 6
9	two-sided p-value associated with Row 8

SetFuzz

```
public void SetFuzz(double fuzz)
```

Description

Sets the nonnegative constant used to determine ties in computing ranks in the combined samples.

Parameter

`fuzz` – A `double` scalar containing the nonnegative constant used to determine ties in computing ranks in the combined samples.

Remarks

A tie is declared when two observations in the combined sample are within `fuzz` of each other. By default, fuzz $= 100 \times 2.2204460492503131e-16 \times \max(|x_{i1}|, |x_{j2}|)$.

Example 1: Wilcoxon Rank Sum Test

The following example is taken from Conover (1980, p. 224). It involves the mixing time of two mixing machines using a total of 10 batches of a certain kind of batter, five batches for each machine. The null hypothesis is not rejected at the 5-percent level of significance.

```
using System;
using Imsl;
using Imsl.Stat;

public class WilcoxonRankSumEx1
{
    public static void  Main(String[] args)
    {
```

```
        double[] x = new double[]{7.3, 6.9, 7.2, 7.8, 7.2};
        double[] y = new double[]{7.4, 6.8, 6.9, 6.7, 7.1};

        WilcoxonRankSum wilcoxon = new WilcoxonRankSum(x, y);
        Console.Out.WriteLine
            ("p-value = " + wilcoxon.Compute().ToString("0.0000"));
    }
}
```

Output

```
p-value = 0.1412
Imsl.Stat.WilcoxonRankSum: "x.Length" = 5 and "y.Length" = 5.
Both sample sizes, "x.Length" and "y.Length", are less than 25.
Significance levels should be obtained from tabled values.
Imsl.Stat.WilcoxonRankSum: At least one tie is detected between the samples.
```

Example 2: Wilcoxon Rank Sum Test

The following example uses the same data as in example 1. Now, all the statistics are displayed.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class WilcoxonRankSumEx2
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{7.3, 6.9, 7.2, 7.8, 7.2};
        double[] y = new double[]{7.4, 6.8, 6.9, 6.7, 7.1};
        String[] labels =new String[]{
                 "Wilcoxon W statistic ......................",
                 "2*E(W) - W ................................",
                 "p-value ..................................  ",
                 "Adjusted Wilcoxon statistic ...............",
                 "Adjusted 2*E(W) - W .......................",
                 "Adjusted p-value ..........................  ",
                 "W statistics for averaged ranks............",
                 "Standard error of W (averaged ranks) ......  ",
                 "Standard normal score of W (averaged ranks) ",
                 "Two-sided p-value of W (averaged ranks) ... "};

        WilcoxonRankSum wilcoxon = new WilcoxonRankSum(x, y);
        wilcoxon.Compute();
        double[] stat = wilcoxon.GetStatistics();

        for (int i = 0; i < 10; i++)
        {
            Console.Out.WriteLine
                (labels[i] + " " + stat[i].ToString("0.000"));
        }
    }
}
```

Output

```
Wilcoxon W statistic ..................... 34.000
2*E(W) - W ............................... 21.000
p-value ...................................  0.110
Adjusted Wilcoxon statistic ..............  35.000
Adjusted 2*E(W) - W ...................... 20.000
Adjusted p-value ..........................  0.075
W statistics for averaged ranks..... 34.500
Standard error of W (averaged ranks) ......  4.758
Standard normal score of W (averaged ranks)  1.471
Two-sided p-value of W (averaged ranks) ...  0.141
Imsl.Stat.WilcoxonRankSum: "x.Length" = 5 and "y.Length" = 5.
Both sample sizes, "x.Length" and "y.Length", are less than 25.
Significance levels should be obtained from tabled values.
Imsl.Stat.WilcoxonRankSum: At least one tie is detected between the samples.
```

Chapter 17: Tests of Goodness of Fit

Types

Usage Notes

The classes in this chapter are used to test for goodness of fit. The goodness-of-fit tests are described in Conover (1980). There is a goodness-of-fit test for general distributions and a chi-squared test. The user supplies the hypothesized cumulative distribution function for the test. There is a class that can be used to test specifically for the normal distribution.

The chi-squared goodness-of-fit test may be used with discrete as well as continuous distributions. The chi-squared goodness-of-fit test allows for missing values (NaN, not a number) in the input data.

The Kolmogorov-Smirnov routines in this chapter compute exact probabilities in small to moderate sample sizes. The chi-squared goodness-of-fit test may be used with discrete as well as continuous distributions.

The Kolmogorov-Smirnov and chi-squared goodness-of-fit test routines allow for missing values (NaN, not a number) in the input data. The routines that test for randomness do not allow for missing values.

ChiSquaredTest Class

```
public class Imsl.Stat.ChiSquaredTest
```

Chi-squared goodness-of-fit test.

`ChiSquaredTest` performs a chi-squared goodness-of-fit test that a random sample of observations is distributed according to a specified theoretical cumulative distribution. The theoretical distribution, which may be continuous, discrete, or a mixture of discrete and continuous distributions, is specified via a user-defined function F where F implements `ICdfFunction`. Because the user is allowed to specify a range for the observations in the `SetRange` method, a test that is conditional upon the specified range is performed.

`ChiSquaredTest` can be constructed in two different ways. The intervals can be specified via the array cutpoints. Otherwise, the number of cutpoints can be given and equiprobable intervals computed by the constructor. The observations are divided into these intervals. Regardless of the method used to obtain them, the intervals are such that the lower endpoint is not included in the interval while the upper endpoint is always included. The user should determine the cutpoints when the cumulative distribution function has discrete elements since `ChiSquaredTest` cannot determine them in this case.

By default, the lower and upper endpoints of the first and last intervals are $-\infty$ and $+\infty$, respectively. The method `SetRange` can be used to change the range.

A tally of counts is maintained for the observations in x as follows:

If the cutpoints are specified by the user, the tally is made in the interval to which x_i belongs, using the user-specified endpoints.

If the cutpoints are determined by the class then the cumulative probability at x_i, $F(x_i)$, is computed using `Cdf`.

The tally for x_i is made in interval number $\lfloor mF(x)+1 \rfloor$, where m is the number of categories and $\lfloor . \rfloor$ is the function that takes the greatest integer that is no larger than the argument of the function. If the cutpoints are specified by the user, the tally is made in the interval to which x_i belongs using the endpoints specified by the user. Thus, if the computer time required to calculate the cumulative distribution function is large, user-specified cutpoints may be preferred in order to reduce the total computing time.

If the expected count in any cell is less than 1, then a rule of thumb is that the chi-squared approximation may be suspect. A warning message to this effect is issued in this case, as well as when an expected value is less than 5.

Properties

ChiSquared

```
public double ChiSquared {get; }
```

Description

The chi-squared statistic.

Property Value

A `double` containing the chi-squared statistic.

DegreesOfFreedom

```
public double DegreesOfFreedom {get; }
```

Description

Returns the degrees of freedom in chi-squared.

Property Value

A `double` containing the degrees of freedom in the chi-squared statistic.

P

```
public double P {get; }
```

Description

The *p*-value for the chi-squared statistic.

Property Value

A `double` containing the *p*-value for the chi-squared statistic.

Constructors

ChiSquaredTest

```
public ChiSquaredTest(Imsl.Stat.ICdfFunction cdf, double[] cutpoints, int
nParameters)
```

Description

Constructor for the Chi-squared goodness-of-fit test.

Parameters

`cdf` – Object that implements the `ICdfFunction` interface.

`cutpoints` – A `double` array containing the cutpoints.

`nParameters` – A `int` which specifies the number of parameters estimated in computing the `Cdf`.

Exception

`Imsl.Stat.NotCDFException` is thrown if the function `cdf.CdfFunction` is not a valid CDF.

ChiSquaredTest

```
public ChiSquaredTest(Imsl.Stat.ICdfFunction cdf, int nCutpoints, int
nParameters)
```

Description

Constructor for the Chi-squared goodness-of-fit test

Parameters

`cdf` – Object that implements the `ICdfFunction` interface.

`nCutpoints` – A `int` which specifies the number of cutpoints.

`nParameters` – A `int` which specifies the number of parameters estimated in computing the `Cdf`.

Exceptions

`Imsl.Stat.NotCDFException` is thrown if the function `cdf.CdfFunction` is not a valid CDF.

`Imsl.Stat.DidNotConvergeException` is thrown if the interation to find the inverse of the CDF did not converge. The inverse CDF is needed to compute the cutpoints.

Methods

GetCellCounts

```
public double[] GetCellCounts()
```

Description

Returns the cell counts.

Returns

A `double` array which contains the number of actual observations in each cell.

GetCutpoints

```
public double[] GetCutpoints()
```

Description

Returns the cutpoints.

Returns

A `double` array which contains the cutpoints.

Remarks

The intervals defined by the cutpoints are such that the lower endpoint is not included while the upper endpoint is included in the interval.

GetExpectedCounts

```
public double[] GetExpectedCounts()
```

Description

Returns the expected counts.

Returns

A `double` array which contains the number of expected observations in each cell.

SetCutpoints

```
public void SetCutpoints(double[] cutpoints)
```

Description

Sets the cutpoints.

Parameter

`cutpoints` – A `double` array which contains the cutpoints.

Remarks

The intervals defined by the cutpoints are such that the lower endpoint is not included while the upper endpoint is included in the interval.

SetRange

```
public void SetRange(double lower, double upper)
```

Description

Sets endpoints of the range of the distribution.

Parameters

`lower` – A `double` which specifies the lower range limit.

`upper` – A `double` which specifies the upper range limit.

Remarks

Points outside of the range are ignored so that distributions conditional on the range can be used. In this case, the point lower is excluded from the first interval, but the point upper is included in the last interval.

By default, a range on the whole real line is used.

Update

```
public void Update(double[] x, double[] freq)
```

Description

Adds new observations to the test.

Parameters

`x` – A `double` array which contains the new observations to be added to the test.

`freq` – A `double` array which contains the frequencies of the corresponding new observations in x.

Update

```
public void Update(double x, double freq)
```

Description

Adds a new observation to the test.

Parameters

`x` – A `double` which specifies the new observation to be added to the test.

`freq` – A `double` which specifies the frequency of the new observation, x.

Example: The Chi-squared Goodness-of-fit Test

In this example, a discrete binomial random sample of size 1000 with binomial parameter $p = 0.3$ and binomial sample size 5 is generated via Random.nextBinomial. Random.setSeed is first used to set the seed. After the ChiSquaredTest constructor is called, the random observations are added to the test one at a time to simulate streaming data. The Chi-squared statistic, *p*-value, and Degrees of freedom are then computed and printed.

```
using System;
using Imsl.Stat;

public class ChiSquaredTestEx1 : ICdfFunction
{
    public double CdfFunction(double x)
    {
        return Cdf.Binomial((int) x, 5, 0.3);
    }

    public static void  Main(String[] args)
    {
        //    Seed the random number generator
        Imsl.Stat.Random rn = new Imsl.Stat.Random(123457);
        rn.Multiplier = 16807;

        //    Construct a ChiSquaredTest object
        ICdfFunction bindf = new ChiSquaredTestEx1();

        double[] cutp = new double[]{0.5, 1.5, 2.5, 3.5, 4.5};
        int nParameters = 0;
        ChiSquaredTest cst =
            new ChiSquaredTest(bindf, cutp, nParameters);
        for (int i = 0; i < 1000; i++)
        {
            cst.Update(rn.NextBinomial(5, 0.3), 1.0);
        }

        //    Print goodness-of-fit test statistics
        Console.Out.WriteLine
            ("The Chi-squared statistic is " + cst.ChiSquared);
        Console.Out.WriteLine("The P-value is " + cst.P);
        Console.Out.WriteLine
            ("The Degrees of freedom are " + cst.DegreesOfFreedom);
    }
}
```

Output

```
The Chi-squared statistic is 4.79629666357389
The P-value is 0.441242957205526
The Degrees of freedom are 5
Imsl.Stat.ChiSquaredTest: An expected value is less than five.
```

NormalityTest Class

`public class Imsl.Stat.NormalityTest`

Performs a test for normality.

Three methods are provided for testing normality: the Shapiro-Wilk W test, the Lilliefors test, and the chi-squared test.

Shapiro-Wilk W Test

The Shapiro-Wilk *W* test is thought by D'Agostino and Stevens (1986, p. 406) to be one of the best omnibus tests of normality. The function is based on the approximations and code given by Royston (1982a, b, c). It can be used in samples as large as 2,000 or as small as 3. In the Shapiro and Wilk test, *W* is given by

$$W = \left(\sum a_i x_{(i)}\right)^2 / \left(\sum (x_i - \bar{x})^2\right)$$

where $x_{(i)}$ is the *i*-th largest order statistic and *x* is the sample mean. Royston (1982) gives approximations and tabled values that can be used to compute the coefficients $a_i, i = 1, \ldots, n$, and obtains the significance level of the *W* statistic.

Lilliefors Test

This function computes Lilliefors test and its *p*-values for a normal distribution in which both the mean and variance are estimated. The one-sample, two-sided Kolmogorov-Smirnov statistic *D* is first computed. The *p*-values are then computed using an analytic approximation given by Dallal and Wilkinson (1986). Because Dallal and Wilkinson give approximations in the range (0.01, 0.10) if the computed probability of a greater *D* is less than 0.01, the *p*-value is set to 0.50. Note that because parameters are estimated, *p*-values in Lilliefors test are not the same as in the Kolmogorov-Smirnov Test.

Observations should not be tied. If tied observations are found, an informational message is printed. A general reference for the Lilliefors test is Conover (1980). The original reference for the test for normality is Lilliefors (1967).

Chi-Squared Test

This function computes the chi-squared statistic, its *p*-value, and the degrees of freedom of the test. Argument *n* finds the number of intervals into which the observations are to be divided. The intervals are equiprobable except for the first and last interval, which are infinite in length.

If more flexibility is desired for the specification of intervals, the same test can be performed with class `ChiSquaredTest`.

Properties

ChiSquared

```
public double ChiSquared {get; }
```

Description

Returns the chi-square statistic for the chi-squared goodness-of-fit test.

Property Value

A `double` scalar containing the chi-square statistic.

Remarks

Returns `Double.NaN` for other tests.

DegreesOfFreedom

```
public double DegreesOfFreedom {get; }
```

Description

Returns the degrees of freedom for the chi-squared goodness-of-fit test.

Property Value

A `double` scalar containing the degrees of freedom.

Remarks

Returns `Double.NaN` for other tests.

MaxDifference

```
public double MaxDifference {get; }
```

Description

Returns the maximum absolute difference between the empirical and the theoretical distributions for the Lilliefors test.

Property Value

A `double` scalar containing the maximum absolute difference between the empirical and the theoretical distributions.

Remarks

Returns `Double.NaN` for other tests.

ShapiroWilkW

```
public double ShapiroWilkW {get; }
```

Description

Returns the Shapiro-Wilk W statistic for the Shapiro-Wilk W test.

Property Value

A `double` scalar containing the Shapiro-Wilk W statistic.

Remarks

Returns `Double.NaN` for other tests.

Constructor

NormalityTest

```
public NormalityTest(double[] x)
```

Description

Constructor for `NormalityTest`.

Parameter

`x` – A `double` array containing the observations.

Remarks

`x.length` must be in the range from 3 to 2,000, inclusive, for the Shapiro-Wilk W test and must be greater than 4 for the Lilliefors test.

Methods

ChiSquaredTest

```
public double ChiSquaredTest(int n)
```

Description

Performs the chi-squared goodness-of-fit test.

Parameter

`n` – A `int` scalar containing the number of cells into which the observations are to be tallied.

Returns

A `double` scalar containing the p-value for the chi-squared goodness-of-fit test.

Exceptions

`Imsl.Stat.NoVariationInputException` is thrown if there is no variation in the input data

`Imsl.Stat.DidNotConvergeException` is thrown if the iteration did not converge

See Also: Imsl.Stat.NormalityTest.ChiSquaredTest(System.Int32) (p. 791)

LillieforsTest

```
public double LillieforsTest()
```

Description

Performs the Lilliefors test.

Returns

A `double` scalar containing the p-value for the Lilliefors test.

Remarks

Probabilities less than 0.01 are reported as 0.01, and probabilities greater than 0.10 for the normal distribution are reported as 0.5. Otherwise, an approximate probability is computed.

Exceptions

`Imsl.Stat.NoVariationInputException` is thrown if there is no variation in the input data

`Imsl.Stat.DidNotConvergeException` is thrown if the iteration did not converge

ShapiroWilkTest

```
public double ShapiroWilkTest()
```

Description

Performs the Shapiro-Wilk W test using Royston's calculation.

Returns

A `double` scalar containing the p-value for the Shapiro-Wilk W test.

Exceptions

`Imsl.Stat.NoVariationInputException` is thrown if there is no variation in the input data

`Imsl.Stat.DidNotConvergeException` is thrown if the iteration did not converge

ShapiroWilkWTest

```
public double ShapiroWilkWTest()
```

Description

Performs the Shapiro-Wilk W test.

Returns

A `double` scalar containing the p-value for the Shapiro-Wilk W test.

Exceptions

`Imsl.Stat.NoVariationInputException` is thrown if there is no variation in the input data

`Imsl.Stat.DidNotConvergeException` is thrown if the iteration did not converge

Example: Shapiro-Wilk W Test

The following example is taken from Conover (1980, pp. 195, 364). The data consists of 50 two-digit numbers taken from a telephone book. The *W* test fails to reject the null hypothesis of normality at the .05 level of significance.

```
using System;
using Imsl;
using Imsl.Stat;

public class NormalityTestEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{
                                    23.0, 36.0, 54.0, 61.0, 73.0, 23.0,
                                    37.0, 54.0, 61.0, 73.0, 24.0, 40.0,
                                    56.0, 62.0, 74.0, 27.0, 42.0, 57.0,
                                    63.0, 75.0, 29.0, 43.0, 57.0, 64.0,
```

```
                                        77.0, 31.0, 43.0, 58.0, 65.0, 81.0,
                                        32.0, 44.0, 58.0, 66.0, 87.0, 33.0,
                                        45.0, 58.0, 68.0, 89.0, 33.0, 48.0,
                                        58.0, 68.0, 93.0, 35.0, 48.0, 59.0,
                                        70.0, 97.0};

        NormalityTest nt = new NormalityTest(x);

        Console.Out.WriteLine
            ("p-value = " + nt.ShapiroWilkWTest().ToString("0.0000"));
        Console.Out.WriteLine("Shapiro Wilk W Statistic = " +
            nt.ShapiroWilkW.ToString("0.0000"));
    }
}
```

Output

```
p-value = 0.2309
Shapiro Wilk W Statistic = 0.9642
```

KolmogorovOneSample Class

`public class Imsl.Stat.KolmogorovOneSample`

The class `KolmogorovOneSample` performs a Kolmogorov-Smirnov goodness-of-fit test in one sample.

The hypotheses tested follow:

$$H_0: F(x) = F^*(x) \quad H_1: F(x) \neq F^*(x)$$
$$H_0: F(x) \geq F^*(x) \quad H_1: F(x) < F^*(x)$$
$$H_0: F(x) \leq F^*(x) \quad H_1: F(x) > F^*(x)$$

where F is the cumulative distribution function (CDF) of the random variable, and the theoretical cdf, F^*, is specified via the user-supplied function cdf. Let n be the number of observations minus the number of missing observations. The test statistics for both one-sided alternatives D_n^+ and D_n^- and the two-sided D_n alternative are computed as well as an asymptotic z-score and p-values associated with the one-sided and two-sided hypotheses. For $n > 80$, asymptotic p-values are used (see Gibbons 1971). For $n \leq 80$, exact one-sided p-values are computed according to a method given by Conover (1980, page 350). An approximate two-sided test p-value is obtained as twice the one-sided p-value. The approximation is very close for one-sided p-values less than 0.10 and becomes very bad as the one-sided p-values get larger.

The theoretical CDF is assumed to be continuous. If the CDF is not continuous, the statistics D_n^* will not be computed correctly.

Estimation of parameters in the theoretical CDF from the sample data will tend to make the p-values associated with the test statistics too liberal. The empirical CDF will tend to be closer to the theoretical CDF than it should be.

No attempt is made to check that all points in the sample are in the support of the theoretical CDF. If all sample points are not in the support of the CDF, the null hypothesis must be rejected.

Properties

MaximumDifference

```
public double MaximumDifference {get; }
```

Description

D^+, the maximum difference between the theoretical and empirical CDF's.

MinimumDifference

```
public double MinimumDifference {get; }
```

Description

D^-, the minimum difference between the theoretical and empirical CDF's.

NumberMissing

```
public int NumberMissing {get; }
```

Description

The number of missing values in the data.

NumberOfTies

```
public int NumberOfTies {get; }
```

Description

The number of ties in the data.

OneSidedPValue

```
public double OneSidedPValue {get; }
```

Description

Probability of the statistic exceeding D under the null hypothesis of equality and against the one-sided alternative. An exact probability is computed if the number of observation is less than or equal to 80, otherwise an approximate probability is computed.

TestStatistic

```
public double TestStatistic {get; }
```

Description

The test statistic, $D = \max(D^+, D^-)$.

TwoSidedPValue

```
public double TwoSidedPValue {get; }
```

Description

Probability of the statistic exceeding *D* under the null hypothesis of equality and against the two-sided alternative.

Remarks

This probability is twice the probability, p_1, reported by `OneSidedPValue`, (or 1.0 if $p_1 \geq 1/2$). This approximation is nearly exact when $p_1 < 0.1$.

Z

```
public double Z {get; }
```

Description

The normalized *D* statistic without the continuity correction applied.

Constructor

KolmogorovOneSample

```
public KolmogorovOneSample(Imsl.Stat.ICdfFunction cdf, double[] x)
```

Description

Constructs a one sample Kolmogorov-Smirnov goodness-of-fit test.

Parameters

`cdf` – is the cdf function, $F(x)$. If must be non-decreasing and its value must be in [0, 1].

`x` – is a `double` array containing the observations.

Example: Kolmogorov One Sample

In this example, a random sample of size 100 is generated using class `Random` for the uniform (0, 1) distribution. We want to test the null hypothesis that the cdf is the standard normal distribution with a mean of 0.5 and a variance equal to the uniform (0, 1) variance (1/12).

```
using System;
using Imsl.Stat;
using Imsl.Math;


public class KolmogorovOneSampleEx1 : ICdfFunction
{
    public static void Main(String[] args)
    {
        double[] x = new double[100];
        Imsl.Stat.Random random = new Imsl.Stat.Random(123457);
        random.Multiplier = 16807;
        for (int i = 0;  i < x.Length;  i++) {
            x[i] = random.NextDouble();
        }
```

```
        ICdfFunction cdf = new KolmogorovOneSampleEx1();
        KolmogorovOneSample kos = new KolmogorovOneSample(cdf, x);
        Console.WriteLine("D  = "+ kos.TestStatistic);
        Console.WriteLine("D+ = " + kos.MaximumDifference);
        Console.WriteLine("D- = " + kos.MinimumDifference);
        Console.WriteLine("Z   = " + kos.Z);
        Console.WriteLine("Prob greater D one sided = " +
                kos.OneSidedPValue);
        Console.WriteLine("Prob greater D two sided = " +
                kos.TwoSidedPValue);
        Console.WriteLine("N missing = " + kos.NumberMissing);
    }

    public double CdfFunction(double x)
    {
        double mean = 0.5;
        double std = 0.2886751;
        double z = (x - mean) / std;
        return Cdf.Normal(z);
    }
}
```

Output

```
D  = 0.121914080665685
D+ = 0.121914080665685
D- = 0.0869429640568776
Z   = 1.21914080665685
Prob greater D one sided = 0.0511696542584409
Prob greater D two sided = 0.102339308516882
N missing = 0
```

KolmogorovTwoSample Class

`public class Imsl.Stat.KolmogorovTwoSample`

Performs a Kolmogorov-Smirnov two-sample test.

Class `KolmogorovTwoSample` computes Kolmogorov-Smirnov two-sample test statistics for testing that two continuous cumulative distribution functions (CDF's) are identical based upon two random samples. One- or two-sided alternatives are allowed. Exact *p*-values are computed for the two-sided test when $nm \leq 104$, where n is the number of non-missing X observations and m the number of non-missing Y observation.

Let $F_n(x)$ denote the empirical CDF in the X sample, let $G_m(y)$ denote the empirical CDF in the Y sample and let the corresponding population distribution functions be denoted by $F(x)$ and $G(y)$, respectively.

Then, the hypotheses tested by `KolmogorovTwoSample` are as follows:

$$\begin{array}{ll} H_0: F(x) = G(x) & H_1: F(x) \neq G(x) \\ H_0: F(x) \geq G(x) & H_1: F(x) < G(x) \\ H_0: F(x) \leq G(x) & H_1: F(x) > G(x) \end{array}$$

The test statistics are given as follows:

$$\begin{array}{ll} D_{mn} & = \max(D^+_{mn}, D^-_{mn}) \\ D^+_{mn} & = \max_x(F_n(x) - G_m(x)) \\ D^-_{mn} & = \max_x(G_m(x) - F_n(x)) \end{array}$$

Asymptotically, the distribution of the statistic

$$Z = D_{mn}\sqrt{\frac{m+n}{mn}}$$

converges to a distribution given by Smirnov (1939).

Exact probabilities for the two-sided test are computed when $nm \leq 104$, according to an algorithm given by Kim and Jennrich (1973). When $nm > 104$, the very good approximations given by Kim and Jennrich are used to obtain the two-sided *p*-values. The one-sided probability is taken as one half the two-sided probability. This is a very good approximation when the *p*-value is small (say, less than 0.10) and not very good for large *p*-values.

Properties

MaximumDifference

```
public double MaximumDifference {get; }
```

Description

D^+, the maximum difference between the theoretical and empirical CDF's.

MinimumDifference

```
public double MinimumDifference {get; }
```

Description

D^-, the minimum difference between the theoretical and empirical CDF's.

NumberMissingX

```
public int NumberMissingX {get; }
```

Description

Returns the number of missing values in the `x` sample.

NumberMissingY

```
public int NumberMissingY {get; }
```

Description

The number of missing values in the y sample.

OneSidedPValue

```
public double OneSidedPValue {get; }
```

Description

Probability of the statistic exceeding D under the null hypothesis of equality and against the one-sided alternative. An exact probability is computed if the number of observation is less than or equal to 80, otherwise an approximate probability is computed.

TestStatistic

```
public double TestStatistic {get; }
```

Description

The test statistic, $D = \max(D^+, D^-)$.

TwoSidedPValue

```
public double TwoSidedPValue {get; }
```

Description

Probability of the statistic exceeding D under the null hypothesis of equality and against the two-sided alternative. This probability is twice the probability, p_1, reported by `OneSidedPValue`, (or 1.0 if $p_1 \geq 1/2$). This approximation is nearly exact when $p_1 < 0.1$.

Z

```
public double Z {get; }
```

Description

The normalized D statistic without the continuity correction applied.

Constructor

KolmogorovTwoSample

```
public KolmogorovTwoSample(double[] x, double[] y)
```

Description

Constructs a two sample Kolmogorov-Smirnov goodness-of-fit test.

Parameters

x – Array containing the observations from the first sample.

y – Array containing the observations from the second sample.

Example: Kolmogorov Two Sample

The following example illustrates the class `KolmogorovTwoSample` routine with two randomly generated samples from a uniform(0,1) distribution. Since the two theoretical distributions are identical, we would not expect to reject the null hypothesis.

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class KolmogorovTwoSampleEx1
{
    public static void Main(String[] args)
    {
        double[] x = new double[100];
        double[] y = new double[60];
        Imsl.Stat.Random random = new Imsl.Stat.Random(123457);
        random.Multiplier = 16807;
        for (int i = 0;  i < x.Length;  i++)
        {
            x[i] = random.NextFloat();
        }
        for (int i = 0;  i < y.Length;  i++)
        {
            y[i] = random.NextFloat();
        }

        KolmogorovTwoSample k2s = new KolmogorovTwoSample(x, y);
        Console.WriteLine("D  = "+ k2s.TestStatistic);
        Console.WriteLine("D+ = " + k2s.MaximumDifference);
        Console.WriteLine("D- = " + k2s.MinimumDifference);
        Console.WriteLine("Z   = " + k2s.Z);
        Console.WriteLine("Prob greater D one sided = " +
                k2s.OneSidedPValue);
        Console.WriteLine("Prob greater D two sided = " +
                k2s.TwoSidedPValue);
        Console.WriteLine("Missing X = " + k2s.NumberMissingX);
        Console.WriteLine("Missing Y = " + k2s.NumberMissingY);
    }
}
```

Output

```
D  = 0.18
D+ = 0.18
D- = 0.01
Z   = 1.10227038425243
Prob greater D one sided = 0.072010607348685
Prob greater D two sided = 0.14402121469737
Missing X = 0
Missing Y = 0
```

Chapter 18: Time Series and Forecasting

Types

Usage Notes

The classes in this chapter allow users to filter and analyze time series data using autoregressive, ARIMA, GARCH and state-space models. Some are designed for automatic model selection, estimation and forecasting using algorithms based upon minimizing AIC (Akaike's Information Criteria).

General Methodology

Model Identification and Data Filtering

All classes in this chapter for time series analysis assume the data are stationary and collected at equally spaced times with no missing observations in the series. In order to prepare a series for analysis, any missing values must be replaced with estimates. The class `ARMAEstimateMissing` can be used to automatically estimate missing values using one of four, user-selected methods, provided the largest time gap with missing values has no more than 4 missing values.

In addition to estimating missing values, if a series is nonstationary or contains seasonal variation, its values should be filtered before fitting a model and preparing forecasts. Class `ARSeasonalFit` evaluates seasonal adjustments prior to modeling the series.Users specify a range of adjustments.`ARSeasonalFit` evaluates each adjustment to identify the one that minimizes the AIC. If a user already knows the series needs to be filtered using differences between consecutive values, the class `Difference` can be used to filter a series into an equivalent series of differences.

There are several classes available for transforming a time series into its autocorrelation matrix: `AutoCorrelation`, `CrossCorrelation` and `MultiCrossCorrelation`. Class `AutoCorrelation` computes the autocorrelation and partial autocorrelation matrices from a time series or its filtered values. Box and Jenkins (1976) describe how to identify the structure of an ARIMA model using the autocorrelation and partial autocorrelation matrices.

Classes `CrossCorrelation` and `MultiCrossCorrelation` are used to calculate the cross-correlation matrix for two univariate or multivariate time series, respectively.

Model Estimation and Forecasting

There are several classes useful for modeling stationary, equally spaced time series - `ARMA`, `ARMAMaxLikelihood`, `GARCH`, and `KalmanFilter`. Class `ARMA` can be used to fit autoregressive, moving-average and ARIMA (autoregressive, integrated moving average) models.Rather than passing the original series to this class, users are required to construct this class using the autocorrlelation matrix of the series. Class `AutoCorrelation` is useful for computing this matrix prior to constructing the `ARMA` class.

The other classes require users to construct the class using the original series or its filtered values. Class `ARMAMaxLikelihood` estimates the maximum likelihood estimates of the ARMA parameters and prepares forecasts from these values. Class `ARAutoUnivariate` selects the best autoregressive model for an equally spaced, stationary time series using AIC. Class `GARCH` estimates the parameters of a GARCH model and the `KalmanFilter` class estimates the parameters for a state-space model using Kalman filtering and calculates one-step ahead forecasts

Classes `ARMA`, `ARMAMaxLikelihood` and `ARAutoUnivariate` are the only classes with methods for obtaining forecasts from an ARIMA model. Class `KalmanFilter` can be used to obtain one-step ahead forecasts for a state-space model.

ARIMA Model (Autoregressive Integrated Moving Average)

A small, yet comprehensive, class of stationary time-series models consists of the nonseasonal ARMA processes defined by

$$\phi(B)(W_t - \mu) = \theta(B)A_t, \quad t \in Z$$

where $Z = \ldots, -2, -1, 0, 1, 2, \ldots$ denotes the set of integers, B is the backward shift operator defined by $B^k W_t = W_{t-k}, \mu$ is the mean of W_t, and the following equations are true:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p, p \geq 0$$

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q, q \geq 0$$

The model is of order (p, q) and is referred to as an ARMA (p, q) model.

An equivalent version of the ARMA (p, q) model is given by

$$\phi(B)W_t = \theta_0 + \theta(B)A_i, \quad t \in Z$$

where θ_0 is an overall constant defined by the following:

$$\theta_0 = \mu\left(1 - \sum_{i=1}^{p} \phi_i\right)$$

See Box and Jenkins (1976, pp. 92-93) for a discussion of the meaning and usefulness of the overall constant.

If the "raw" data, $\{Z_t\}$, are homogeneous and nonstationary, then differencing using the Difference class induces stationarity, and the model is called ARIMA (AutoRegressive Integrated Moving Average). Parameter estimation is performed on the stationary time series $W_t, = \Delta^d Z_t$, where $\Delta^d = (1 - B)^d$ is the backward difference operator with period 1 and order $d, \mathrm{d} > 0$.

There are two classes for estimating the parameters in an ARMA model and preparing forecasts: `ARMA` and `ARMAMaxLikelihood`. Class `ARMA` estimates model parameters using either method of moments or least squares. The method of moments is selected by default or by setting property `Method` to `MethodOfMoments`. Method of moment estimates are good initial values for obtaining the least-squares estimates by setting property `Method` to `LeastSquares` . Other initial estimates provided by the user can be used. The least-squares procedure can be used to compute conditional or unconditional least-squares estimates of the parameters, depending on the choice of the backcasting length. The parameter estimates from either the method of moments or least-squares procedures can be used in the `Forecast` method. The functions for preliminary parameter estimation, least-squares parameter estimation, and forecasting follow the approach of Box and Jenkins (1976, Programs 2-4, pp. 498-509).

Class `ARMAMaxLikelihood` estimates model parameters using maximum likelihood. Users can tailor the maximum likelihood algorithm by setting tolerances and providing initial estimates for the parameters. By default, `ARMAMaxLikelihood` uses method of moment estimates to initialize the parameters. However, this can be overridden with the `SetAR` method in `ARMAMaxLikelihood`.

AutoCorrelation Class

`public class Imsl.Stat.AutoCorrelation`

Computes the sample autocorrelation function of a stationary time series.

`AutoCorrelation` estimates the autocorrelation function of a stationary time series given a sample of *n* observations $\{X_t\}$ for t = $1, 2, \ldots,$ n.

Let

$$\hat{\mu} = \text{xmean}$$

be the estimate of the mean μ of the time series $\{X_t\}$ where

$$\hat{\mu} = \begin{cases} \mu & \text{for } \mu \text{ known} \\ pa \ \frac{1}{n}\sum_{t=1}^{n} X_t & \text{for } \mu \text{ unknown} \end{cases}$$

The autocovariance function $\sigma(k)$ is estimated by

$$\hat{\sigma}(k) = \frac{1}{n}\sum_{t=1}^{n-k}(X_t - \hat{\mu})(X_{t+k} - \hat{\mu}), \qquad \text{k=0,1,...,K}$$

where K = `maximumLag`. Note that $\hat{\sigma}(0)$ is an estimate of the sample variance. The autocorrelation function $\rho(k)$ is estimated by

$$\hat{\rho}(k) = \frac{\hat{\sigma}(k)}{\hat{\sigma}(0)}, \qquad k = 0, 1, \ldots, K$$

Note that $\hat{\rho}(0) \equiv 1$ by definition.

The standard errors of sample autocorrelations may be optionally computed according to the *GetStandardErrors* method argument `stderrMethod`. One method (Bartlett 1946) is based on a general asymptotic expression for the variance of the sample autocorrelation coefficient of a stationary time series with independent, identically distributed normal errors. The theoretical formula is

$$\text{var}\{\hat{\rho}(k)\} = \frac{1}{n}\sum_{i=-\infty}^{\infty}\left[\rho^2(i) + \rho(i-k)\rho(i+k) - 4\rho(i)\rho(k)\rho(i-k) + 2\rho^2(i)\rho^2(k)\right]$$

where $\hat{\rho}(k)$ assumes μ is unknown. For computational purposes, the autocorrelations $\rho(k)$ are replaced by their estimates $\hat{\rho}(k)$ for $|k| \leq K$, and the limits of summation are bounded because of the assumption that $\rho(k) = 0$ for all k such that $|k| > K$.

A second method (Moran 1947) utilizes an exact formula for the variance of the sample autocorrelation coefficient of a random process with independent, identically distributed normal errors. The theoretical formula is

$$var\{\hat{\rho}(k)\} = \frac{n-k}{n(n+2)}$$

where μ is assumed to be equal to zero. Note that this formula does not depend on the autocorrelation function.

The method `GetPartialAutoCorrelations` returns the estimated partial autocorrelations of the stationary time series given K = `maximumLag` sample autocorrelations $\hat{\rho}(k)$ for k=0,1,...,K. Consider the AR(k) process defined by

$$X_t = \phi_{k1}X_{t-1} + \phi_{k2}X_{t-2} + \cdots + \phi_{kk}X_{t-k} + A_t$$

where ϕ_{kj} denotes the j-th coefficient in the process. The set of estimates $\{\hat{\phi}_{kk}\}$ for k = 1, ..., K is the sample partial autocorrelation function. The autoregressive parameters $\{\hat{\phi}_{kj}\}$ for j = 1, ..., k are approximated by Yule-Walker estimates for successive AR(k) models where k = 1, ..., K. Based on the sample Yule-Walker equations

$$\hat{\rho}(j) = \hat{\phi}_{k1}\hat{\rho}(j-1) + \hat{\phi}_{k2}\hat{\rho}(j-2) + \cdots + \hat{\phi}_{kk}\hat{\rho}(j-k), \qquad \text{j = 1,2,...,k}$$

a recursive relationship for k=1, ..., K was developed by Durbin (1960). The equations are given by

$$\hat{\phi}_{kk} = \begin{cases} \hat{\rho}(1) & \text{for k} = 1 \\ \dfrac{\hat{\rho}(k) - \sum_{j=1}^{k-1} \hat{\phi}_{k-1,j}\hat{\rho}(k-j)}{1 - \sum_{j=1}^{k-1} \hat{\phi}_{k-1,j}\hat{\rho}(j)} & \text{for k} = 2, \ldots, \text{K} \end{cases}$$

and

$$\hat{\phi}_{kj} = \begin{cases} \hat{\phi}_{k-1,j} - \hat{\phi}_{kk}\hat{\phi}_{k-1,k-j} & \text{for j} = 1, 2, \ldots, \text{k}-1 \\ \hat{\phi}_{kk} & \text{for j} = \text{k} \end{cases}$$

This procedure is sensitive to rounding error and should not be used if the parameters are near the nonstationarity boundary. A possible alternative would be to estimate $\{\phi_{kk}\}$ for successive AR(k) models using least or maximum likelihood. Based on the hypothesis that the true process is AR(p), Box and Jenkins (1976, page 65) note

$$\text{var}\{\hat{\phi}_{kk}\} \simeq \frac{1}{n} \quad \text{k} \geq \text{p}+1$$

See Box and Jenkins (1976, pages 82-84) for more information concerning the partial autocorrelation function.

Properties

Mean

```
public double Mean {get; set; }
```

Description

The mean of the time series `x`.

Property Value

A `double` containing the mean of the time series `x`.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Variance

```
public double Variance {get; }
```

Description

Returns the variance of the time series `x`.

Property Value

A `double` containing the variance of the time series `x`.

Constructor

AutoCorrelation

```
public AutoCorrelation(double[] x, int maximumLag)
```

Description

Constructor to compute the sample autocorrelation function of a stationary time series.

Parameters

`x` – A one-dimensional `double` array containing the stationary time series.

`maximumLag` – An `int` containing the maximum lag of autocovariance, autocorrelations, and standard errors of autocorrelations to be computed.

Remarks

`maximumLag` must be greater than or equal to 1 and less than the number of observations in `x`.

Methods

GetAutoCorrelations

```
public double[] GetAutoCorrelations()
```

Description

Returns the autocorrelations of the time series `x`.

Returns

A `double` array of length `maximumLag` +1 containing the autocorrelations of the time series `x`.

Remarks

The *0*-th element of this array is 1. The *k*-th element of this array contains the autocorrelation of lag k where *k* = 1, ..., `maximumLag`.

GetAutoCovariances

```
public double[] GetAutoCovariances()
```

Description

Returns the variance and autocovariances of the time series `x`.

Returns

A `double` array of length `maximumLag` + 1 containing the variances and autocovariances of the time series `x`.

Remarks

The *0*-th element of the array contains the variance of the time series `x`. The *k*-th element contains the autocovariance of lag k where *k* = 1, ..., `maximumLag`.

Exception

`Imsl.Stat.NonPosVarianceException` is thrown if the problem is ill-conditioned.

GetPartialAutoCorrelations

```
public double[] GetPartialAutoCorrelations()
```

Description

Returns the sample partial autocorrelation function of the stationary time series `x`.

Returns

A `double` array of length `maximumLag` containing the partial autocorrelations of the time series `x`.

GetStandardErrors

```
public double[] GetStandardErrors(Imsl.Stat.AutoCorrelation.StdErr
stderrMethod)
```

Description

Returns the standard errors of the autocorrelations of the time series `x`.

Parameter

`stderrMethod` – An `int` specifying the method to compute the standard errors of autocorrelations of the time series `x`.

Returns

A `double` array of length `maximumLag` containing the standard errors of the autocorrelations of the time series `x`.

Remarks

Method of computation for standard errors of the autocorrelation is chosen by the `stderrMethod` parameter.

If `stderrMethod` is set to `Bartletts`, Bartlett's formula is used to compute the standard errors of autocorrelations.

If `stderrMethod` is set to `Morans`, Moran's formula is used to compute the standard errors of autocorrelations.

Example 1: AutoCorrelation

Consider the Wolfer Sunspot Data (Anderson 1971, p. 660) consisting of the number of sunspots observed each year from 1749 through 1924. The data set for this example consists of the number of sunspots observed from 1770 through 1869. This example computes the estimated autocovariances, estimated autocorrelations, and estimated standard errors of the autocorrelations using both Barlett and Moran formulas.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class AutoCorrelationEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{   100.8, 81.6, 66.5, 34.8, 30.6,
                                     7, 19.8, 92.5, 154.4, 125.9,
                                     84.8, 68.1, 38.5, 22.8, 10.2,
                                     24.1, 82.9, 132, 130.9, 118.1,
                                     89.9, 66.6, 60, 46.9, 41,
                                     21.3, 16, 6.4, 4.1, 6.8,
                                     14.5, 34, 45, 43.1, 47.5,
                                     42.2, 28.1, 10.1, 8.1, 2.5,
                                     0, 1.4, 5, 12.2, 13.9,
                                     35.4, 45.8, 41.1, 30.4, 23.9,
                                     15.7, 6.6, 4, 1.8, 8.5,
                                     16.6, 36.3, 49.7, 62.5, 67,
                                     71, 47.8, 27.5, 8.5, 13.2,
                                     56.9, 121.5, 138.3, 103.2, 85.8,
                                     63.2, 36.8, 24.2, 10.7, 15,
                                     40.1, 61.5, 98.5, 124.3, 95.9,
                                     66.5, 64.5, 54.2, 39, 20.6,
                                     6.7, 4.3, 22.8, 54.8, 93.8,
```

```
                                    95.7, 77.2, 59.1, 44, 47,
                                    30.5, 16.3, 7.3, 37.3, 73.9};

        AutoCorrelation ac = new AutoCorrelation(x, 20);

        new PrintMatrix
            ("AutoCovariances are: ").Print(ac.GetAutoCovariances());
        Console.Out.WriteLine();
        new PrintMatrix
            ("AutoCorrelations are:  ").Print(ac.GetAutoCorrelations());
        Console.Out.WriteLine("Mean = " + ac.Mean);
        Console.Out.WriteLine();
        new PrintMatrix
            ("Standard Error using Bartlett are:  ").Print
            (ac.GetStandardErrors(AutoCorrelation.StdErr.Bartletts));
        Console.Out.WriteLine();
        new PrintMatrix
            ("Standard Error using Moran are:  ").Print
            (ac.GetStandardErrors(AutoCorrelation.StdErr.Morans));
        Console.Out.WriteLine();
        new PrintMatrix
            ("Partial AutoCovariances:  ").
            Print(ac.GetPartialAutoCorrelations());
        ac.Mean = 50;
        new PrintMatrix
            ("AutoCovariances are:  ").Print
            (ac.GetAutoCovariances());
        Console.Out.WriteLine();
        new PrintMatrix
            ("AutoCorrelations are:  ").
            Print(ac.GetAutoCorrelations());
        Console.Out.WriteLine();
        new PrintMatrix
            ("Standard Error using Bartlett are:  ").Print
            (ac.GetStandardErrors(AutoCorrelation.StdErr.Bartletts));
    }
}
```

Output

```
AutoCovariances are:
          0
 0  1382.908024
 1  1115.02915024
 2   592.00446848
 3    95.29741072
 4  -235.95179904
 5  -370.0108088
 6  -294.25541456
 7   -60.44237232
 8   227.63259792
 9   458.38076816
10   567.8407384
11   546.12202864
12   398.93728688
```

```
13   197.75742912
14    26.89107936
15   -77.2807224
16  -143.73279616
17  -202.04799792
18  -245.37223168
19  -230.81567344
20  -142.8788232


 AutoCorrelations are:
             0
 0   1
 1   0.806293065691258
 2   0.428086653780237
 3   0.0689108813212006
 4  -0.170620023128885
 5  -0.267559955093586
 6  -0.212780177317129
 7  -0.043706718936501
 8   0.164604293249802
 9   0.331461500117813
10   0.410613524938228
11   0.394908424249623
12   0.288477093166393
13   0.143001143740562
14   0.0194453129877855
15  -0.0558827637549379
16  -0.10393518127421
17  -0.146103713633525
18  -0.17743206881559
19  -0.166906019369514
20  -0.103317661565611

Mean = 46.976

Standard Error using Bartlett are:
            0
 0  0.0347838253702384
 1  0.0962419914340011
 2  0.156783378574532
 3  0.205766777086907
 4  0.230955675779118
 5  0.228994712235613
 6  0.208621905639667
 7  0.178475936561125
 8  0.145727084432033
 9  0.134405581638002
10  0.150675803916788
11  0.174348147103935
12  0.190619474429408
13  0.195490061669564
14  0.195892530944597
15  0.196285328179458
16  0.196020624500033
17  0.198716030900604
```

```
18  0.205358590947539
19  0.2093868822353


Standard Error using Moran are:
            0
 0  0.0985184366143778
 1  0.0980196058819607
 2  0.0975182235357506
 3  0.0970142500145332
 4  0.0965076447241154
 5  0.0959983659991659
 6  0.0954863710632231
 7  0.0949716159867634
 8  0.094454055643212
 9  0.0939336436627724
10  0.0934103323839415
11  0.0928840728025648
12  0.0923548145182799
13  0.0918225056781811
14  0.0912870929175277
15  0.090748521297303
16  0.0902067342384192
17  0.0896616734523426
18  0.0891132788679007
19  0.0885614885540095


Partial AutoCovariances:
              0
 0   0.806293065691258
 1  -0.634544877310468
 2   0.0782508772709519
 3  -0.0585660846582815
 4  -0.00094221571933657
 5   0.171719898229681
 6   0.108591873581717
 7   0.11000138764865
 8   0.0785374339029981
 9   0.0791563332964613
10   0.0687065876031485
11  -0.0378019674610775
12   0.0811184838397538
13   0.0334124214991749
14  -0.0348467839607946
15  -0.130648157884444
16  -0.154900984829049
17  -0.119085063160732
18  -0.0161889037437313
19  -0.00385175459253345

AutoCovariances are:
        0
 0  1392.0526
 1  1126.5241
 2   604.1624
```

```
 3   106.7545
 4  -225.882
 5  -361.0259
 6  -286.5701
 7   -53.7603
 8   235.9665
 9   470.7857
10   584.0143
11   564.7639
12   418.3631
13   216.1044
14    43.125
15   -63.4683
16  -131.5012
17  -189.0627
18  -229.6888
19  -212.1559
20  -121.5693


 AutoCorrelations are:
             0
 0   1
 1   0.809253975029392
 2   0.434008312616923
 3   0.0766885532917362
 4  -0.162265420142888
 5  -0.259347886710603
 6  -0.205861545749061
 7  -0.0386194458456527
 8   0.16950975846746
 9   0.338195338308337
10   0.419534649768263
11   0.405705861976767
12   0.300536847530043
13   0.155241547625427
14   0.0309794328174093
15  -0.04559332025241
16  -0.0944656832651295
17  -0.135815773053403
18  -0.165000086922003
19  -0.152405088715757
20  -0.0873309672350025


Standard Error using Bartlett are:
             0
 0  0.0344591054641365
 1  0.0972222809088609
 2  0.15947410033087
 3  0.209799660647689
 4  0.235599778243579
 5  0.233236443705991
 6  0.211657508693781
 7  0.180412936841618
 8  0.14689653606348
```

```
 9  0.133747601649498
10  0.148150190923942
11  0.172282351100035
12  0.190275929042947
13  0.196791614240352
14  0.197983743593071
15  0.198474748794747
16  0.198318159677368
17  0.201022833791806
18  0.207071652966429
19  0.210217650328868
```

AutoCorrelation.StdErr Enumeration

`public enumeration Imsl.Stat.AutoCorrelation.StdErr`

Standard Error computation method.

Fields

Bartletts

`public Imsl.Stat.AutoCorrelation.StdErr Bartletts`

Description

Indicates standard error computation using Bartlett's formula.

Morans

`public Imsl.Stat.AutoCorrelation.StdErr Morans`

Description

Indicates standard error computation using Moran's formula.

CrossCorrelation Class

`public class Imsl.Stat.CrossCorrelation`

Computes the sample cross-correlation function of two stationary time series.

`CrossCorrelation` estimates the cross-correlation function of two jointly stationary time series given a sample of n = `x.Length` observations $\{X_t\}$ and $\{Y_t\}$ for t = 1,2, ..., n.

Let

$$\hat{\mu}_x = \text{xmean}$$

be the estimate of the mean μ_X of the time series $\{X_t\}$ where

$$\hat{\mu}_X = \begin{cases} \mu_X & \text{for } \mu_X \text{ known} \\ \frac{1}{n}\sum_{t=1}^{n} X_t & \text{for } \mu_X \text{ unknown} \end{cases}$$

The autocovariance function of $\{X_t\}$, $\sigma_X(k)$, is estimated by

$$\hat{\sigma}_X(k) = \frac{1}{n}\sum_{t=1}^{n-k}(X_t - \hat{\mu}_X)(X_{t+k} - \hat{\mu}_X), \qquad \text{k=0,1,...,K}$$

where K = `maximumLag`. Note that $\hat{\sigma}_X(0)$ is equivalent to the sample variance of `x` returned by property `VarianceX`. The autocorrelation function $\rho_X(k)$ is estimated by

$$\hat{\rho}_X(k) = \frac{\hat{\sigma}_X(k)}{\hat{\sigma}_X(0)}, \qquad k = 0, 1, \ldots, K$$

Note that $\hat{\rho}_x(0) \equiv 1$ by definition. Let

$$\hat{\mu}_Y = \text{ymean}, \hat{\sigma}_Y(k), \text{and} \hat{\rho}_Y(k)$$

be similarly defined.

The cross-covariance function $\sigma_{XY}(k)$ is estimated by

$$\hat{\sigma}_{XY}(k) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-k}(X_t - \hat{\mu}_X)(Y_{t+k} - \hat{\mu}_Y) & k = 0, 1, \ldots, K \\ \frac{1}{n}\sum_{t=1-k}^{n}(X_t - \hat{\mu}_X)(Y_{t+k} - \hat{\mu}_Y) & k = -1, -2, \ldots, -K \end{cases}$$

The cross-correlation function $\rho_{XY}(k)$ is estimated by

$$\hat{\rho}_{XY}(k) = \frac{\hat{\sigma}_{XY}(k)}{[\hat{\sigma}_X(0)\hat{\sigma}_Y(0)]^{\frac{1}{2}}} \quad k = 0, \pm 1, \ldots, \pm K$$

The standard errors of the sample cross-correlations may be optionally computed according to the `GetStandardErrors` method argument `stderrMethod`. One method is based on a general asymptotic expression for the variance of the sample cross-correlation coefficient of two jointly stationary time series with independent, identically distributed normal errors given by Bartlet (1978, page 352). The theoretical formula is

$$\begin{aligned} \text{var}\{\hat{\rho}_{XY}(k)\} = \frac{1}{n-k}\sum_{i=-\infty}^{\infty} [\, & \rho_X(i) + \rho_{XY}(i-k)\rho_{XY}(i+k) \\ & -2\rho_{XY}(k)\{\rho_X(i)\rho_{XY}(i+k) + \rho_{XY}(-i)\rho_Y(i+k)\} \\ & +\rho_{XY}^2(k)\{\rho_X(i) + \tfrac{1}{2}\rho_X^2(i) + \tfrac{1}{2}\rho_Y^2(i)\}\,] \end{aligned}$$

For computational purposes, the autocorrelations $\rho_X(k)$ and $\rho_Y(k)$ and the cross-correlations $\rho_{XY}(k)$ are replaced by their corresponding estimates for $|k| \leq K$, and the limits of summation are equal to zero for all k such that $|k| > K$.

A second method evaluates Bartlett's formula under the additional assumption that the two series have no cross-correlation. The theoretical formula is

$$\text{var}\{\hat{\rho}_{XY}(k)\} = \frac{1}{n-k} \sum_{i=-\infty}^{\infty} \rho_X(i)\rho_Y(i) \quad k \geq 0$$

For additional special cases of Bartlett's formula, see Box and Jenkins (1976, page 377).

An important property of the cross-covariance coefficient is $\sigma_{XY}(k) = \sigma_{YX}(-k)$ for $k \geq 0$. This result is used in the computation of the standard error of the sample cross-correlation for lag $k < 0$. In general, the cross-covariance function is not symmetric about zero so both positive and negative lags are of interest.

Properties

MeanX

```
public double MeanX {get; set; }
```

Description

Estimate of the mean of time series `x`.

Property Value

A `double` containing the estimate of the mean of time series `x`.

MeanY

```
public double MeanY {get; set; }
```

Description

Estimate of the mean of time series `y`.

Property Value

A `double` containing the estimate mean of the time series `y`.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

VarianceX

```
public double VarianceX {get; }
```

Description

Returns the variance of time series `x`.

Property Value

A `double` containing the variance of the time series `x`.

Exception

`Imsl.Stat.NonPosVarianceXYException` is thrown if the problem is ill-conditioned. The variance is too small to work with.

VarianceY

```
public double VarianceY {get; }
```

Description

Returns the variance of time series `y`.

Property Value

A `double` containing the variance of the time series `y`.

Exception

`Imsl.Stat.NonPosVarianceXYException` is thrown if the problem is ill-conditioned. The variance is too small to work with.

Constructor

CrossCorrelation

```
public CrossCorrelation(double[] x, double[] y, int maximumLag)
```

Description

Constructor to compute the sample cross-correlation function of two stationary time series.

Parameters

`x` – A one-dimensional `double` array containing the first stationary time series.

`y` – A one-dimensional `double` array containing the second stationary time series.

`maximumLag` – An `int` containing the maximum lag of the cross-covariance and cross-correlations to be computed.

Remarks

`maximumLag` must be greater than or equal to 1 and less than the minimum of the number of observations of `x` and `y`.

Methods

GetAutoCorrelationX

```
public double[] GetAutoCorrelationX()
```

Description

Returns the autocorrelations of the time series *x*.

Returns

A `double` array of length `maximumLag` + 1 containing the autocorrelations of the time series `x`.

Remarks

The *0*-th element of this array is 1. The *k*-th element of this array contains the autocorrelation of lag *k* where *k* = 1, ..., `maximumLag`.

Exception

`Imsl.Stat.NonPosVarianceException` is thrown if the problem is ill-conditioned.

GetAutoCorrelationY

```
public double[] GetAutoCorrelationY()
```

Description

Returns the autocorrelations of the time series *y*.

Returns

A `double` array of length `maximumLag` + 1 containing the autocorrelations of the time series `y`.

Remarks

The *0*-th element of this array is 1. The *k*-th element of this array contains the autocorrelation of lag *k* where *k* = 1, ..., `maximumLag`.

Exception

`Imsl.Stat.NonPosVarianceException` is thrown if the problem is ill-conditioned.

GetAutoCovarianceX

```
public double[] GetAutoCovarianceX()
```

Description

Returns the autocovariances of the time series `x`.

Returns

A `double` array of length `maximumLag` + 1 containing the variances and autocovariances of the time series `x`.

Remarks

The *0*-th element of the array contains the variance of the time series `x`. The *k*-th elements contains the autocovariance of lag *k* where *k* = 1, ..., `maximumLag`.

Exception

`Imsl.Stat.NonPosVarianceException` is thrown if the problem is ill-conditioned.

GetAutoCovarianceY

```
public double[] GetAutoCovarianceY()
```

Description

Returns the autocovariances of the time series `y`.

Returns

A `double` array of length `maximumLag` + 1 containing the variances and autocovariances of the time series `y`.

Remarks

The *0*-th element of the array contains the variance of the time series `y`. The *k*-th elements contains the autocovariance of lag *k* where *k* = 1, ..., `maximumLag`.

Exception

`Imsl.Stat.NonPosVarianceException` is thrown if the problem is ill-conditioned.

GetCrossCorrelations

```
public double[] GetCrossCorrelations()
```

Description

Returns the cross-correlations between the time series `x` and `y`.

Returns

A `double` array of length 2 * `maximumLag` + 1 containing the cross-correlations between the time series `x` and `y`.

Remarks

The cross-correlation between `x` and `y` at lag *k*, where *k* = -`maximumLag`,..., 0, 1,...,`maximumLag`, corresponds to output array indices `0, 1,..., (2*maximumLag)`.

Exception

`Imsl.Stat.NonPosVarianceXYException` is thrown if the problem is ill-conditioned. The variance is too small to work with.

GetCrossCovariances

```
public double[] GetCrossCovariances()
```

Description

Returns the cross-covariances between the time series `x` and `y`.

Returns

A `double` array of length 2 * `maximumLag` + 1 containing the cross-covariances between the time series `x` and `y`.

Remarks

The cross-covariance between `x` and `y` at lag *k*, where *k* = -`maximumLag`,..., 0, 1,...,`maximumLag`, corresponds to output array indices `0, 1,..., (2*maximumLag)`.

GetStandardErrors

```
public double[] GetStandardErrors(Imsl.Stat.CrossCorrelation.StdErr
stderrMethod)
```

Description

Returns the standard errors of the cross-correlations between the time series `x` and `y`.

Parameter

`stderrMethod` – An `int` specifying the method to compute the standard errors of cross-correlations between the time series `x` and `y`.

Returns

A `double` array of length 2 * `maximumLag` + 1 containing the standard errors of the cross-correlations between the time series `x` and `y`.

Remarks

The standard error of cross-correlations between `x` and `y` at lag *k*, where *k* = -`maximumLag`,..., 0, 1,..., `maximumLag`, corresponds to output array indices 0, 1,..., `(2*maximumLag)`.

Method of computation for standard errors of the cross-correlation is determined by the `stderrMethod` parameter. If `stderrMethod` is set to `Bartletts`, Bartlett's formula is used to compute the standard errors of cross-correlations. If `stderrMethod` is set to `BartlettsNoCC`, Bartlett's formula is used to compute the standard errors of cross-correlations, with the assumption of no cross-correlation.

Exception

`Imsl.Stat.NonPosVarianceException` is thrown if the problem is ill-conditioned.

Example 1: CrossCorrelation

Consider the Gas Furnace Data (Box and Jenkins 1976, pages 532-533) where *X* is the input gas reate in cubic feet/minute and *Y* is the percent CO_2 in the outlet gas. The `CrossCorrelation` methods `GetCrossCovariance` and `GetCrossCorrelation` are used to compute the cross-covariances and cross-correlations between time series *X* and *Y* with lags from −`maximumLag` = -10 through lag `maximumLag` = 10. In addition, the estimated standard errors of the estimated cross-correlations are computed. In the first invocation of method `GetStandardErrors` stderrMethod = `Bartletts`, the standard errors are based on the assumption that autocorrelations and cross-correlations for lags greater than `maximumLag` or less than −`maximumLag` are zero. In the second invocation of method `GetStandardErrors` with `stderrMethod` = `BartlettsNoCC`, the standard errors are based on the additional assumption that all cross-correlations for `X` and `Y` are zero.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class CrossCorrelationEx1
{
    public static void  Main(String[] args)
    {
        double[] x2 = new double[]{100.8, 81.6, 66.5, 34.8, 30.6,
                                   7, 19.8, 92.5, 154.4, 125.9,
                                   84.8, 68.1, 38.5, 22.8, 10.2,
                                   24.1, 82.9, 132, 130.9, 118.1,
                                   89.9, 66.6, 60, 46.9, 41,
                                   21.3, 16, 6.4, 4.1, 6.8,
                                   14.5, 34, 45, 43.1, 47.5,
                                   42.2, 28.1, 10.1, 8.1, 2.5,
                                   0, 1.4, 5, 12.2, 13.9,
                                   35.4, 45.8, 41.1, 30.4, 23.9,
                                   15.7, 6.6, 4, 1.8, 8.5,
                                   16.6, 36.3, 49.7, 62.5, 67,
                                   71, 47.8, 27.5, 8.5, 13.2,
                                   56.9, 121.5,138.3, 103.2, 85.8,
                                   63.2, 36.8, 24.2, 10.7, 15,
                                   40.1, 61.5, 98.5, 124.3, 95.9,
                                   66.5, 64.5, 54.2, 39, 20.6,
                                   6.7, 4.3, 22.8, 54.8, 93.8,
                                   95.7, 77.2, 59.1, 44, 47,
                                   30.5, 16.3, 7.3, 37.3, 73.9};
        double[] x = new double[]{- 0.109, 0.0, 0.178, 0.339, 0.373,
                                  0.441, 0.461, 0.348, 0.127,
                                  - 0.18, - 0.588, - 1.055, - 1.421,
                                  - 1.52, - 1.302, - 0.814, - 0.475,
                                  - 0.193, 0.088, 0.435, 0.771,
                                  0.866, 0.875, 0.891, 0.987, 1.263,
                                  1.775, 1.976, 1.934, 1.866, 1.832,
                                  1.767, 1.608, 1.265, 0.79, 0.36,
                                  0.115, 0.088, 0.331, 0.645, 0.96,
                                  1.409, 2.67, 2.834, 2.812, 2.483,
                                  1.929, 1.485, 1.214, 1.239, 1.608,
                                  1.905, 2.023, 1.815, 0.535, 0.122,
                                  0.009, 0.164, 0.671, 1.019, 1.146,
                                  1.155, 1.112, 1.121, 1.223, 1.257,
                                  1.157, 0.913, 0.62, 0.255, - 0.28,
                                  - 1.08, - 1.551, - 1.799, - 1.825,
                                  - 1.456, - 0.944, - 0.57, - 0.431,
                                  - 0.577, - 0.96, - 1.616, - 1.875,
                                  - 1.891, - 1.746, - 1.474, - 1.201,
                                  - 0.927, - 0.524, 0.04, 0.788,
                                  0.943, 0.93, 1.006, 1.137, 1.198,
                                  1.054, 0.595, - 0.08, - 0.314,
                                  - 0.288, - 0.153, - 0.109, - 0.187,
                                  - 0.255, - 0.229, - 0.007, 0.254,
                                  0.33, 0.102, - 0.423, - 1.139,
                                  - 2.275, - 2.594, - 2.716, - 2.51,
                                  - 1.79, - 1.346, - 1.081, - 0.91,
                                  - 0.876, - 0.885, - 0.8, - 0.544,
```

```
                                - 0.416, - 0.271, 0.0, 0.403,
                                0.841,    1.285, 1.607, 1.746, 1.683,
                                1.485, 0.993, 0.648, 0.577, 0.577,
                                0.632, 0.747, 0.9, 0.993, 0.968,
                                0.79, 0.399, - 0.161, - 0.553,
                                - 0.603, - 0.424, - 0.194, - 0.049,
                                0.06, 0.161, 0.301, 0.517, 0.566,
                                0.56, 0.573, 0.592, 0.671, 0.933,
                                1.337, 1.46, 1.353, 0.772, 0.218,
                                - 0.237, - 0.714, - 1.099, -1.269,
                                - 1.175, - 0.676, 0.033, 0.556,
                                0.643, 0.484, 0.109, - 0.31, -0.697,
                                - 1.047, - 1.218, - 1.183, -0.873,
                                -0.336, 0.063, 0.084, 0.0, 0.001,
                                0.209, 0.556, 0.782, 0.858, 0.918,
                                0.862, 0.416, - 0.336, - 0.959,
                                - 1.813, - 2.378, - 2.499, -2.473,
                                - 2.33, - 2.053, - 1.739, - 1.261,
                                - 0.569, - 0.137, - 0.024, - 0.05,
                                - 0.135, - 0.276, - 0.534, -0.871,
                                - 1.243, - 1.439, - 1.422, -1.175,
                                - 0.813, - 0.634, - 0.582, -0.625,
                                - 0.713, - 0.848, - 1.039, -1.346,
                                - 1.628, - 1.619, - 1.149, -0.488,
                                - 0.16, - 0.007, - 0.092, - 0.62,
                                - 1.086, - 1.525, - 1.858, -2.029,
                                - 2.024, - 1.961, - 1.952, -1.794,
                                - 1.302, - 1.03, - 0.918, - 0.798,
                                - 0.867, - 1.047,- 1.123, - 0.876,
                                - 0.395, 0.185, 0.662, 0.709,
                                0.605, 0.501, 0.603, 0.943, 1.223,
                                1.249, 0.824, 0.102, 0.025, 0.382,
                                0.922, 1.032, 0.866, 0.527, 0.093,
                                - 0.458, - 0.748, - 0.947, -1.029,
                                - 0.928, - 0.645, - 0.424, -0.276,
                                - 0.158, - 0.033, 0.102, 0.251,
                                0.28, 0.0, -0.493, -0.759, -0.824,
                                - 0.74, - 0.528, - 0.204, 0.034,
                                0.204, 0.253, 0.195, 0.131, 0.017,
                                - 0.182, - 0.262};
double[] y = new double[]{53.8, 53.6, 53.5, 53.5, 53.4, 53.1,
                                52.7, 52.4, 52.2, 52.0, 52.0,
                                52.4, 53.0, 54.0, 54.9, 56.0,
                                56.8, 56.8, 56.4, 55.7, 55.0,
                                54.3, 53.2, 52.3, 51.6, 51.2,
                                50.8, 50.5, 50.0, 49.2, 48.4,
                                47.9, 47.6, 47.5, 47.5, 47.6,
                                48.1, 49.0, 50.0, 51.1, 51.8,
                                51.9, 51.7, 51.2, 50.0, 48.3,
                                47.0, 45.8, 45.6, 46.0, 46.9,
                                47.8, 48.2, 48.3, 47.9, 47.2,
                                47.2, 48.1, 49.4, 50.6, 51.5,
                                51.6, 51.2, 50.5, 50.1, 49.8,
                                49.6, 49.4, 49.3, 49.2, 49.3,
                                49.7, 50.3, 51.3, 52.8, 54.4,
                                56.0, 56.9, 57.5, 57.3, 56.6,
```

```
                                56.0, 55.4, 55.4, 56.4, 57.2,
                                58.0, 58.4, 58.4, 58.1, 57.7,
                                57.0, 56.0, 54.7, 53.2, 52.1,
                                51.6, 51.0, 50.5, 50.4, 51.0,
                                51.8, 52.4, 53.0, 53.4, 53.6,
                                53.7, 53.8, 53.8, 53.8, 53.3,
                                53.0, 52.9, 53.4, 54.6, 56.4,
                                58.0, 59.4, 60.2, 60.0, 59.4,
                                58.4, 57.6, 56.9, 56.4, 56.0,
                                55.7, 55.3, 55.0, 54.4, 53.7,
                                52.8, 51.6, 50.6, 49.4, 48.8,
                                48.5, 48.7, 49.2, 49.8, 50.4,
                                50.7, 50.9, 50.7, 50.5, 50.4,
                                50.2, 50.4, 51.2, 52.3, 53.2,
                                53.9, 54.1, 54.0, 53.6, 53.2,
                                53.0, 52.8, 52.3, 51.9, 51.6,
                                51.6, 51.4, 51.2, 50.7, 50.0,
                                49.4, 49.3, 49.7, 50.6, 51.8,
                                53.0, 54.0, 55.3, 55.9, 55.9,
                                54.6, 53.5, 52.4, 52.1, 52.3,
                                53.0, 53.8, 54.6, 55.4, 55.9,
                                55.9, 55.2, 54.4, 53.7, 53.6,
                                53.6, 53.2, 52.5, 52.0, 51.4,
                                51.0, 50.9, 52.4, 53.5, 55.6,
                                58.0, 59.5, 60.0, 60.4, 60.5,
                                60.2, 59.7, 59.0, 57.6, 56.4,
                                55.2, 54.5, 54.1, 54.1, 54.4,
                                55.5, 56.2, 57.0, 57.3, 57.4,
                                57.0, 56.4, 55.9, 55.5, 55.3,
                                55.2, 55.4, 56.0, 56.5, 57.1,
                                57.3, 56.8, 55.6, 55.0, 54.1,
                                54.3, 55.3, 56.4, 57.2, 57.8,
                                58.3, 58.6, 58.8, 58.8, 58.6,
                                58.0, 57.4, 57.0, 56.4, 56.3,
                                56.4, 56.4, 56.0, 55.2, 54.0,
                                53.0, 52.0, 51.6, 51.6, 51.1,
                                50.4, 50.0, 50.0, 52.0, 54.0,
                                55.1, 54.5, 52.8, 51.4, 50.8,
                                51.2, 52.0, 52.8, 53.8, 54.5,
                                54.9, 54.9, 54.8, 54.4, 53.7,
                                53.3, 52.8, 52.6, 52.6, 53.0,
                                54.3, 56.0, 57.0, 58.0, 58.6,
                                58.5, 58.3, 57.8, 57.3, 57.0};

CrossCorrelation cc = new CrossCorrelation(x, y, 10);
Console.Out.WriteLine("Mean = " + cc.MeanX);
Console.Out.WriteLine("Mean = " + cc.MeanY);
Console.Out.WriteLine("Xvariance  = " + cc.VarianceX);
Console.Out.WriteLine("Yvariance  = " + cc.VarianceY);
new PrintMatrix
    ("CrossCovariances are:  ").Print(cc.GetCrossCovariances());
new PrintMatrix
    ("CrossCorrelations are:  ").Print(cc.GetCrossCorrelations());

double[] stdErrors =
    cc.GetStandardErrors(CrossCorrelation.StdErr.Bartletts);
```

```
        new PrintMatrix
            ("Standard Errors using Bartlett are:  ").Print(stdErrors);

        stdErrors =
            cc.GetStandardErrors(CrossCorrelation.StdErr.BartlettsNoCC);
        new PrintMatrix("Standard Errors using Bartlett #2 are:  ").Print
            (stdErrors);

        new PrintMatrix("AutoCovariances of X are:  ").Print
            (cc.GetAutoCovarianceX());
        new PrintMatrix("AutoCovariances of Y are:  ").Print
            (cc.GetAutoCovarianceY());
        new PrintMatrix("AutoCorrelations of X are:  ").Print
            (cc.GetAutoCorrelationX());
        new PrintMatrix("AutoCorrelations of Y are:  ").Print
            (cc.GetAutoCorrelationY());
    }
}
```

Output

```
Mean = -0.0568344594594595
Mean = 53.5091216216216
Xvariance  = 1.14693790165038
Yvariance  = 10.2189370662893
CrossCovariances are:
            0
 0  -0.404501563294314
 1  -0.508490782763824
 2  -0.614369467627782
 3  -0.705476130258359
 4  -0.776166564117932
 5  -0.831473609098764
 6  -0.891315326970392
 7  -0.980605209560792
 8  -1.12477059434257
 9  -1.34704305203341
10  -1.65852650999817
11  -2.04865124574232
12  -2.48216585776478
13  -2.88541054192018
14  -3.16536049680239
15  -3.25343758942199
16  -3.13112860301494
17  -2.83919398544463
18  -2.45302186901565
19  -2.05268794195849
20  -1.6946546517713

CrossCorrelations are:
            0
 0  -0.118153717307789
 1  -0.148528662561878
 2  -0.179455515102209
 3  -0.206067503381416
```

```
 4  -0.226715971265165
 5  -0.242870996488244
 6  -0.260350586329711
 7  -0.286431898500946
 8  -0.3285421835153
 9  -0.39346731487308
10  -0.484450717109386
11  -0.598405005361053
12  -0.725033348897091
13  -0.842819935503927
14  -0.924592494205792
15  -0.950319553992448
16  -0.914593458680361
17  -0.829320215245049
18  -0.716520475473708
19  -0.599584112456951
20  -0.495003641096017

Standard Errors using Bartlett are:
            0
 0  0.158147783754555
 1  0.155750271182418
 2  0.152735096430409
 3  0.149086745416716
 4  0.145054998300008
 5  0.141300099196058
 6  0.138420534019813
 7  0.136074039397204
 8  0.132158917844376
 9  0.123531347020305
10  0.107879045104545
11  0.0873410658167485
12  0.0641407975847026
13  0.0469456102701398
14  0.0440970262220149
15  0.0482335854893665
16  0.0491545707033738
17  0.0475621871011123
18  0.0534780426550682
19  0.0715660938138719
20  0.0939330263600716

Standard Errors using Bartlett #2 are:
            0
 0  0.162753654681801
 1  0.162469864309526
 2  0.162187553298139
 3  0.161906708839297
 4  0.161627318279375
 5  0.161349369117073
 6  0.16107284900106
 7  0.160797745727675
 8  0.160524047238664
 9  0.160251741618955
10  0.159980817094486
11  0.160251741618955
```

```
12  0.160524047238664
13  0.160797745727675
14  0.16107284900106
15  0.161349369117073
16  0.161627318279375
17  0.161906708839297
18  0.162187553298139
19  0.162469864309526
20  0.162753654681801

AutoCovariances of X are:
            0
 0  1.14693790165038
 1  1.09242958215267
 2  0.956651878489968
 3  0.782050821478561
 4  0.609290776371371
 5  0.467379623909361
 6  0.36495658921123
 7  0.298426970727032
 8  0.260942845999682
 9  0.244377603086156
10  0.238942463361545

AutoCovariances of Y are:
            0
 0  10.2189370662893
 1   9.92010118439122
 2   9.15657243817617
 3   8.09900196442277
 4   6.94850770962479
 5   5.87055032023953
 6   4.96076244327211
 7   4.25188969596136
 8   3.73611877936647
 9   3.37615547781905
10   3.13231605775447

AutoCorrelations of X are:
            0
 0  1
 1  0.952474916541448
 2  0.834092130980584
 3  0.681859776674248
 4  0.531232576318765
 5  0.407502117801518
 6  0.31820082733867
 7  0.26019453215175
 8  0.227512619143721
 9  0.213069602752258
10  0.208330776250152

AutoCorrelations of Y are:
            0
 0  1
 1  0.970756656983059
```

```
 2  0.896039615351222
 3  0.79254837483442
 4  0.67996384208558
 5  0.574477588242088
 6  0.485447988483746
 7  0.416079448222429
 8  0.365607377277169
 9  0.330382255602345
10  0.306520730819207
```

CrossCorrelation.StdErr Enumeration

`public enumeration Imsl.Stat.CrossCorrelation.StdErr`

Standard Error computation method.

Fields

Bartletts

`public Imsl.Stat.CrossCorrelation.StdErr Bartletts`

Description

Indicates standard error computation using Bartlett's formula.

BartlettsNoCC

`public Imsl.Stat.CrossCorrelation.StdErr BartlettsNoCC`

Description

Indicates standard error computation using Bartlett's formula with the assumption of no cross-correlation.

MultiCrossCorrelation Class

`public class Imsl.Stat.MultiCrossCorrelation`

Computes the multichannel cross-correlation function of two mutually stationary multichannel time series.

`MultiCrossCorrelation` estimates the multichannel cross-correlation function of two mutually stationary multichannel time series. Define the multichannel time series X by

$$X = (X_1, X_2, \ldots, X_p)$$

where

$$X_j = (X_{1j}, X_{2j}, \ldots, X_{nj})^T, \quad j = 1, 2, \ldots, p$$

with n = `x.GetLength(0)` and p = `x.GetLength(1)`. Similarly, define the multichannel time series Y by

$$Y = (Y_1, Y_2, \ldots, Y_q)$$

where

$$Y_j = (Y_{1j}, Y_{2j}, \ldots, Y_{mj})^T, \quad j = 1, 2, \ldots, q$$

with m = `y.GetLength(0)` and q = `y.GetLength(1)`. The columns of X and Y correspond to individual channels of multichannel time series and may be examined from a univariate perspective. The rows of X and Y correspond to observations of p-variate and q-variate time series, respectively, and may be examined from a multivariate perspective. Note that an alternative characterization of a multivariate time series X considers the columns to be observations of the multivariate time series while the rows contain univariate time series. For example, see Priestley (1981, page 692) and Fuller (1976, page 14).

Let $\hat{\mu}_X$ = `xmean` be the row vector containing the means of the channels of X. In particular,

$$\hat{\mu}_X = (\hat{\mu}_{X_1}, \hat{\mu}_{X_2}, \ldots, \hat{\mu}_{X_p})$$

where for j = 1, 2, ..., p

$$\hat{\mu}_{X_j} = \begin{cases} \mu_{X_j} & \text{for } \mu_{X_j} \text{ known} \\ \frac{1}{n} \sum_{t=1}^{n} X_{tj} & \text{for } \mu_{X_j} \text{ unknown} \end{cases}$$

Let $\hat{\mu}_Y$ = `ymean` be similarly defined. The cross-covariance of lag k between channel i of X and channel j of Y is estimated by

$$\hat{\sigma}_{X_i Y_j}(k) = \begin{cases} \frac{1}{N} \sum_t (X_{ti} - \hat{\mu}_{X_i})(Y_{t+k,j} - \hat{\mu}_{Y_j}) & k = 0, 1, \ldots, K \\ \frac{1}{N} \sum_t (X_{ti} - \hat{\mu}_{X_i})(Y_{t+k,j} - \hat{\mu}_{Y_j}) & k = -1, -2, \ldots, -K \end{cases}$$

where i = 1, ..., p, j = 1, ..., q, and K = `maximumLag`. The summation on t extends over all possible cross-products with N equal to the number of cross-products in the sum.

Let $\hat{\sigma}_X(0)$ = `xvar`, where `xvar` is the variance of X, be the row vector consisting of estimated variances of the channels of X. In particular,

$$\hat{\sigma}_X(0) = (\hat{\sigma}_{X_1}(0), \hat{\sigma}_{X_2}(0), \ldots, \hat{\sigma}_{X_p}(0))$$

where

$$\hat{\sigma}_{X_j}(0) = \frac{1}{n} \sum_{t=1}^{n} \left(X_{tj} - \hat{\mu}_{X_j}\right)^2, \qquad \text{j=0,1,...,p}$$

Let $\hat{\sigma}_Y(0)$ = `yvar`, where `yvar` is the variance of Y, be similarly defined. The cross-correlation of lag k between channel i of X and channel j of Y is estimated by

$$\hat{\rho}_{X_j Y_j}(k) = \frac{\hat{\sigma}_{X_j Y_j(k)}}{[\hat{\sigma}_{X_i}(0)\hat{\sigma}_{X_j}(0)]^{\frac{1}{2}}} \quad k = 0, \pm 1, \ldots, \pm K$$

Property

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Constructor

MultiCrossCorrelation

```
public MultiCrossCorrelation(double[,] x, double[,] y, int maximumLag)
```

Description

Constructor to compute the multichannel cross-correlation function of two mutually stationary mulitchannel time series.

Parameters

`x` – A two-dimensional `double` array containing the first multichannel stationary time series. Each row of `x` corresponds to an observation of a multivariate time series and each column of `x` corresponds to a univariate time series.

`y` – A two-dimensional `double` array containing the second multichannel stationary time series. Each row of `y` corresponds to an observation of a multivariate time series and each column of `y` corresponds to a univariate time series.

`maximumLag` – A `int` containing the maximum lag of the cross-covariance and cross-correlations to be computed. `maximumLag` must be greater than or equal to 1 and less than the minimum number of observations of `x` and `y`.

Methods

GetCrossCorrelation

```
public double[,,] GetCrossCorrelation()
```

Description

Returns the cross-correlations between the channels of `x` and `y`.

Returns

A `double` array of size 2 * `maximumLag` +1 by `x.GetLength(1)` by `y.GetLength(1)` containing the cross-correlations between the time series `x` and `y`.

Remarks

The cross-correlation between channel *i* of the `x` series and channel *j* of the `y` series at lag *k*, where *k* = -`maximumLag`, ..., 0, 1, ..., `maximumLag`, corresponds to output array, CC[k,i,j] where *k*= 0, 1, ..., `(2*maximumLag)`, *i* = 1, ..., `x.GetLength(1)`, and *j* = 1, ..., `y.GetLength(1)`.

Exception

`Imsl.Stat.NonPosVarianceXYException` is thrown if the problem is ill-conditioned. The variance is too small to work with.

GetCrossCovariance

```
public double[,] GetCrossCovariance()
```

Description

Returns the cross-covariances between the channels of `x` and `y`.

Returns

A `double` array of size 2 * `maximumLag` +1 by `x.GetLength(1)` by `y.GetLength(1)` containing the cross-covariances between the time series `x` and `y`.

Remarks

The cross-covariances between channel *i* of the `x` series and channel *j* of the `y` series at lag *k* where *k* = -`maximumLag`, ..., 0, 1, ..., `maximumLag`, corresponds to output array, CCV[k,i,j] where *k*= 0, 1, ..., `(2*maximumLag)`, *i* = 1, ..., `x.GetLength(1)`, and *j* = 1, ..., `y.GetLength(1)`.

Exception

`Imsl.Stat.NonPosVarianceXYException` is thrown if the problem is ill-conditioned. The variance is too small to work with.

GetMeanX

```
public double[] GetMeanX()
```

Description

Returns an estimate of the mean of each channel of *x*.

Returns

A one-dimensional `double` containing the estimate of the mean of each channel in time series *x*.

GetMeanY

```
public double[] GetMeanY()
```

Description

Returns an estimate of the mean of each channel of *y*.

Returns

A one-dimensional `double` containing the estimate of the mean of each channel in the time series *y*.

GetVarianceX

```
public double[] GetVarianceX()
```

Description

Returns the variances of the channels of `x`.

Returns

A one-dimensional `double` containing the variances of each channel in the time series `x`.

Exception

`Imsl.Stat.NonPosVarianceXYException` is thrown if the problem is ill-conditioned. The variance is too small to work with.

GetVarianceY

```
public double[] GetVarianceY()
```

Description

Returns the variances of the channels of `y`.

Returns

A one-dimensional `double` containing the variances of each channel in the time series `y`.

Exception

`Imsl.Stat.NonPosVarianceXYException` is thrown if the problem is ill-conditioned. The variance is too small to work with.

Example 1: MultiCrossCorrelation

Consider the Wolfer Sunspot Data (Y) (Box and Jenkins 1976, page 530) along with data on northern light activity (X1) and earthquake activity (X2) (Robinson 1967, page 204) to be a three-channel time series. Methods `GetCrossCovariance` and `GetCrossCorrelation` are used to compute the cross-covariances and cross-correlations between X1 and Y and between X2 and Y with lags from -`maximumLag` = -10 through lag `maximumLag` = 10.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;
using Matrix = Imsl.Math.Matrix;

public class MultiCrossCorrelationEx1
{
    public static void  Main(String[] args)
    {
        int i;
        double[,] x = {
            {155.0, 66.0}, {113.0, 62.0},
            {3.0, 66.0}, {10.0, 197.0},
```

```
    {0.0, 63.0}, {0.0, 0.0},
    {12.0, 121.0}, {86.0, 0.0},
    {102.0, 113.0}, {20.0, 27.0},
     {98.0, 107.0}, {116.0, 50.0},
    {87.0, 122.0}, {131.0, 127.0},
    {168.0, 152.0}, {173.0, 216.0},
    {238.0, 171.0}, {146.0, 70.0},
    {0.0, 141.0}, {0.0, 69.0},
    {0.0, 160.0}, {0.0, 92.0},
    {12.0, 70.0}, {0.0, 46.0},
    {37.0, 96.0}, {14.0, 78.0},
    {11.0, 110.0}, {28.0, 79.0},
    {19.0, 85.0}, {30.0, 113.0},
    {11.0, 59.0}, {26.0, 86.0},
    {0.0, 199.0}, {29.0, 53.0},
    {47.0, 81.0}, {36.0, 81.0},
    {35.0, 156.0}, {17.0, 27.0},
    {0.0, 81.0}, {3.0, 107.0},
    {6.0, 152.0}, {18.0, 99.0},
    {15.0, 177.0},{0.0, 48.0},
    {3.0, 70.0}, {9.0, 158.0},
    {64.0, 22.0}, {126.0, 43.0},
    {38.0, 102.0}, {33.0, 111.0},
    {71.0, 90.0}, {24.0, 86.0},
    {20.0, 119.0}, {22.0, 82.0},
    {13.0, 79.0}, {35.0, 111.0},
    {84.0, 60.0}, {119.0, 118.0},
    {86.0, 206.0}, {71.0, 122.0},
    {115.0, 134.0}, {91.0, 131.0},
    {43.0, 84.0}, {67.0, 100.0},
    {60.0, 99.0}, {49.0, 99.0},
    {100.0, 69.0}, {150.0, 67.0},
    {178.0, 26.0}, {187.0, 106.0},
    {76.0, 108.0}, {75.0, 155.0},
    {100.0, 40.0}, {68.0, 75.0},
    {93.0, 99.0}, {20.0, 86.0},
    {51.0, 127.0}, {72.0, 201.0},
    {118.0, 76.0}, {146.0, 64.0},
    {101.0, 31.0}, {61.0, 138.0},
    {87.0, 163.0}, {53.0, 98.0},
    {69.0, 70.0}, {46.0, 155.0},
    {47.0, 97.0}, {35.0, 82.0},
    {74.0, 90.0}, {104.0, 122.0},
    {97.0, 70.0}, {106.0, 96.0},
    {113.0, 111.0}, {103.0, 42.0},
    {68.0, 97.0}, {67.0, 91.0},
    {82.0, 64.0}, {89.0, 81.0},
    {102.0, 162.0}, {110.0, 137.0}};

double[,] y = {{101.0}, {82.0},
                    {66.0}, {35.0},
                    {31.0}, {7.0},
                    {20.0}, {92.0},
                    {154.0}, {126.0},
                    {85.0}, {68.0},
                    {38.0}, {23.0},
```

```
                    {10.0}, {24.0},
                    {83.0}, {132.0},
                    {131.0}, {118.0},
                    {90.0}, {67.0},
                    {60.0}, {47.0},
                    {41.0}, {21.0},
                    {16.0}, {6.0},
                    {4.0}, {7.0},
                    {14.0}, {34.0},
                    {45.0}, {43.0},
                    {48.0}, {42.0},
                    {28.0}, {10.0},
                    {8.0}, {2.0},
                    {0.0}, {1.0},
                    {5.0}, {12.0},
                    {14.0}, {35.0},
                    {46.0}, {41.0},
                    {30.0}, {24.0},
                    {16.0}, {7.0},
                    {4.0}, {2.0},
                    {8.0}, {17.0},
                    {36.0}, {50.0},
                    {62.0}, {67.0},
                    {71.0}, {48.0},
                    {28.0}, {8.0},
                    {13.0}, {57.0},
                    {122.0}, {138.0},
                    {103.0}, {86.0},
                    {63.0}, {37.0},
                    {24.0}, {11.0},
                    {15.0}, {40.0},
                    {62.0}, {98.0},
                    {124.0}, {96.0},
                    {66.0}, {64.0},
                    {54.0}, {39.0},
                    {21.0}, {7.0},
                    {4.0}, {23.0},
                    {55.0}, {94.0},
                    {96.0}, {77.0},
                    {59.0}, {44.0},
                    {47.0}, {30.0},
                    {16.0}, {7.0},
                    {37.0}, {74.0}};

MultiCrossCorrelation mcc =
    new MultiCrossCorrelation(x, y, 10);

new PrintMatrix("Mean of X :  ").Print(mcc.GetMeanX());
new PrintMatrix("Variance of X :  ").Print(mcc.GetVarianceX());
new PrintMatrix("Mean of Y :  ").Print(mcc.GetMeanY());
new PrintMatrix("Variance of Y :  ").Print(mcc.GetVarianceY());

double[,] tmpArr = new double[x.GetLength(1), y.GetLength(1)];
double[,,] ccv = mcc.GetCrossCovariance();
Console.Out.WriteLine
    ("Multichannel cross-covariance between X and Y");
```

```
        for (i = 0; i < 21; i++)
        {
            for (int j=0;j<x.GetLength(1);j++)
                for (int k=0;k<y.GetLength(1);k++)
                    tmpArr[j,k] = ccv[i,j,k];
            Console.Out.WriteLine("Lag K = " + (i - 10));
            new PrintMatrix("CrossCovariances : ").Print(tmpArr);
        }

        double[,,] cc = mcc.GetCrossCorrelation();
        Console.Out.WriteLine
            ("Multichannel cross-correlation between X and Y");
        for (i = 0; i < 21; i++)
        {
            for (int j=0;j<x.GetLength(1);j++)
                for (int k=0;k<y.GetLength(1);k++)
                    tmpArr[j,k] = cc[i,j,k];
            Console.Out.WriteLine("Lag K = " + (i - 10));
            new PrintMatrix("CrossCorrelations : ").Print(tmpArr);
        }
    }
}
```

Output

```
Mean of X :
     0
0  63.43
1  97.97

Variance of X :
       0
0  2643.6851
1  1978.4291

Mean of Y :
     0
0  46.94

Variance of Y :
       0
0  1383.7564

Multichannel cross-covariance between X and Y
Lag K = -10
 CrossCovariances :
           0
0  -20.5123555555557
1   70.7132444444444

Lag K = -9
 CrossCovariances :
          0
0  65.0243098901099
1  38.1363054945055
```

```
Lag K = -8
 CrossCovariances :
          0
0  216.637243478261
1  135.57832173913

Lag K = -7
 CrossCovariances :
          0
0  246.793769892473
1  100.362230107527

Lag K = -6
 CrossCovariances :
           0
0  142.127923404255
1   44.9678638297872

Lag K = -5
 CrossCovariances :
           0
0   50.6970421052632
1  -11.8094631578948

Lag K = -4
 CrossCovariances :
           0
0  72.6846166666667
1  32.6926333333334

Lag K = -3
 CrossCovariances :
           0
0  217.854096907217
1  -40.1185092783505

Lag K = -2
 CrossCovariances :
           0
0   355.820628571429
1  -152.649118367347

Lag K = -1
 CrossCovariances :
           0
0   579.653492929293
1  -212.95022020202

Lag K = 0
CrossCovariances :
       0
0   821.6258
1  -104.7518

Lag K = 1
```

```
 CrossCovariances :
           0
0  810.131371717171
1   55.1601838383839

Lag K = 2
 CrossCovariances :
           0
0  628.385118367347
1   84.7751673469388

Lag K = 3
 CrossCovariances :
           0
0  438.271931958763
1   75.9630371134021

Lag K = 4
 CrossCovariances :
           0
0  238.792741666667
1  200.383466666667

Lag K = 5
 CrossCovariances :
           0
0  143.621147368421
1  282.986431578947

Lag K = 6
 CrossCovariances :
           0
0  252.973774468085
1  234.393289361702

Lag K = 7
 CrossCovariances :
           0
0  479.468286021505
1  223.033735483871

Lag K = 8
 CrossCovariances :
           0
0  724.912243478261
1  124.456582608696

Lag K = 9
 CrossCovariances :
           0
0  924.971232967034
1  -79.5174307692309

Lag K = 10
 CrossCovariances :
           0
```

```
0   922.759311111112
1  -279.286422222222

Multichannel cross-correlation between X and Y
Lag K = -10
  CrossCorrelations :
             0
0  -0.0107245938219684
1   0.0427376557935899

Lag K = -9
 CrossCorrelations :
           0
0  0.0339970370656115
1  0.023048812287829

Lag K = -8
 CrossCorrelations :
           0
0  0.113265706453004
1  0.0819407975561327

Lag K = -7
 CrossCorrelations :
           0
0  0.129032618058936
1  0.0606569035081169

Lag K = -6
 CrossCorrelations :
           0
0  0.074309566502109
1  0.0271776680765982

Lag K = -5
  CrossCorrelations :
             0
0   0.0265062285548632
1  -0.00713740085770933

Lag K = -4
 CrossCorrelations :
           0
0  0.0380021196855836
1  0.0197587668528454

Lag K = -3
  CrossCorrelations :
             0
0   0.11390192098873
1  -0.0242468161934945

Lag K = -2
  CrossCorrelations :
             0
0   0.186035762912295
```

```
1  -0.0922580420292281

Lag K = -1
 CrossCorrelations :
           0
0   0.303063597562697
1  -0.128702809263875

Lag K = 0
  CrossCorrelations :
            0
0   0.429575382251174
1  -0.0633098708358119

Lag K = 1
 CrossCorrelations :
           0
0  0.423565683647071
1  0.0333377002981115

Lag K = 2
 CrossCorrelations :
           0
0  0.328542235922487
1  0.051236397797642

Lag K = 3
 CrossCorrelations :
           0
0  0.22914425606054
1  0.0459105243818767

Lag K = 4
 CrossCorrelations :
           0
0  0.124849394067548
1  0.121107717407232

Lag K = 5
 CrossCorrelations :
           0
0  0.075090277447643
1  0.171031279954621

Lag K = 6
 CrossCorrelations :
           0
0  0.132263745693782
1  0.141662566889261

Lag K = 7
 CrossCorrelations :
           0
0  0.250683184784367
1  0.134797082107539
```

```
Lag K = 8
 CrossCorrelations :
           0
0  0.37901007257894
1  0.0752190432013873

Lag K = 9
  CrossCorrelations :
            0
0   0.48360807434863
1  -0.0480587280714567

Lag K = 10
 CrossCorrelations :
           0
0   0.48245160241607
1  -0.168795069078383
```

LackOfFit Class

`public class Imsl.Stat.LackOfFit`

Performs lack-of-fit test for a univariate time series or transfer function given the appropriate correlation function.

`LackOfFit` may be used to diagnose lack of fit in both ARMA and transfer function models. Typical arguments for these situations are:

Model	`lagMin`	`lagMax`	`npFree`
ARMA (*p*, *q*)	1	$\sqrt{\texttt{nObservations}}$	*p* + *q*
Transfer function	0	$\sqrt{\texttt{nObservations}}$	*r* + *s*

`LackOfFit` performs a portmanteau lack of fit test for a time series or transfer function containing `nObservations` observations given the appropriate sample correlation function $\hat{\rho}(k)$ for $k = L, L+1,...,K$ where L = `lagMin` and K = `lagMax`.

The basic form of the test statistic Q is

$$Q = n(n+2)\sum_{k=L}^{K}(n-k)^{-1}\hat{\rho}(k)$$

with $L = 1$ if $\hat{\rho}(k)$ is an autocorrelation function. Given that the model is adequate, Q has a chi-squared distribution with $K - L + 1 - m$ degrees of freedom where m = `npFree` is the number of parameters estimated in the model. If the mean of the time series is estimated, Woodfield (1990) recommends not including this in the count of the parameters estimated in the model. Thus, for an ARMA(*p*, *q*) model set

npFree = $p + q$ regardless of whether the mean is estimated or not. The original derivation for time series models is due to Box and Pierce (1970) with the above modified version discussed by Ljung and Box (1978). The extension of the test to transfer function models is discussed by Box and Jenkins (1976, pages 394-395).

Methods

Compute

```
static public double[] Compute(int nObservations, double[] correlations, int
npFree, int lagMax)
```

Description

Performs lack-of-fit test for a univariate time series or transfer function given the appropriate correlation function using a minimum lag of 1.

Parameters

`nObservations` – An `int` containing the number of observations of the stationary time series.

`correlations` – A `double` array of length `lagMax+1` containing the correlation function.

`npFree` – An `int` scalar specifying the number of free parameters in the formulation of the time series model. `npfree` must be greater than or equal to zero and less than `lagMax`. Woodfield (1990) recommends `npFree = p + q`.

`lagMax` – An `int` scalar specifying the maximum lag of the correlation function.

Returns

A `double` array of length 2 with the test statistic, Q, and its *p*-value, *p*. Under the null hypothesis, Q has an approximate chi-squared distribution with `lagMax-lagMin+1-npFree` degrees of freedom.

Compute

```
static public double[] Compute(int nObservations, double[] correlations, int
npFree, int lagMax, int lagMin)
```

Description

Performs lack-of-fit test for a univariate time series or transfer function given the appropriate correlation function.

Parameters

`nObservations` – An `int` containing the number of observations of the stationary time series.

`correlations` – A `double` array of length `lagMax+1` containing the correlation function.

`npFree` – An `int` scalar specifying the number of free parameters in the formulation of the time series model. `npfree` must be greater than or equal to zero and less than `lagMax`. Woodfield (1990) recommends `npFree = p + q`.

`lagMax` – An `int` scalar specifying the maximum lag of the correlation function.

`lagMin` – An `int` scalar specifying the minimum lag of the correlation function. `lagMin` corresponds to the lower bound of summation in the lack of fit test statistic.

Default: `lagMin` = 1.

Returns

A `double` array of length 2 with the test statistic, Q, and its *p*-value, *p*. Under the null hypothesis, Q has an approximate chi-squared distribution with `lagMax-lagMin+1-npFree` degrees of freedom.

Example: Lack Of Fit

Consider the Wölfer Sunspot Data (Anderson 1971, page 660) consisting of the number of sunspots observed each year from 1749 through 1924. The data set for this example consists of the number of sunspots observed from 1770 through 1869. An ARMA(2,1) with nonzero mean is fitted using the `ARMA` class. The autocorrelations of the residuals are estimated using the `AutoCorrelation` class. Class `LackOfFit` is used to compute a portmanteau lack of fit test using 10 lags. The warning messages from `ARMA` in the output can be ignored. (See Example 2 in `ARMA` for a full explanation of the warning messages.)

```
using System;
using Imsl.Stat;

public class LackOfFitEx1
{

    public static void  Main(String[] args)
    {
        int lagmax = 10;
        int npfree = 4;
        double tolerance = 0.125;

        //  sunspot data for 1770 through 1869
        double[] x = { 100.8, 81.6, 66.5, 34.8, 30.6, 7, 19.8,
            92.5, 154.4, 125.9, 84.8, 68.1, 38.5, 22.8, 10.2, 24.1, 82.9,
            132, 130.9, 118.1, 89.9, 66.6, 60, 46.9, 41, 21.3, 16, 6.4,
            4.1, 6.8, 14.5, 34, 45, 43.1, 47.5, 42.2, 28.1, 10.1, 8.1,
            2.5, 0, 1.4, 5, 12.2, 13.9, 35.4, 45.8, 41.1, 30.4, 23.9,
            15.7, 6.6, 4, 1.8, 8.5, 16.6, 36.3, 49.7, 62.5, 67, 71, 47.8,
            27.5, 8.5, 13.2, 56.9, 121.5, 138.3, 103.2, 85.8, 63.2, 36.8,
            24.2, 10.7, 15, 40.1, 61.5, 98.5, 124.3, 95.9, 66.5, 64.5,
            54.2, 39, 20.6, 6.7, 4.3, 22.8, 54.8, 93.8, 95.7, 77.2, 59.1,
            44, 47, 30.5, 16.3, 7.3, 37.3, 73.9};

        // Get residuals from ARMA(2,1) for autocorrelation/lack of fit
        ARMA arma = new ARMA(2, 1, x);
        arma.Method = ARMA.ParamEstimation.LeastSquares;
        arma.ConvergenceTolerance = tolerance;
        arma.Compute();
        double[] residuals = arma.GetResidual();

        // Get autocorrelations from residuals for lack of fit test
        AutoCorrelation ac = new AutoCorrelation(residuals, lagmax);
        double[] correlations = ac.GetAutoCorrelations();
```

```
        // Get lack of fit test statistic and p-value
        double[] result = LackOfFit.Compute(x.Length, correlations,
            npfree, lagmax);

        // Print parameter estimates, test statistic, and p-value
        // NOTE: Test Statistic Q follows a Chi-squared dist.
        Console.WriteLine("Lack of Fit Statistic,  Q = " + result[0]);
        Console.WriteLine("P-value of Q = " + result[1]);
    }
}
```

Output

```
Lack of Fit Statistic,  Q = 14.6060180633065
P-value of Q = 0.97644740013821
Imsl.Stat.ARMA: Relative function convergence - Both the scaled actual and
predicted reductions in the function are less than or equal to the relative
function convergence tolerance "ConvergenceTolerance" = 0.0645856533065147.
Imsl.Stat.ARMA: Least squares estimation of the parameters has failed to
converge.  Increase "length" and/or "tolerance" and/or "convergence_tolerance".
The estimates of the parameters at the last iteration may be used as new
starting values.
```

ARAutoUnivariate Class

`public class Imsl.Stat.ARAutoUnivariate`

Automatically determines the best autoregressive time series model using Akaike's Information Criterion.

`ARAutoUnivariate` automatically selects the order of the AR model that best fits the data and then computes the AR coefficients. The algorithm used in `ARAutoUnivariate` is derived from the work of Akaike, H., et. al (1979) and Kitagawa and Akaike (1978). This code was adapted from the UNIMAR procedure published as part of the TIMSAC-78 Library.

The best fit AR model is determined by successively fitting AR models with $0, 1, 2, \ldots,$ maxlag autoregressive coefficients. For each model, Akaike's Information Criterion (AIC) is calculated based on the formula

$$\text{AIC} = -2\ln(\textit{likelihood}) + 2p$$

Class `ARAutoUnivariate` uses the approximation to this formula developed by Ozaki and Oda (1979),

$$\text{AIC} \approx (n - \text{maxlag})\ln(\hat{\sigma}^2) + 2(p+1) + (n - \text{maxlag})(\ln(2\pi) + 1)$$

where $\hat{\sigma}^2$ is an estimate of the residual variance of the series, commonly known in time series analysis as the innovation variance and n is the number of observations in the time series `z`, n=`z.Length`. By dropping the constant

$$(n - \text{maxlag})(\ln(2\pi) + 1),$$

the calculation is simplified to

$$\text{AIC} \approx (n - \text{maxlag}) \ln(\hat{\sigma}^2) + 2(p+1),$$

The best fit model is the model with minimum AIC. If the number of parameters in this model selected by `ARAutoUnivariate` is equal to the highest order autoregressive model fitted, i.e., *p*=`maxlag`, then a model with smaller AIC might exist for larger values of `maxlag`. In this case, increasing `maxlag` to explore AR models with additional autoregressive parameters might be warranted.

Property `EstimationMethod` can be used to specify the method used to calculate the AR coeficients. If `EstimationMethod` is set to `MethodOfMoments`, estimates of the autoregressive coefficients for the model with minimum AIC are calculated using method of moments as described in the `ARMA` class. If `LeastSquares` is specified, the coefficients are determined by the method of least squares applied in the form described by Kitagawa and Akaike (1978). If `MaximumLikelihood` is specified, the coefficients are estimated using maximum likelihood as described in the `ARMAMaxLikelihood` class.

Properties

AIC

```
public double AIC {get; }
```

Description

The final estimate for Akaike's Information Criterion (AIC) at the optimum.

Property Value

A `double` scalar value which is an approximation to $\text{AIC} = -2\ln(L) + 2p$, where L is the value of the maximum likelihood function evaluated at the parameter estimates. The approximation uses the calculation

$$\text{AIC} \approx (\text{n} - \text{maxlag}) \ln(\hat{\sigma}^2) + 2(\text{p}+1) + (\text{n} - \text{maxlag})(\ln(2\pi) + 1),$$

where $\hat{\sigma}^2$ is an estimate of the residual variance of the series, commonly known in time series analysis as the innovation variance, and `n` is the number of observations in the time series `z (n=z.Length)`.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

BackwardOrigin

```
public int BackwardOrigin {get; set; }
```

Description

The maximum backward origin used in calculating the forecasts.

Property Value

An `int` scalar containing the maximum backward origin.

Remarks

`BackwardOrigin` must be greater than or equal to 0 and less than or equal to `z.Length - p`, where *p* is the optimum number of AR coefficients determined by `ARAutoUnivariate`. The forecasted values returned will be `z.Length - BackwardOrigin` through `z.Length`. By default, `BackwardOrigin = 0`.

Confidence

```
public double Confidence {get; set; }
```

Description

The confidence level for calculating confidence limit deviations returned from `GetDeviations`.

Property Value

A `double` scalar value representing the confidence level used in computing forecast confidence intervals.

Remarks

Typical choices for `Confidence` are 0.90, 0.95, and 0.99. `Confidence` must be greater than 0.0 and less than 1.0. By default, `Confidence` = 0.95.

Constant

```
public double Constant {get; }
```

Description

The estimate for the constant parameter in the ARMA series.

Property Value

A `double` scalar equal to the estimate for the constant parameter in the ARMA series.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

ConvergenceTolerance

```
public double ConvergenceTolerance {get; set; }
```

Description

The tolerance level used to determine convergence of the nonlinear least-squares and maximum likelihood algorithms.

Property Value

A `double` scalar containing the tolerance level used to determine convergence of the nonlinear least-squares and maximum likelihood algorithms. `ConvergenceTolerance` represents the minimum relative decrease in sum of squares between two iterations required to determine convergence.

Remarks

`ConvergenceTolerance` must be greater than or equal to 0. By default, ConvergenceTolerance = 10^{-10}.

EstimationMethod

```
public Imsl.Stat.ARAutoUnivariate.ParamEstimation EstimationMethod {get; set; }
```

Description

The estimation method used for estimating the final estimates for the autoregressive coefficients.

Property Value

A `ParamEstimation` specifying the method used to estimate the autoregressive parameters estimates.

Remarks

ARAutoUnivariate.ParamEstimation	*Method Description*
MethodOfMoments	Autoregressive parameters are estimated by a method of moments procedure.
LeastSquares	Autoregressive parameters are estimated by a least-squares procedure
MaxLikelihood	Autoregressive parameters are estimated by a maximum likelihood procedure.

By default, EstimationMethod = ARAutoUnivariate.ParamEstimation.LeastSquares (p. 862).

InnovationVariance

```
public double InnovationVariance {get; }
```

Description

The final estimate for the innovation variance.

Property Value

A double scalar value equal to the estimate for the innovation variance.

Exceptions

Imsl.Stat.MatrixSingularException is thrown if the input matrix is singular.

Imsl.Stat.TooManyCallsException is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

Imsl.Stat.IncreaseErrRelException is thrown if the bound for the relative error is too small.

Imsl.Stat.NewInitialGuessException is thrown if the iteration has not made good progress.

Imsl.Stat.IllConditionedException is thrown if the problem is ill-conditioned.

Imsl.Stat.TooManyIterationsException is thrown if the maximum number of iterations is exceeded.

Imsl.Stat.TooManyFunctionEvaluationsException is thrown if the maximum number of function evaluations is exceeded.

Imsl.Stat.TooManyJacobianEvalException is thrown if the maximum number of Jacobian evaluations is exceeded.

Imsl.Stat.SingularTriangularMatrixException is thrown if the input triangular matrix is singular.

Imsl.Stat.NonStationaryException is thrown if the final maximum likelihood estimates for the time series are nonstationary.

Likelihood

```
public double Likelihood {get; }
```

Description

The final estimate for $L \approx e^{-(\text{AIC}-2p)/2}$, where p is the AR order, AIC is the value of Akaike's Information Criterion, and L is the likelihood function evaluated for the optimum autoregressive model.

Property Value

A `double` scalar equal to the estimate for the $L \approx e^{-(\text{AIC}-2p)/2}$.

Remarks

If `MaximumLikelihood` is chosen as the `EstimationMethod`, the exact likelihood evaluated for the optimum autoregressive model will be returned instead. Otherwise it is calculated using the approximation to the AIC.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations used for estimating the autoregressive coefficients.

Property Value

An `int` scalar containing the maximum number of iterations allowed in the nonlinear equation solver used in both the method of moments and least-squares algorithms.

Remarks

`MaxIterations` must be greater than 0. By default, `MaxIterations` = 300.

Maxlag

```
public int Maxlag {get; }
```

Description

The current value used to represent the maximum number of autoregressive lags to achieve the minimum AIC.

Property Value

An `int` containing the maximum number of autoregressive lags evaluated.

Mean

```
public double Mean {get; set; }
```

Description

The mean used to center the time series `z`.

Property Value

A `double` containing the value used to center the series before searching for the optimum number of AR lags. This is equal to the mean of the time series `z` unless the `Mean` is set.

Remarks

By default, the time series `z` is centered about its sample mean.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Order

```
public int Order {get; }
```

Description

The order of the AR model selected with the minimum AIC.

Property Value

An `int` containing the optimum AR order selected by `ARAutoUnivariate`'s minimum AIC selection.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

TimsacConstant

```
public double TimsacConstant {get; }
```

Description

The estimate for the constant parameter in the ARMA series.

Property Value

A `double` scalar equal to the estimate for the constant parameter in the ARMA series.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

TimsacVariance

```
public double TimsacVariance {get; }
```

Description

The final estimate for the innovation variance calculated by the TIMSAC automatic AR modeling routine (UNIMAR).

Property Value

A `double` scalar value equal to the estimate for the innovation variance.

Remarks

This variance depends upon the value of `maxlag` since the series length in this computation depends on its value. Use `InnovationVariance` to return the innovation variance calculated using the maximum series length.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

Constructor

ARAutoUnivariate

```
public ARAutoUnivariate(int maxlag, double[] z)
```

Description

`ARAutoUnivariate` constructor.

Parameters

`maxlag` – An `int` scalar specifying the maximum number of autoregressive lags to evaluate.

`z` – A `double` array containing the time series.

Methods

Compute

```
public void Compute()
```

Description

Determines the autoregressive model with the minimum AIC by fitting autoregressive models from 0 to `maxlag` lags using the method of moments or an estimation method specified by the user through `EstimationMethod`.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

Forecast

```
public double[,] Forecast(int nForecast)
```

Description

Returns forecasts and associated confidence interval offsets.

Parameter

`nForecast` – An `int` representing the number of requested forecasts.

Returns

A `double` matrix of dimension `nForecast` by `BackwardOrigin + 1` containing the forecasts. The forecasts are for lead times $l = 1, 2, \ldots,$ nForecast at origins `z.Length-BackwardOrigin-1+j` where $j = 1, \ldots,$ BackwardOrigin $+ 1$.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

GetAR

```
public double[] GetAR()
```

Description

Returns the final autoregressive parameter estimates at the optimum AIC using the estimation method specified in `EstimationMethod`.

Returns

A `double` array containing the estimates for the autoregressive parameters.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

GetDeviations

```
public double[] GetDeviations()
```

Description

Returns the deviations for each forecast used for calculating the forecast confidence limits.

Returns

A `double` array of length `BackwardOrigin+nForecast` containing the deviations for calculating forecast confidence intervals. The confidence level is specified in the `Confidence` property.

Remarks

Note that the `Forecast` or `GetForecast`method must be invoked first before invoking this method. Otherwise, the method returns `null`.

GetForecast

```
public double[] GetForecast(int nForecast)
```

Description

Returns a specified number of forecasts beyond the last value in the series.

Parameter

`nForecast` – An `int` representing the number of requested forecasts beyond the last value in the series.

Returns

A `double` array containing the `nForecast+BackwardOrigin` forecasts. The first `BackwardOrigin` forecasts are one-step ahead forecasts for the last `BackwardOrigin` values in the series. The next `nForecast` values in the returned series are forecasts for the next values beyond the series.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

GetResiduals

```
public double[] GetResiduals()
```

Description

Returns the current values of the vector of residuals.

Returns

A `double` array of length `BackwardOrigin` containing the residuals for the last `BackwardOrigin` values in the time series. This array will be `null` unless `Compute` and either the `Forecast` or `GetForecast` methods are called previously.

GetTimeSeries

```
public double[] GetTimeSeries()
```

Description

Returns the time series used for estimating the minimum AIC and the autoregressive coefficients.

Returns

A `double` array containing the time series values set in the constructor.

GetTimsacAR

```
public double[] GetTimsacAR()
```

Description

Returns the final auto regressive parameter estimates at the optimum AIC estimated by the original TIMSAC routine (UNIMAR).

Returns

A `double` array of length *p* containing the estimates for the autoregressive parameters.

Remarks

These estimates vary depending upon the value of `maxlag` since its value changes the length of the series used to estimate these parameters. Use the `GetAR` method to obtain the best estimates from the estimation method specified by `EstimationMethod` calculated using the maximum number of available observations.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

Example 1: ARAutoUnivariate

Consider the Wolfer Sunspot Data (Anderson 1971, p. 660) consisting of the number of sunspots observed each year from 1749 through 1924. The data set for this example consists of the number of sunspots observed from 1770 through 1869. In this example, `ARAutoUnivariate` found the minimum AIC fit is an autoregressive model with 3 lags:

$$\tilde{W}_t = \phi_1 \tilde{W_{t-1}} + \phi_2 \tilde{W_{t-2}} + \phi_3 \tilde{W_{t-3}} + a_t,$$

where

$$\tilde{W_t} := W_t - \mu$$

μ is the sample mean of the time series $\{W_t\}$. Defining the overall constant θ_0 by $\theta_0 := \mu(1 - \sum_{i=1}^{3} \phi_i)$, we obtain the following equivalent representation:

$$W_t = \theta_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \phi_3 W_{t-3} + a_t.$$

The example computes estimates for $\theta_0, \phi_1, \phi_2, \phi_3$ for each of the three parameter estimation methods available.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class ARAutoUnivariateEx1
{
    public static void  Main(String[] args)
    {
        double[] z = new double[]{100.8, 81.6, 66.5, 34.8, 30.6,
                                  7, 19.8, 92.5, 154.4, 125.9,
                                  84.8, 68.1, 38.5, 22.8, 10.2,
                                  24.1, 82.9, 132, 130.9, 118.1,
                                  89.9, 66.6, 60, 46.9, 41,
```

```
                                21.3, 16, 6.4, 4.1, 6.8,
                                14.5, 34, 45, 43.1, 47.5,
                                42.2, 28.1, 10.1, 8.1, 2.5,
                                0, 1.4, 5, 12.2, 13.9,
                                35.4, 45.8, 41.1, 30.4, 23.9,
                                15.7, 6.6, 4, 1.8, 8.5,
                                16.6, 36.3, 49.7, 62.5, 67,
                                71, 47.8, 27.5, 8.5, 13.2,
                                56.9, 121.5, 138.3, 103.2, 85.8,
                                63.2, 36.8, 24.2, 10.7, 15,
                                40.1, 61.5, 98.5, 124.3, 95.9,
                                66.5, 64.5, 54.2, 39, 20.6,
                                6.7, 4.3, 22.8, 54.8, 93.8,
                                95.7, 77.2, 59.1, 44, 47,
                                30.5, 16.3, 7.3, 37.3, 73.9};

        ARAutoUnivariate autoAR = new ARAutoUnivariate(20, z);
        Console.Out.WriteLine("");
        Console.Out.WriteLine("    Method of Moments ");
        Console.Out.WriteLine("");

        autoAR.EstimationMethod =
            Imsl.Stat.ARAutoUnivariate.ParamEstimation.MethodOfMoments;
        autoAR.Compute();

        Console.Out.WriteLine("Order Selected: " + autoAR.Order);
        Console.Out.WriteLine("AIC = " + autoAR.AIC
            + "       Variance = " + autoAR.InnovationVariance);
        Console.Out.WriteLine("Constant = " + autoAR.Constant);
        Console.Out.WriteLine("");
        new PrintMatrix("AR estimates are:  ").Print(autoAR.GetAR());


        /* try another method */
        Console.Out.WriteLine("");
        Console.Out.WriteLine("");
        Console.Out.WriteLine("    Least Squares ");
        Console.Out.WriteLine("");
        autoAR.EstimationMethod =
            Imsl.Stat.ARAutoUnivariate.ParamEstimation.LeastSquares;
        autoAR.Compute();

        Console.Out.WriteLine("Order Selected: " + autoAR.Order);
        Console.Out.WriteLine("AIC = " + autoAR.AIC
            + "       Variance = " + autoAR.InnovationVariance);
        Console.Out.WriteLine("Constant = " + autoAR.Constant);
        Console.Out.WriteLine();
        new PrintMatrix("AR estimates are:  ").Print(autoAR.GetAR());

        Console.Out.WriteLine("");
        Console.Out.WriteLine("");
        Console.Out.WriteLine("    Maximum Likelihood ");
        Console.Out.WriteLine("");
        autoAR.EstimationMethod =
            Imsl.Stat.ARAutoUnivariate.ParamEstimation.MaximumLikelihood;
        autoAR.Compute();
```

```
        Console.Out.WriteLine("Order Selected: " + autoAR.Order);
        Console.Out.WriteLine("AIC = " + autoAR.AIC
            + "       Variance = " + autoAR.InnovationVariance);
        Console.Out.WriteLine("Constant = " + autoAR.Constant);
        Console.Out.WriteLine();
        new PrintMatrix("AR estimates are:  ").Print(autoAR.GetAR());
    }
}
```

Output

```
    Method of Moments

Order Selected: 3
AIC = 405.981241965022       Variance = 287.269907744347
Constant = 13.7097894010953

  AR estimates are:
             0
0   1.36757589345393
1  -0.737673445646019
2   0.0782508772709516



    Least Squares

Order Selected: 3
AIC = 405.981241965022       Variance = 221.995196605174
Constant = 11.6104946249815

  AR estimates are:
            0
0   1.55022677857954
1  -1.00394753883711
2   0.205447994348897



    Maximum Likelihood

Order Selected: 3
AIC = 405.981241965022       Variance = 218.848375884726
Constant = 11.4178509825012

  AR estimates are:
            0
0   1.55139136379628
1  -0.998935931377056
2   0.204487453682164
```

Example 2 (Forecasting): ARAutoUnivariate

Using the Canadian Lynx data included in TIMSAC-78, `ARAutoUnivariate` is used to find the minimum AIC autoregressive model using a maximum number of lags of `maxlag`=20.

This example compares the three different methods for estimating the autoregressive coefficients, and it illustrates the relationship between these estimates and those calculated within the TIMSAC UNIMAR code. As illustrated, the UNIMAR code estimates the coefficients and innovation variance using only the last N-maxlag values in the time series. The other estimation methods use all *N-k* values, where *k* is the number of lags with minimum AIC selected by `ARAutoUnivariate`.

This example also illustrates how to generate forecasts for the observed series values and beyond by setting the backward orgin for the forecasts.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using PrintMatrix = Imsl.Math.PrintMatrix;
public class ARAutoUnivariateEx2
{
    public static void  Main(String[] args)
    {
        /* THE CANDIAN LYNX DATA AS USED IN TIMSAC 1821-1934 */
        double[] y = new double[]{
                   0.24300e01, 0.25060e01, 0.27670e01, 0.29400e01, 0.31690e01,
                   0.34500e01, 0.35940e01, 0.37740e01, 0.36950e01, 0.34110e01,
                   0.27180e01, 0.19910e01, 0.22650e01, 0.24460e01, 0.26120e01,
                   0.33590e01, 0.34290e01, 0.35330e01, 0.32610e01, 0.26120e01,
                   0.21790e01, 0.16530e01, 0.18320e01, 0.23280e01,  0.27370e01,
                   0.30140e01, 0.33280e01, 0.34040e01, 0.29810e01, 0.25570e01,
                   0.25760e01, 0.23520e01, 0.25560e01, 0.28640e01, 0.32140e01,
                   0.34350e01,  0.34580e01, 0.33260e01, 0.28350e01, 0.24760e01,
                   0.23730e01, 0.23890e01,  0.27420e01, 0.32100e01, 0.35200e01,
                   0.38280e01, 0.36280e01, 0.28370e01,  0.24060e01, 0.26750e01,
                   0.25540e01, 0.28940e01, 0.32020e01, 0.32240e01,  0.33520e01,
                   0.31540e01, 0.28780e01, 0.24760e01, 0.23030e01, 0.23600e01,
                   0.26710e01, 0.28670e01, 0.33100e01, 0.34490e01, 0.36460e01,
                   0.34000e01,  0.25900e01, 0.18630e01, 0.15810e01, 0.16900e01,
                   0.17710e01, 0.22740e01,  0.25760e01, 0.31110e01, 0.36050e01,
                   0.35430e01, 0.27690e01, 0.20210e01,  0.21850e01, 0.25880e01,
                   0.28800e01, 0.31150e01, 0.35400e01, 0.38450e01,  0.38000e01,
                   0.35790e01, 0.32640e01, 0.25380e01, 0.25820e01, 0.29070e01,
                   0.31420e01, 0.34330e01, 0.35800e01, 0.34900e01, 0.34750e01,
                   0.35790e01,  0.28290e01, 0.19090e01, 0.19030e01, 0.20330e01,
                   0.23600e01, 0.26010e01, 0.30540e01, 0.33860e01, 0.35530e01,
                   0.34680e01, 0.31870e01, 0.27230e01, 0.26860e01, 0.28210e01,
                   0.30000e01, 0.32010e01, 0.34240e01, 0.35310e01};
        double[][] printOutput = null;
        double[] timsacAR, mmAR, mleAR, lsAR;
        double[] forecasts, residuals;
        double timsacConstant, mmConstant, mleConstant, lsConstant;
        double timsacVar, timsacEquivalentVar, mmVar, mleVar, lsVar;
        int maxlag = 20;
        String[] colLabels = new String[]{"TIMSAC",
                                          "Method of Moments",
```

```
                                    "Least Squares",
                                    "Maximum Likelihood"};
String[] colLabels2 = new String[]{"Observed", "Forecast", "Residual"};
PrintMatrixFormat pmf = new PrintMatrixFormat();
PrintMatrix pm = new PrintMatrix();
pmf.SetColumnLabels(colLabels);
pmf.NumberFormat = "0.0000";
Console.Out.WriteLine
    ("Automatic Selection of Minimum AIC AR Model");
Console.Out.WriteLine("");

ARAutoUnivariate autoAR = new ARAutoUnivariate(maxlag, y);
autoAR.Compute();
int orderSelected = autoAR.Order;
Console.Out.WriteLine("Minimum AIC Selected=" + autoAR.AIC
    + " with an optimum lag of k= " + autoAR.Order);
Console.Out.WriteLine("");

timsacAR = autoAR.GetTimsacAR();
timsacConstant = autoAR.TimsacConstant;
timsacVar = autoAR.TimsacVariance;
lsAR = autoAR.GetAR();
lsConstant = autoAR.Constant;
lsVar = autoAR.InnovationVariance;

autoAR.EstimationMethod =
    Imsl.Stat.ARAutoUnivariate.ParamEstimation.MethodOfMoments;
autoAR.Compute();
mmAR = autoAR.GetAR();
mmConstant = autoAR.Constant;
mmVar = autoAR.InnovationVariance;

autoAR.EstimationMethod =
    Imsl.Stat.ARAutoUnivariate.ParamEstimation.MaximumLikelihood;
autoAR.Compute();
mleAR = autoAR.GetAR();
mleConstant = autoAR.Constant;
mleVar = autoAR.InnovationVariance;

printOutput = new double[orderSelected + 1][];
for (int i = 0; i < orderSelected + 1; i++)
{
    printOutput[i] = new double[4];
}
printOutput[0][0] = timsacConstant;
for (int i = 0; i < orderSelected; i++)
    printOutput[i + 1][0] = timsacAR[i];
printOutput[0][1] = mmConstant;
for (int i = 0; i < orderSelected; i++)
    printOutput[i + 1][1] = mmAR[i];
printOutput[0][2] = lsConstant;
for (int i = 0; i < orderSelected; i++)
    printOutput[i + 1][2] = lsAR[i];
printOutput[0][3] = mleConstant;
for (int i = 0; i < orderSelected; i++)
    printOutput[i + 1][3] = mleAR[i];
```

```
pm.SetTitle("Comparison of AR Estimates");
pm.Print(pmf, printOutput);

/* calculation of equivalent innovation variance using TIMSAC
coefficients.  The Timsac innovation variance is calculated using
only N-maxlag observations in the series.  The following code
calculates the innovation variance using N-k observations in the
series with the Timsac coefficient.  This illustrates that the
least squares Timsac coefficients will not have the least value for
the sum of squared residuals, which is calculated using all N-k
observations. */
ARMA armaLS = new ARMA(orderSelected, 0, y);
armaLS.SetARMAInfo(autoAR.TimsacConstant, autoAR.GetTimsacAR(),
    new double[0], autoAR.TimsacVariance);
armaLS.BackwardOrigin = y.Length - orderSelected;
forecasts = armaLS.GetForecast(1);
double sumResiduals = 0.0;
for (int i = 0; i < y.Length - orderSelected; i++)
{
    sumResiduals += (y[i + orderSelected] - forecasts[i]) *
        (y[i + orderSelected] - forecasts[i]);
}
timsacEquivalentVar = sumResiduals / (y.Length - orderSelected - 1);
printOutput = new double[1][];
for (int i2 = 0; i2 < 1; i2++)
{
    printOutput[i2] = new double[4];
}
printOutput[0][0] = timsacEquivalentVar;
/* the method of moments variance */
printOutput[0][1] = mmVar;
/* the least squares variance */
printOutput[0][2] = lsVar;
/* the maximum likelihood estimate of the variance */
printOutput[0][3] = mleVar;
pm.SetTitle("Comparison of Equivalent Innovation Variances");
pm.Print(pmf, printOutput);

/* FORECASTING - An example of forecasting using the maximum
* likelihood estimates for the minimum AIC AR model.  In this example,
* forecasts are returned for the last 10 values in the series followed
* by the forecasts for the next 5 values.
*/
autoAR.BackwardOrigin = 10;
forecasts = autoAR.GetForecast(15);
residuals = autoAR.GetResiduals();
printOutput = new double[15][];
for (int i3 = 0; i3 < 15; i3++)
{
    printOutput[i3] = new double[3];
}
for (int i = 0; i < 10; i++)
{
    printOutput[i][0] = y[y.Length - 10 + i];
    printOutput[i][1] = forecasts[i];
    printOutput[i][2] = residuals[i];
```

```
        }
        for (int i = 10; i < 15; i++)
        {
            printOutput[i][0] = Double.NaN;
            printOutput[i][1] = forecasts[i];
            printOutput[i][2] = Double.NaN;
        }
        pmf.FirstRowNumber = 105;
        pmf.SetColumnLabels(colLabels2);
        pm.SetTitle("Maximum Likelihood Forecasts of Last 10 Values");
        pm.Print(pmf, printOutput);
    }
}
```

Output

```
Automatic Selection of Minimum AIC AR Model

Minimum AIC Selected=-296.130132635624 with an optimum lag of k= 11

                    Comparison of AR Estimates
    TIMSAC   Method of Moments  Least Squares  Maximum Likelihood
 0   1.0427        1.1679           1.1144           1.1186
 1   1.1813        1.1381           1.1481           1.1664
 2  -0.5516       -0.5061          -0.5331          -0.5419
 3   0.2314        0.2098           0.2757           0.2624
 4  -0.1780       -0.2672          -0.3263          -0.3052
 5   0.0199        0.1112           0.1685           0.1519
 6  -0.0626       -0.1246          -0.1643          -0.1460
 7   0.0286        0.0693           0.0728           0.0582
 8  -0.0507       -0.0419          -0.0305          -0.0310
 9   0.1999        0.1366           0.1509           0.1379
10   0.1618        0.1828           0.1935           0.1995
11  -0.3391       -0.3101          -0.3414          -0.3375

          Comparison of Equivalent Innovation Variances
   TIMSAC  Method of Moments  Least Squares  Maximum Likelihood
0  0.0377       0.0427           0.0369           0.0362

Maximum Likelihood Forecasts of Last 10 Values
     Observed  Forecast  Residual
105   3.5530    3.4387    0.1143
106   3.4680    3.4801   -0.0121
107   3.1870    2.9244    0.2626
108   2.7230    2.7026    0.0204
109   2.6860    2.5558    0.1302
110   2.8210    2.7852    0.0358
111   3.0000    2.9492    0.0508
112   3.2010    3.1861    0.0149
113   3.4240    3.3855    0.0385
114   3.5310    3.5272    0.0038
115   NaN       3.4465    NaN
116   NaN       3.1947    NaN
117   NaN       2.8289    NaN
118   NaN       2.4918    NaN
```

```
119   NaN       2.4143    NaN
```

ARAutoUnivariate.ParamEstimation Enumeration

```
public enumeration Imsl.Stat.ARAutoUnivariate.ParamEstimation
```

Parameter Estimation procedures.

Fields

LeastSquares

```
public Imsl.Stat.ARAutoUnivariate.ParamEstimation LeastSquares
```

Description

Indicates autoregressive parameters are estimated by a least-squares procedure.

MaximumLikelihood

```
public Imsl.Stat.ARAutoUnivariate.ParamEstimation MaximumLikelihood
```

Description

Indicates autoregressive parameters are estimated by a maximum likelihood procedure.

MethodOfMoments

```
public Imsl.Stat.ARAutoUnivariate.ParamEstimation MethodOfMoments
```

Description

Indicates autoregressive parameters are estimated by a method of moments procedure.

ARSeasonalFit Class

```
public class Imsl.Stat.ARSeasonalFit
```

Estimates the optimum seasonality parameters for a time series using an autoregressive model, AR(p), to represent the time series.

ARMA time series modeling assumes the time series is stationary. Seasonal trends and cycles violate this assumption, which can lead to inaccurate predictions. However, in many cases the nonstationary series

can be transformed into a stationary series by first differencing the series. For example, if the correlation is strong from one period to the next, the series might be differenced by a lag of 1. Instead of fitting a model to the original series Z_t, the model is fitted to the transformed series: $W_t = Z_t - Z_{t-1}$. Higher order lags or differences are warranted if the series has cycles every 4 or 13 weeks. Class `ARSeasonalFit` is designed to help identify the optimum differencing for a series with seasonal trends or cycles.

`ARSeasonalFit` assumes the original series has no missing values, is equally spaced in time and is not centered before computing the optimum differencing. However, by default the transformed series is centered using the mean of that series. Users can change this default setting with the `Center` (p. 864) property. If `Center` is set to `None` (p. 873) the series is not centered, if set to `Mean` (p. 873) the series is centered using the mean of the series, and if it is set to `Median` (p. 873), the series is centered using the median of the series. If `Center` is set to `Mean` or `Median` then the differenced series, W_t is centered before determination of minimum AIC and optimum lag.

For every combination of rows in `SInitial` and `DInitial`, the series Z_t is converted to the seasonally adjusted series using the following computation

$$W_t(s,d) = \Delta_{s_1}^{d_1}\Delta_{s_2}^{d_2}\cdots\Delta_{s_m}^{d_m}Z_t = \prod_{i=1}^{m}\sum_{j=0}^{d_i}\binom{d_i}{j}(-1)^j B^{j\cdot s_i}Z_t$$

where $s := (s_1,\ldots,s_m)$, $d := (d_1,\ldots,d_m)$ represent specific rows of arrays `SInitial` and `DInitial` respectively, and m is the number of differences, or m=`SInitial.GetLength(1)`.

This transformation of the series Z_t to $W_t(s,d)$ is computed using the `Difference` class. After this transformation the transformed series

$$W_t(s,d)$$

is centered, unless `None` is specified, and the `ARAutoUnivariate` (p. 841) class is used to automatically determine the optimum lag for an AR(p) representation for $W_t(s,d)$.

This procedure is repeated for every possible combination of rows in `SInitial` and `DInitial`. The series with the minimum AIC is identified as the optimum representation and returned in the methods and properties `AROrder` (p. 864), `GetOptimumS` (p. 868), `GetOptimumD` (p. 867), `AIC` (p. 863), and `GetAR` (p. 866). The transformed series with the minium AIC can be retrieved from the `GetTransformedTimeSeries` (p. 869) method.

Properties

AIC

```
public double AIC {get; }
```

Description

The final estimate for Akaike's Information Criterion (AIC) at the optimum.

Property Value

A `double` containing the optimum AIC estimate. AIC $\approx -2\ln(L)+2p$, where L is the value of the maximum likelihood function evalutated at the parameter estimates.

AROrder

```
public int AROrder {get; }
```

Description

The optimum number of lags, p, for the optimum autoregressive AR(p) model. This is the value of p for the transformed series, W_t.

Property Value

An `int` containing the optimum number of lags in the autoregressive model used to fit the transformed series W_t.

Center

```
public Imsl.Stat.ARSeasonalFit.CenterMethod Center {get; set; }
```

Description

The setting for centering the input time series.

Property Value

A `CenterMethod` containing the setting for Center, either `None` (p. 873), `Mean` (p. 873), or `Median` (p. 873).

Remarks

By default, `Center=CenterMethod.Mean`

Exclude

```
public bool Exclude {get; set; }
```

Description

Controls whether to exclude or replace the intial values in the transformed series.

Property Value

A `bool` value that controls whether values in W_t that cannot be calculated are dropped or set to missing. Missing values are set to `Double.NaN`.

Remarks

If `Exclude` is `true`, then intial values in the transformed series that cannot be computed are set to missing, `NaN`. This ensures that the length of the transformed series W_t is equal to the length of the time series, `z.Length`. If `Exclude` is set to `false`, then intial values in the transformed series W_t that cannot be computed are removed. This makes the length of the transformed series W_t equal to `z.Length-NLost` where `NLost` is the number of lost values obtained from property `NLost` (p. 865). By default, `Exclude=true`.

Maxlag

```
public int Maxlag {get; }
```

Description

The maximum lag used to fit the AR(p) model.

Property Value

An `int` scalar containing the maximum lag allowed when fitting an AR(p) model.

NLost

```
public int NLost {get; }
```

Description

The number of values in the initial part of the series lost to differencing.

Property Value

An `int` containing the number of values in the initial part of the series lost to differencing. These lost values will be set to missing or dropped in the transformed series, depending upon the setting for exclude.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

Constructor

ARSeasonalFit

```
public ARSeasonalFit(int maxlag, int[,] SInitial, double[] z)
```

Description

Constructor for `ARSeasonalFit`.

Parameters

`maxlag` – An `int` scalar specifying the maximum lag allowed when fitting an AR(p) model.

`SInitial` – An `int` matrix where each row represents seasonal differences to evaluate. The number of columns in `SInitial` represent the number of differences to perform. All values of `SInitial` must be greater than zero.

`z` – A `double` array containing the time series.

Methods

Compute

```
public void Compute()
```

Description

Computes the minimum AIC and optimum values for s and d based upon the candidates provided in `SInitial` and `DInitial`, and computes the values for the transformed series, $W_t(s,d)$.

Remarks

Warnings are printed if a row in `SInitial` is skipped due to too many observations lost in the differenced series or if problems occurred computing the optimum AIC using the differenced series.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

GetAR

```
public double[] GetAR()
```

Description

Returns the final autoregressive parameter estimates at the optimum in the transformed series W_t.

Returns

A `double` array containing the estimates for the autoregressive parameters.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

GetDInitial

```
public int[,] GetDInitial()
```

Description

Returns the candidate values for *d* to evaluate.

Returns

An `int` matrix containing the candidate values for *d* to evaluate.

GetOptimumD

```
public int[] GetOptimumD()
```

Description

Returns the optimum values for *d* selected among the candidates in `DInitial`.

Returns

An `int` array of length `DInitial.GetLength(1)` containing the optimum values for *d* selected among the candidates in `dInitial`.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

GetOptimumS

```
public int[] GetOptimumS()
```

Description

Returns the optimum values for *s* selected among the candidates in `SInitial`.

Returns

An `int` array of length `SInitial.GetLength(1)` containing the optimum values for *s* selected among the candidates in `SInitial` .

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

GetSInitial

```
public int[,] GetSInitial()
```

Description

Returns the candidate values for s to evaluate.

Returns

An `int` matrix containing the candidate values for s to evaluate.

GetTimeSeries

```
public double[] GetTimeSeries()
```

Description

Returns the time series.

Returns

A `double` array containing the time series values.

GetTransformedTimeSeries

```
public double[] GetTransformedTimeSeries()
```

Description

Returns the transformed series, $W_t(s,d)$.

Returns

A `double` array of length `z.Length` or `z.Length-NLost`, depending upon the setting for the `Exclude` property.

Remarks

$W_t(s,d)$ is an array of length `z.Length` or `z.Length-NLost` containing the optimum seasonally adjusted, autoregressive series, where `NLost` is the first lost observations in this series that are dropped due to differencing. If the missing values are not dropped the first `NLost` values of W_t will be set to `Double.NaN`. The `NLost` (p. 865) property can be used to obtain the number of lost observations.

The seasonal adjustment is done by selecting optimum values for $d_1,\ldots,d_m$, $s_1,\ldots,s_m$ and p in the AR model, where m is number of differences, m=`sIinitial.GetLength(1)`:

$$\phi_p(B)(\Delta_{s_1}^{d_1}\Delta_{s_2}^{d_2}\ldots\Delta_{s_m}^{d_m}Z_t-\mu)=a_t,$$

where $\{Z_t\}$ is the original time series, B is the backward shift operator defined by $B^kZ_t=Z_{t-k}$, with k ≥ 0, a_t is Gaussian white noise with $E[a_t]=0$ and $\text{Var}[a_t]=\sigma^2$, $\phi_p(B)=1-\phi_1B-\phi_2B^2-\cdots-\phi_pB^p$, $0\leq p\leq$ maxlag, $\Delta_s^d=(1-B^s)^d$, with $s>0, d\geq 0$, and μ is a centering parameter for the differenced series.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input triangular matrix is singular.

SetDInitial

```
public void SetDInitial(int[,] DInitial)
```

Description

Sets the candidate values for selecting the optimum seasonal adjustment prior to calling the compute method.

Parameter

`DInitial` – An `int` array of candidate values for *d* to evaluate. All values must be non-negative. `DInitial` must have the same number of differences (columns) as `SInitial`. By default, `DInitial` is initialized to all ones.

Example: ARSeasonalFit

Consider the Airline Data (Box, Jenkins and Reinsel 1994, p. 547) consisting of the monthly total number of international airline passengers from January 1949 through December 1960. Class `ARSeasonalFit` is used to compute the optimum seasonality representation of the adjusted series

$$W_t(s,d) = \Delta_{s_1}^{d_1}\Delta_{s_2}^{d_2}Z_t = (1-B^{s_1})^{d_1}(1-B^{s_2})^{d_2}Z_t,$$

where

$$s = (1,1)$$

or

$$s = (1,12)$$

and

$$d = (1,1).$$

As differenced series with minimum AIC,

$$W_t = \Delta_1^1 \Delta_{12}^2 Z_t = (Z_t - Z_{t-12}) - (Z_{t-1} - Z_{t-13}),$$

is identified.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using PrintMatrix = Imsl.Math.PrintMatrix;
using Imsl;

public class ARSeasonalFitEx1
{

    public static void  Main(String[] args)
    {
        double[] z =
            new double[]{112, 118, 132, 129, 121, 135, 148, 148, 136, 119,
                         104, 118, 115, 126, 141, 135, 125, 149, 170,
                         170, 158, 133, 114, 140, 145, 150, 178, 163,
                         172, 178, 199, 199, 184, 162, 146, 166, 171,
                         180, 193, 181, 183, 218, 230, 242, 209, 191,
                         172, 194, 196, 196, 236, 235, 229, 243, 264,
                         272, 237, 211, 180, 201, 204, 188, 235, 227,
                         234, 264, 302, 293, 259, 229, 203, 229, 242,
                         233, 267, 269, 270, 315, 364, 347, 312, 274,
                         237, 278, 284, 277, 317, 313, 318, 374, 413,
                         405, 355, 306, 271, 306, 315, 301, 356, 348,
                         355, 422, 465, 467, 404, 347, 305, 336, 340,
                         318, 362, 348, 363, 435, 491, 505, 404, 359,
                         310, 337, 360, 342, 406, 396, 420, 472, 548,
                         559, 463, 407, 362, 405, 417, 391, 419, 461,
                         472, 535, 622, 606, 508, 461, 390, 432};

        int[,] sInit = {{1, 1}, {1, 12}};

        ARSeasonalFit seasFit = new ARSeasonalFit(10, sInit, z);
        seasFit.Compute();
        Console.Out.WriteLine("NLost = " + seasFit.NLost);
        Console.Out.WriteLine("aic =" + seasFit.AIC);
        new PrintMatrix("Best Periods (Optimum S)").Print(seasFit.GetOptimumS());
        new PrintMatrix("Best Orders (Optimum D)").Print(seasFit.GetOptimumD());
        Console.Out.WriteLine("Best AR order selected = " + seasFit.AROrder);
        new PrintMatrix("AR Coefficients").Print(seasFit.GetAR());

        Console.Out.WriteLine("");
        double[] w = seasFit.GetTransformedTimeSeries();
        double[][] pack = new double[30][];
        for (int i = 0; i < 30; i++)
        {
            pack[i] = new double[2];
```

```
            pack[i][0] = z[i];
            pack[i][1] = w[i];
        }
        PrintMatrix pm = new PrintMatrix();
        String str = "First 30 elements of the original time series and " +
         "differenced series";
        pm.SetTitle(str);
        PrintMatrixFormat fmt = new PrintMatrixFormat();
        fmt.SetColumnLabels(new String[]{"Original", "Differenced"});
        pm.Print(fmt, pack);
    }
}
```

Output

```
NLost = 13
aic =606.397245653491
Best Periods (Optimum S)
   0
0   1
1  12

Best Orders (Optimum D)
   0
0  1
1  1

Best AR order selected = 1
    AR Coefficients
           0
0  -0.309834454941166


First 30 elements of the original time series and differenced series
    Original  Differenced
 0    112          NaN
 1    118          NaN
 2    132          NaN
 3    129          NaN
 4    121          NaN
 5    135          NaN
 6    148          NaN
 7    148          NaN
 8    136          NaN
 9    119          NaN
10    104          NaN
11    118          NaN
12    115          NaN
13    126          5
14    141          1
15    135         -3
16    125         -2
17    149         10
18    170          8
19    170          0
```

```
20    158          0
21    133         -8
22    114         -4
23    140         12
24    145          8
25    150         -6
26    178         13
27    163         -9
28    172         19
29    178        -18
```

ARSeasonalFit.CenterMethod Enumeration

`public enumeration Imsl.Stat.ARSeasonalFit.CenterMethod`

Methods for centering the input time series.

Fields

Mean

`public Imsl.Stat.ARSeasonalFit.CenterMethod Mean`

Description

Indicates the transformed series should be centered using the average of the differenced series.

Median

`public Imsl.Stat.ARSeasonalFit.CenterMethod Median`

Description

Indicates the transformed series should be centered using the median of the differenced series.

None

`public Imsl.Stat.ARSeasonalFit.CenterMethod None`

Description

Indicates the transformed series should not be centered.

ARMA Class

`public class Imsl.Stat.ARMA`

Computes least-square estimates of parameters for an ARMA model.

Class `ARMA` computes estimates of parameters for a nonseasonal ARMA model given a sample of observations, $\{W_t\}$, for $t = 1, 2, \ldots, n$, where n = `z.Length`. There are two methods, method of moments and least squares, from which to choose. The default is method of moments.

Two methods of parameter estimation, method of moments and least squares, are provided. The user can choose a method using the `Method` property. If the user wishes to use the least-squares algorithm, the preliminary estimates are the method of moments estimates by default. Otherwise, the user can input initial estimates by using the `SetInitialEstimates` method. The following table lists the appropriate methods and properties for both the method of moments and least-squares algorithm:

Least Squares	Both Method of Moment and Least Squares
	`Center`
`ARLags`	`Method`
`MALags`	`RelativeError`
`Backcasting`	`MaxIterations`
`ConvergenceTolerance`	`Mean`
`SetInitialEstimates`	`Mean`
`Residual`	`AutoCovariance`
`SSResidual`	`Variance`
`ParamEstimatesCovariance`	`Constant`
	`AR`
	`MA`

Method of Moments Estimation

Suppose the time series $\{Z_t\}$ is generated by an ARMA (p, q) model of the form

$$\phi(B)Z_t = \theta_0 + \theta(B)A_t$$

$$\text{for } t \in \{0, \pm 1, \pm 2, \ldots\}$$

Let $\hat{\mu} = \text{Mean}$ be the estimate of the mean μ of the time series $\{Z_t\}$, where $\hat{\mu}$ equals the following:

$$\hat{\mu} = \begin{cases} \mu & \text{for } \mu \text{ known} \\ \frac{1}{n}\sum_{t=1}^{n} Z_t & \text{for } \mu \text{ unknown} \end{cases}$$

The autocovariance function is estimated by

$$\hat{\sigma}(k) = \frac{1}{n}\sum_{t=1}^{n-k}(Z_t - \hat{\mu})(Z_{t+k} - \hat{\mu})$$

for $k = 0, 1, \ldots, K$, where $K = p + q$. Note that $\hat{\sigma}(0)$ is an estimate of the sample variance.

Given the sample autocovariances, the function computes the method of moments estimates of the autoregressive parameters using the extended Yule-Walker equations as follows:

$$\hat{\Sigma}\hat{\phi} = \hat{\sigma}$$

where

$$\hat{\phi} = \left(\hat{\phi}_1, \ldots, \hat{\phi}_p\right)^T$$

$$\hat{\Sigma}_{ij} = \hat{\sigma}(|q + i - j|), \; i, j = 1, \ldots, p$$

$$\hat{\sigma}_i = \hat{\sigma}(q + i), \; i = 1, \ldots, p$$

The overall constant θ_0 is estimated by the following:

$$\hat{\theta}_0 = \begin{cases} \hat{\mu} \text{ for } p = 0 \\ \hat{\mu}\left(1 - \sum_{i=1}^{p}\hat{\phi}_i\right) \text{ for } p > 0 \end{cases}$$

The moving average parameters are estimated based on a system of nonlinear equations given $K = p + q + 1$ autocovariances, $\sigma(k)$ for $k = 1, \ldots, K$, and p autoregressive parameters ϕ_i for $i = 1, \ldots, p$.

Let $Z_t' = \phi(B)Z_t$. The autocovariances of the derived moving average process $Z_t' = \theta(B)A_t$ are estimated by the following relation:

$$\hat{\sigma}'(k) = \begin{cases} \hat{\sigma}(k) \text{ for } p = 0 \\ \sum_{i=0}^{p}\sum_{j=0}^{p}\hat{\phi}_i\hat{\phi}_j(\hat{\sigma}(|k + i - j|)) \text{ for } p \geq 1, \hat{\phi}_0 \equiv -1 \end{cases}$$

The iterative procedure for determining the moving average parameters is based on the relation

$$\sigma(k) = \begin{cases} \left(1 + \theta_1^2 + \ \ldots \ + \theta_q^2\right) \sigma_A^2 \ \text{ for } k = 0 \\ \left(-\theta_k + \theta_1 \theta_{k+1} + \ \ldots \ + \theta_{q-k} \theta_q\right) \sigma_A^2 \ \text{ for } k \geq 1 \end{cases}$$

where $\sigma(k)$ denotes the autocovariance function of the original Z_t process.

Let $\tau = (\tau_0, \tau_1, \ldots, \tau_q)^T$ and $f = (f_0, f_1, \ldots, f_q)^T$, where

$$\tau_j = \begin{cases} \sigma_A \ \text{ for } \ j = 0 \\ -\theta_j / \tau_0 \ \text{ for } \ j = 1, \ \ldots, \ q \end{cases}$$

and

$$f_j = \sum_{i=0}^{q-j} \tau_i \tau_{i+j} - \hat{\sigma}'(j) \ \text{ for } \ j = 0, \ 1, \ \ldots, \ q$$

Then, the value of τ at the (i + 1)-th iteration is determined by the following:

$$\tau^{i+1} = \tau^i - \left(T^i\right)^{-1} f^i$$

The estimation procedure begins with the initial value

$$\tau^0 = (\sqrt{\hat{\sigma}'(0)}, \quad 0, \ \ldots, \ 0)^T$$

and terminates at iteration i when either $\left\| f^i \right\|$ is less than `RelativeError` or i equals `MaxIterations`. The moving average parameter estimates are obtained from the final estimate of τ by setting

$$\hat{\theta}_j = -\tau_j / \tau_0 \ \text{ for } \ j = 1, \ \ldots, \ q$$

The random shock variance is estimated by the following:

$$\hat{\sigma}_A^2 = \begin{cases} \hat{\sigma}(0) - \sum_{i=1}^{p} \hat{\phi}_i \hat{\sigma}(i) \ \text{ for } q = 0 \\ \tau_0^2 \ \text{ for } q \geq 0 \end{cases}$$

See Box and Jenkins (1976, pp. 498-500) for a description of a function that performs similar computations.

Least-squares Estimation

Suppose the time series $\{Z_t\}$ is generated by a nonseasonal ARMA model of the form,

$$\phi(B)(Z_t - \mu) = \theta(B)A_t \quad \text{for } t \in \{0, \pm 1, \pm 2, \ldots\}$$

where B is the backward shift operator, μ is the mean of Z_t, and

$$\phi(B) = 1 - \phi_1 B^{l_\phi(1)} - \phi_2 B^{l_\phi(2)} - \ \ldots\ - \phi_p B^{l_\phi(p)} \qquad \text{for } p \geq 0$$

$$\theta(B) = 1 - \theta_1 B^{l_\theta(1)} - \theta_2 B^{l_\theta(2)} - \ \ldots\ - \theta_q B^{l_\theta(q)} \qquad \text{for } q \geq 0$$

with p autoregressive and q moving average parameters. Without loss of generality, the following is assumed:

$$1 \leq l_\phi(1) \leq l_\phi(2) \leq \ldots \leq l_\phi(p)$$

$$1 \leq l_\theta(1) \leq l_\theta(2) \leq \ldots \leq l_\theta(q)$$

so that the nonseasonal ARMA model is of order (p', q'), where $p' = l_\theta(p)$ and $q' = l_\theta(q)$. Note that the usual hierarchical model assumes the following:

$$l_\phi(i) = i, 1 \leq i \leq p$$

$$l_\theta(j) = j, 1 \leq j \leq q$$

Consider the sum-of-squares function

$$S_T(\mu, \phi, \theta) = \sum_{-T+1}^{n} [A_t]^2$$

where

$$[A_t] = E[A_t \,|(\mu, \phi, \theta, Z)]$$

and T is the backward origin. The random shocks $\{A_t\}$ are assumed to be independent and identically distributed

$$N\left(0, \sigma_A^2\right)$$

random variables. Hence, the log-likelihood function is given by

$$l\left(\mu, \phi, \theta, \sigma_A\right) = f\left(\mu, \phi, \theta\right) - n\ln\left(\sigma_A\right) - \frac{S_T\left(\mu, \phi, \theta\right)}{2\sigma_A^2}$$

where $f(\mu, \phi, \theta)$ is a function of $\mu, \phi,$ and θ.

For $T = 0$, the log-likelihood function is conditional on the past values of both Z_t and A_t required to initialize the model. The method of selecting these initial values usually introduces transient bias into the model (Box and Jenkins 1976, pp. 210-211). For $T = \infty$, this dependency vanishes, and estimation problem concerns maximization of the unconditional log-likelihood function. Box and Jenkins (1976, p. 213) argue that

$$S_\infty\left(\mu, \phi, \theta\right) / \left(2\sigma_A^2\right)$$

dominates

$$l\left(\mu, \phi, \theta, \sigma_A^2\right)$$

The parameter estimates that minimize the sum-of-squares function are called least-squares estimates. For large n, the unconditional least-squares estimates are approximately equal to the maximum likelihood-estimates.

In practice, a finite value of T will enable sufficient approximation of the unconditional sum-of-squares function. The values of $[A_T]$ needed to compute the unconditional sum of squares are computed iteratively with initial values of Z_t obtained by back forecasting. The residuals (including backcasts), estimate of random shock variance, and covariance matrix of the final parameter estimates also are computed. ARIMA parameters can be computed by using `Difference` with `ARMA`.

Forecasting

The Box-Jenkins forecasts and their associated probability limits for a nonseasonal ARMA model are computed given a sample of n = `z.Length`, $\{Z_t\}$ for $t = 1, 2, \ldots, n$.

Suppose the time series Z_t is generated by a nonseasonal ARMA model of the form

$$\phi(B)Z_t = \theta_0 + \theta(B)A_t$$

for $t \in \{0, \pm 1, \pm 2, \ldots\}$, where B is the backward shift operator, θ_0 is the constant, and

$$\phi(B) = 1 - \phi_1 B^{l_\phi(1)} - \phi_2 B^{l_\phi(2)} - \ \ldots \ - \phi_p B^{l_\phi(p)}$$

$$\theta(B) = 1 - \theta_1 B^{l_\theta(1)} - \theta_2 B^{l_\theta(2)} - \ \ldots \ - \theta_q B^{l_\theta(q)}$$

with p autoregressive and q moving average parameters. Without loss of generality, the following is assumed:

$$1 \leq l_\phi(1) \leq l_\phi(2) \leq \ldots l_\phi(p)$$

$$1 \leq l_\theta(1) \leq l_\theta(2) \leq \ldots \leq l_\theta(q)$$

so that the nonseasonal ARMA model is of order (p', q'), where $p' = l_\theta(p)$ and $q' = l_\theta(q)$. Note that the usual hierarchical model assumes the following:

$$l_\phi(i) = i, 1 \leq i \leq p$$

$$l_\theta(j) = j, 1 \leq j \leq q$$

The Box-Jenkins forecast at origin t for lead time l of Z_{t+1} is defined in terms of the difference equation

$$\hat{Z}_t(l) = \theta_0 + \phi_1 \left[Z_{t+l-l_\phi(1)} \right] + \ \ldots \ + \phi_p \left[Z_{t+l-l_\phi(p)} \right]$$

$$+ \left[A_{t+l} \right] - \theta_1 \left[A_{t+l-l_\theta(1)} \right] \ - \ \ldots \ - \theta_q \left[A_{t+l-l_\theta(q)} \right]$$

where the following is true:

$$\left[Z_{t+k} \right] = \begin{cases} Z_{t+k} & \text{for } k = 0, -1, -2, \ldots \\ \hat{Z}_t(k) & \text{for } k = 1, 2, \ldots \end{cases}$$

$$[A_{t+k}] = \begin{cases} Z_{t+k} - \hat{Z}_{t+k-1}(1) & \text{for } k = 0, -1, -2, \ldots \\ 0 & \text{for } k = 1, 2, \ldots \end{cases}$$

The $100(1-\alpha)$ percent probability limits for Z_{t+l} are given by

$$\hat{Z}_t(l) \pm z_{\alpha/2} \left\{ 1 + \sum_{j=1}^{l-1} \psi_j^2 \right\}^{1/2} \sigma_A$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percentile of the standard normal distribution

$$\sigma_A^2$$

and

$$\left\{ \psi_j^2 \right\}$$

are the parameters of the random shock form of the difference equation. Note that the forecasts are computed for lead times $l = 1, 2, \ldots, L$ at origins $t = (n-b), (n-b+1), \ldots, n$, where $L =$ nForecast and $b =$ BackwardOrigin.

The Box-Jenkins forecasts minimize the mean-square error

$$E\left[Z_{t+l} - \hat{Z}_t(l)\right]^2$$

Also, the forecasts can be easily updated according to the following equation:

$$\hat{Z}_{t+1}(l) = \hat{Z}_t(l+1) + \psi_l A_{t+1}$$

This approach and others are discussed in Chapter 5 of Box and Jenkins (1976).

Properties

BackwardOrigin

```
public int BackwardOrigin {get; set; }
```

Description

The maximum backward origin.

Property Value

An `int` specifying the maximum backward origin.

Remarks

`BackwardOrigin` must be greater than or equal to 0 and less than or equal to `z.Length - max(maxar, maxma)`, where *maxar* = *max*(`ARLags[i]`), *maxma* = *max*(`MALags[j]`), and forecasts at origins `z.Length - BackwardOrigin` through `z.Length` are generated. By default, `BackwardOrigin` = 0.

Center

```
public bool Center {get; set; }
```

Description

The center option.

Property Value

A `bool` value specifying whether the time series center occurs about its mean.

Remarks

If `Center` is set to `false`, the time series is not centered about its mean. If `Center` is set to `true`, the time series is centered about its mean. By Default, `Center = false`.

Confidence

```
public double Confidence {get; set; }
```

Description

The confidence level for calculating confidence limit deviations returned from `GetDeviations`.

Property Value

A `double` scalar specifying the confidence level used in computing forecasts confidence intervals.

Remarks

Typical choices for `Confidence` are 0.90, 0.95, and 0.99. `Confidence` must be greater than 0.0 and less than 1.0. By default, `Confidence` = 0.95.

Constant

```
public double Constant {get; }
```

Description

The constant parameter estimate.

Property Value

A `double` scalar containing the constant parameter estimate.

Remarks

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `NaN`.

ConvergenceTolerance

```
public double ConvergenceTolerance {get; set; }
```

Description

The tolerance level used to determine convergence of the nonlinear least-squares algorithm.

Property Value

A `double` scalar containing the tolerance level used to determine convergence of the nonlinear least-squares algorithm.

Remarks

`ConvergenceTolerance` represents the minimum relative decrease in sum of squares between two iterations required to determine convergence. Hence, `ConvergenceTolerance` must be greater than or equal to 0. By default `ConvergenceTolerance` = 10^{-20}.

InnovationVariance

```
public double InnovationVariance {get; }
```

Description

The variance of the random shock.

Property Value

A `double` scalar equal to the variance of the random shock.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations.

Property Value

An `int` scalar specifying the maximum number of iterations allowed in the nonlinear equation solver used in both the method of moments and least-squares algorithms.

Remarks

By default, `MaxIterations` = 200.

Mean

```
public double Mean {get; set; }
```

Description

An update of the mean of the time series z.

Property Value

A `double` scalar containing the mean of the time series `z`.

Remarks

If the time series is not centered about its mean, and the least-squares algorithm is used, the mean is not used in parameter estimation. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Method

```
public Imsl.Stat.ARMA.ParamEstimation Method {get; set; }
```

Description

The method used to estimate the autoregressive and moving average parameters estimates.

Property Value

A `ParamEstimation` specifying the method used to estimate the autoregressive and moving average parameters estimates.

Remarks

If `ARMA.ParamEstimation.MethodOfMoments` is specified, the autoregressive and moving average parameters are estimated by a method of moments procedure.

If `ARMA.ParamEstimation.LeastSquares` is specified, the autoregressive and moving average parameters are estimated by a least-squares procedure. By default, `Method = ARMA.ParamEstimation.MethodOfMoments`.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

RelativeError

```
public double RelativeError {get; set; }
```

Description

The stopping criterion for use in the nonlinear equation solver.

Property Value

A `double` scalar containing the stopping criterion for use in the nonlinear equation solver used in both the method of moments and least-squares algorithms.

Remarks

By default, `RelativeError` = 100 * 2.2204460492503131e-16.

SSResidual

```
public double SSResidual {get; }
```

Description

The sum of squares of the random shock.

Property Value

A `double` scalar containing the sum of squares of the random shock, $\text{residual}[0]^2 + \ldots + \text{residual}[\text{na} - 1]^2$, where `residual` is the array returned from the `GetResidual` method and `na = residual.Length`.

Remarks

The `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`. This property is only applicable using least-squares algorithm.

Variance

```
public double Variance {get; }
```

Description

The variance of the time series `z`.

Property Value

A `double` scalar containing the variance of the time series `z`.

Remarks

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `NaN`.

Constructor

ARMA

```
public ARMA(int p, int q, double[] z)
```

Description

Constructor for `ARMA`.

Parameters

`p` – An `int` scalar containing the number of autoregressive (AR) parameters.

`q` – An `int` scalar containing the number of moving average (MA) parameters.

`z` – A `double` array containing the observations.

Exception

`System.ArgumentException` is thrown if `p`, `q`, and `z.Length` are not consistent.

Methods

Compute

```
public void Compute()
```

Description

Computes least-square estimates of parameters for an ARMA model.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NoProgressException` is thrown if the algorithm is not making any progress.

Forecast

```
public double[,] Forecast(int nForecast)
```

Description

Computes forecasts and their associated probability limits for an ARMA model.

Parameter

`nForecast` – An `int` scalar containing the maximum lead time for forecasts. `nForecast` must be greater than 0.

Returns

A `double` matrix of dimensions of `nForecast` by `BackwardOrigin+1` containing the forecasts. Return `null` if the least-square estimates of parameters is not computed.

GetAR

```
public double[] GetAR()
```

Description

Returns the final autoregressive parameter estimates.

Returns

A `double` array of length `p` containing the final autoregressive parameter estimates.

Remarks

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

GetAutoCovariance

```
public double[] GetAutoCovariance()
```

Description

Returns the autocovariances of the time series `z`.

Returns

A `double` array containing the autocovariances of lag `k`, where `k = 1, ..., p + q + 1`.

Remarks

Note that the `Compute` method must be invoked before this method. Otherwise, the method throws a `NullReferenceException` exception.

GetDeviations

```
public double[] GetDeviations()
```

Description

Returns the deviations for each forecast used for calculating the forecast confidence limits.

Returns

A `double` array of length `nForecast` containing the deviations for calculating forecast confidence intervals. The confidence level is specified in the `Confidence` property.

Remarks

Note that the `Forecast` method must be invoked first before invoking this method. Otherwise, the method returns `null`.

GetForecast

```
public double[] GetForecast(int nForecast)
```

Description

Returns forecasts

Parameter

`nForecast` – An input `int` representing the number of requested forecasts beyond the last value in the series.

Returns

A `double` array containing the `nForecast+BackwardOrigin` forecasts. The first `BackwardOrigin` forecasts are one-step ahead forecasts for the last `BackwardOrigin` values in the series. The next `nForecast` values in the returned series are forecasts for the next values beyond the series.

GetMA

```
public double[] GetMA()
```

Description

Returns the final moving average parameter estimates.

Returns

A `double` array of length `q` containing the final moving average parameter estimates.

Remarks

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

GetNumberOfBackcasts

```
public int GetNumberOfBackcasts()
```

Description

Returns the number of backcasts used to calculate the AR coefficients for the time series `z`.

Returns

An `int` containing the number of backcasts calculated.

Remarks

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

GetParamEstimatesCovariance

```
public double[,] GetParamEstimatesCovariance()
```

Description

Returns the covariances of parameter estimates.

Returns

A `double` matrix of *np* by *np* dimensions, where *np* `= p + q + 1` if `z` is centered about `Mean`, and *np* `= p + q` if `z` is not centered, containing the covariances of parameter estimates.

Remarks

The ordering of variables is `mean`, `AR`, and `MA`.

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

GetPsiWeights

```
public double[] GetPsiWeights()
```

Description

Returns the psi weights of the infinite order moving average form of the model.

Returns

A `double` array of length `nForecast` containing the psi weights of the infinite order moving average form of the model.

Remarks

Note that the `forecast` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

GetResidual

```
public double[] GetResidual()
```

Description

Returns the residuals at the final parameter estimate.

Returns

A `double` array of length `z.Length - max(arLags[i]) + length` containing the residuals (including backcasts) at the final parameter estimate point in the first `z.Length - max(arLags[i]) + nb`, where `nb` is the number of values backcast.

Remarks

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.This method is only applicable using least-squares algorithm.

SetARLags

```
public void SetARLags(int[] arLags)
```

Description

The order of the autoregressive parameters.

Parameter

`arLags` – An `int` array of length `p` containing the order of the autoregressive parameters.

Remarks

The elements of `arLags` must be greater than or equal to 1. By default, `arLags = [1, 2, ..., p]`.

SetARMAInfo

```
public void SetARMAInfo(double constant, double[] ar, double[] ma, double var)
```

Description

Sets the ARMAInfo object to previously determined values

Parameters

`constant` – A `double` scalar equal to the constant term in the ARMA model.

`ar` – A `double` array of length `p` containing estimates of the autoregressive parameters.

`ma` – A `double` array of length `q` containing estimates of the moving average parameters.

`var` – A `double` scalar equal to the innovation variance

SetBackcasting

```
public void SetBackcasting(int maxBackcast, double tolerance)
```

Description

Sets backcasting option.

Parameters

`maxBackcast` – An `int` scalar containing the maximum length of backcasting and must be greater than or equal to 0. By default, `maxBackcast` = 10.

`tolerance` – A `double` scalar containing the tolerance level used to determine convergence of the backcast algorithm. Typically, `tolerance` is set to a fraction of an estimate of the standard deviation of the time series. By default, `tolerance` = 0.01 * standard deviation of `z`.

SetInitialAREstimates

```
public void SetInitialAREstimates(double[] ar)
```

Description

Sets preliminary autoregressive estimates.

Parameter

`ar` – A `double` array of length `p` containing preliminary estimates of the autoregressive parameters.

Remarks

`ar` and `ma` are computed internally if this method is not used. This method is only applicable using least-squares algorithm.

SetInitialEstimates

```
public void SetInitialEstimates(double[] ar, double[] ma)
```

Description

Sets preliminary estimates.

Parameters

`ar` – A `double` array of length `p` containing preliminary estimates of the autoregressive parameters.

`ma` – A `double` array of length `q` containing preliminary estimates of the moving average parameters.

Remarks

`ar` and `ma` are computed internally if this method is not used. This method is only applicable using least-squares algorithm.

SetInitialMAEstimates

```
public void SetInitialMAEstimates(double[] ma)
```

Description

Sets preliminary moving average estimates.

Parameter

`ma` – A `double` array of length `q` containing preliminary estimates of the moving average parameters.

Remarks

`ma` are computed internally if this method is not used. This method is only applicable using least-squares algorithm.

SetMALags

```
public void SetMALags(int[] maLags)
```

Description

Sets the order of the moving average parameters.

Parameter

`maLags` – An `int` array of length `q` containing the order of the moving average parameters.

Remarks

The `maLags` elements must be greater than or equal to 1. By default, `maLags = [1, 2, ..., q]`.

Example 1: ARMA

Consider the Wolfer Sunspot Data (Anderson 1971, p. 660) consisting of the number of sunspots observed each year from 1749 through 1924. The data set for this example consists of the number of sunspots observed from 1770 through 1869. The method of moments estimates

$$\hat{\theta}_0, \hat{\phi}_1, \hat{\phi}_2, \text{and } \hat{\theta}_1$$

for the ARMA(2, 1) model

$$z_t = \theta_0 + \phi_1 z_{t-1} + \phi_2 z_{t-2} - \theta_1 A_{t-1} + A_t$$

where the errors A_t are independently normally distributed with mean zero and variance

$$\sigma_A^2$$

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class ARMAEx1
{
    public static void  Main(String[] args)
    {
        double[] z = new double[]{  100.8, 81.6, 66.5, 34.8, 30.6,
                                     7, 19.8, 92.5, 154.4, 125.9,
                                     84.8, 68.1, 38.5, 22.8, 10.2,
                                     24.1, 82.9, 132, 130.9, 118.1,
                                     89.9, 66.6, 60, 46.9, 41,
                                     21.3, 16, 6.4, 4.1, 6.8,
                                     14.5, 34, 45, 43.1, 47.5,
                                     42.2, 28.1, 10.1, 8.1, 2.5,
```

```
                                    0, 1.4, 5, 12.2, 13.9,
                                    35.4, 45.8, 41.1, 30.4, 23.9,
                                    15.7, 6.6, 4, 1.8, 8.5,
                                    16.6, 36.3, 49.7, 62.5, 67,
                                    71, 47.8, 27.5, 8.5, 13.2,
                                    56.9, 121.5, 138.3, 103.2, 85.8,
                                    63.2, 36.8, 24.2, 10.7, 15,
                                    40.1, 61.5, 98.5, 124.3, 95.9,
                                    66.5, 64.5, 54.2, 39, 20.6,
                                    6.7, 4.3,  22.8, 54.8, 93.8,
                                    95.7, 77.2, 59.1, 44, 47,
                                    30.5, 16.3, 7.3, 37.3, 73.9};

        ARMA arma = new ARMA(2, 1, z);
        arma.Compute();

        new PrintMatrix("AR estimates are:  ").Print(arma.GetAR());
        Console.Out.WriteLine();
        new PrintMatrix("MA estimate is:  ").Print(arma.GetMA());
    }
}
```

Output

```
  AR estimates are:
           0
0   1.24425777984372
1  -0.575149766040151


   MA estimate is:
           0
0  -0.124089747872598
```

Example 2: ARMA

The data for this example are the same as that for Example 1. Preliminary method of moments estimates are computed by default, and the method of least squares is used to find the final estimates. Note that at the end of the output, a warning message appears. In most cases, this warning message can be ignored. There are three general reasons this warning can occur:

1. Convergence is declared using the criterion based on tolerance, but the gradient of the residual sum-of-squares function is nonzero. This occurs in this example. Either the message can be ignored or `ConvergenceTolerance` can be reduced to allow more iterations and a slightly more accurate solution.
2. Convergence is declared based on the fact that a very small step was taken, but the gradient of the residual sum-of-squares function was nonzero. This message can usually be ignored. Sometimes, however, the algorithm is making very slow progress and is not near a minimum.
3. Convergence is not declared after 100 iterations.

Trying a smaller value for `ConvergenceTolerance` can help determine what caused the error message.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class ARMAEx2
{
    public static void  Main(String[] args)
    {
        double[] arInit = new double[]{1.24426e0, - 5.75149e-1};
        double[] maInit = new double[]{- 1.24094e-1};
        double[] z = new double[]{  100.8, 81.6, 66.5, 34.8, 30.6,
                                    7, 19.8, 92.5, 154.4, 125.9,
                                    84.8, 68.1, 38.5, 22.8, 10.2,
                                    24.1, 82.9, 132, 130.9, 118.1,
                                    89.9, 66.6, 60, 46.9, 41,
                                    21.3, 16, 6.4, 4.1, 6.8,
                                    14.5, 34, 45, 43.1, 47.5,
                                    42.2, 28.1, 10.1, 8.1, 2.5,
                                    0, 1.4, 5, 12.2, 13.9,
                                    35.4, 45.8, 41.1, 30.4, 23.9,
                                    15.7, 6.6, 4, 1.8, 8.5,
                                    16.6, 36.3, 49.7, 62.5, 67,
                                    71, 47.8, 27.5, 8.5, 13.2,
                                    56.9, 121.5, 138.3, 103.2, 85.8,
                                    63.2, 36.8, 24.2, 10.7, 15,
                                    40.1, 61.5, 98.5, 124.3, 95.9,
                                    66.5, 64.5, 54.2, 39, 20.6,
                                    6.7, 4.3,  22.8, 54.8, 93.8,
                                    95.7, 77.2, 59.1, 44, 47,
                                    30.5, 16.3, 7.3, 37.3, 73.9};

        ARMA arma = new ARMA(2, 1, z);
        arma.Method = Imsl.Stat.ARMA.ParamEstimation.LeastSquares;
        arma.SetInitialEstimates(arInit, maInit);
        arma.ConvergenceTolerance = 0.125;
        arma.Mean = 46.976;
        arma.Compute();

        new PrintMatrix("AR estimates are:  ").Print(arma.GetAR());
        Console.Out.WriteLine();
        new PrintMatrix("MA estimate is:  ").Print(arma.GetMA());
    }
}
```

Output

```
  AR estimates are:
           0
0   1.39325700313638
1  -0.733660553488482
```

```
   MA estimate is:
           0
0  -0.137145395974998

Imsl.Stat.ARMA: Relative function convergence - Both the scaled actual and
predicted reductions in the function are less than or equal to the relative
function convergence tolerance "ConvergenceTolerance" = 0.0645856533065147.
Imsl.Stat.ARMA: Least squares estimation of the parameters has failed to
converge.  Increase "length" and/or "tolerance" and/or "convergence_tolerance".
The estimates of the parameters at the last iteration may be used as new
starting values.
```

Example 3: Forecasting

Consider the Wolfer Sunspot Data (Anderson 1971, p. 660) consisting of the number of sunspots observed each year from 1749 through 1924. The data set for this example consists of the number of sunspots observed from 1770 through 1869. An ARMA(2,1) model is fitted to these data using the Method of Moments. With `BackwardOrigin` = 3, the `Forecast` method is used to obtain forecasts starting from 1866, 1867, 1868, and 1869, respectively. Note that the values in the first row of the matrix returned by this method are the one-step ahead forecasts for 1867, 1868, ..., 1870. The values in the second row are the two-step ahead forecasts for 1868, 1869, ..., 1871, etc.

Method `GetForecast` is used to compute the one-step ahead forecasts setting `BackwardOrigin` = 10. This obtains the one-step ahead forecasts for the last 10 observations in the series, i.e. years 1860-1869, plus the next 5 years. The upper 90% confidence limits are computed for these forecasts using the `GetDeviations` method.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class ARMAEx3
{
    public static void  Main(String[] args)
    {
        double[] z = new double[]{  100.8, 81.6, 66.5, 34.8, 30.6,
                                     7, 19.8, 92.5, 154.4, 125.9,
                                     84.8, 68.1, 38.5, 22.8, 10.2,
                                     24.1, 82.9, 132, 130.9, 118.1,
                                     89.9, 66.6, 60, 46.9, 41,
                                     21.3, 16, 6.4, 4.1, 6.8,
                                     14.5, 34, 45, 43.1, 47.5,
                                     42.2, 28.1, 10.1, 8.1, 2.5,
                                     0, 1.4, 5, 12.2, 13.9,
                                     35.4, 45.8, 41.1, 30.4, 23.9,
                                     15.7, 6.6, 4, 1.8, 8.5,
                                     16.6, 36.3, 49.7, 62.5, 67,
                                     71, 47.8, 27.5, 8.5, 13.2,
                                     56.9, 121.5, 138.3, 103.2, 85.8,
                                     63.2, 36.8, 24.2, 10.7, 15,
                                     40.1, 61.5, 98.5, 124.3, 95.9,
                                     66.5, 64.5, 54.2, 39, 20.6,
```

```
                            6.7, 4.3,  22.8, 54.8, 93.8,
                            95.7, 77.2, 59.1, 44, 47,
                            30.5, 16.3, 7.3, 37.3, 73.9};
double[,] printEstimates = new double[1,4];
PrintMatrixFormat pmf = new PrintMatrixFormat();
PrintMatrix pm        = new PrintMatrix();
pm.SetColumnSpacing(3);

ARMA arma = new ARMA(2, 1, z);
arma.Compute();

printEstimates[0,0] = arma.Constant;
double[] ar = arma.GetAR();
printEstimates[0,1] = ar[0];
printEstimates[0,2] = ar[1];
double[] ma = arma.GetMA();
printEstimates[0,3] = ma[0];
String[] estimateLabels = {"Constant", "AR(1)", "AR(2)", "MA(1)"};
pmf.SetColumnLabels(estimateLabels);
pmf.NumberFormat = "0.0000";
pm.SetTitle("ARMA ESTIMATES");
pm.Print(pmf, printEstimates);

String[] labels = new String[]{"From 1866",
                               "From 1867",
                               "From 1868",
                               "From 1869"};
pmf.SetColumnLabels(labels);
pmf.FirstRowNumber = 1;
pmf.NumberFormat = "00.0";
arma.BackwardOrigin = 3;
new PrintMatrix("FORECASTS").Print(pmf, arma.Forecast(5));

double[,] printTable = new double[15,4];
/* FORECASTING - An example of forecasting using the ARMA estimates
 * In this case, forecasts are returned for the last 10 values in the
 * series followed by the forecasts for the next 5 values.
 */
String[] forecastLabels={"Observed", "Forecast",
                         "Residual", "UCL(90%)"};
pmf.SetColumnLabels(forecastLabels);
int backOrigin = 10;
int n_forecast = 5;
arma.BackwardOrigin = backOrigin;
arma.Confidence = 0.9;
double[] forecasts  = arma.GetForecast(n_forecast);
double[] deviations = arma.GetDeviations();
for(int i=0; i<backOrigin; i++)
{
    printTable[i,0] = z[z.Length-backOrigin+i];
    printTable[i,1] = forecasts[i];
    printTable[i,2] = z[z.Length-backOrigin+i]-forecasts[i];
    printTable[i,3] = forecasts[i] + deviations[0];
}
for(int i=backOrigin; i<n_forecast+backOrigin; i++)
{
```

```
            printTable[i,0] = Double.NaN;
            printTable[i,1] = forecasts[i];
            printTable[i,2] = Double.NaN;
            printTable[i,3] = forecasts[i] + deviations[i-backOrigin];
        }
        pmf.FirstRowNumber = 1869-backOrigin+1;
        pmf.NumberFormat = "000.0";
        pm.SetTitle("ARMA ONE-STEP AHEAD FORECASTS");
        pm.Print(pmf, printTable);

    }
}
```

Output

```
               ARMA ESTIMATES
    Constant    AR(1)    AR(2)     MA(1)
0   15.5440    1.2443   -0.5751   -0.1241

                   FORECASTS
   From 1866  From 1867  From 1868  From 1869
1    17.3       14.0       60.6       87.7
2    27.7       28.8       69.6       82.1
3    40.1       43.3       67.2       67.3
4    49.5       52.9       59.2       52.1
5    54.0       56.4       50.5       41.6

           ARMA ONE-STEP AHEAD FORECASTS
       Observed   Forecast   Residual   UCL(90%)
1860    095.7      103.4      -007.7     131.3
1861    077.2      079.7      -002.5     107.6
1862    059.1      056.2       002.9     084.1
1863    044.0      045.0      -001.0     072.9
1864    047.0      036.2       010.8     064.0
1865    030.5      050.1      -019.6     077.9
1866    016.3      024.0      -007.7     051.9
1867    007.3      017.3      -010.0     045.2
1868    037.3      014.0       023.3     041.9
1869    073.9      060.6       013.3     088.5
1870    NaN        087.7       NaN       115.6
1871    NaN        082.1       NaN       129.4
1872    NaN        067.3       NaN       124.1
1873    NaN        052.1       NaN       111.3
1874    NaN        041.6       NaN       101.0
```

ARMA.ParamEstimation Enumeration

```
public enumeration Imsl.Stat.ARMA.ParamEstimation
```

Parameter Estimation procedures.

Fields

LeastSquares

```
public Imsl.Stat.ARMA.ParamEstimation LeastSquares
```

Description

Indicates autoregressive and moving average parameters are estimated by a least-squares procedure.

MethodOfMoments

```
public Imsl.Stat.ARMA.ParamEstimation MethodOfMoments
```

Description

Indicates autoregressive and moving average parameters are estimated by a method of moments procedure.

ARMAEstimateMissing Class

```
public class Imsl.Stat.ARMAEstimateMissing
```

Estimates missing values in a time series collected with equal spacing. Missing values can be replaced by these estimates prior to fitting a time series using the `ARMA` class.

Traditional time series analysis as described by Box, Jenkins and Reinsel (1994) requires the observations be made at equidistant time points $t_0, t_1, \ldots, t_n$ where $t_i = t_0 + i$. When observations are missing, `ARMA` requires that they be replaced with suitable estimates. Class `ARMAEstimateMissing` offers 4 methods for estimating missing values: `Median, CubicSpline, AR_1,` and `AR_p`

The centering method `Median` estimates the missing observations in a gap by the median of the last four time series values before and the first four values after the gap. If not enough values are available before or after the gap then the number is reduced accordingly. This method is very fast and simple, but its use is limited to stationary ergodic series without outliers and level shifts.

Centering method `CubicSpline` uses a cubic spline interpolation method to estimate missing values. Here the interpolation is again done over the last four time series values before and the first four values after the gap. The missing values are estimated by the resulting interpolant. This method gives smooth transitions across missing values.

Method `AR_1` assumes that the time series before the gap can be approximated using an AR(1) process. If the last observation prior to the gap is made at time point t_m then this method uses values at $t_0, t_1, \ldots, t_m$ to compute the one-step-ahead forecast at origin t_m. This value is used to estimate the missing value at time point $t_m + 1$. If the value at $t_m + 2$ is also missing then the values at time points $t_0, t_1, \ldots, t_m + 1$ are

used to recompute the AR(1) model, and then estimate the value at $t_m + 2$ and so on. The coefficient ϕ_1 in the AR(1) model is computed internally by the method of least squares from class ARMA.

Finally, method `AR_p` uses an AR(p) model to estimate missing values using a one-step-ahead forecast similar to method `AR_1`. First, class `ARAutoUnivariate`, is applied to the time series values just prior to the missing values to determine the optimum p from the set $\{0, 1, \ldots, \text{maxlag}\}$ of possible values and to compute the parameters $\phi_1, \ldots, \phi_p$ of the resulting AR(p) model. The parameters are estimated by the least squares method based on Householder transformations as described in Kitagawa and Akaike (1978). Denoting the mean of the series $y_{t_0}, y_{t_1}, \ldots, y_{t_m}$ by μ the one-step-ahead forecast at origin t_m, $\hat{y_{t_m}}(1)$, can be computed by the formula

$$\hat{y_{t_m}}(1) = \mu(1 - \sum_{j=1}^{p} \phi_j) + \sum_{j=1}^{p} \phi_j y_{t_m+1-j}.$$

This value is used as an estimate for the missing value at t_{m+1}. The procedure starting with `ARAutoUnivariate` is then repeated for every further missing value in the gap. All four estimation methods treat gaps of missing values in increasing time order.

Properties

ConvergenceTolerance

```
public double ConvergenceTolerance {get; set; }
```

Description

The covergence tolerance used by the `AR_1` and `AR_p` missing value estimation methods.

Property Value

A `double` scalar value. By default, `ConvergenceTolerance` = 1.0e-09.

EstimationMethod

```
public Imsl.Stat.ARAutoUnivariate.ParamEstimation EstimationMethod {get; set; }
```

Description

The method used for estimating the final autoregressive coefficients for missing value estimation methods `AR_1` and `AR_p`.

Property Value

An `ARAutoUnivariate.ParamEstimation` scalar specifying the method used to estimate the autoregressive coefficients. Valid methods are `MethodOfMoments`, `LeastSquares`, and `MaxLikelihood`.

Remarks

By default, `EstimationMethod=ARAutoUnivariate.ParamEstimation.LeastSquares`.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of estimation iterations for missing value estimation methods AR_1 and AR_p.

Property Value

An int specifying the maximum number of iterations for the maximum likelihood estimation.

Remarks

If this limit is exceeded ARMAEstimateMissing stops execution during the Compute method and issues an Imsl.Stat.IterationLimitExceededException. By default, MaxIterations=200.

Maxlag

```
public int Maxlag {get; set; }
```

Description

The maximum number of autoregressive lags when method AR_p is selected as the missing value estimation method.

Property Value

An int scalar value equal to the maximum number of autoregressive lags. Maxlag must be greater than z.Length-5. By default Maxlag=10.

Mean

```
public double Mean {get; set; }
```

Description

The mean value used to center the series.

Property Value

A double scalar value used to center the series.

Remarks

By default the median of the series is used for centering.

MissingValueMethod

```
public Imsl.Stat.ARMAEstimateMissing.MissingValueEstimation MissingValueMethod
{get; set; }
```

Description

The current missing value estimation method.

Property Value

A MissingValueEstimation representing the estimation method used for estimating the missing values in the time series. Available methods are Median, CubicSpline, AR_1 or AR_p.

Remarks

By default, MissingValueMethod=ARMAEstimateMissing.MissingValueEstimation.AR_p.

NumberMissing

```
public int NumberMissing {get; }
```

Description

The number of missing values in the original series

Property Value

An `int` scalar containing the number of missing values in the time series.

RelativeError

```
public double RelativeError {get; set; }
```

Description

The relative error used for the `MethodOfMoments` and `LeastSquares` estimation methods.

Property Value

A `double` scalar containing the stopping criterion for use in the nonlinear equation solver used in both the method of moments and least-squares algorithm.

Remarks

By default, `RelativeError` = 100 * 2.2204460492503131e-16.

Constructor

ARMAEstimateMissing

```
public ARMAEstimateMissing(int[] tpoints, double[] z)
```

Description

Constructor for `ARMAEstimateMissing`.

Parameters

`tpoints` – An `int` array containing the times at which the series values were observed. The values must be strictly increasing. Times for missing values are identified as non-incremental gaps in this series. A gap of missing values in `z` is assumed when the difference between two consecutive values is greater than one, i.e. $t_{i+1} - t_i > 1$. The difference is the number of missing values in the gap. The series can have multiple gaps with missing values, but any one gap can have no more than three missing values.

`z` – a `double` array containing the values for the time series observed at the times given in the vector `tpoints`.

Methods

GetCompleteTimes

```
public int[] GetCompleteTimes()
```

Description

Returns an `int` array of all time points, including values for times with missing values in `z`.

Returns

An `int` array of all times from `tpoints[0]=1` to `tpoints.Length+NumberMissing`, where `NumberMissing` is the number of values removed from the original time series.

GetCompleteTimeSeries

```
public double[] GetCompleteTimeSeries()
```

Description

Returns a `double` precision vector of length `tpoints[tpoints.Length-1]-tpoints[0]+1` containing the observed values in the time series `z` plus estimates for missing values in gaps identified in `tpoints`.

Returns

A `double` array of length `tpoints[tpoints.Length-1]-tpoints[0]+1` containing the observed values in the time series `z` plus estimates for missing values in gaps identified in `tpoints`.

Exceptions

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input matrix to ARAutoUnivariate is singular. This can only occur with estimation method `AR_p`.

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

GetMissingTimes

```
public int[] GetMissingTimes()
```

Description

Returns the times at which missing values were estimated.

Returns

An `int` array containing the times at which missing values were estimated. If there are no missing values a `null` array is returned.

Example: ARMAEstimateMissing

The data in this example was artificially generated using an autoregressive time series with a lag of 1, i.e., AR(1). The constant term in the model was set to zero and -0.7 was used for the autoregressive coefficient. The data were generated from a random gaussian distribution with a mean of zero and an innovation variance of 0.51. This series is stationary with Var(Y) = 1.0.

Two hundred values were generated. For this example, six values at times t=130, t=140, t=141, t=160, t=175, and t=176 are removed and designated as missing. `ARMAEstimateMissing` is used to estimate these missing values using each of its estimation methods. The missing value estimates are compared to the actual values generated in the full series.

As expected, the AR(1) method produced the best missing value estimates in this example, closely followed by the AR(p) method.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using PrintMatrix = Imsl.Math.PrintMatrix;
using Imsl;
public class ARMAEstimateMissingEx1
{
    public static void  Main(String[] args)
    {
        int i, k;
        int maxlag = 20;
        int n_obs = 194, n_miss = 6;
        int[] missing_index = null;
        int[] tpointsMiss = null;
        int[] tpointsComplete = null;
        double missVar = 0;
        double sum = 0;
        double variance = 0;
        double[] y = null;
        double[] yMiss = null;
        double[] yComplete =
            {
                1.30540, -1.37166, 1.47905, -0.91059, 1.36191,
                -2.16966, 3.11254, -1.99536, 2.29740, -1.82474,
                -0.25445, 0.33519, -0.25480, -0.50574, -0.21429,
                -0.45932, -0.63813, 0.25646, -0.46243, -0.44104,
                0.42733, 0.61102, -0.82417, 1.48537, -1.57733,
                -0.09846, 0.46311, 0.49156, -1.66090, 2.02808,
                -1.45768, 1.36115, -0.65973, 1.13332, -0.86285,
                1.23848, -0.57301, -0.28210, 0.20195, 0.06981,
                0.28454, 0.19745, -0.16490, -1.05019, 0.78652,
                -0.40447, 0.71514, -0.90003, 1.83604, -2.51205,
                1.00526, -1.01683, 1.70691, -1.86564, 1.84912,
                -1.33120, 2.35105, -0.45579, -0.57773, -0.55226,
```

```
            0.88371, 0.23138, 0.59984, 0.31971, 0.59849,
            0.41873, -0.46955, 0.53003, -1.17203, 1.52937,
            -0.48017, -0.93830, 1.00651, -1.41493, -0.42188,
            -0.67010, 0.58079, -0.96193, 0.22763, -0.92214,
            1.35697, -1.47008, 2.47841, -1.50522, 0.41650,
            -0.21669, -0.90297, 0.00274, -1.04863, 0.66192,
            -0.39143, 0.40779, -0.68174, -0.04700, -0.84469,
            0.30735, -0.68412, 0.25888, -1.08642, 0.52928,
            0.72168, -0.18199, -0.09499, 0.67610, 0.14636,
            0.46846, -0.13989, 0.50856, -0.22268, 0.92756,
            0.73069, 0.78998, -1.01650, 1.25637, -2.36179,
            1.99616, -1.54326, 1.38220, 0.19674, -0.85241,
            0.40463, 0.39523, -0.60721, 0.25041, -1.24967,
            0.26727, 1.40042, -0.66963, 1.26049, -0.92074,
            0.05909, -0.61926, 1.41550, 0.25537, -0.13240,
            -0.07543, 0.10413, 1.42445, -1.37379, 0.44382,
            -1.57210, 2.04702, -2.22450, 1.27698, 0.01073,
            -0.88459, 0.88194, -0.25019, 0.70224, -0.41855,
            0.93850, 0.36007, -0.46043, 0.18645, 0.06337,
            0.29414, -0.20054, 0.83078, -1.62530, 2.64925,
            -1.25355, 1.59094, -1.00684, 1.03196, -1.58045,
            2.04295, -2.38264, 1.65095, -0.33273, -1.29092,
            0.14020, -0.11434, 0.04392, 0.05293, -0.42277,
            0.59143, -0.03347, -0.58457, 0.87030, 0.19985,
            -0.73500, 0.73640, 0.29531, 0.22325, -0.60035,
            1.42253, -1.11278, 1.30468, -0.41923, -0.38019,
            0.50937, 0.23051, 0.46496, 0.02459, -0.68478,
            0.25821, 1.17655, -2.26629, 1.41173, -0.68331};

        ARMAEstimateMissing estMiss = null;
        ARAutoUnivariate arAuto = null;
        String title = " ";
        String[] colLabels = new
            String[]{"TIME", "ACTUAL", "PREDICTED", "DIFFERENCE"};
        PrintMatrixFormat pmf = new PrintMatrixFormat();
        PrintMatrix pm = new PrintMatrix();
        pmf.NumberFormat = "0.000";
        pmf.SetColumnLabels(colLabels);
        pmf.FirstRowNumber = 1;
        pm.SetColumnSpacing(3);

        /* setup missing data arrays */
        tpointsComplete = new int[200];
        tpointsMiss = new int[194];
        yMiss = new double[194];
        for (i = 1; i <= 200; i++)
            tpointsComplete[i - 1] = i;
        tpointsMiss[0] = tpointsComplete[0];
        yMiss[0] = yComplete[0];
        k = 0;
        for (i = 1; i < 200; i++)
        {
            /* Generate series with missing values */
            if (i != 129 && i != 139 && i != 140 && i
                != 159 && i != 174 && i != 175)
            {
```

```
        k += 1;
        tpointsMiss[k] = tpointsComplete[i];
        yMiss[k] = yComplete[i];
    }
}
n_obs = k + 1;

foreach (ARMAEstimateMissing.MissingValueEstimation j in
    Enum.GetValues(typeof(ARMAEstimateMissing.MissingValueEstimation)))
{
    estMiss = new ARMAEstimateMissing(tpointsMiss, yMiss);
    estMiss.MissingValueMethod = j;

    switch (j)
    {


        case ARMAEstimateMissing.MissingValueEstimation.Median:
            title = "MEDIAN ESTIMATES";
            break;

        case ARMAEstimateMissing.MissingValueEstimation.CubicSpline:
            title = "CUBIC SPLINE ESTIMATES";
            break;

        case ARMAEstimateMissing.MissingValueEstimation.AR_1:
            title = "AR(1) ESTIMATES";
            break;

        case ARMAEstimateMissing.MissingValueEstimation.AR_p:
            estMiss.Maxlag = maxlag;
            estMiss.EstimationMethod =
                ARAutoUnivariate.ParamEstimation.MethodOfMoments;
            title = "AR(P) ESTIMATES";
            break;
        }
    /* For some data it is useful to turn off warnings produced by
    * the ARMA estimation process.  This is only necessary for
    * the AR_1 and AR_P estimation methods
    */
    WarningObject w = Warning.WarningObject;
    Warning.WarningObject = null;
    y = estMiss.GetCompleteTimeSeries();
    Warning.WarningObject = w; // turn on warnings
    missing_index = estMiss.GetMissingTimes();
    n_miss = y.Length - yMiss.Length;
    double[][] printOutput = new double[n_miss][];
    for (int i2 = 0; i2 < n_miss; i2++)
    {
        printOutput[i2] = new double[4];
    }
    sum = 0;
    for (i = 0; i < n_miss; i++)
    {
        k = missing_index[i];
        printOutput[i][0] = tpointsComplete[k];
```

```
                printOutput[i][1] = yComplete[k];
                printOutput[i][2] = y[k];
                printOutput[i][3] = Math.Abs(yComplete[k] - y[k]);
                sum += Math.Pow(printOutput[i][3], 2);
            }
            arAuto = new ARAutoUnivariate(maxlag, y);
            arAuto.Compute();
            variance = arAuto.InnovationVariance;
            pm.SetTitle(title);
            pm.Print(pmf, printOutput);
            missVar = sum / n_miss;
            Console.Out.WriteLine
                ("Innovation Variance Analysis - Estimate (percent of actual)");
            Console.Out.WriteLine("         Missing Values(only): "
                + missVar + " ("
                + (long) Math.Round(100.0 * missVar / 0.51) + "%)");
            Console.Out.WriteLine("         Entire  Series:       "
                + variance + " ("
                + (long) Math.Round(100.0 * variance / 0.51) + "%)");
        }
    }
}
```

Output

```
                MEDIAN ESTIMATES
      TIME    ACTUAL   PREDICTED   DIFFERENCE
1   130.000   -0.921     0.261       1.182
2   140.000    0.444     0.057       0.386
3   141.000   -1.572     0.057       1.630
4   160.000    2.649     0.047       2.602
5   175.000   -0.423     0.048       0.471
6   176.000    0.591     0.048       0.543

Innovation Variance Analysis - Estimate (percent of actual)
         Missing Values(only): 1.91525937619167 (376%)
         Entire  Series:       0.535039841503757 (105%)
             CUBIC SPLINE ESTIMATES
      TIME    ACTUAL   PREDICTED   DIFFERENCE
1   130.000   -0.921     1.541       2.462
2   140.000    0.444    -0.407       0.851
3   141.000   -1.572     2.497       4.069
4   160.000    2.649    -2.947       5.596
5   175.000   -0.423     0.251       0.673
6   176.000    0.591     0.380       0.211

Innovation Variance Analysis - Estimate (percent of actual)
         Missing Values(only): 9.19346459339994 (1803%)
         Entire  Series:       0.75913799041326 (149%)
                AR(1) ESTIMATES
      TIME    ACTUAL   PREDICTED   DIFFERENCE
1   130.000   -0.921    -0.930       0.009
2   140.000    0.444     1.028       0.584
3   141.000   -1.572    -0.745       0.827
4   160.000    2.649     1.229       1.420
```

```
5   175.000   -0.423     0.010       0.433
6   176.000    0.591     0.037       0.555

Innovation Variance Analysis - Estimate (percent of actual)
         Missing Values(only): 0.58975292115581 (116%)
         Entire  Series:       0.501310666025287 (98%)
                AR(P) ESTIMATES
      TIME    ACTUAL   PREDICTED   DIFFERENCE
1   130.000   -0.921    -0.889       0.032
2   140.000    0.444     1.009       0.565
3   141.000   -1.572    -0.688       0.884
4   160.000    2.649     1.210       1.439
5   175.000   -0.423    -0.002       0.421
6   176.000    0.591     0.038       0.553

Innovation Variance Analysis - Estimate (percent of actual)
         Missing Values(only): 0.609151365834584 (119%)
         Entire  Series:       0.501790370468904 (98%)
```

ARMAEstimateMissing.MissingValueEstimation Enumeration

`public enumeration Imsl.Stat.ARMAEstimateMissing.MissingValueEstimation`

Missing value estimation methods.

Fields

AR_1

`public Imsl.Stat.ARMAEstimateMissing.MissingValueEstimation AR_1`

Description

Indicates that missing values should be estimated using an autoregressive time series with 1 lag.

AR_p

`public Imsl.Stat.ARMAEstimateMissing.MissingValueEstimation AR_p`

Description

Indicates that missing values should be estimated using an autoregressive time series with a maximum lag of `Maxlag`. By default `Maxlag=10`, but this can be changed using the `Maxlag` property.

CubicSpline

`public Imsl.Stat.ARMAEstimateMissing.MissingValueEstimation CubicSpline`

Description

Indicates that missing values should be estimated using cublic spline interpolation.

Median

```
public Imsl.Stat.ARMAEstimateMissing.MissingValueEstimation Median
```

Description

Indicates that missing values should be estimated using the median of the values just before and after the missing value gap.

ARMAMaxLikelihood Class

```
public class Imsl.Stat.ARMAMaxLikelihood
```

Computes maximum likelihood estimates of parameters for an ARMA model with p and q autoregressive and moving average terms respectively.

`ARMAMaxLikelihood` computes estimates of parameters for a nonseasonal ARMA model given a sample of observations, W_t, for $t = 1, 2, \ldots, n$, where $n =$ `z.Length`. The class is derived from the maximum likelihood estimation algorithm described by Akaike, Kitagawa, Arahata and Tada (1979), and the XSARMA routine published in the TIMSAC-78 Library.

The stationary time series W_t with mean μ can be represented by the nonseasonal autoregressive moving average (ARMA) model by the following relationship:

$$\phi(B)(W_t - \mu) = \theta(B)a_t$$

where

$$t \in Z, \;\; Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

B is the backward shift operator defined by $B^k W_t = W_{t-k}$,

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p, \;\; p \geq 0.$$

and

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q, \;\; q \geq 0.$$

The `ARMAMaxLikelihood` class estimates the AR coefficients $\phi_1, \phi_2, \ldots, \phi_p$ and the MA coefficients $\theta_1, \theta_2, \ldots, \theta_q$ using maximum likelihood estimation.

`ARMAMaxLikelihood` checks the initial estimates for both the autoregressive and moving average coefficients to ensure that they represent a stationary and invertible series respectively.

If

$$\phi_1, \phi_2, \ldots, \phi_p$$

are the initial estimates for a stationary series then all (complex) roots of the following polynomial will fall outside the unit circle:

$$1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p.$$

If

$$\theta_1, \theta_2, \ldots, \theta_q$$

are initial estimates for an invertible series then all (complex) roots of the polynomial

$$1 - \theta_1 z - \theta_2 z^2 - \ldots - \theta_q z^q$$

will fall outside the unit circle.

By default, the order of the lags for the autoregressive terms is $1, 2, \ldots, p$ and $1, 2, \ldots, q$ for the moving average terms. However, this cannot be overridden.

Initial values for the `AR` and `MA` coefficients can be supplied via the `SetAR` and `SetMA` methods. Otherwise, initial estimates are computed internally by the method of moments. The class computes the roots of the associated polynomials. If the `AR` estimates represent a nonstationary series, `ARMAMaxLikelihood` issues a warning message and replaces the intial `AR` estimates with initial values that are stationary. If the `MA` estimates represent a noninvertible series, a terminal error is issued and new initial values must be sought.

`ARMAMaxLikelihood` also validates the final estimates of the `AR` coefficients to ensure that they too represent a stationary series. This is done to guard against the possibility that the internal log-likelihood optimizer converged to a nonstationary solution. If nonstationary estimates are encountered, an exception will be thrown.

For model selection, the `ARMA` model with the minimum value for AIC might be preferred, $\text{AIC} = -2\ln(L) + 2(p+q)$, where L is the value of the maximum likelihood function evaluated at the parameter estimates.

`ARMAMaxLikelihood` can also handle white noise processes, i.e. $ARMA(0,0)$ processes.

Properties

BackwardOrigin

```
public int BackwardOrigin {get; set; }
```

Description

The maximum backward origin.

Property Value

An `int` scalar containing the maximum backward origin used in forecasting. `BackwardOrigin` must be greater than or equal to 0 and less than or equal to `z.Length - Math.Max(p, q)`.

Remarks

By default, `BackwardOrigin = 0`.

Confidence

```
public double Confidence {get; set; }
```

Description

The confidence level for calculating confidence limit deviations returned from `GetDeviations`.

Property Value

A `double` scalar specifying the confidence level used in computing forecast confidence intervals.

Remarks

The confidence limits must be greater than 0.0 and less than 1.0. Suggested confidence limits are 0.90, 0.95, and 0.99. By default, `Confidence = 0.95`.

Constant

```
public double Constant {get; set; }
```

Description

The constant parameter in the ARMA series.

Property Value

A `double` scalar containing the constant term in the ARMA model.

Remarks

By default, the constant term is initially estimated using ARMA method of moments estimation.

GradientTolerance

```
public double GradientTolerance {get; set; }
```

Description

The gradient tolerance for the convergence algorithm.

Property Value

A `double` scalar containing the tolerance used for numerically estimating the gradient by differences.

Remarks

By default, `GradientTolerance = 1e-04`.

InnovationVariance

```
public double InnovationVariance {get; }
```

Description

The estimated innovation variance of this series.

Property Value

A `double` scalar containing the estimated innovation variance for the time series.

Likelihood

```
public double Likelihood {get; }
```

Description

The final estimate for $-2\ln(L)$, where L is equal to the likelihood function evaluated using the final parameter estimates.

Property Value

A `double` scalar equal to the log likelihood function, $-2\ln(L)$.

Exceptions

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.NonInvertibleException` is thrown if the final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms using `SetMA` are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations.

Property Value

An `int` scalar containing the maximum number of iterations allowed in the nonlinear equation solver used in both the method of moments and least-squares algorithms.

Remarks

By default, `MaxIterations` = 300.

Mean

```
public double Mean {get; set; }
```

Description

The mean used for centering the series.

Property Value

A `double` scalar containing the value for centering the time series. By default the series is not centered.

P

```
public int P {get; }
```

Description

The number of autoregressive terms in the ARMA model

Property Value

An `int` scalar value of `p`, containing the number of autoregressive terms in the ARMA model.

Q

```
public int Q {get; }
```

Description

The number of moving average terms in the ARMA model

Property Value

An `int` scalar value of `q`, containing the number of moving average terms in the ARMA model.

Tolerance

```
public double Tolerance {get; set; }
```

Description

The tolerance for the convergence algorithm.

Property Value

A `double` scalar containing the value to use for the convergence tolerance during maximum likelihood estimation.

Remarks

By default, `Tolerance = 2.220446049e-016`.

Constructor

ARMAMaxLikelihood

```
public ARMAMaxLikelihood(int p, int q, double[] z)
```

Description

Constructor for `ARMAMaxLikelihood`.

Parameters

`p` – An `int` scalar equal to the number of autoregressive (AR) parameters.

`q` – A `int` scalar equal to the number of moving average (MA) parameters.

`z` – A `double` array containing the time series.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

Methods

Compute

```
public void Compute()
```

Description

Computes the exact maximum likelihood estimates for the autoregressive and moving average parameters of an ARMA time series.

Exceptions

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.NonInvertibleException` is thrown if the final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms using `SetMA` are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

Forecast

```
public double[,] Forecast(int nForecast)
```

Description

Returns forecasts for lead times $l = 1, 2, \ldots,$ nForecast at origins `z.Length-BackwardOrigin-1+j` where $j = 1, \ldots,$ BackwardOrigin $+ 1$.

Parameter

`nForecast` – An `int` scalar equal to the number of requested forecasts.

Returns

A `double` matrix of dimensions of `nForecast` by `BackwardOrigin + 1` containing the forecasts. The forecasts are for lead times $l = 1, 2, \ldots,$ nForecast at origins `z.Length-BackwardOrigin-1+j` where $j = 1, \ldots,$ BackwardOrigin $+ 1$.

Exceptions

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.NonInvertibleException` is thrown if the final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms using `SetMA` are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

GetAR

```
public double[] GetAR()
```

Description

Returns the final autoregressive parameter estimates.

Returns

A `double` array of length `p` containing the final autoregressive parameter estimates.

GetDeviations

```
public double[] GetDeviations()
```

Description

Returns the deviations for each forecast used for calculating the forecast confidence limits.

Returns

A `double` array of length `nForecast+BackwardOrigin` containing the deviations for calculating forecast confidence intervals. The confidence level is specified in `Confidence`.

Remarks

Note that the `Forecast` or `GetForecast` method must be invoked first before invoking this method. Otherwise, the method returns `null`.

GetForecast

```
public double[] GetForecast(int nForecast)
```

Description

Returns forecasts

Parameter

`nForecast` – An input `int` representing the number of requested forecasts beyond the last value in the series.

Returns

A `double` array containing the `nForecast+BackwardOrigin` forecasts. The first `BackwardOrigin` forecasts are one-step ahead forecasts for the last `BackwardOrigin` values in the series. The next `nForecast` values in the returned series are forecasts for the next values beyond the series.

Exceptions

`Imsl.Stat.NonStationaryException` is thrown if the final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.NonInvertibleException` is thrown if the final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms using `SetMA` are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

GetGradients

```
public double[] GetGradients()
```

Description

Returns the gradients for the final parameter estimates.

Returns

A `double` array of length `p+q` containing the gradients of the final parameter estimates.

GetMA

```
public double[] GetMA()
```

Description

Returns the final moving average parameter estimates.

Returns

A `double` array of length `q` containing the final moving average parameter estimates.

GetPsiWeights

```
public double[] GetPsiWeights()
```

Description

Returns the psi weights used for calculating forecasts from the infinite order moving average form of the ARMA model.

Returns

A `double` array of length `nForecast` containing the psi weights of the infinite order moving average form of the model.

GetResiduals

```
public double[] GetResiduals()
```

Description

The current values of the vector of residuals.

Returns

A `double` array of length `BackwardOrigin` containing the residuals for the last `BackwardOrigin` values in the time series. The `Compute` and either the `Forecast` or `GetForecast` methods must be called before calling this method.

GetTimeSeries

```
public double[] GetTimeSeries()
```

Description

Returns the time series used to construct `ARMAMaxLikelihood`.

Returns

A `double` containing the values of the time series passed to the class constructor.

IsInvertible

```
public bool IsInvertible(double[] ma)
```

Description

Tests whether the coefficients in `ma` are invertible.

Parameter

`ma` – A `double` array containing the coefficients for the moving average terms in an ARMA model.

Returns

A `bool` scalar equal to `true` if the coefficients in `ma` are invertible and `false` otherwise.

IsStationary

```
public bool IsStationary(double[] ar)
```

Description

Tests whether the coefficients in `ar` are stationary.

Parameter

`ar` – A `double` array containing the coefficients for the autoregressive terms in an ARMA model.

Returns

A `bool` scalar equal to `true` if the coefficients in `ar` are stationary and `false` otherwise.

SetAR

```
public void SetAR(double[] ar)
```

Description

Sets the initial values for the autoregressive terms to the `p` values in `ar`.

Parameter

`ar` – An input `double` array of length `p` containing the initial values for the autoregressive terms. If this method is not called, initial values are computed by method of moments in the `ARMA` class.

SetMA

```
public void SetMA(double[] ma)
```

Description

Sets the initial values for the moving average terms to the `q` values in `ma`.

Parameter

`ma` – A `double` array of length `q` containing the initial values for the moving average terms. If this method is not called, initial values are computed by method of moments in the `ARMA` class.

Example: ARMAMaxLikelihood

Consider the Wolfer Sunspot Data (Anderson 1971, p. 660) consisting of the number of sunspots observed each year from 1749 through 1924. The data set for this example consists of the number of sunspots observed from 1770 through 1869. The maximum likelihood estimates

$$\hat{\theta}_0, \hat{\phi}_1, \hat{\phi}_2, \text{and } \hat{\theta}_1$$

are computed for the ARMA(2, 1) model

$$z_t = \theta_0 + \phi_1 z_{t-1} + \phi_2 z_{t-2} - \theta_1 A_{t-1} + A_t$$

where the errors A_t are independently normally distributed with mean zero and variance σ_A^2. The maximum likelihood estimates from `ARMAMaxLikelihood` are compared to the same estimates using the method of moments and least squares from the `ARMA` class. For each method, the coefficients and forecasts for the last ten years, 1860-1869, are compared. The method of moments and maximum likelihood estimates produced similar results. However, the least squares method does not converge using the default relative error convergence criteria. As a result, a warning is produced and the estimates are very different from the method of moments and maximum likelihood estimates.

```
using System;
using Imsl.Stat;
using Imsl.Math;
public class ARMAMaxLikelihoodEx1
{
```

```
public static void  Main(String[] args)
{
    int backwardOrigin = 0, n_forecast = 10, n_series = 100;
    double[] sunspots = {
                            100.8, 81.6, 66.5, 34.8, 30.6, 7, 19.8, 92.5,
                            154.4, 125.9, 84.8, 68.1, 38.5, 22.8, 10.2,
                            24.1, 82.9, 132, 130.9, 118.1, 89.9, 66.6,
                            60, 46.9, 41, 21.3, 16, 6.4, 4.1, 6.8, 14.5,
                            34, 45, 43.1, 47.5, 42.2, 28.1, 10.1, 8.1,
                            2.5, 0, 1.4, 5, 12.2, 13.9, 35.4, 45.8, 41.1,
                            30.4, 23.9, 15.7, 6.6, 4, 1.8, 8.5, 16.6,
                            36.3, 49.7, 62.5, 67, 71, 47.8, 27.5, 8.5,
                            13.2, 56.9, 121.5, 138.3, 103.2, 85.8, 63.2,
                            36.8, 24.2, 10.7, 15, 40.1, 61.5, 98.5, 124.3,
                            95.9, 66.5, 64.5, 54.2, 39, 20.6, 6.7, 4.3,
                            22.8, 54.8, 93.8, 95.7, 77.2, 59.1, 44, 47,
                            30.5, 16.3, 7.3, 37.3, 73.9};
    double[] z = null;
    double[] arMM, maMM;
    double constantMM;
    double[] arLS, maLS;
    double constantLS;
    double[] ar, ma;
    double constant;
    double[,] forecastMM, forecastLS, forecast;
    double[] deviations;
    double likelihood, var, varMM, varLS;
    double[] avgDev = new double[]{0.0, 0.0, 0.0};
    ARMA armaMM = null;
    ARMA armaLS = null;
    ARMAMaxLikelihood maxArma = null;
    double[,] printOutput = null;
    double[,] printOutput2 = null;
    String[] colLabels = {"Method of Moments"
                            , "Least Squares"
                            , "Maximum Likelihood"};
    String[] colLabels1 = {"Least Squares", "Maximum Likelihood"};
    String[] colLabels2 = {"Observed Sunspots", "Method of Moments"
                            , "Least Squares", "Maximum Likelihood"};
    String[] colLabels3 = {"Lower Confidence Limit", "Forecast"
                            , "Upper Confidence Limit"};
    PrintMatrixFormat pmf = new PrintMatrixFormat();
    PrintMatrix pm = new PrintMatrix();
    pm.SetColumnSpacing(3);
    pmf.SetNoRowLabels();
    pmf.SetColumnLabels(colLabels);
    printOutput = new double[1,3];
    printOutput2 = new double[n_forecast,4];
    z = new double[n_series];
    for (int i = 0; i < n_series; i++)
        z[i] = sunspots[i];
    /* Method of Moments ARMA(2,1) Estimation */
    armaMM = new ARMA(2, 1, z);
    armaMM.Method = Imsl.Stat.ARMA.ParamEstimation.MethodOfMoments;
    armaMM.Compute();
    armaMM.BackwardOrigin = backwardOrigin;
```

```
arMM = armaMM.GetAR();
maMM = armaMM.GetMA();
constantMM = armaMM.Constant;
forecastMM = armaMM.Forecast(n_forecast);
varMM = armaMM.InnovationVariance;

/* Least Squares ARMA(2,1) Estimation */
armaLS = new ARMA(2, 1, z);
armaLS.Method = Imsl.Stat.ARMA.ParamEstimation.LeastSquares;
armaLS.Compute();
armaLS.BackwardOrigin = backwardOrigin;
arLS = armaLS.GetAR();
maLS = armaLS.GetMA();
constantLS = armaLS.Constant;
varLS = armaLS.InnovationVariance;
forecastLS = armaLS.Forecast(n_forecast);

/* Maximum Likelihood ARMA(2,1) Estimation */
maxArma = new ARMAMaxLikelihood(2, 1, z);
maxArma.Compute();
maxArma.BackwardOrigin = backwardOrigin;
ar = maxArma.GetAR();
ma = maxArma.GetMA();
constant = maxArma.Constant;
likelihood = maxArma.Likelihood;
var = maxArma.InnovationVariance;
maxArma.Confidence = 0.9;
forecast = maxArma.Forecast(n_forecast);
deviations = maxArma.GetDeviations();

printOutput[0,0] = constantMM;
printOutput[0,1] = constantLS;
printOutput[0,2] = constant;
pm.SetTitle("ARMA(2,1) - Constant Term");
pm.Print(pmf, printOutput);
printOutput[0,0] = arMM[0];
printOutput[0,1] = arLS[0];
printOutput[0,2] = ar[0];
pm.SetTitle("ARMA(2,1) - AR(1) Coefficient");
pm.Print(pmf, printOutput);
printOutput[0,0] = arMM[1];
printOutput[0,1] = arLS[1];
printOutput[0,2] = ar[1];
pm.SetTitle("ARMA(2,1) - AR(2) Coefficient");
pm.Print(pmf, printOutput);
printOutput[0,0] = maMM[0];
printOutput[0,1] = maLS[0];
printOutput[0,2] = ma[0];
pm.SetTitle("ARMA(2,1) - MA(1) Coefficient");
pm.Print(pmf, printOutput);
Console.Out.WriteLine("INNOVATION VARIANCE:");
Console.Out.WriteLine("Method of Moments  " + varMM);
Console.Out.WriteLine("Least Squares      " + varLS);
Console.Out.WriteLine("Maximum Likelihood " + var);
Console.Out.WriteLine("");
```

```
        for (int i = 0; i < n_forecast; i++)
        {
            printOutput2[i,0] = sunspots[90 + i];
            printOutput2[i,1] = forecastMM[i,backwardOrigin];
            printOutput2[i,2] = forecastLS[i,backwardOrigin];
            printOutput2[i,3] = forecast[i,backwardOrigin];
        }
        pm.SetTitle("SUNSPOT FORECASTS FOR 1860-1869");
        pmf.SetColumnLabels(colLabels2);
        pmf.FirstRowNumber = 1860;
        pm.Print(pmf, printOutput2);
        /* Get Confidence Interval Deviations */
        printOutput2 = new double[n_forecast,3];
        for (int i = 0; i < n_forecast; i++)
        {
            printOutput2[i,0] = Math.Max(0, forecast[i,backwardOrigin]
                - deviations[i + backwardOrigin]);
            printOutput2[i,1] = forecast[i,backwardOrigin];
            printOutput2[i,2] = forecast[i,backwardOrigin] + deviations[i];
        }
        pmf.SetColumnLabels(colLabels3);
        pmf.FirstRowNumber = 1860;
        pm.SetTitle("SUNSPOT MAX. LIKELIHOOD 90% CONFIDENCE INTERVALS");
        pm.Print(pmf, printOutput2);
    }
}
```

Output

```
                  ARMA(2,1) - Constant Term
   Method of Moments     Least Squares    Maximum Likelihood
15.5439819435637    17.9316722242554    15.7580039183081

                 ARMA(2,1) - AR(1) Coefficient
   Method of Moments     Least Squares    Maximum Likelihood
1.24425777984372    1.53128221858841    1.22506595122835

                  ARMA(2,1) - AR(2) Coefficient
    Method of Moments      Least Squares     Maximum Likelihood
-0.575149766040151   -0.894433318954122   -0.56051392292258

                  ARMA(2,1) - MA(1) Coefficient
    Method of Moments      Least Squares     Maximum Likelihood
-0.124089747872598   -0.132018760615022   -0.382891621968495

INNOVATION VARIANCE:
Method of Moments  287.242403738122
Least Squares      239.68797223859
Maximum Likelihood 214.508788425629

                           SUNSPOT FORECASTS FOR 1860-1869
       Observed Sunspots   Method of Moments      Least Squares     Maximum Likelihood
1860         95.7          87.6861734610048     98.0265069437306     87.9339016395945
1861         77.2          82.1446177467774    101.939296986813      82.0608538716595
1862         59.1          67.3203794962267     86.351331124405      66.9997857593059
```

```
1863         44            52.0624301952585     58.982026390731      51.8409090696475
1864         47            41.6037652345956     31.0142927589859     41.7122237493372
1865         30.5          37.3660959612162     12.6678176248844     37.8006776731389
1866         16.3          38.1086417046268      9.58945929412723    38.6860449014629
1867          7.3          41.469854513895      21.2853225650988     41.9631541830784
1868         37.3          45.2249946909406     41.9483762816881     45.4815685240649
1869         73.9          47.9641563097715     63.1281732161496     47.9549927562368

              SUNSPOT MAX. LIKELIHOOD 90% CONFIDENCE INTERVALS
      Lower Confidence Limit       Forecast       Upper Confidence Limit
1860      63.8431804305709      87.9339016395945     112.024622848618
1861      36.4438944595113      82.0608538716595     127.677813283808
1862      10.134623371631       66.9997857593059     123.864948146981
1863       0                    51.8409090696475     112.081240519376
1864       0                    41.7122237493372     102.18743849516
1865       0                    37.8006776731389      98.4521257842839
1866       0                    38.6860449014629      99.9504700070846
1867       0                    41.9631541830784     103.74778523107
1868       0                    45.4815685240649     107.46464011018
1869       0                    47.9549927562368     109.958371498445

Imsl.Stat.ARMA: Relative function convergence - Both the scaled actual and
predicted reductions in the function are less than or equal to the relative
function convergence tolerance "ConvergenceTolerance" = 5.00000041370186E-10.
Imsl.Stat.ARMA: Least squares estimation of the parameters has failed to
converge.  Increase "length" and/or "tolerance" and/or "convergence_tolerance".
The estimates of the parameters at the last iteration may be used as new
starting values.
```

ARMAOutlierIdentification Class

```
public class Imsl.Stat.ARMAOutlierIdentification
```

Detects and determines outliers and simultaneously estimates the model parameters in a time series whose underlying outlier free series follows a general seasonal or nonseasonal ARMA model. This class also allows computation of forecasts.

Consider a univariate time series $\{Y_t\}$ that can be described by the following multiplicative seasonal ARIMA model of order $(p,0,q)\times(0,d,0)_s$:

$$Y_t - \mu = \frac{\theta(B)}{\Delta_s^d \phi(B)} a_t, t = 1,\ldots,n$$

Here, $\Delta_s^d = (1-B^s)^d$, $\theta(B) = 1-\theta_1 B - \ldots - \theta_q B^q$, $\phi(B) = 1-\phi_1 B - \ldots - \phi_p B^p$. B is the lag operator, $B^k Y_t = Y_{t-k}$, $\{a_t\}$ is a white noise process, and μ denotes the mean of the series $\{Y_t\}$.

Outlier detection and parameter estimation

In general, $\{Y_t\}$ is not directly observable due to the influence of outliers. Chen and Liu (1993) distinguish between four types of outliers: innovational outliers (IO), additive outliers (AO), temporary

changes (TC) and level shifts (LS). If an outlier occurs as the last observation of the series, then Chen and Liu's algorithm is unable to determine the outlier's classification. In class `ARMAOutlierIdentification`, such an outlier is called a UI (unable to identify) and is treated as an innovational outlier.

In order to take the effects of multiple outliers occurring at time points $t_1, \ldots, t_m$ into account, Chen and Liu consider the following model:

$$Y_t^* - \mu = \sum_{j=1}^{m} \omega_j L_j(B) I_t(t_j) + \frac{\theta(B)}{\Delta_s^d \phi(B)} a_t$$

Here, $\{Y_t^*\}$ is the observed outlier contaminated series, and ω_j and $L_j(B)$ denote the magnitude and dynamic pattern of outlier j, respectively. $I_t(t_j)$ is an indicator function that determines the temporal course of the outlier effect, $I_{t_j}(t_j) = 1, I_t(t_j) = 0$ otherwise. Note that $L_j(B)$ operates on I_t via $B^k I_t = I_{t-k}$, $k = 0, 1, \ldots$.

The last formula shows that the outlier free series $\{Y_t\}$ can be obtained from the original series $\{Y_t^*\}$ by removing all occurring outlier effects:

$$Y_t = Y_t^* - \sum_{j=1}^{m} \omega_j L_j(B) I_t(t_j)$$

The different types of outliers are characterized by different values for $L_j(B)$:

1. $L_j(B) = \frac{\theta(B)}{\Delta_s^d \phi(B)}$ for an innovational outlier,
2. $L_j(B) = 1$ for an additive outlier,
3. $L_j(B) = (1-B)^{-1}$ for a level shift outlier and
4. $L_j(B) = (1-\delta B)^{-1}, 0 < \delta < 1$, for a temporary change outlier.

Class `ARMAOutlierIdentification` is an implementation of Chen and Liu's algorithm. It determines the coefficients in $\phi(B)$ and $\theta(B)$ and the outlier effects in the model for the observed series jointly in three stages. The magnitude of the outlier effects is determined by least squares estimates. Outlier detection itself is realized by examination of the maximum value of the standardized statistics of the outlier effects. For a detailed description, see Chen and Liu's original paper (1993).

Intermediate and final estimates for the coefficients in $\phi(B)$ and $\theta(B)$ are computed by the `Compute` methods from classes `ARMA` and `ARMAMaxLikelihood`. If the roots of $\phi(B)$ or $\theta(B)$ lie on or within the unit circle, then the algorithm stops with an appropriate exception. In this case, different values for p and q should be tried.

Forecasting

From the relation between original and outlier free series,

$$Y_t^* = Y_t + \sum_{j=1}^{m} \omega_j L_j(B) I_t(t_j)$$

it follows that the Box-Jenkins forecast at origin t for lead time l, $\hat{Y}_t^*(l)$, can be computed as

$$\hat{Y}_t^*(l) = \hat{Y}_t(l) + \sum_{j=1}^{m} \omega_j L_j(B) I_{t+l}(t_j),\ l = 1, \ldots, \text{nForecast}$$

Therefore, computation of the forecasts for $\{Y_t^*\}$ is done in two steps:

1. Computation of the forecasts for the outlier free series $\{Y_t\}$.
2. Computation of the forecasts for the original series $\{Y_t^*\}$ by adding the multiple outlier effects to the forecasts for $\{Y_t\}$.

Step 1: Computation of the forecasts for the outlier free series $\{Y_t\}$

Since

$$\varphi(B)(Y_t - \mu) = \theta(B) a_t$$

where

$$\varphi(B) := \Delta_s^d \phi(B) = 1 - \varphi_1 B - \ldots - \varphi_{p+sd} B^{p+sd}$$

the Box-Jenkins forecast at origin t for lead time l, $\hat{Y}_t(l)$, can be computed recursively as

$$\hat{Y}_t(l) = (1 - \sum_{j=1}^{p+sd} \varphi_j)\mu + \sum_{j=1}^{p+sd} \varphi_j \hat{Y}_t(l-j) - \sum_{j=1}^{q} \theta_j a_{t+l-j}$$

Here,

$$\hat{Y}_t(l-j) = \begin{cases} Y_{t+l-j} & \text{for} \quad l - j \leq 0 \\ \hat{Y}_t(l-j) & \text{for} \quad l - j > 0 \end{cases}$$

and

$$a_k = \begin{cases} 0 & \text{for} \quad k \leq \max\{1, p+sd\} \\ Y_k - \hat{Y}_{k-1}(1) & \text{for} \quad k = \max\{1, p+sd\} + 1, \ldots, n \end{cases}$$

Step 2: Computation of the forecasts for the original series $\{Y_t^*\}$ by adding the multiple outlier effects to the forecasts for $\{Y_t\}$

The formulas for $L_j(B)$ for the different types of outliers are as follows:

Innovational outlier (IO)

$$L_j(B) = \frac{\theta(B)}{\Delta_s^d \phi(B)} := \psi(B) = \sum_{k=0}^{\infty} \psi_k B^k,\ \psi_0 = 1$$

Additive outliers (AO)

$$L_j(B) = 1$$

Level shifts (LS)

$$L_j(B) = \frac{1}{1-B} = \sum_{k=0}^{\infty} B^k$$

Temporary changes (TC)

$$L_j(B) = \frac{1}{1-\delta B} = \sum_{k=0}^{\infty} \delta^k B^k$$

Assuming the outlier occurs at time point t_j, the outlier impact is therefore:

Innovational outliers (IO)

$$\omega_j L_j(B) I_t(t_j) = \begin{cases} 0 & \text{for} \quad t < t_j \\ \omega_j \psi_k & \text{for} \quad t = t_j + k, k \geq 0 \end{cases}$$

Additive outliers (AO)

$$\omega_j L_j(B) I_t(t_j) = \begin{cases} 0 & \text{for} \quad t \neq t_j \\ \omega_j & \text{for} \quad t = t_j \end{cases}$$

Level shifts (LS)

$$\omega_j L_j(B) I_t(t_j) = \begin{cases} 0 & \text{for} \quad t < t_j \\ \omega_j & \text{for} \quad t = t_j + k, k \geq 0 \end{cases}$$

Temporary changes (TC)

$$\omega_j L_j(B) I_t(t_j) = \begin{cases} 0 & \text{for} \quad t < t_j \\ \omega_j \delta^k & \text{for} \quad t = t_j + k, k \geq 0 \end{cases}$$

From these formulas, the forecasts $\hat{Y}_t^*(l)$ can be computed easily. The $100(1-\alpha)$ percent probability limits for Y_{t+l}^* and Y_{t+l} are given by

$$\hat{Y}_t^*(l) \,(\text{or } \hat{Y}_t(l), \text{resp.}) \pm u_{\alpha/2}(1 + \sum_{j=1}^{l-1} \psi_j^2)^{1/2} s_a$$

where $u_{\alpha/2}$ is the $100(1-\alpha/2)$ percentile of the standard normal distribution, s_a^2 is an estimate of the variance σ_a^2 of the random shocks, and the ψ weights $\{\psi_j\}$ are the coefficients in

$$\psi(B) := \sum_{k=0}^{\infty} \psi_k B^k := \frac{\theta(B)}{\Delta_s^d \phi(B)}, \; \psi_0 = 1\,.$$

For a detailed explanation of these concepts, see chapter 5:“Forecasting” in Box, Jenkins and Reinsel (1994).

Fields

ADDITIVE

```
public int ADDITIVE
```

Description

Indicates detection of an additive outlier.

INNOVATIONAL

```
public int INNOVATIONAL
```

Description

Indicates detection of an innovational outlier.

LEVEL_SHIFT

```
public int LEVEL_SHIFT
```

Description

Indicates detection of a level shift outlier.

TEMPORARY_CHANGE

```
public int TEMPORARY_CHANGE
```

Description

Indicates detection of a temporary change outlier.

UNABLE_TO_IDENTIFY

```
public int UNABLE_TO_IDENTIFY
```

Description

Indicates detection of an outlier that cannnot be categorized.

Properties

AccuracyTolerance

```
public double AccuracyTolerance {get; set; }
```

Description

The tolerance value controlling the accuracy of the parameter estimates.

Property Value

A `double` scalar, a positive tolerance value controlling the accuracy of parameter estimates during outlier detection.

Default: `AccuracyTolerance` = 0.001.

AIC

```
public double AIC {get; }
```

Description

Returns Akaike's information criterion (AIC).

Property Value

A `double` scalar containing Akaike's information criterion (AIC) for the outlier free series.

Remarks

The `Compute` method must be called before using this property. Otherwise, an `InvalidOperationException` exception is thrown.

AICC

```
public double AICC {get; }
```

Description

Returns Akaike's Corrected Information Criterion (AICC).

Property Value

A `double` scalar containing Akaike's Corrected Information Criterion (AICC) for the outlier free series.

Remarks

The `Compute` method must be called before using this property. Otherwise, an `InvalidOperationException` exception is thrown.

BIC

```
public double BIC {get; }
```

Description

Returns the Bayesian Information Criterion (BIC).

Property Value

A `double` scalar containing the Bayesian Information Criterion (BIC) for the outlier free series.

Remarks

The `Compute` method must be called before using this property. Otherwise, an `InvalidOperationException` exception is thrown.

Confidence

```
public double Confidence {get; set; }
```

Description

The confidence level for calculating confidence limit deviations via method `GetDeviations`.

Property Value

A `double` scalar specifying the confidence level used in computing forecast confidence intervals.

Default: `Confidence` = 0.95.

Remarks

The confidence levels must be greater than 0.0 and less than 1.0. Typical level choices are 0.90, 0.95, and 0.99.

Constant

```
public double Constant {get; }
```

Description

Returns the constant parameter estimate.

Remarks

The `Compute` method must be called before using this property. Otherwise, an `InvalidOperationException` exception is thrown.

CriticalValue

```
public double CriticalValue {get; set; }
```

Description

The critical value used as a threshold during outlier detection.

Property Value

A `double` scalar, the critical value used as a threshold for the statistics used in the outlier detection.

Default: `CriticalValue` = 3.0.

Remarks

The critical value must be greater than zero.

Delta

```
public double Delta {get; set; }
```

Description

The dampening effect parameter.

Property Value

A `double` scalar, the dampening effect parameter used in the detection of a Temporary Change Outlier (TC).

Default: `Delta` = 0.

Remarks

The dampening effect parameter must be greater than 0 and less than 1.

NumberOfOutliers

```
public int NumberOfOutliers {get; }
```

Description

Returns the number of outliers detected.

Remarks

The `Compute` method must be called before using this property. Otherwise, an `InvalidOperationException` exception is thrown.

RelativeError

```
public double RelativeError {get; set; }
```

Description

The stopping criterion for use in the nonlinear equation solver.

Property Value

A `double` positive scalar containing the stopping criterion for use in the nonlinear equation solver used in the least-squares algorithm.

Default: `RelativeError`=1.0e-10.

ResidualStandardError

```
public double ResidualStandardError {get; }
```

Description

Returns the residual standard error of the outlier free series.

Property Value

A `double` scalar containing the standard error of the outlier free series.

Remarks

The `Compute` method must be called before using this property. Otherwise, an `InvalidOperationException` exception is thrown.

Constructor

ARMAOutlierIdentification

```
public ARMAOutlierIdentification(double[] z)
```

Description

Constructor for `ARMAOutlierIdentification`.

Parameter

`z` – A `double` array containing the observations.

Methods

Compute

```
public void Compute(int[] model)
```

Description

Detects and determines outliers and simultaneously estimates the model parameters for the given time series.

Parameter

`model` – An `int` array of length 4 containing the numbers *p*, *q*, *s*, *d* of the $\mathrm{ARIMA}(p,0,q)\times(0,d,0)_s$ model the outlier free series is following. It is required that *p*, *q* and *d* are non-negative and *s* is positive and consistent with `z.Length`.

Exceptions

`Imsl.Stat.NonStationaryException` is thrown if the intermediate or final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.NonInvertibleException` is thrown if the intermediate or final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

`Imsl.Math.DidNotConvergeException` is thrown if the algorithm computing the roots of the AR- or MA- polynomial does not converge.

`Imsl.Stat.MatrixSingularException` is thrown if one of the matrices used in classes `ARMA` or `ARMAMaxLikelihood` is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Math.SingularMatrixException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is singular.

`Imsl.Math.NotSPDException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is not positive definite.

ComputeForecasts

```
public void ComputeForecasts(int nForecast)
```

Description

Computes forecasts, associated probability limits and ψ weights for an outlier contaminated time series whose underlying outlier free series obeys a general seasonal or non-seasonal ARMA model.

Parameter

`nForecast` – An `int` scalar containing the maximum lead time for forecasts. `nForecast` must be greater than 0. Forecast origin is the time point of the last observed value in the time series, n. Forecasts are computed for lead times $1, 2, \ldots,$ nForecast, i.e. time points $n+1, n+2, \ldots, n+$ nForecast.

Remarks

The `Compute` method must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetAR

```
public double[] GetAR()
```

Description

Returns the final autoregressive parameter estimates.

Returns

A `double` array of length p = `model[0]` containing the final autoregressive parameter estimates.

Remarks

The `Compute` method must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetDeviations

```
public double[] GetDeviations()
```

Description

Returns the deviations used for calculating the forecast confidence limits.

Returns

A `double` array of length `nForecast` containing the deviations from each forecast for calculating forecast confidence intervals.

Remarks

The confidence level is specified by the `Confidence` property. Method `ComputeForecasts` has to be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetForecast

```
public double[] GetForecast()
```

Description

Returns forecasts for the original outlier contaminated series.

Returns

A `double` array of length `nForecast` containing the forecasts for the original series.

Remarks

Method `ComputeForecasts` must be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetMA

```
public double[] GetMA()
```

Description

Returns the final moving average parameter estimates.

Returns

a `double` array of length q = `model[1]` containing the final moving average parameter estimates.

Remarks

Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetOmegaWeights

```
public double[] GetOmegaWeights()
```

Description

Returns the ω weights for the detected outliers.

Returns

A `double` array containing the computed ω weights for the detected outliers.

Remarks

If the number of detected outliers equals zero, then an array of length zero is returned. The `Compute` method must be invoked before using this method. Otherwise, an `InvalidOperationException` exception is thrown.

GetOutlierFreeForecast

```
public double[] GetOutlierFreeForecast()
```

Description

Returns forecasts for the outlier free series.

Returns

A `double` array of length `nForecast` containing the forecasts for the outlier free series.

Remarks

Method `ComputeForecasts` has to be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetOutlierFreeSeries

```
public double[] GetOutlierFreeSeries()
```

Description

Returns the outlier free series.

Returns

A `double` array containing the outlier free series.

Remarks

The `Compute` method must be called before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

GetOutlierStatistics

```
public int[,] GetOutlierStatistics()
```

Description

Returns the outlier statistics.

Returns

An `int` matrix of length `nOutliers` by 2, where `nOutliers` is the number of detected outliers, containing the outlier statistics. The first column contains the time at which the outlier was observed (time ranging from 1 to `z.Length`, the number of observations in the time series) and the second column contains an identifier indicating the type of outlier observed. Outlier types fall into one of five categories:

Identifier	Outlier Type
`INNOVATIONAL=0`	Innovational Outliers (IO)
`ADDITIVE=1`	Additive Outliers (AO)
`LEVEL_SHIFT=2`	Level Shift Outliers (LS)
`TEMPORARY_CHANGE=3`	Temporary Change Outliers (TC)
`UNABLE_TO_IDENTIFY=4`	Unable to Identify (UI)

Remarks

If the number of detected outliers equals zero, then an `int` array of size 0 is returned.

The `Compute` method must be invoked first before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

GetPsiWeights

```
public double[] GetPsiWeights()
```

Description

Returns the ψ weights of the infinite order moving average form of the model.

Returns

A `double` array of length `nForecast` containing the ψ weights of the infinite order moving average form of the model for the outlier free series.

Remarks

Method `ComputeForecasts` must be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetResidual

```
public double[] GetResidual()
```

Description

Returns the residuals.

Returns

A `double` array containing the residuals for the outlier free series at the final parameter estimation point.

Remarks

The `Compute` method must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetTauStatistics

```
public double[] GetTauStatistics()
```

Description

Returns the *t* value for each detected outlier.

Returns

A `double` array containing the *t* statistics for each detected outlier.

Remarks

If the number of detected outliers equals zero, then a vector of length 0 is returned. The `Compute` method must be invoked before using this method. Otherwise, an `InvalidOperationException` exception is thrown.

Example 1: Parameter estimation and outlier detection for a time series from biology

This example is based on estimates of the Canadian lynx population. Class `ARMAOutlierIdentification` is used to fit an $\mathrm{ARIMA}(2,2,0)$ model of the form

$$(1-B)(1-\phi_1 B-\phi_2 B^2)Y_t = a_t, t = 1,2,\ldots,144,$$

$\{a_t\}$ Gaussian White noise, to the given series. Method `Compute` determines parameters $\phi_1 = 0.10682$ and $\phi_2 = -0.19666$ and identifies a LS outlier at time point $t = 16$.

```
using System;
using Imsl.Stat;

public class ARMAOutlierIdentificationEx1
{
   public static void Main(String[] args)
   {

     double[] series = {
      0.24300e01, 0.25060e01, 0.27670e01, 0.29400e01, 0.31690e01, 0.34500e01,
      0.35940e01, 0.37740e01, 0.36950e01, 0.34110e01, 0.27180e01, 0.19910e01,
      0.22650e01, 0.24460e01, 0.26120e01, 0.33590e01, 0.34290e01, 0.35330e01,
      0.32610e01, 0.26120e01, 0.21790e01, 0.16530e01, 0.18320e01, 0.23280e01,
      0.27370e01, 0.30140e01, 0.33280e01, 0.34040e01, 0.29810e01, 0.25570e01,
      0.25760e01, 0.23520e01, 0.25560e01, 0.28640e01, 0.32140e01, 0.34350e01,
      0.34580e01, 0.33260e01, 0.28350e01, 0.24760e01, 0.23730e01, 0.23890e01,
      0.27420e01, 0.32100e01, 0.35200e01, 0.38280e01, 0.36280e01, 0.28370e01,
      0.24060e01, 0.26750e01, 0.25540e01, 0.28940e01, 0.32020e01, 0.32240e01,
      0.33520e01, 0.31540e01, 0.28780e01, 0.24760e01, 0.23030e01, 0.23600e01,
      0.26710e01, 0.28670e01, 0.33100e01, 0.34490e01, 0.36460e01, 0.34000e01,
      0.25900e01, 0.18630e01, 0.15810e01, 0.16900e01, 0.17710e01, 0.22740e01,
      0.25760e01, 0.31110e01, 0.36050e01, 0.35430e01, 0.27690e01, 0.20210e01,
```

```
      0.21850e01, 0.25880e01, 0.28800e01, 0.31150e01, 0.35400e01, 0.38450e01,
      0.38000e01, 0.35790e01, 0.32640e01, 0.25380e01, 0.25820e01, 0.29070e01,
      0.31420e01, 0.34330e01, 0.35800e01, 0.34900e01, 0.34750e01, 0.35790e01,
      0.28290e01, 0.19090e01, 0.19030e01, 0.20330e01, 0.23600e01, 0.26010e01,
      0.30540e01, 0.33860e01, 0.35530e01, 0.34680e01, 0.31870e01, 0.27230e01,
      0.26860e01, 0.28210e01, 0.30000e01, 0.32010e01, 0.34240e01, 0.35310e01};

      int[] model = { 2, 0, 1, 2 };
      double[] outlierFreeSeries;
      double resStdErr, aic, constant;
      double[] ar;
      int[,] outlierStatistics;
      int numOutliers;

      ARMAOutlierIdentification armaOutlier = new ARMAOutlierIdentification(series);

      armaOutlier.CriticalValue = 3.5;
      armaOutlier.Compute(model);

      outlierFreeSeries = armaOutlier.GetOutlierFreeSeries();
      numOutliers = armaOutlier.NumberOfOutliers;
      outlierStatistics = armaOutlier.GetOutlierStatistics();
      constant = armaOutlier.Constant;
      ar = armaOutlier.GetAR();
      resStdErr = armaOutlier.ResidualStandardError;
      aic = armaOutlier.AIC;

      Console.Out.WriteLine();
      Console.Out.WriteLine();
      Console.Out.WriteLine("   ARMA parameters:");
      Console.Out.WriteLine("constant:{0,9:f6}", constant);
      Console.Out.WriteLine("ar[0]:{0,12:f6}", ar[0]);
      Console.Out.WriteLine("ar[1]:{0,12:f6}", ar[1]);
      Console.Out.WriteLine();
      Console.Out.WriteLine("Number of outliers:{0,3:d}", numOutliers);
      Console.Out.WriteLine("\n   Outlier statistics:");
      Console.Out.WriteLine("Time point      Outlier type");
      for (int i = 0; i < numOutliers; i++)
        Console.Out.WriteLine("{0,10:d}{1,18:d}", outlierStatistics[i,0],
                              outlierStatistics[i,1]);
      Console.Out.WriteLine("\nRSE:{0,11:f6}", resStdErr);
      Console.Out.WriteLine("AIC:{0,11:f6}", aic);
      Console.Out.WriteLine("\n\n   Extract from the series:");
      Console.Out.WriteLine("time   original   outlier free");
      for (int i = 0; i < 36; i++)
          Console.Out.WriteLine("{0,4:d}{1,11:f4}{2,15:f4}", i + 1,
                                series[i], outlierFreeSeries[i]);
    }
}
```

Output

```
   ARMA parameters:
constant: 0.000000
```

```
ar[0]:    0.106532
ar[1]:   -0.195856

Number of outliers:  1

   Outlier statistics:
Time point      Outlier type
        16                 2

RSE:   0.319542
AIC: 282.918191


   Extract from the series:
time   original   outlier free
   1     2.4300         2.4300
   2     2.5060         2.5060
   3     2.7670         2.7670
   4     2.9400         2.9400
   5     3.1690         3.1690
   6     3.4500         3.4500
   7     3.5940         3.5940
   8     3.7740         3.7740
   9     3.6950         3.6950
  10     3.4110         3.4110
  11     2.7180         2.7180
  12     1.9910         1.9910
  13     2.2650         2.2650
  14     2.4460         2.4460
  15     2.6120         2.6120
  16     3.3590         2.6997
  17     3.4290         2.7697
  18     3.5330         2.8737
  19     3.2610         2.6017
  20     2.6120         1.9527
  21     2.1790         1.5197
  22     1.6530         0.9937
  23     1.8320         1.1727
  24     2.3280         1.6687
  25     2.7370         2.0777
  26     3.0140         2.3547
  27     3.3280         2.6687
  28     3.4040         2.7447
  29     2.9810         2.3217
  30     2.5570         1.8977
  31     2.5760         1.9167
  32     2.3520         1.6927
  33     2.5560         1.8967
  34     2.8640         2.2047
  35     3.2140         2.5547
  36     3.4350         2.7757
```

Example 2: Parameter estimation and outlier detection for an artificial time series

This example is an artificial realization of an ARMA$(1,1)$ process via formula

$$Y_t - 0.8Y_{t-1} = 10.0 + a_t + 0.5a_{t-1}, t = 1,\ldots,300,$$

$\{a_t\}$ Gaussian white noise, $E[Y_t] = 50.0$. An additive outlier with $\omega_1 = 4.5$ was added at time point $t = 150$, a temporary change outlier with $\omega_2 = 3.0$ was added at time point $t = 200$.

```
using System;
using Imsl.Stat;

public class ARMAOutlierIdentificationEx2
{
    public static void Main(String[] args)
    {
        double resStdErr, aic;
        int[,] outlierStatistics;
        int numOutliers;
        double[] omegaWeights;
        int[] model = { 1, 1, 1, 0 };
        double[] outlierFreeSeries;
        double constant;
        double[] ar, ma;

        double[] series = {
         50.0000000, 50.2728081, 50.6242599, 51.0373917, 51.9317627, 50.3494759,
         51.6597252, 52.7004929, 53.5499802, 53.1673279, 50.2373505, 49.3373871,
         49.5516472, 48.6692696, 47.6606636, 46.8774185, 45.7315445, 45.6469727,
         45.9882355, 45.5216560, 46.0479660, 48.1958656, 48.6387749, 49.9055367,
         49.8077278, 47.7858467, 47.9386749, 49.7691956, 48.5425873, 49.1239853,
         49.8518791, 50.3320694, 50.9146347, 51.8772049, 51.8745689, 52.3394470,
         52.7273712, 51.4310036, 50.6727448, 50.8370399, 51.2843437, 51.8162918,
         51.6933670, 49.7038231, 49.0189247, 49.455703, 50.2718010, 49.9605980,
         51.3775749, 50.2285385, 48.2692299, 47.6495590, 49.2938499, 49.1924858,
         49.6449242, 50.0446815, 51.9972496, 54.2576981, 52.9835434, 50.4193535,
         50.3617897, 51.8276901, 53.1239929, 54.0682144, 54.9238319, 55.6877632,
         54.8896332, 54.0701065, 52.2754097, 52.2522354, 53.1248703, 51.1287193,
         50.5003815, 49.6504173, 47.2453079, 45.4555626, 45.8449707, 45.9765129,
         45.7682228, 45.2343674, 46.6496811, 47.0894432, 49.3368340, 50.8058052,
         49.9132500, 49.5893288, 48.2470627, 46.9779968, 45.6760864, 45.7070389,
         46.6158409, 47.5303612, 47.5630417, 47.0389214, 46.0352287, 45.8161545,
         45.7974396, 46.0015373, 45.3796463, 45.3461685, 47.6444016, 49.3327446,
         49.3810692, 50.2027817, 51.4567032, 52.3986320, 52.5819206, 52.7721825,
         52.6919098, 53.3274345, 55.1345940, 56.8962631, 55.7791634, 55.0616989,
         52.3551178, 51.3264084, 51.0968323, 51.1980476, 52.8001442, 52.0545082,
         50.8742943, 51.5150337, 51.2242050, 50.5033989, 48.7760124, 47.4179192,
         49.7319527, 51.3320541, 52.3918304, 52.4140434, 51.0845947, 49.6485748,
         50.6893463, 52.9840813, 53.3246994, 52.4568024, 51.9196091, 53.6683121,
         53.4555359, 51.7755814, 49.2915611, 49.8755112, 49.4546776, 48.6171913,
         49.9643021, 49.3766441, 49.2551308, 50.1021881, 51.0769119, 55.8328133,
         52.0212708, 53.4930801, 53.2147255, 52.2356453, 51.9648819, 52.1816330,
         51.9898071, 52.5623627, 51.0717278, 52.2431946, 53.6943054, 54.3752098,
         54.1492615, 53.8523254, 52.1093712, 52.3982697, 51.2405128, 50.3018112,
```

```
 51.3819618, 49.5479546, 47.5024452, 47.4447708, 47.8939056, 48.4070015,
 48.2440681, 48.7389755, 49.7309227, 49.1998024, 49.5798340, 51.1196213,
 50.6288414, 50.3971405, 51.6084099, 52.4564743, 51.6443901, 52.4080658,
 52.4643364, 52.6257210, 53.1604691, 51.9309731, 51.4137230, 52.1233368,
 52.9867249, 53.3180733, 51.9647636, 50.7947655, 52.3815842, 50.8353729,
 49.4136009, 52.8355217, 52.2234840, 51.1392517, 48.5245132, 46.8700218,
 46.1607285, 45.2324257, 47.4157829, 48.9989090, 49.6230736, 50.4352913,
 51.1652985, 50.2588654, 50.7820129, 51.0448799, 51.2880516, 49.6898804,
 49.0288200, 49.9338837, 48.2214432, 46.2103348, 46.9550171, 47.5595894,
 47.7176018, 48.4502945, 50.9816895, 51.6950073, 51.6973495, 52.1941261,
 51.8988075, 52.5617599, 52.0218391, 49.5236053, 47.9684906, 48.2445183,
 48.8275146, 49.7176971, 51.5649338, 52.5627213, 52.0182419, 50.9688835,
 51.5846901, 50.9486771, 48.8685837, 48.5600624, 48.4760094, 48.5348396,
 50.4187813, 51.2542381, 50.1872864, 50.4407692, 50.6222687, 50.4972000,
 51.0036087, 51.3367500, 51.7368202, 53.0463791, 53.6261253, 52.0728683,
 48.9740753, 49.3280830, 49.2733917, 49.8519020, 50.8562126, 49.5594254,
 49.6109200, 48.3785629, 48.0026474, 49.4874268, 50.1596375, 51.8059540,
 53.0288620, 51.3321075, 49.3114815, 48.7999306, 47.7201881, 46.3433914,
 46.5303612, 47.6294632, 48.6012459, 47.8567657, 48.0604057, 47.1352806,
 49.5724792, 50.5566483, 49.4182968, 50.5578079, 50.6883736, 50.6333389,
 51.9766159, 51.0595245, 49.3751640, 46.9667702, 47.1658173, 47.4411278,
 47.5360374, 48.9914742, 50.4747620, 50.2728043, 51.9117165, 53.7627792};

ARMAOutlierIdentification armaOutlier = new ARMAOutlierIdentification(series);

armaOutlier.RelativeError = 1.0e-5;
armaOutlier.Compute(model);

outlierFreeSeries = armaOutlier.GetOutlierFreeSeries();
numOutliers = armaOutlier.NumberOfOutliers;
outlierStatistics = armaOutlier.GetOutlierStatistics();
omegaWeights = armaOutlier.GetOmegaWeights();
constant = armaOutlier.Constant;
ar = armaOutlier.GetAR();
ma = armaOutlier.GetMA();
resStdErr = armaOutlier.ResidualStandardError;
aic = armaOutlier.AIC;

Console.Out.WriteLine("\n\n   ARMA parameters:");
Console.Out.WriteLine("constant:{0, 11:f6}", constant);
Console.Out.WriteLine("ar[0]:{0,14:f6}", ar[0]);
Console.Out.WriteLine("ma[0]:{0,14:f6}", ma[0]);
Console.Out.WriteLine();
Console.Out.WriteLine("Number of outliers:{0,3:d}", numOutliers);
Console.Out.WriteLine();
Console.Out.WriteLine("   Outlier statistics:");
Console.Out.WriteLine("Time point      Outlier type");
for (int i = 0; i < numOutliers; i++)
    Console.Out.WriteLine("{0,10:d}{1,18:d}", outlierStatistics[i,0],
                          outlierStatistics[i,1]);
Console.Out.WriteLine("\n   Omega statistics:");
Console.Out.WriteLine("Time point      Omega");
for (int i = 0; i < numOutliers; i++)
    Console.Out.WriteLine("{0,10:d}{1,11:f6}", outlierStatistics[i,0],
                          omegaWeights[i]);
Console.Out.WriteLine("\nRSE:{0,12:f6}", resStdErr);
```

```
        Console.Out.WriteLine("AIC:{0,12:f6}", aic);
    }
}
```

Output

```
   ARMA parameters:
constant:  10.829171
ar[0]:      0.785217
ma[0]:     -0.496449

Number of outliers:  2

   Outlier statistics:
Time point      Outlier type
       150                 1
       200                 3

   Omega statistics:
Time point      Omega
       150   4.477853
       200   3.381622

RSE:    1.007223
AIC: 1417.044262
```

Example 3: Forecasting an outlier contaminated time series

This example is a realization of an ARMA$(2,1)$ process described by the model

$$Y_t - Y_{t-1} + 0.24Y_{t-2} = 10.0 + a_t + 0.5a_{t-1},$$

$\{a_t\}$ a Gaussian White noise process. An additive outlier with $\omega_1 = 4.5$ was added at time point $t = 150$, a temporary change outlier with $\omega_2 = 3.0$ was added at time point $t = 200$.

Outliers were artificially added to the outlier free series $\{Y - t\}_{t=1,\ldots,280}$ at time points $t = 150$ (level shift with $\omega_1 = +2.5$) and $t = 200$ (additive outlier with $\omega_2 = +3.2$), resulting in the outlier contaminated series $\{Z_t\}_{t=1,\ldots,280}$. For both series, forecasts were determined for time points $t = 281,\ldots,290$ and compared with the actual values of the series.

```
using System;
using Imsl.Stat;

public class ARMAOutlierIdentificationEx3
{
    public static void  Main(String[] args)
    {

        double resStdErr, aic;
        int[,] outlierStatistics;
        int numOutliers;
        int[] model = { 2, 1, 1, 0 };
```

```
double constant;
double[] ar, ma;
int nForecast = 10;
double[] outlierContaminatedForecast;
double[] outlierFreeForecast;
double[] probabilityLimits;
double[] psiWeights;

double[] outlierContaminatedSeries = {
  41.6699982, 41.6699982, 42.0752144, 42.6123962, 43.6161919, 42.1932831,
  43.1055450, 44.3518715, 45.3961258, 45.0790215, 41.8874397, 40.2159805,
  40.2447319, 39.6208458, 38.6873589, 37.9272423, 36.8718872, 36.8310852,
  37.4524879, 37.3440933, 37.9861374, 40.3810501, 41.3464622, 42.6495285,
  42.6096764, 40.3134537, 39.7971268, 41.5401535, 40.7160759, 41.0363541,
  41.8171883, 42.4190292, 43.0318832, 43.9968109, 44.0419617, 44.3225212,
  44.6082611, 43.2199631, 42.0419197, 41.9679718, 42.4926224, 43.2091255,
  43.2512283, 41.2301674, 40.1057358, 40.4510574, 41.5329170, 41.5678177,
  43.0090141, 42.1592140, 39.9234505, 38.8394127, 40.4319878, 40.8679352,
  41.4551926, 41.9756317, 43.9878922, 46.5736389, 45.5939293, 42.4487762,
  41.5325394, 42.8830910, 44.5771217, 45.8541985, 46.8249474, 47.5686378,
  46.6700745, 45.4120026, 43.2305107, 42.7635345, 43.7112923, 42.0768661,
  41.1835632, 40.3352280, 37.9761467, 35.9550056, 36.3212509, 36.9925880,
  37.2625008, 37.0040665, 38.5232544, 39.4119797, 41.8316803, 43.7091446,
  42.9381447, 42.1066780, 40.3771248, 38.6518707, 37.0550499, 36.9447708,
  38.1017685, 39.4727097, 39.8670387, 39.3820763, 38.2180786, 37.7543488,
  37.7265244, 38.0290642, 37.5531158, 37.4685936, 39.8233147, 42.0480766,
  42.4053535, 43.0117416, 44.1289330, 45.0393829, 45.1114540, 45.0086479,
  44.6560631, 45.0278931, 46.7830849, 48.7649765, 47.7991905, 46.5339661,
  43.3679199, 41.6420822, 41.2694893, 41.5959740, 43.5330009, 43.3643608,
  42.1471291, 42.5552788, 42.4521446, 41.7629128, 39.9476891, 38.3217010,
  40.5318718, 42.8811569, 44.4796944, 44.6887932, 43.1670265, 41.2226143,
  41.8330154, 44.3721924, 45.2697029, 44.4174194, 43.5068550, 44.9793015,
  45.0585403, 43.2746620, 40.3317070, 40.3880501, 40.2627106, 39.6230278,
  41.0305252, 40.9262009, 40.8326912, 41.7084885, 42.9038048, 45.8650513,
  46.5231590, 47.9916115, 47.8463135, 46.5921936, 45.8854408, 45.9130440,
  45.7450371, 46.2964249, 44.9394569, 45.8141251, 47.5284042, 48.5527802,
  48.3950577, 47.8753052, 45.8880005, 45.7086983, 44.6174774, 43.5567932,
  44.5891113, 43.1778679, 40.9405632, 40.6206894, 41.3330421, 42.2759552,
  42.4744949, 43.0719833, 44.2178459, 43.8956337, 44.1033440, 45.6241455,
  45.3724861, 44.9167595, 45.9180603, 46.9077835, 46.1666603, 46.6013489,
  46.6592331, 46.7291603, 47.1908340, 45.9784355, 45.1215782, 45.6791115,
  46.7379875, 47.3036957, 45.9968834, 44.4669495, 45.7734680, 44.6315041,
  42.9911766, 46.3842583, 43.7214432, 43.5276833, 41.3946495, 39.7013168,
  39.1033401, 38.5292892, 41.0096245, 43.4535828, 44.6525154, 45.5725899,
  46.2815285, 45.2766647, 45.3481712, 45.5039482, 45.6745682, 44.0144806,
  42.9305000, 43.6785469, 42.2500534, 40.0007210, 40.4477005, 41.4432716,
  42.0058670, 42.9357758, 45.6758842, 46.8809929, 46.8601494, 47.0449791,
  46.5420647, 46.8939934, 46.2963371, 43.5479164, 41.3864059, 41.4046364,
  42.3037987, 43.6223717, 45.8602371, 47.3016396, 46.8632469, 45.4651413,
  45.6275482, 44.9968376, 42.7558670, 42.0218239, 41.9883728, 42.2571678,
  44.3708687, 45.7483635, 44.8832512, 44.7945862, 44.8922577, 44.7409401,
  45.1726494, 45.5686874, 45.9946709, 47.3151054, 48.0654068, 46.4817467,
  42.8618279, 42.4550323, 42.5791168, 43.4230957, 44.7787971, 43.8317108,
  43.6481781, 42.4183960, 41.8426285, 43.3475227, 44.4749908, 46.3498306,
  47.8599319, 46.2449913, 43.6044006, 42.4563484, 41.2715340, 39.8492508,
  39.9997292, 41.4410820, 42.9388237, 42.5687332};
```

```
// Actual values of the outlier contaminated series for
// t = 181,...,190
double[] exactForecastOutlierContaminatedSeries = {
   42.6384087, 41.7088661, 43.9399033, 45.4284401, 44.4558411,
   45.1761856, 45.3489113, 45.1892662, 46.3754730, 45.6082802};

// Actual values of the outlier free series for
// t = 181,...,190. This are the values of the outlier contaminated
// series - 2.5
double[] exactForecastOutlierFreeSeries = {
   40.1384087, 39.2088661, 41.4399033, 42.9284401, 41.9558411,
   42.6761856, 42.8489113, 42.6892662, 43.8754730, 43.1082802};

ARMAOutlierIdentification armaOutlier =
          new ARMAOutlierIdentification(outlierContaminatedSeries);

armaOutlier.RelativeError = 1.0e-5;
armaOutlier.Compute(model);
armaOutlier.ComputeForecasts(nForecast);

numOutliers = armaOutlier.NumberOfOutliers;
outlierStatistics = armaOutlier.GetOutlierStatistics();
constant = armaOutlier.Constant;
ar = armaOutlier.GetAR();
ma = armaOutlier.GetMA();
resStdErr = armaOutlier.ResidualStandardError;
aic = armaOutlier.AIC;

outlierContaminatedForecast = armaOutlier.GetForecast();
outlierFreeForecast = armaOutlier.GetOutlierFreeForecast();
probabilityLimits = armaOutlier.GetDeviations();
psiWeights = armaOutlier.GetPsiWeights();

Console.Out.WriteLine("\n\n   ARMA parameters:");
Console.Out.WriteLine("constant: {0,9:f6}", constant);
Console.Out.WriteLine("ar[0]: {0,12:f6}", ar[0]);
Console.Out.WriteLine("ar[1]: {0,12:f6}", ar[1]);
Console.Out.WriteLine("ma[0]: {0,12:f6}", ma[0]);
Console.Out.WriteLine();
Console.Out.WriteLine("Number of outliers: {0,4:d}", numOutliers);
Console.Out.WriteLine();
Console.Out.WriteLine("   Outlier statistics:");
Console.Out.WriteLine("Time point        Outlier type");
for (int i = 0; i < numOutliers; i++)
    Console.Out.WriteLine("{0,10:d}{1,20:d}", outlierStatistics[i,0],
                          outlierStatistics[i,1]);
Console.Out.WriteLine("\nRSE:{0,12:f6}", resStdErr);
Console.Out.WriteLine("AIC:{0,12:f6}", aic);
Console.Out.WriteLine();
Console.Out.WriteLine("     * * * Forecast Table for outlier "
                      + "contaminated series * * *");
Console.Out.WriteLine();
Console.Out.WriteLine("          Exact     forecast"+
                      "       limit         psi");
for (int i = 0; i < nForecast; i++)
```

```
            Console.Out.WriteLine("       {0,8:f4}{1,13:f4}{2,12:f4}{3,12:f4}",
                                  exactForecastOutlierContaminatedSeries[i],
                                  outlierContaminatedForecast[i],
                                  probabilityLimits[i],
                                  psiWeights[i]);

        Console.Out.WriteLine("\n\n     * * * Forecast Table for outlier " +
                              "free series * * *");
        Console.Out.WriteLine();
        Console.Out.WriteLine("          Exact     forecast       " +
                              "limit         psi");
        for (int i = 0; i < nForecast; i++)
          Console.Out.WriteLine("       {0,8:f4}{1,13:f4}{2,12:f4}{3,12:f4}",
                                exactForecastOutlierFreeSeries[i],
                                outlierFreeForecast[i],
                                probabilityLimits[i],
                                psiWeights[i]);
    }
}
```

Output

```
   ARMA parameters:
constant:  8.891421
ar[0]:     0.944052
ar[1]:    -0.150404
ma[0]:    -0.558928

Number of outliers:    2

   Outlier statistics:
Time point        Outlier type
       150                   2
       200                   1

RSE:    1.004306
AIC: 1323.617443

     * * * Forecast Table for outlier contaminated series * * *

          Exact     forecast       limit         psi
        42.6384      42.3158      1.9684      1.5030
        41.7089      42.7933      3.5535      1.2685
        43.9399      43.2822      4.3430      0.9715
        45.4284      43.6718      4.7453      0.7263
        44.4558      43.9662      4.9560      0.5396
        45.1762      44.1854      5.0686      0.4002
        45.3489      44.3482      5.1294      0.2966
        45.1893      44.4688      5.1625      0.2198
        46.3755      44.5582      5.1806      0.1629
        45.6083      44.6245      5.1906      0.1207


     * * * Forecast Table for outlier free series * * *
```

```
   Exact     forecast       limit         psi
40.1384      40.5904      1.9684      1.5030
39.2089      41.0679      3.5535      1.2685
41.4399      41.5567      4.3430      0.9715
42.9284      41.9464      4.7453      0.7263
41.9558      42.2407      4.9560      0.5396
42.6762      42.4600      5.0686      0.4002
42.8489      42.6227      5.1294      0.2966
42.6893      42.7434      5.1625      0.2198
43.8755      42.8328      5.1806      0.1629
43.1083      42.8991      5.1906      0.1207
```

AutoARIMA Class

```
public class Imsl.Stat.AutoARIMA
```

Automatically identifies time series outliers, determines parameters of a multiplicative seasonal ARIMA$(p,0,q)\times(0,d,0)_s$ model and produces forecasts that incorporate the effects of outliers whose effects persist beyond the end of the series.

Class `AutoARIMA` determines the parameters of a multiplicative seasonal ARIMA$(p,0,q)\times(0,d,0)_s$ model, and then uses the fitted model to identify outliers and prepare forecasts. The order of this model can be specified or automatically determined through use of an overloaded `Compute` method. Potential missing values in the time series are estimated prior to the parameter and outlier computations.

The ARIMA$(p,0,q)\times(0,d,0)_s$ model handled by class `AutoARIMA` has the following form:

$$\phi(B)\Delta_s^d(Y_t-\mu)=\theta(B)a_t,\ t=1,2,\ldots,n,$$

where

$$\phi(B)=1-\phi_1B-\phi_2B^2-\ldots-\phi_pB^p,\ \theta(B)=1-\theta_1B-\theta_2B^2-\ldots-\theta_qB^q,\ \Delta_s^d=(1-B_s)^d$$

and

$$B^kY_t=Y_{t-k}$$

It is assumed that all roots of $\phi(B)$ and $\theta(B)$ lie outside the unit circle. Clearly, if $s=1$ the model reduces to the traditional ARIMA(p,d,q) model.

Y_t is the unobserved, outlier-free time series with mean μ, and white noise a_t. This model is referred to as the underlying, outlier-free model. Class `AutoARIMA` does not assume that this series is observable. It assumes that the observed values might be contaminated by one or more outliers, whose effects are added to the underlying outlier-free series:

$$Y_t^*=Y_t+outlierEffect_t$$

Outlier identification uses the algorithm developed by Chen and Liu (1993). Outliers are classified into 1 of 5 types:

1. innovational
2. additive
3. level shift
4. temporary change and
5. unable to identify

Once the model parameters are estimated and the outliers are identified, class `AutoARIMA` estimates Y_t, the outlier-free series representation of the data, by removing the estimated outlier effects. Parameter estimation and outlier detection are based on methods from class `ARMAOutlierIdentification`.

Using the information about the adjusted ARIMA$(p,0,q)\times(0,d,0)_s$ model and the removed outliers, forecasts are then prepared for the outlier-free series. Outlier effects are added to these forecasts to produce a forecast for the observed series, Y_t^*. If there are no outliers, then the forecasts for the outlier-free series and the observed series will be identical.

Model selection techniques

Users have an option of either specifying specific values for p, q, s, d or have class `AutoARIMA` automatically select best fit values. Model selection can be conducted in one of three ways listed below depending upon which `Compute` method of class `AutoARIMA` is invoked.

Technique 1: Automatic ARIMA$(p,0,0)\times(0,d,0)_s$ **Selection**

This technique, chosen by use of method `Compute(int maxlag)`, tries to fit a model of the form

$$\phi(B)\Delta_s^d(Y_t-\mu)=a_t$$

to the outlier free series Y_t.

It initially searches for the AR(p) representation with minimum value of the chosen information criterion (AIC, AICC or BIC) for the noisy data, where $p=0,\ldots,$maxlag.

If the user calls methods `SetPeriods` and `SetDifferenceOrders` prior to invoking the `Compute` method, then the values in arrays `periods` and `orders` are included in the search to find an optimum ARIMA$(p,0,0)\times(0,d,0)_s$ representation of the series. Here, every possible combination of values for s, d in `periods` and `orders`, respectively, are examined. The best found model order is then used as input for the parameter and outlier detection routine.

The optimum values for p, q, s and d are returned through method `GetOptimumModelOrder`.

Technique 2: Grid Search

This technique, chosen by means of method `Compute(int[] arOrders, int[] maOrders)`, conducts a grid search for p and q using all possible combinations of candidate values in `arOrders` and `maOrders`.

If methods `SetPeriods` and `SetDifferenceOrders` are called prior to invoking the `Compute` method, then the grid search is extended to include the candidate values for *s* and *d* given in arrays `periods` and `orders`, respectively.

If method `SetDifferenceOrders` is not called prior to `Compute`, then $d = 0$ by default, and therefore no seasonal adjustment is attempted. The grid search is then restricted to searching for optimum values of *p* and *q* only.

The optimum values for *p*, *q*, *s* and *d* are contained in the array returned by method `GetOptimumModelOrder`.

Technique 3: Specified $\text{ARIMA}(p,0,q) \times (0,d,0)_s$ **Model**

In the third technique, selectable by means of method `Compute(int p, int q, int s, int d)`, specific values for *p*, *q*, *s* and *d* are given. This technique has essentially the same functionality as class `ARMAOutlierIdentification` but with the additional option of missing value estimation.

Outliers

The algorithm of Chen and Liu (1993) is used to identify outliers. The number of outliers identified is returned via the `NumberOfOutliers` property. Both the time and classification for these outliers are contained in the matrix returned by method `GetOutlierStatistics`. Outliers are classified into one of five categories based upon the standardized statistic for each outlier type. The time at which the outlier occurred is given in the first column of the returned matrix. The outlier identifier returned in the second column is according to the descriptions in the following table:

Outlier Identifier	Name	General Description
`INNOVATIONAL = 0`	Innovational Outlier (IO)	Innovational outliers persist. That is, there is an initial impact at the time the outlier occurs. This effect continues in a lagged fashion with all future observations. The lag coefficients are determined by the coefficients of the underlying ARIMA$(p,0,q) \times (0,d,0)_s$ model.
`ADDITIVE = 1`	Additive Outlier (AO)	Additive outliers do not persist. As the name implies, an additive outlier affects only the observation at the time the outlier occurs. Hence additive outliers have no effect on future forecasts.
`LEVEL_SHIFT = 2`	Level Shift (LS)	Level shift outliers persist. They have the effect of either raising or lowering the mean of the series starting at the time the outlier occurs. This shift in the mean is abrupt and permanent.
`TEMPORARY_CHANGE = 3`	Temporary Change (TC)	Temporary change outliers persist and are similar to level shift outliers with one major exception. Like level shift outliers, there is an abrupt change in the mean of the series at the time this outlier occurs. However, unlike level shift outliers, this shift is not permanent. The TC outlier gradually decays, eventually bringing the mean of the series back to its original value. The rate of this decay is modeled using property `Delta`. The default of `Delta = 0.7` is the value recommended for general use by Chen and Liu (1993).
`UNABLE_TO_IDENTIFY = 4`	Unable to Identify (UI)	If an outlier is identified as the last observation, then the algorithm is unable to determine the outlier's classification. For forecasting, a UI outlier is treated as an IO outlier. That is, its effect is lagged into the forecasts.

Except for additive outliers (AO), the effect of an outlier persists to observations following that outlier. Forecasts produced by methods of class `AutoARIMA` take this into account.

For more information on forecasting an outlier contaminated series, see the description of class `ARMAOutlierIdentification`.

Fields

ADDITIVE

```
public int ADDITIVE
```

Description

Indicates detection of an additive outlier.

INNOVATIONAL

```
public int INNOVATIONAL
```

Description

Indicates detection of an innovational outlier.

LEVEL_SHIFT

```
public int LEVEL_SHIFT
```

Description

Indicates detection of a level shift outlier.

TEMPORARY_CHANGE

```
public int TEMPORARY_CHANGE
```

Description

Indicates detection of a temporary change outlier.

UNABLE_TO_IDENTIFY

```
public int UNABLE_TO_IDENTIFY
```

Description

Indicates detection of an outlier that cannnot be categorized.

Properties

AccuracyTolerance

```
virtual public double AccuracyTolerance {get; set; }
```

Description

The tolerance value controlling the accuracy of the parameter estimates.

Property Value

A `double` scalar containing a positive tolerance value controlling the accuracy of parameter estimates during outlier detection.

Default: `AccuracyTolerance` = 0.001.

AIC

```
public double AIC {get; }
```

Description

Akaike's information criterion (AIC) for the optimum model.

Property Value

A `double` scalar containing Akaike's information criterion (AIC) for the optimum outlier free series.

Remarks

One of the `Compute` methods must be called before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

AICC

```
public double AICC {get; }
```

Description

Akaike's Corrected Information Criterion (AICC) for the optimum model.

Property Value

A `double` scalar containing Akaike's Corrected Information Criterion (AICC) for the optimum outlier free series.

Remarks

One of the `Compute` methods must be called before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

BIC

```
public double BIC {get; }
```

Description

The Bayesian Information Criterion (BIC) for the optimum model.

Property Value

A `double` scalar containing the Bayesian Information Criterion (BIC) for the optimum outlier free series.

Remarks

One of the `Compute` methods must be called before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

Confidence

```
public double Confidence {get; set; }
```

Description

The confidence level used in the calculation of confidence limit deviations via method `GetDeviations`.

Property Value

A `double` scalar specifying the confidence level used in computing forecast confidence intervals.

Default: `Confidence` = 0.95.

Remarks

The confidence levels must be greater than 0.0 and less than 1.0. Typical choices are 0.90, 0.95, and 0.99.

Constant

```
public double Constant {get; }
```

Description

The constant parameter estimate for the optimum model.

Property Value

A `double` scalar containing the constant parameter estimate for the optimum model.

Remarks

Note that one of the `Compute` methods must be invoked first before using this property. Otherwise, the method throws an `InvalidOperationException` exception.

CriticalValue

```
public double CriticalValue {get; set; }
```

Description

The critical value used as a threshold during outlier detection.

Property Value

A `double`, strictly positive scalar, the critical value used as a threshold for the statistics used in the outlier detection.

Default: `CriticalValue` = 3.0.

Delta

```
public double Delta {get; set; }
```

Description

The dampening effect parameter.

Property Value

A `double` scalar, the dampening effect parameter used in the detection of a Temporary Change Outlier (TC). The scalar must be greater than 0 and less than 1.

Default: `Delta` = 0.7.

MaximumARLag

```
public int MaximumARLag {get; set; }
```

Description

The maximum AR lag used in the determination of the optimum (s,d) combination of method `Compute(int[] arOrders, int[] maOrders)`.

Property Value

A scalar `int`, the maximum AR lag used in the computation of the optimum (s,d) combination for method `Compute(int[] arOrders, int[] maOrders)`.

Default: `MaximumARLag` = 10.

Remarks

It is required that the maximum AR lag is greater than zero and smaller than the original series after replacement of potential missing values.

ModelSelectionCriterion

```
public Imsl.Stat.AutoARIMA.InformationCriterion ModelSelectionCriterion {get;
set; }
```

Description

The model selection criterion used in the optimum model search.

Property Value

An `InformationCriterion` value specifying the model selection criterion to be used in the search for the optimum model.

Default: `InformationCriterion.Akaike` is chosen.

Remarks

InformationCriterion	Criterion Used
`Akaike`	Use Akaike's information criterion (AIC).
`AkaikeCorrected`	Use Akaike's corrected information criterion (AICC).
`Bayes`	Use the Bayesian information criterion (BIC).

NumberOfOutliers

```
public int NumberOfOutliers {get; }
```

Description

The number of detected outliers.

Property Value

An `int` scalar containing the number of outliers detected.

Remarks

One of the `Compute` methods must be invoked first before using this property. Otherwise, the method throws an `InvalidOperationException` exception.

RelativeError

```
public double RelativeError {get; set; }
```

Description

The stopping criterion for use in the nonlinear equation solver.

Property Value

A `double` positive scalar containing the stopping criterion for use in the nonlinear equation solver used in the least-squares algorithm.

Default: `RelativeError` = 2.2204460492503131e-012.

ResidualStandardError

```
public double ResidualStandardError {get; }
```

Description

The residual standard error of the outlier free series.

Property Value

A `double` scalar containing the standard error of the outlier free series.

Remarks

Note that one of the `Compute` methods must be invoked first before using this property. Otherwise, an `InvalidOperationException` exception is thrown.

Constructor

AutoARIMA

```
public AutoARIMA(int[] times, double[] x)
```

Description

Constructor for `AutoARIMA`.

Parameters

`times` – An `int` array of length `nObs`, where `nObs` is the number of observed time series values, containing the time points $t_1, \ldots, t_{\text{nObs}}$ at which the time series was observed. It is required that $t_1, \ldots, t_{\text{nObs}}$ are in strictly increasing order. Times for missing values are identified as non-incremental gaps in this series. A gap of missing values in `x` is assumed if the difference between two consecutive values is greater than 1, i.e. $t_{i+1} - t_i > 1$. The difference is the number of missing values in the gap.

`x` – A `double` array containing the observations $Y_1^*, Y_2^*, \ldots, Y_{\text{nObs}}^*$ at the times given in array `times`. This series can contain outliers and missing observations.

Methods

Compute

```
public void Compute(int maxlag)
```

Description

Estimates potential missing values, detects and determines outliers and simultaneously fits an optimum model from a set of different ARIMA$(p,0,0)\times(0,d,0)_s$ models to the outlier free time series.

Parameter

`maxlag` – The maximum value for p allowed when fitting ARIMA$(p,0,0)\times(0,d,0)_s$ models to the given series, $0 \leq p \leq$ maxlag. It is required that $1 \leq$ maxlag $\leq$ `x.Length`. The optimum ARIMA$(p,0,0)\times(0,d,0)_s$ model is determined according to the model selection criterion chosen by the user, see property `ModelSelectionCriterion`.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input matrix to ARAutoUnivariate is singular.

`Imsl.Stat.NonInvertibleException` is thrown if the intermediate or final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.NonStationaryException` is thrown if the intermediate or final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

`Imsl.Math.DidNotConvergeException` is thrown if the algorithm computing the roots of the AR- or MA- polynomial does not converge.

`Imsl.Math.SingularMatrixException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is singular.

`Imsl.Math.NotSPDException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is not positive definite.

`Imsl.Stat.NoAcceptableModelFoundException` is thrown if no appropriate ARIMA model for the given time series could be found.

Compute

```
public void Compute(int[] arOrders, int[] maOrders)
```

Description

Estimates potential missing values, detects and determines outliers and simultaneously fits an optimum model from a set of different ARIMA$(p,0,q)\times(0,d,0)_s$ models to the outlier free time series.

Parameters

`arOrders` – An `int` array containing all possible AR orders to consider in the optimum model search. It is required that all values in `arOrders` are greater than or equal to zero.

`maOrders` – An `int` array containing all possible MA orders to consider in the optimum model search. It is required that all values in `maOrders` are greater than or equal to zero.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input matrix to ARAutoUnivariate is singular.

`Imsl.Stat.NonInvertibleException` is thrown if the intermediate or final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.NonStationaryException` is thrown if the intermediate or final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

`Imsl.Math.DidNotConvergeException` is thrown if the algorithm computing the roots of the AR- or MA- polynomial does not converge.

`Imsl.Math.SingularMatrixException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is singular.

`Imsl.Math.NotSPDException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is not positive definite.

`Imsl.Stat.NoAcceptableModelFoundException` is thrown if no appropriate ARIMA model for the given time series could be found.

Compute

```
public void Compute(int p, int q, int s, int d)
```

Description

Estimates potential missing values, detects and determines outliers and simultaneously fits an ARIMA$(p,0,q)\times(0,d,0)_s$ model to the outlier free time series.

Parameters

`p` – A non-negative scalar `int`, the order of the AR part of the model.

`q` – A non-negative scalar `int`, the order of the MA part of the model.

`s` – A positive scalar `int`, the period of the difference used in the model.

`d` – A non-negative scalar `int`, the order of the difference used in the model.

Exceptions

`Imsl.Stat.MatrixSingularException` is thrown if the input matrix is singular.

`Imsl.Stat.TooManyCallsException` is thrown if the number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters + 1.

`Imsl.Stat.IncreaseErrRelException` is thrown if the bound for the relative error is too small.

`Imsl.Stat.NewInitialGuessException` is thrown if the iteration has not made good progress.

`Imsl.Stat.IllConditionedException` is thrown if the problem is ill-conditioned.

`Imsl.Stat.TooManyIterationsException` is thrown if the maximum number of iterations is exceeded.

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the maximum number of function evaluations is exceeded.

`Imsl.Stat.TooManyJacobianEvalException` is thrown if the maximum number of Jacobian evaluations is exceeded.

`Imsl.Stat.SingularTriangularMatrixException` is thrown if the input matrix to ARAutoUnivariate is singular.

`Imsl.Stat.NonInvertibleException` is thrown if the intermediate or final maximum likelihood estimates for the time series are noninvertible.

`Imsl.Stat.NonStationaryException` is thrown if the intermediate or final maximum likelihood estimates for the time series are nonstationary.

`Imsl.Stat.InitialMAException` is thrown if the initial values provided for the moving average terms are noninvertible. In this case, `ARMAMaxLikelihood` terminates and does not compute the time series estimates.

`Imsl.Math.DidNotConvergeException` is thrown if the algorithm computing the roots of the AR- or MA- polynomial does not converge.

`Imsl.Math.SingularMatrixException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is singular.

`Imsl.Math.NotSPDException` is thrown if during the computation of a small perturbation of the matrix product A^TA, it is found that A, the matrix used in the determination of the ω weights, is not positive definite.

`Imsl.Stat.NoAcceptableModelFoundException` is thrown if no appropriate ARIMA model for the given time series could be found.

Forecast

```
public void Forecast(int nForecast)
```

Description

Computes forecasts, associated probability limits and ψ weights for the given outlier contaminated time series.

Parameter

`nForecast` – An `int` scalar representing the number of forecasts that will be computed. `nForecast` must be greater than 0. Forecast origin is the time point of the last observed value in the time series, t_{nObs}. Forecasts are computed for lead times $1, 2, \ldots,$ nForecast, i.e. time points $t_{\text{nObs}}+1, t_{\text{nObs}}+2, \ldots, t_{\text{nObs}}+\text{nForecast}$.

Remarks

One of the `Compute` methods must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetAR

```
public double[] GetAR()
```

Description

Returns the final autoregressive parameter estimates of the optimum model.

Returns

A `double` array containing the final autoregressive parameter estimates. If the optimum model has no AR component then a zero-length array is returned.

Remarks

Note that one of the `Compute` methods must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetCompleteTimes

```
public int[] GetCompleteTimes()
```

Description

Returns all time points at which the original series was observed, including values for times with missing values in `x`.

Returns

An `int` array of length `times[times.Length-1]-times[0]+1` containing the times at which the time series (including missing values) was observed.

Remarks

One of the `Compute` methods must be called before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

GetCompleteTimeSeries

```
public double[] GetCompleteTimeSeries()
```

Description

Returns the original series with potentially missing values replaced by estimates.

Returns

A `double` array containing the original time series with missing values replaced by estimates.

Remarks

One of the `Compute` methods must be called before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

GetDeviations

```
public double[] GetDeviations()
```

Description

Returns the deviations used for calculating the forecast confidence limits.

Returns

A `double` array of length `nForecast` containing the deviations from each forecast for calculating forecast confidence intervals. The confidence level is specified by the `Confidence` property.

Remarks

Method `Forecast` must be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetForecast

```
public double[] GetForecast()
```

Description

Returns forecasts for the original outlier contaminated series.

Returns

A `double` array of length `nForecast` containing the forecasts for the original series. Forecast origin is the time point of the last observed value in the time series, t_{nObs}. Forecasts are returned for lead times $1, 2, \ldots,$ nForecast, i.e. time points $t_{\text{nObs}}+1, t_{\text{nObs}}+2, \ldots, t_{\text{nObs}}+$ nForecast.

Remarks

Method `Forecast` must be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetMA

```
public double[] GetMA()
```

Description

Returns the final moving average parameter estimates of the optimum model.

Returns

A `double` array containing the final moving average parameter estimates. If the optimum model has no MA component then a zero-length array is returned.

Remarks

Note that one of the `Compute` methods must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetOptimumModelOrder

```
public int[] GetOptimumModelOrder()
```

Description

Returns the order $(p,0,q)\times(0,d,0)_s$ of the optimum model.

Returns

An `int` array of length 4 containing the values p, q, s and d for the optimum model.

Remarks

One of the `Compute` methods must be invoked first before calling this method. Otherwise, an `InvalidOperationException` exception is thrown.

GetOutlierFreeForecast

```
public double[] GetOutlierFreeForecast()
```

Description

Returns forecasts for the outlier free series.

Returns

A `double` array of length `nForecast` containing the forecasts for the outlier free series. Forecast origin is the time point of the last observed value in the time series, t_{nObs}. Forecasts are returned for lead times $1, 2, \ldots,$ nForecast, i.e. time points $t_{\text{nObs}}+1, t_{\text{nObs}}+2, \ldots, t_{\text{nObs}}+$ nForecast.

Remarks

Method `Forecast` must be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetOutlierFreeSeries

```
public double[] GetOutlierFreeSeries()
```

Description

Returns the outlier free series.

Returns

A `double` array containing the original time series with estimated missing values after removal of any outlier effects.

Remarks

One of the `Compute` methods must be called before invoking this method. Otherwise, an `InvalidOperationException` exception is thrown.

GetOutlierStatistics

```
public int[,] GetOutlierStatistics()
```

Description

Returns the outlier statistics.

Returns

A `double` array of length `nOutliers` by 2, where `nOutliers` is the number of detected outliers, containing the outlier statistics. The first column contains the time at which the outlier was observed (time ranging from `times[0]` to `times[times.Length - 1]`) and the second column contains an identifier indicating the type of outlier observed.

Remarks

Outlier types fall into one of five categories:

Identifier	Outlier type
`INNOVATIONAL = 0`	Innovational Outliers (IO)
`ADDITIVE = 1`	Additive Outliers (AO)
`LEVEL_SHIFT = 2`	Level Shift Outliers (LS)
`TEMPORARY_CHANGE = 3`	Temporary Change Outliers (TC)
`UNABLE_TO_IDENTIFY = 4`	Unable to Identify (UI)

If the number of detected outliers equals zero, then an array of length zero is returned.

One of the `Compute` methods must be invoked first before invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

GetPsiWeights

```
public double[] GetPsiWeights()
```

Description

Returns the ψ weights of the infinite order moving average form of the model.

Returns

A `double` array of length `nForecast` containing the ψ weights of the infinite order moving average form of the optimum model for the outlier free series.

Remarks

Method `Forecast` must be invoked before this method is called. Otherwise, an `InvalidOperationException` exception is thrown.

GetResiduals

```
public double[] GetResiduals()
```

Description

Returns the residuals.

Returns

A `double` array containing the residuals for the outlier and gap free series at the final parameter estimation point.

Remarks

Note that one of the `Compute` methods must be invoked first before invoking this method. Otherwise, `GetResiduals` throws an `InvalidOperationException` exception.

SetDifferenceOrders

```
public void SetDifferenceOrders(int[] orders)
```

Description

Defines the orders of the periodic differences used in the determination of the optimum model.

Parameter

`orders` – An `int` array containing all possible orders for each difference given in `periods`. All elements in `orders` must be non-negative.

Default: `orders` is a one-element array with `orders[0]` = 0.

SetPeriods

```
public void SetPeriods(int[] periods)
```

Description

Defines the periods used in the determination of the optimum model.

Parameter

`periods` – An `int` array containing all possible periods that can be applied to the original series after insertion of missing values. All elements of `periods` must be positive.

Default: `periods` is a one-element array with `periods[0]` = 1.

Example 1: Determination of an optimum AR(p) model

This example uses time series LNU03327709 from the US Department of Labor, Bureau of Labor Statistics. It contains the unadjusted special unemployment rate, taken monthly from January 1994 through September 2005. The values 01/2004 - 03/2005 are used by class `AutoARIMA` for outlier detection and parameter estimation. In this example, method 1, without seasonal adjustment, is chosen to find an appropriate AR(p) model. A forecast is done for the following six months and compared with the actual values 04/2005 - 09/2005.

```
using System;
using Imsl.Stat;

public class AutoARIMAEx1
{
   public static void Main(String[] args)
   {
      int nOutliers;
      double aic, RSE, constant;
      int[] optimumModel;
      int[,] outlierStatistics;
      double[] outlierForecast, ar, ma;
      double[] psiWeights, probabilityLimits;


      double[] x =
      {
            12.8, 12.2, 11.9, 10.9, 10.6, 11.3, 11.1, 10.4, 10.0, 9.7, 9.7,
            9.7, 11.1, 10.5, 10.3, 9.8, 9.8, 10.4, 10.4, 10.0, 9.7, 9.3, 9.6,
            9.7, 10.8, 10.7, 10.3, 9.7, 9.5, 10.0, 10.0, 9.3, 9.0, 8.8, 8.9,
            9.2, 10.4, 10.0, 9.6, 9.0, 8.5, 9.2, 9.0, 8.6, 8.3, 7.9, 8.0,
            8.2, 9.3, 8.9, 8.9, 7.7, 7.6, 8.4, 8.5, 7.8, 7.6, 7.3, 7.2, 7.3,
            8.5, 8.2, 7.9, 7.4, 7.1, 7.9, 7.7, 7.2, 7.0, 6.7, 6.8, 6.9, 7.8,
            7.6, 7.4, 6.6, 6.8, 7.2, 7.2, 7.0, 6.6, 6.3, 6.8, 6.7, 8.1, 7.9,
            7.6, 7.1, 7.2, 8.2, 8.1, 8.1, 8.2, 8.7, 9.0, 9.3, 10.5, 10.1,
            9.9, 9.4, 9.2, 9.8, 9.9, 9.5, 9.0, 9.0, 9.4, 9.6, 11.0, 10.8,
            10.4, 9.8, 9.7, 10.6, 10.5, 10.0, 9.8, 9.5, 9.7, 9.6, 10.9, 10.3,
            10.4, 9.3, 9.3, 9.8, 9.8, 9.3, 8.9, 9.1, 9.1, 9.1, 10.2, 9.9, 9.4
      };

      double[] exactForecast = { 8.7, 8.6, 9.3, 9.1, 8.8, 8.5 };

      int[] times =
      {
         1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
         20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
         37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53,
         54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,
         71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87,
```

```
         88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103,
         104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116,
         117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129,
         130, 131, 132, 133, 134, 135
      };

      AutoARIMA autoArima = new AutoARIMA(times, x);
      autoArima.CriticalValue = 3.8;
      autoArima.Compute(5);
      autoArima.Forecast(6);

      nOutliers = autoArima.NumberOfOutliers;
      aic = autoArima.AIC;
      optimumModel = autoArima.GetOptimumModelOrder();
      outlierStatistics = autoArima.GetOutlierStatistics();
      RSE = autoArima.ResidualStandardError;
      outlierForecast = autoArima.GetForecast();
      psiWeights = autoArima.GetPsiWeights();
      probabilityLimits = autoArima.GetDeviations();
      constant = autoArima.Constant;
      ar = autoArima.GetAR();
      ma = autoArima.GetMA();

      Console.Out.WriteLine("\nMethod 1: Automatic AR model selection"
                            + ", no differencing");
      Console.Out.WriteLine(
         "\nOptimum Model: p={0,1:d}, q={1,1:d}, s={2,1:d}, d={3,1:d}",
         optimumModel[0], optimumModel[1], optimumModel[2], optimumModel[3]);
      Console.Out.WriteLine("\nNumber of outliers:{0,3:d}", nOutliers);
      Console.Out.WriteLine();
      Console.Out.WriteLine("Outlier statistics:");
      Console.Out.WriteLine(" Time    Type");
      for (int i = 0; i < nOutliers; i++)
         Console.Out.WriteLine("{0,5:d}{1,8:d}", outlierStatistics[i, 0],
            outlierStatistics[i, 1]);
      Console.Out.WriteLine("\nAIC:{0,12:f6}", aic);
      Console.Out.WriteLine("RSE:{0,12:f6}", RSE);
      Console.Out.WriteLine();
      Console.Out.WriteLine("      Parameters");
      Console.Out.WriteLine(" constant:{0,12:f6}", constant);
      for (int i = 0; i < ar.Length; i++)
         Console.Out.WriteLine(" ar[{0,1:d}]:{1,15:f6}", i, ar[i]);
      for (int i = 0; i < ma.Length; i++)
         Console.Out.WriteLine(" ma[{0,1:d}]:{1,15:f6}", i, ma[i]);

      Console.Out.WriteLine();
      Console.Out.WriteLine();
      Console.Out.WriteLine("      * * * Forecast Table * * *");
      Console.Out.WriteLine("  Exact   forecast     limits        psi");
      for (int i = 0; i < outlierForecast.Length; i++)
         Console.Out.WriteLine("{0,7:f4}{1,11:f4}{2,11:f4}{3,11:f4}",
                               exactForecast[i], outlierForecast[i],
                               probabilityLimits[i], psiWeights[i]);
   }
}
```

Output

```
Method 1: Automatic AR model selection, no differencing

Optimum Model: p=5, q=0, s=1, d=0

Number of outliers:  7

Outlier statistics:
 Time    Type
    8       2
   13       0
   37       3
   85       0
   97       0
  109       0
  121       0

AIC:  371.104666
RSE:    0.359632

      Parameters
 constant:    0.097542
 ar[0]:       0.891871
 ar[1]:      -0.123831
 ar[2]:      -0.138262
 ar[3]:       0.135621
 ar[4]:       0.224111


      * * * Forecast Table * * *
  Exact   forecast     limits        psi
 8.7000     9.1076     0.7049     0.8919
 8.6000     9.0993     0.9445     0.6716
 9.3000     9.4032     1.0565     0.3503
 9.1000     9.5806     1.0849     0.2416
 8.8000     9.5506     1.0982     0.4243
 8.5000     9.3932     1.1382     0.5910
```

Example 2: Determination of an optimum ARIMA model via Grid search

This is the same as Example 1, except now class `AutoARIMA` uses Method 2 with a possible seasonal adjustment. As a result, the unadjusted model with `p = 3, q = 2, s = 1, d = 0` is chosen as optimum.

```
using System;
using Imsl.Stat;

public class AutoARIMAEx2
{
   public static void Main(String[] args)
   {
      int nOutliers;
      double aic, RSE, constant;
```

```
int[] optimumModel;
int[,] outlierStatistics;
double[] outlierForecast, ar, ma;
double[] psiWeights, probabilityLimits;
int[] arOrders = { 0, 1, 2, 3 };
int[] maOrders = { 0, 1, 2, 3 };
int[] periods =  { 1, 2 };
int[] orders =  { 0, 1, 2 };

double[] x =
{
      12.8, 12.2, 11.9, 10.9, 10.6, 11.3, 11.1, 10.4, 10.0, 9.7, 9.7,
      9.7, 11.1, 10.5, 10.3, 9.8, 9.8, 10.4, 10.4, 10.0, 9.7, 9.3, 9.6,
      9.7, 10.8, 10.7, 10.3, 9.7, 9.5, 10.0, 10.0, 9.3, 9.0, 8.8, 8.9,
      9.2, 10.4, 10.0, 9.6, 9.0, 8.5, 9.2, 9.0, 8.6, 8.3, 7.9, 8.0,
      8.2, 9.3, 8.9, 8.9, 7.7, 7.6, 8.4, 8.5, 7.8, 7.6, 7.3, 7.2, 7.3,
      8.5, 8.2, 7.9, 7.4, 7.1, 7.9, 7.7, 7.2, 7.0, 6.7, 6.8, 6.9, 7.8,
      7.6, 7.4, 6.6, 6.8, 7.2, 7.2, 7.0, 6.6, 6.3, 6.8, 6.7, 8.1, 7.9,
      7.6, 7.1, 7.2, 8.2, 8.1, 8.1, 8.2, 8.7, 9.0, 9.3, 10.5, 10.1,
      9.9, 9.4, 9.2, 9.8, 9.9, 9.5, 9.0, 9.0, 9.4, 9.6, 11.0, 10.8,
      10.4, 9.8, 9.7, 10.6, 10.5, 10.0, 9.8, 9.5, 9.7, 9.6, 10.9, 10.3,
      10.4, 9.3, 9.3, 9.8, 9.8, 9.3, 8.9, 9.1, 9.1, 9.1, 10.2, 9.9, 9.4
};

double[] exactForecast = { 8.7, 8.6, 9.3, 9.1, 8.8, 8.5 };

int[] times =
{
   1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
   20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
   37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53,
   54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,
   71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87,
   88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103,
   104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116,
   117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129,
   130, 131, 132, 133, 134, 135
};

AutoARIMA autoArima = new AutoARIMA(times, x);
autoArima.CriticalValue = 3.8;
autoArima.MaximumARLag = 5;
autoArima.SetPeriods(periods);
autoArima.SetDifferenceOrders(orders);
autoArima.Compute(arOrders, maOrders);
autoArima.Forecast(6);

nOutliers = autoArima.NumberOfOutliers;
aic = autoArima.AIC;
optimumModel = autoArima.GetOptimumModelOrder();
outlierStatistics = autoArima.GetOutlierStatistics();
RSE = autoArima.ResidualStandardError;
outlierForecast = autoArima.GetForecast();
psiWeights = autoArima.GetPsiWeights();
probabilityLimits = autoArima.GetDeviations();
constant = autoArima.Constant;
```

```
      ar = autoArima.GetAR();
      ma = autoArima.GetMA();

      Console.Out.WriteLine("\nMethod 2: Grid search with " +
                            "differencing");
      Console.Out.WriteLine();
      Console.Out.WriteLine(
         "Optimum Model: p={0,1:d}, q={1,1:d}, s={2,1:d}, d={3,1:d}",
         optimumModel[0], optimumModel[1], optimumModel[2], optimumModel[3]);
      Console.Out.WriteLine("\nNumber of outliers:{0,2:d}", nOutliers);
      Console.Out.WriteLine();
      Console.Out.WriteLine("Outlier statistics:");
      Console.Out.WriteLine(" Time    Type");
      for (int i = 0; i < nOutliers; i++)
         Console.Out.WriteLine("{0,5:d}{1,8:d}", outlierStatistics[i, 0],
            outlierStatistics[i, 1]);
      Console.Out.WriteLine("\nAIC:{0,12:f6}", aic);
      Console.Out.WriteLine("RSE:{0,12:f6}", RSE);
      Console.Out.WriteLine();
      Console.Out.WriteLine("     Parameters");
      Console.Out.WriteLine(" constant:{0,11:f6}", constant);
      for (int i = 0; i < ar.Length; i++)
         Console.Out.WriteLine(" ar[{0,1:d}]: {1,13:f6}", i, ar[i]);
      for (int i = 0; i < ma.Length; i++)
         Console.Out.WriteLine(" ma[{0,1:d}]: {1,13:f6}", i, ma[i]);
      Console.Out.WriteLine("\n\n      * * * Forecast Table * * *");
      Console.Out.WriteLine("  Exact   forecast     limits        psi");
      for (int i = 0; i < outlierForecast.Length; i++)
         Console.Out.WriteLine("{0,7:f4}{1,11:f4}{2,11:f4}{3,11:f4}",
            exactForecast[i], outlierForecast[i],
            probabilityLimits[i], psiWeights[i]);
   }
}
```

Output

```
Method 2: Grid search with differencing

Optimum Model: p=3, q=2, s=1, d=0

Number of outliers: 1

Outlier statistics:
 Time    Type
  109       0

AIC:  408.108176
RSE:    0.412456

     Parameters
 constant:   0.554459
 ar[0]:      1.940615
 ar[1]:     -1.898025
 ar[2]:      0.897791
 ma[0]:      1.115803
```

```
 ma[1]:     -0.911902


     * * * Forecast Table * * *
  Exact   forecast     limits        psi
 8.7000     9.1085     0.8084     0.8248
 8.6000     9.1715     1.0479     0.6145
 9.3000     9.5039     1.1597     0.5248
 9.1000     9.7677     1.2349     0.5926
 8.8000     9.7051     1.3245     0.7056
 8.5000     9.3817     1.4421     0.7157
```

Example 3: Specified ARIMA model

This example is the same as example 2 but now method 3 with the optimum model parameters `p = 3, q = 2, s = 1, d = 0` from Example 2 is chosen for outlier detection and forecasting.

```
using System;
using Imsl.Stat;

public class AutoARIMAEx3
{
   public static void Main(String[] args)
   {
      int nOutliers;
      double aic, RSE, constant;
      int[] optimumModel;
      int[,] outlierStatistics;
      double[] outlierForecast, ar, ma;
      double[] psiWeights, probabilityLimits;

      double[] x =
      {
            12.8, 12.2, 11.9, 10.9, 10.6, 11.3, 11.1, 10.4, 10.0, 9.7, 9.7,
            9.7, 11.1, 10.5, 10.3, 9.8, 9.8, 10.4, 10.4, 10.0, 9.7, 9.3, 9.6,
            9.7, 10.8, 10.7, 10.3, 9.7, 9.5, 10.0, 10.0, 9.3, 9.0, 8.8, 8.9,
            9.2, 10.4, 10.0, 9.6, 9.0, 8.5, 9.2, 9.0, 8.6, 8.3, 7.9, 8.0,
            8.2, 9.3, 8.9, 8.9, 7.7, 7.6, 8.4, 8.5, 7.8, 7.6, 7.3, 7.2, 7.3,
            8.5, 8.2, 7.9, 7.4, 7.1, 7.9, 7.7, 7.2, 7.0, 6.7, 6.8, 6.9, 7.8,
            7.6, 7.4, 6.6, 6.8, 7.2, 7.2, 7.0, 6.6, 6.3, 6.8, 6.7, 8.1, 7.9,
            7.6, 7.1, 7.2, 8.2, 8.1, 8.1, 8.2, 8.7, 9.0, 9.3, 10.5, 10.1,
            9.9, 9.4, 9.2, 9.8, 9.9, 9.5, 9.0, 9.0, 9.4, 9.6, 11.0, 10.8,
            10.4, 9.8, 9.7, 10.6, 10.5, 10.0, 9.8, 9.5, 9.7, 9.6, 10.9, 10.3,
            10.4, 9.3, 9.3, 9.8, 9.8, 9.3, 8.9, 9.1, 9.1, 9.1, 10.2, 9.9, 9.4
      };

      double[] exactForecast = { 8.7, 8.6, 9.3, 9.1, 8.8, 8.5 };

      int[] times =
      {
         1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
         20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
         37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53,
         54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,
         71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87,
```

```
         88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103,
         104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116,
         117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129,
         130, 131, 132, 133, 134, 135
      };

      AutoARIMA autoArima = new AutoARIMA(times, x);
      autoArima.CriticalValue = 3.8;
      autoArima.Compute(3, 2, 1, 0);
      autoArima.Forecast(6);

      nOutliers = autoArima.NumberOfOutliers;
      aic = autoArima.AIC;
      optimumModel = autoArima.GetOptimumModelOrder();
      outlierStatistics = autoArima.GetOutlierStatistics();
      RSE = autoArima.ResidualStandardError;
      outlierForecast = autoArima.GetForecast();
      psiWeights = autoArima.GetPsiWeights();
      probabilityLimits = autoArima.GetDeviations();
      constant = autoArima.Constant;
      ar = autoArima.GetAR();
      ma = autoArima.GetMA();

      Console.Out.WriteLine("\nMethod 3: Specified ARIMA model");
      Console.Out.WriteLine();
      Console.Out.WriteLine(
         "Optimum Model: p={0,1:d}, q={1,1:d}, s={2,1:d}, d={3,1:d}",
         optimumModel[0], optimumModel[1], optimumModel[2], optimumModel[3]);
      Console.Out.WriteLine("\nNumber of outliers:{0,2:d}\n", nOutliers);
      Console.Out.WriteLine("Outlier statistics:");
      Console.Out.WriteLine(" Time    Type");
      for (int i = 0; i < nOutliers; i++)
         Console.Out.WriteLine("{0,5:d}{1,8:d}", outlierStatistics[i, 0],
            outlierStatistics[i, 1]);
      Console.Out.WriteLine("\nAIC:{0,13:f6}", aic);
      Console.Out.WriteLine("RSE:{0,13:f6}", RSE);
      Console.Out.WriteLine();
      Console.Out.WriteLine("     Parameters");
      Console.Out.WriteLine(" constant:{0,11:f6}", constant);
      for (int i = 0; i < ar.Length; i++)
         Console.Out.WriteLine(" ar[{0,2:d}]:{1,13:f6}", i, ar[i]);
      for (int i = 0; i < ma.Length; i++)
         Console.Out.WriteLine(" ma[{0,2:d}]:{1,13:f6}", i, ma[i]);
      Console.Out.WriteLine("\n\n      * * * Forecast Table * * *");
      Console.Out.WriteLine("  Exact   forecast     limits        psi");
      for (int i = 0; i < outlierForecast.Length; i++)
         Console.Out.WriteLine("{0,7:f4}{1,11:f4}{2,11:f4}{3,11:f4}",
            exactForecast[i], outlierForecast[i],
            probabilityLimits[i], psiWeights[i]);
   }
}
```

Output

```
Method 3: Specified ARIMA model
```

```
Optimum Model: p=3, q=2, s=1, d=0

Number of outliers: 1

Outlier statistics:
 Time    Type
  109       0

AIC:   408.108176
RSE:     0.412456

     Parameters
 constant:   0.554459
 ar[ 0]:     1.940615
 ar[ 1]:    -1.898025
 ar[ 2]:     0.897791
 ma[ 0]:     1.115803
 ma[ 1]:    -0.911902


      * * * Forecast Table * * *
  Exact   forecast     limits        psi
 8.7000     9.1085     0.8084     0.8248
 8.6000     9.1715     1.0479     0.6145
 9.3000     9.5039     1.1597     0.5248
 9.1000     9.7677     1.2349     0.5926
 8.8000     9.7051     1.3245     0.7056
 8.5000     9.3817     1.4421     0.7157
```

AutoARIMA.InformationCriterion Enumeration

`public enumeration Imsl.Stat.AutoARIMA.InformationCriterion`

Indicates which information criterion is used in the optimum model search.

Fields

Akaike

`public Imsl.Stat.AutoARIMA.InformationCriterion Akaike`

Description

Indicates that Akaike's information criterion (AIC) is used in the optimum model determination.

AkaikeCorrected

`public Imsl.Stat.AutoARIMA.InformationCriterion AkaikeCorrected`

Description

Indicates that Akaike's corrected information criterion (AICC) is used in the optimum model determination.

Bayes

```
public Imsl.Stat.AutoARIMA.InformationCriterion Bayes
```

Description

Indicates that the Bayesian information criterion (BIC) is used in the optimum model determination.

Difference Class

```
public class Imsl.Stat.Difference
```

Differences a seasonal or nonseasonal time series.

Class `Difference` performs m = `periods.Length` successive backward differences of period $s_i = \text{periods}[i-1]$ and order $d_i = \text{orders}[i-1]$ for $i = 1, \ldots, m$ on the n = `z.Length` observations $\{Z_t\}$ for $t = 1, 2, \ldots, n$.

Consider the backward shift operator B given by

$$B^k Z_t = Z_{t-k}$$

for all k. Then, the *backward difference operator* with period s is defined by the following:

$$\Delta_s Z_t = (1 - B^s) Z_t = Z_t - Z_{t-s} \quad \text{for} \, s \geq 0$$

Note that $B_s Z_t$ and $\Delta_s Z_t$ are defined only for $t = (s+1), \ldots, n$. Repeated differencing with period s is simply

$$\Delta_s^d Z_t = (1 - B^s)^d Z_t = \sum_{j=0}^{d} \frac{d!}{j!\,(d-j)!} (-1)^j B^{sj} Z_t$$

where $d \geq 0$ is the order of differencing. Note that

$$\Delta_s^d Z_t$$

is defined only for $t = (sd+1), \ldots, n$.

The general difference formula used in the class `Difference` is given by

$$W_T = \begin{cases} \text{NaN} & \text{for } t = 1, \ldots, n_L \\ \Delta_{s_1}^{d_1} \Delta_{s_2}^{d2} \ldots \Delta_{s_m}^{d_m} Z_t & \text{for } t = n_L + 1, \ldots, n \end{cases}$$

where n_L represents the number of observations "lost" because of differencing and NaN represents the missing value code. Note that

$$n_L = \sum_j s_j d_j$$

A homogeneous, stationary time series can be arrived at by appropriately differencing a homogeneous, nonstationary time series (Box and Jenkins 1976, p. 85). Preliminary application of an appropriate transformation followed by differencing of a series can enable model identification and parameter estimation in the class of homogeneous stationary autoregressive moving average models.

Property

ObservationsLost

```
public int ObservationsLost {get; }
```

Description

Returns the number of observations lost because of differencing the time series. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

An `int` containing the number of observations lost because of differencing the time series `z`.

Constructor

Difference

```
public Difference()
```

Description

Constructor for `Difference`.

Methods

Compute

```
public double[] Compute(double[] z, int[] periods)
```

Description

Computes a Difference series.

Parameters

`z` – A `double` array containing the time series.

`periods` – A `int` array containing the periods at which z is to be differenced.

Returns

A `double` array containing the differenced series.

ExcludeFirst

```
public void ExcludeFirst(bool exclude)
```

Description

Excludes observations lost due to differencing.

Parameter

`exclude` – A `bool` specifying whether or not to exclude lost observations due to differencing.

Remarks

If set to `true`, the observations lost due to differencing will be excluded. The differenced series will be the length of the number of observations minus the number of observations lost. If set to `false`, the observations lost due to differencing will be set to `NaN` (Not a number) and included in the differenced series. The default is to set the lost observations to `NaN`.

SetOrders

```
public void SetOrders(int[] orders)
```

Description

Sets the orders for the Difference object.

Parameter

`orders` – An `int` array of length equal to length of `periods`, containing the order of each difference given in periods.

Remarks

The elements of `orders` must be greater than or equal to 0.

Example 1: Difference

This example uses the Airline Data (Box and Jenkins 1976, p. 531) consisting of the monthly total number of international airline passengers from January 1949 through December 1960. Difference is used to compute ...

$$W_t = \Delta_1\Delta_{12}Z_t = (Z_t - Z_{t-12}) - (Z_{t-1} - Z_{t-13})$$

for t= 14, 15, ...,24.

```
using System;
using Imsl.Stat;

public class DifferenceEx1
{
    public static void  Main(String[] args)
    {

        int[] periods = new int[]{1, 12};
        int nLost;
        double[] z = new double[]{112.0, 118.0, 132.0, 129.0, 121.0,
                                  135.0, 148.0, 148.0, 136.0, 119.0,
                                  104.0, 118.0, 115.0, 126.0, 141.0,
                                  135.0, 125.0, 149.0, 170.0, 170.0,
                                  158.00, 133.0, 114.0, 140.0};

        Difference diff = new Difference();
        double[] output = diff.Compute(z, periods);
        nLost = diff.ObservationsLost;

        Console.Out.WriteLine("Observations Lost = " + nLost);

        for (int i = 0; i < output.Length; i++)
            Console.Out.WriteLine(output[i]);
    }
}
```

Output

```
Observations Lost = 13
NaN
NaN
NaN
NaN
NaN
NaN
NaN
NaN
NaN
NaN
NaN
NaN
NaN
```

```
5
1
-3
-2
10
8
0
0
-8
-4
12
```

Example 2: Difference

This example uses the same data as Example 1. The first number of lost observations are excluded from W due to differencing, and the number of lost observations is also output.

```
using System;
using Imsl.Stat;

public class DifferenceEx2
{
    public static void  Main(String[] args)
    {
        int[] periods = new int[]{1, 12};
        int nLost;
        double[] z = new double[]{112.0, 118.0, 132.0, 129.0, 121.0,
                                  135.0, 148.0, 148.0, 136.0, 119.0,
                                  104.0, 118.0, 115.0, 126.0, 141.0,
                                  135.0, 125.0, 149.0, 170.0, 170.0,
                                  158.00, 133.0, 114.0, 140.0};

        Difference diff = new Difference();
        diff.ExcludeFirst(true);
        double[] output = diff.Compute(z, periods);
        nLost = diff.ObservationsLost;

        Console.Out.WriteLine
            ("The number of observation lost = " + nLost);
        for (int i = 0; i < output.Length; i++)
            Console.Out.WriteLine(output[i]);
    }
}
```

Output

```
The number of observation lost = 13
5
1
-3
-2
10
8
0
```

```
0
-8
-4
12
```

GARCH Class

```
public class Imsl.Stat.GARCH
```

Computes estimates of the parameters of a GARCH(p,q) model.

The Generalized Autoregressive Conditional Heteroskedastic (GARCH) model is defined as

$$y_t = z_t \sigma_t$$

$$\sigma_t^2 = \sigma^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i y_{t-i}^2$$

where z_t's are independent and identically distributed standard normal random variables,

$$\sigma > 0, \beta_i \geq 0, \alpha_i \geq 0$$

and

$$\sum_{i=1}^{p} \beta_i + \sum_{i=1}^{q} \alpha_i < 1$$

The above model is denoted as GARCH(p, q). The p is the autoregressive lag and the q is the moving average lag. When $\beta_i = 0, i = 1, 2, \ldots, p$, the above model reduces to ARCH(q) which was proposed by Engle (1982). The nonnegativity conditions on the parameters implied a nonnegative variance and the condition on the sum of the β_i's and α_i's is required for wide sense stationarity.

In the empirical analysis of observed data, GARCH(1,1) or GARCH(1,2) models have often found to appropriately account for conditional heteroskedasticity (Palm 1996). This finding is similar to linear time series analysis based on ARMA models.

It is important to notice that for the above models positive and negative past values have a symmetric impact on the conditional variance. In practice, many series may have strong asymmetric influence on the conditional variance. To take into account this phenomena, Nelson (1991) put forward Exponential

GARCH (EGARCH). Lai (1998) proposed and studied some properties of a general class of models that extended linear relationship of the conditional variance in ARCH and GARCH into nonlinear fashion.

The maximum likelihood method is used in estimating the parameters in GARCH(p,q). The log-likelihood of the model for the observed series $\{Y_t\}$ with length m is

$$\log(L) = \frac{m}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{m} y_t^2/\sigma_t^2 - \frac{1}{2}\sum_{t=1}^{m}\log\sigma_t^2,$$

$$\text{where } \sigma_t^2 = \sigma^2 + \sum_{i=1}^{p}\beta_i\sigma_{t-i}^2 + \sum_{i=1}^{q}\alpha_i y_{t-i}^2.$$

In the model, if $q = 0$, the model GARCH is singular such that the estimated Hessian matrix H is singular.

The initial values of the parameter array $x[\,]$ entered in array `xguess[ ]` must satisfy certain constraints. The first element of `xguess` refers to sigma and must be greater than zero and less than `MaxSigma`. The remaining $p+q$ initial values must each be greater than or equal to zero but less than one.

To guarantee stationarity in model fitting,

$$\sum_{i=1}^{p+q} x(i) < 1,$$

is checked internally. The initial values should be selected from the values between zero and one. The value of Akaike Information Criterion is computed by

$$2 \times \log(\mathrm{L}) + 2 \times (\mathrm{p} + \mathrm{q} + 1),$$

where log(L) is the value of the log-likelihood function at the estimated parameters.

In fitting the optimal model, the class Imsl.Math.MinConGenLin (p. 391), is modified to find the maximal likelihood estimates of the parameters in the model. Statistical inferences can be performed outside of the class `GARCH` based on the output of the log-likelihood function (`LogLikelihood` property), the Akaike Information Criterion (`Akaike` property), and the variance-covariance matrix (`GetVarCovarMatrix` method).

Properties

Akaike

```
public double Akaike {get; }
```

Description

Returns the value of Akaike Information Criterion evaluated at the estimated parameter array. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

A `double` scalar containing the value of Akaike Information Criterion evaluated at the estimated parameter array.

LogLikelihood

```
public double LogLikelihood {get; }
```

Description

Returns the value of Log-likelihood function evaluated at the estimated parameter array. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `0`.

Property Value

A `double` scalar containing the value of Log-likelihood function evaluated at the estimated parameter array.

MaxSigma

```
public double MaxSigma {get; set; }
```

Description

The value of the upperbound on the first element (sigma) of the array of returned estimated coefficients.

Property Value

A `double` scalar containing the value of the upperbound on the first element (sigma) of the array of returned estimated coefficients.

Remarks

By default, `MaxSigma`= 10.

Sigma

```
public double Sigma {get; }
```

Description

Returns the estimated value of sigma squared. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the return value is `NaN`.

Property Value

A `double` scalar containing the estimated value of sigma squared.

Constructor

GARCH

```
public GARCH(int p, int q, double[] y, double[] xguess)
```

Description

Constructor for `GARCH`.

Parameters

`p` – A `int` scalar containing the number of autoregressive (AR) parameters.

`q` – A `int` scalar containing the number of moving average (MA) parameters.

`y` – A `double` array containing the observed time series data.

`xguess` – A `double` array of length `p` + `q` + 1 containing the initial values for the parameter array.

Exception

`System.ArgumentException` is thrown if the dimensions of `y`, and `xguess` are not consistent

Methods

Compute

```
public void Compute()
```

Description

Computes estimates of the parameters of a GARCH(p,q) model.

Exceptions

`Imsl.Stat.ConstrInconsistentException` is thrown if the equality constraints are inconsistent

`Imsl.Stat.EqConstrInconsistentException` is thrown if the equality constraints and the bounds on the variables are found to be inconsistent

`Imsl.Stat.NoVectorXException` is thrown if no vector X satisfies all of the constraints

`Imsl.Stat.TooManyFunctionEvaluationsException` is thrown if the number of function evaluations exceeded 1000

`Imsl.Stat.VarsDeterminedException` is thrown if the variables are determined by the equality constraints

GetAR

```
public double[] GetAR()
```

Description

Returns the estimated values of autoregressive (AR) parameters. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` array of size `p` containing the estimated values of autoregressive (AR) parameters.

GetMA

```
public double[] GetMA()
```

Description

Returns the estimated values of moving average (MA) parameters. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` array of size `q` containing the estimated values of moving average (MA) parameters.

GetVarCovarMatrix

```
public double[,] GetVarCovarMatrix()
```

Description

Returns the variance-covariance matrix. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` matrix of size `p + q + 1` by `p + q + 1` containing the variance-covariance matrix.

GetX

```
public double[] GetX()
```

Description

Returns the estimated parameter array, `x`. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` array of size `p + q + 1` containing the estimated values of sigma squared, the AR parameters, and the MA parameters.

Example: GARCH

The data for this example are generated to follow a GARCH(p,q) process by using a random number generation function *sgarch*. The data set is analyzed and estimates of sigma, the AR parameters, and the MA parameters are returned. The values of the Log-likelihood function and the Akaike Information Criterion are returned.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class GARCHEx1
{
    static private void  sgarch(int p, int q, int m, double[] x,
        double[] y, double[] z, double[] y0, double[] sigma)
    {
        int i, j, l;
        double s1, s2, s3;
        Imsl.Stat.Random rand = new Imsl.Stat.Random(182198625);
        rand.Multiplier = 16807;
```

```
    for (i = 0; i < m + 1000; i++)
        z[i] = rand.NextNormal();

    l = System.Math.Max(p, q);
    l = System.Math.Max(l, 1);
    for (i = 0; i < l; i++)
        y0[i] = z[i] * x[0];

    /* COMPUTE THE INITIAL VALUE OF SIGMA */
    s3 = 0.0;
    if (System.Math.Max(p, q) >= 1)
    {
        for (i = 1; i < (p + q + 1); i++)
            s3 += x[i];
    }
    for (i = 0; i < l; i++)
        sigma[i] = x[0] / (1.0 - s3);
    for (i = l; i < (m + 1000); i++)
    {
        s1 = 0.0;
        s2 = 0.0;
        if (q >= 1)
        {
            for (j = 0; j < q; j++)
                s1 += x[j + 1] * y0[i - j - 1] * y0[i - j - 1];
        }
        if (p >= 1)
        {
            for (j = 0; j < p; j++)
                s2 += x[q + 1 + j] * sigma[i - j - 1];
        }
        sigma[i] = x[0] + s1 + s2;
        y0[i] = z[i] * Math.Sqrt(sigma[i]);
    }
    /*
     * DISCARD THE FIRST 1000 SIMULATED OBSERVATIONS
     */
    for (i = 0; i < m; i++)
        y[i] = y0[1000 + i];
    return ;
}


public static void  Main(String[] args)
{
    int n, p, q, m;
    double[] x = new double[]{1.3, 0.2, 0.3, 0.4};
    double[] xguess = new double[]{1.0, 0.1, 0.2, 0.3};
    double[] y = new double[1000];
    double[] wk1 = new double[2000];
    double[] wk2 = new double[2000];
    double[] wk3 = new double[2000];

    m = 1000;
    p = 2;
    q = 1;
```

```
        n = p + q + 1;
        sgarch(p, q, m, x, y, wk1, wk2, wk3);

        GARCH garch = new GARCH(p, q, y, xguess);
        garch.Compute();

        Console.Out.WriteLine
            ("Sigma estimate is " + garch.Sigma.ToString("0.000"));
        Console.Out.WriteLine();
        new PrintMatrix("AR estimate is ").Print(garch.GetAR());
        new PrintMatrix("MR estimate is ").Print(garch.GetMA());
        Console.Out.WriteLine("Log-likelihood function value is " +
            garch.LogLikelihood.ToString("0.000"));
        Console.Out.WriteLine("Akaike Information Criterion value is "
             + garch.Akaike.ToString("0.000"));
    }
}
```

Output

```
Sigma estimate is 1.692

   AR estimate is
            0
0  0.244996841351061
1  0.337228450714669

  MR estimate is
           0
0  0.3095927608719

Log-likelihood function value is -2707.073
Akaike Information Criterion value is 5422.146
```

KalmanFilter Class

`public class Imsl.Stat.KalmanFilter`

Performs Kalman filtering and evaluates the likelihood function for the state-space model.

Class `KalmanFilter` is based on a recursive algorithm given by Kalman (1960), which has come to be known as the `KalmanFilter`. The underlying model is known as the state-space model. The model is specified stage by stage where the stages generally correspond to time points at which the observations become available. `KalmanFilter` avoids many of the computations and storage requirements that would be necessary if one were to process all the data at the end of each stage in order to estimate the state vector. This is accomplished by using previous computations and retaining in storage only those items essential for processing of future observations.

The notation used here follows that of Sallas and Harville (1981). Let y_k (input in y using method

`Update`) be the $n_k \times 1$ vector of observations that become available at time *k*. The subscript *k* is used here rather than *t*, which is more customary in time series, to emphasize that the model is expressed in stages $k = 1, 2, \ldots$ and that these stages need not correspond to equally spaced time points. In fact, they need not correspond to time points of any kind. The *observation equation* for the state-space model is

$$y_k = Z_k b_k + e_k \quad k = 1, 2, \ldots$$

Here, Z_k (input in `z` using method `update`) is an $n_k \times q$ known matrix and b_k is the $q \times 1$ state vector. The state vector b_k is allowed to change with time in accordance with the *state equation*

$$b_{k+1} = T_{k+1} b_k + w_{k+1} \quad k = 1, 2, \ldots$$

starting with $b_1 = \mu_1 + w_1$.

The change in the state vector from time *k* to *k + 1* is explained in part by the *transition matrix* T_{k+1} (the identity matrix by default, or optionally using method `SetTransitionMatrix`), which is assumed known. It is assumed that the *q*-dimensional $w_k s (k = 1, 2, \ldots)$ are independently distributed multivariate normal with mean vector 0 and variance-covariance matrix $\sigma^2 Q_k$, that the n_k-dimensional $e_k s (k = 1, 2, \ldots)$ are independently distributed multivariate normal with mean vector 0 and variance-covariance matrix $\sigma^2 R_k$, and that the $w_k s$ and $e_k s$ are independent of each other. Here, $\mu_1$is the mean of b_1 and is assumed known, σ^2 is an unknown positive scalar. Q_{k+1} (input in `Q`) and R_k (input in `R`) are assumed known.

Denote the estimator of the realization of the state vector b_k given the observations $y_1, y_2, \ldots, y_j$ by

$$\hat{\beta}_{k|j}$$

By definition, the mean squared error matrix for

$$\hat{\beta}_{k|j}$$

is

$$\sigma^2 C_{k|j} = E(\hat{\beta}_{k|j} - b_k)(\hat{\beta}_{k|j} - b_k)^T$$

At the time of the *k*-th invocation, we have

$$\hat{\beta}_{k|k-1}$$

and

$C_{k|k-1}$, which were computed from the *k*-1-st invocation, input in `b` and `covb`, respectively. During the *k*-th invocation, `KalmanFilter` computes the filtered estimate

$$\hat{\beta}_{k|k}$$

along with $C_{k|k}$. These quantities are given by the *update equations*:

$$\hat{\beta}_{k|k} = \hat{\beta}_{k|k-1} + C_{k|k-1} Z_k^T H_k^{-1} v_k$$

$$C_{k|k} = C_{k|k-1} - C_{k|k-1} Z_k^T H_k^{-1} Z_k C_{k|k-1}$$

where

$$v_k = y_k - Z_k \hat{\beta}_{k|k-1}$$

and where

$$H_k = R_k + Z_k C_{k|k-1} Z_k^T$$

Here, v_k (stored in *v*) is the one-step-ahead prediction error, and $\sigma^2 H_k$ is the variance-covariance matrix for v_k. H_k is stored in `covv`. The "start-up values" needed on the first invocation of `KalmanFilter` are

$$\hat{\beta}_{1|0} = \mu_1$$

and $C_{1|0} = Q_1$ input via `b` and `covb`, respectively. Computations for the *k*-th invocation are completed by `KalmanFilter` computing the one-step-ahead estimate

$$\hat{\beta}_{k+1|k}$$

along with $C_{k+1|k}$ given by the *prediction equations*:

$$\hat{\beta}_{k+1|k} = T_{k+1} \hat{\beta}_{k|k}$$

$$C_{k+1|k} = T_{k+1} C_{k|k} T_{k+1}^T + Q_{k+1}$$

If both the filtered estimates and one-step-ahead estimates are needed by the user at each time point, `KalmanFilter` can be used twice for each time point-first without methods `SetTransitionMatrix` and `SetQ` to produce

$$\hat{\beta}_{k|k}$$

and $C_{k|k}$, and second without method `Update` to produce

$$\hat{\beta}_{k+1|k}$$

and $C_{k+1|k}$ (Without methods `SetTransitionMatrix` and `SetQ`, the prediction equations are skipped. Without method `update`, the update equations are skipped.).

Often, one desires the estimate of the state vector more than one-step-ahead, i.e., an estimate of

$$\hat{\beta}_{k|j}$$

is needed where $k > j+1$. At time j, `KalmanFilter` is invoked with method `Update` to compute

$$\hat{\beta}_{j+1|j}$$

Subsequent invocations of `KalmanFilter` without method `Update` can compute

$$\hat{\beta}_{j+2|j}, \hat{\beta}_{j+3|j}, \ldots, \hat{\beta}_{k|j}$$

Computations for

$$\hat{\beta}_{k|j}$$

and $C_{k|j}$ assume the variance-covariance matrices of the errors in the observation equation and state equation are known up to an unknown positive scalar multiplier, σ^2. The maximum likelihood estimate of σ^2 based on the observations $y_1, y_2, \ldots, y_m$, is given by

$$\hat{\sigma}^2 = SS/N$$

where

$$N = \sum_{k=1}^{m} n_k \text{ and } SS = \sum_{k=1}^{m} v_k^T H_k^{-1} v_k$$

N and SS are the input/output arguments `n` and `sumOfSquares`.

If σ^2 is known, the R_ks and Q_ks can be input as the variance-covariance matrices exactly. The earlier discussion is then simplified by letting $\sigma^2 = 1$.

In practice, the matrices T_k, Q_k, and R_k are generally not completely known. They may be known functions of an unknown parameter vector θ. In this case, `KalmanFilter` can be used in conjunction with an optimization class (see class `MinUnconMultiVar`, IMSL C# Library Math namespace), to obtain a maximum likelihood estimate of θ. The natural logarithm of the likelihood function for $y_1, y_2, \ldots, y_m$ differs by no more than an additive constant from

$$L(\theta, \sigma^2; y_1, y_2, \ldots, y_m) = -\frac{1}{2} N \ln \sigma^2 - \frac{1}{2} \sum_{k=1}^{m} \ln[\det(H_k)] - \frac{1}{2} \sigma^{-2} \sum_{k=1}^{m} v_k^T H_k^{-1} v_k$$

(Harvey 1981, page 14, equation 2.21).

Here,

$$\sum_{k=1}^{m} \ln[\det(H_k)]$$

(stored in logDeterminant) is the natural logarithm of the determinant of V where $\sigma^2 V$ is the variance-covariance matrix of the observations.

Minimization of $-2L(\theta, \sigma^2; y_1, y_2, \ldots, y_m)$ over all θ and σ^2 produces maximum likelihood estimates. Equivalently, minimization of $-2L_c(\theta; y_1, y_2, \ldots, y_m)$ where

$$L_c(\theta; y_1, y_2, \ldots, y_m) = -\frac{1}{2} N \ln \left(\frac{SS}{N} \right) - \frac{1}{2} \sum_{k=1}^{m} \ln[\det(H_k)]$$

produces maximum likelihood estimates

$$\hat{\theta} \text{ and } \hat{\sigma}^2 = SS/N$$

Minimization of $-2L_c(\theta; y_1, y_2, \ldots, y_m)$ instead of $-2L(\theta, \sigma^2; y_1, y_2, \ldots, y_m)$, reduces the dimension of the minimization problem by one. The two optimization problems are equivalent since

$$\hat{\sigma}^2(\theta) = SS(\theta)/N$$

minimizes $-2L(\theta,\sigma^2;y_1,y_2,\ldots,y_m)$ for all θ, consequently,

$$\hat{\sigma}^2(\theta)$$

can be substituted for σ^2 in $L(\theta,\sigma^2;y_1,y_2,\ldots,y_m)$ to give a function that differs by no more than an additive constant from $L_c(\theta;y_1,y_2,\ldots,y_m)$.

The earlier discussion assumed H_k to be nonsingular. If H_k is singular, a modification for singular distributions described by Rao (1973, pages 527-528) is used. The necessary changes in the preceding discussion are as follows:

1. Replace H_k^{-1} by a generalized inverse.
2. Replace $det(H_k)$ by the product of the nonzero eigenvalues of H_k.
3. Replace N by $\sum_{k=1}^{m} \text{rank}(H_k)$

Maximum likelihood estimation of parameters in the Kalman filter is discussed by Sallas and Harville (1988) and Harvey (1981, pages 111-113).

Properties

LogDeterminant

```
public double LogDeterminant {get; }
```

Description

Returns the natural log of the product of the nonzero eigenvalues of *P* where *P * sigma*^{2} is the variance-covariance matrix of the observations.

Property Value

A `double` scalar containing the natural log of the product of the nonzero eigenvalues of P where `P` * σ^2 is the variance-covariance matrix of the observations.

Remarks

In the usual case when P is nonsingular, `LogDeterminant` is the natural log of the determinant of P.

Rank

```
public int Rank {get; }
```

Description

Returns the rank of the variance-covariance matrix for all the observations.

Property Value

A `int` scalar containing the rank of the variance-covariance matrix for all the observations.

SumOfSquares

```
public double SumOfSquares {get; }
```

Description

Returns the generalized sum of squares.

Property Value

A `double` scalar containing the generalized sum of squares.

Remarks

The estimate of σ^2 is given by `sumOfSquares / n`.

Tolerance

```
public double Tolerance {get; set; }
```

Description

The tolerance used in determining linear dependence.

Property Value

A `double` scalar containing the tolerance used in determining linear dependence.

Remarks

By default, `Tolerance` = 100.0*2.2204460492503131e-16.

Constructor

KalmanFilter

```
public KalmanFilter(double[] b, double[,] covb, int rank, double sumOfSquaress,
double logDeterminant)
```

Description

Constructor for `KalmanFilter`.

Parameters

`b` – A `double` array containing the estimated state vector.

`covb` – A `double` matrix of size `b.Length` by `b.Length` such that `covb` * σ^2 is the mean squared error matrix for `b`.

`rank` – An `int` scalar containing the rank of the variance-covariance matrix for all the observations.

`sumOfSquaress` – A `double` scalar containing the generalized sum of squares.

`logDeterminant` – A `double` scalar containing the natural log of the product of the nonzero eigenvalues of P where `P` * σ^2 is the variance-covariance matrix of the observations.

Remarks

`b` is the estimated state vector at time `k` given the observations through time `k`-1.

Exception

`System.ArgumentException` is thrown if the dimensions of `b`, and `covb` are not consistent

Methods

Filter

```
public void Filter()
```

Description

Performs Kalman filtering and evaluates the likelihood function for the state-space model.

GetCovB

```
public double[,] GetCovB()
```

Description

Returns the mean squared error matrix for `b` divided by sigma squared.

Returns

A `double` matrix of size `b.Length` by `b.Length` such that `covb` * σ^2 is the mean squared error matrix for `b`.

GetCovV

```
public double[,] GetCovV()
```

Description

Returns the variance-covariance matrix of *v* dividied by sigma squared.

Returns

A `double` matrix containing a `y.length by y.Length` matrix such that `covv` * σ^2 is the variance-covariance matrix of `v`.

GetPredictionError

```
public double[] GetPredictionError()
```

Description

Returns the one-step-ahead prediction error.

Returns

A `double` array of size `y.Length` containing the one-step-ahead prediction error.

GetStateVector

```
public double[] GetStateVector()
```

Description

Returns the estimated state vector at time k + 1 given the observations through time k.

Returns

A `double` array containing the estimated state vector at time k + 1 given the observations through time k.

ResetQ

```
public void ResetQ()
```

Description

Removes the Q matrix.

ResetTransitionMatrix

```
public void ResetTransitionMatrix()
```

Description

Removes the transition matrix.

ResetUpdate

```
public void ResetUpdate()
```

Description

Do not perform computation of the update equations.

SetQ

```
public void SetQ(double[,] q)
```

Description

Sets the Q matrix.

Parameter

`q` – A `double` matrix containing the `b.Length` by `b.Length` matrix such that `q` * σ^2 is the variance-covariance matrix of the error vector in the state equation.

Remarks

By default, there is no error term in the state equation.

SetTransitionMatrix

```
public void SetTransitionMatrix(double[,] t)
```

Description

Sets the transition matrix.

Parameter

`t` – A `double` matrix containing the `b.Length by b.Length` transition matrix in the state equation.

Remarks

By default, `t` = identity matrix.

Update

```
public void Update(double[] y, double[,] z, double[,] r)
```

Description

Performs computation of the update equations.

Parameters

`y` – A `double` array containing the observations.

`z` – A `double` matrix containing the `y.Length by b.Length` matrix relating the observations to the state vector in the observation equation.

`r` – A `double` matrix containing the `y.Length by y.Length` matrix such that `r` * σ^2 is the variance-covariance matrix of errors in the observation equation.

Remarks

σ^2 is a positive unknown scalar. Only elements in the upper triangle of `r` are referenced.

Example 1: Kalman Filter

KalmanFilter is used to compute the filtered estimates and one-step-ahead estimates for a scalar problem discussed by Harvey (1981, pages 116-117). The observation equation and state equation are given by

$$y_k = b_k + e_k$$

$$b_{k+1} = b_k + w_{k+1}$$

k = 1, 2, 3, 4

where the e_ks are identically and independently distributed normal with mean 0 and variance σ^2, the w_ks are identically and independently distributed normal with mean 0 and variance $4\sigma^2$, and b_1 is distributed normal with mean 4 and variance $16\sigma^2$. Two KalmanFilter objects are needed for each time point in order to compute the filtered estimate and the one-step-ahead estimate. The first object does not use the methods `SetTransitionMatrix` and `SetQ` so that the prediction equations are skipped in the computations. The update equations are skipped in the computations in the second object.

This example also computes the one-step-ahead prediction errors. Harvey (1981, page 117) contains a misprint for the value v_4 that he gives as 1.197. The correct value of v_4 = 1.003 is computed by KalmanFilter.

```
using System;
using Imsl.Stat;

public class KalmanFilterEx1
{
```

```
private static readonly String format =
    "{0}/{1}\t{2:0.000}\t{3:0.000}\t{4}\t{5:0.000}" +
    "\t{6:0.000}\t{7:0.000}\t{8:0.000}";

public static void  Main(String[] args)
{
    int nobs = 4;
    int rank = 0;
    double logDeterminant = 0.0;
    double ss = 0.0;
    double[] b = new double[]{4};
    double[,] covb = new double[1,1]{{16}};
    double[,] q = {{4}};
    double[,] r = {{1}};
    double[,] t = {{1}};
    double[,] z = {{1}};
    double[] ydata = new double[]{4.4, 4.0, 3.5, 4.6};

    System.Object[] argFormat =
        new System.Object[]{"k", "j", "b", "cov(b)", "rank", "ss",
                            "ln(det)", "v", "cov(v)"};
    Console.Out.WriteLine(String.Format(format, argFormat));

    KalmanFilter kalman =
        new KalmanFilter(b, covb, rank, ss, logDeterminant);

    for (int i = 0; i < nobs; i++)
    {
        double[] y = new double[]{ydata[i]};
        kalman.Update(y, z, r);
        kalman.Filter();
        double[] v = kalman.GetPredictionError();
        double[,] covv = kalman.GetCovV();
        argFormat[0] = i;
        argFormat[1] = i;
        argFormat[2] = kalman.GetStateVector()[0];
        argFormat[3] = kalman.GetCovB()[0,0];
        argFormat[4] = kalman.Rank;
        argFormat[5] = kalman.SumOfSquares;
        argFormat[6] = kalman.LogDeterminant;
        argFormat[7] = v[0];
        argFormat[8] = covv[0,0];
        Console.Out.WriteLine(String.Format(format, argFormat));
        kalman.ResetUpdate();

        kalman.SetTransitionMatrix(t);
        kalman.SetQ(q);
        kalman.Filter();
        argFormat[0] = i + 1;
        argFormat[1] = i;
        argFormat[2] = kalman.GetStateVector()[0];
        argFormat[3] = kalman.GetCovB()[0,0];
        argFormat[4] = kalman.Rank;
        argFormat[5] = kalman.SumOfSquares;
        argFormat[6] = kalman.LogDeterminant;
        argFormat[7] = v[0];
```

```
            argFormat[8] = covv[0,0];
            Console.Out.WriteLine(String.Format(format, argFormat));
            kalman.ResetTransitionMatrix();
            kalman.ResetQ();
        }
    }
}
```

Output

```
k/j b cov(b) rank ss ln(det) v cov(v)
0/0 4.376 0.941 1 0.009 2.833 0.400 17.000
1/0 4.376 4.941 1 0.009 2.833 0.400 17.000
1/1 4.063 0.832 2 0.033 4.615 -0.376 5.941
2/1 4.063 4.832 2 0.033 4.615 -0.376 5.941
2/2 3.597 0.829 3 0.088 6.378 -0.563 5.832
3/2 3.597 4.829 3 0.088 6.378 -0.563 5.832
3/3 4.428 0.828 4 0.260 8.141 1.003 5.829
4/3 4.428 4.828 4 0.260 8.141 1.003 5.829
```

Example 2: Kalman Filter Maximum Likelihood Estimate

KalmanFilter is used with the `MinUnconMultivar` class to find a maximum likelihood estimate of the parameter θ in an MA(1) time series represented by $y_k = \varepsilon_k - \theta\varepsilon_{k-1}$. The input is 200 observations from an MA(1) time series with $\theta = 0.5$ and $\sigma^2 = 1$. The MA(1) time series is cast as a state-space model of the following form (see Harvey 1981, pages 103-104, 112):

$$y_k = \begin{pmatrix} 1 & 0 \end{pmatrix} b_k$$

$$b_k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b_{k-1} + w_k$$

where the two-dimensional w_ks are independently distributed bivariate normal with mean 0 and variance $\sigma^2 Q_k$ and

$$Q_1 = \begin{pmatrix} 1+\theta^2 & -\theta \\ -\theta & \theta^2 \end{pmatrix}$$

$$Q_k = \begin{pmatrix} 1 & -\theta \\ -\theta & \theta^2 \end{pmatrix} \quad k = 2, 3, \ldots, 200$$

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class KalmanFilterEx2 : MinUnconMultiVar.IFunction
{
```

```
public double F(double[] theta)
{
    int nobs = 200;
    int rank = 0;
    double ss = 0.0;
    double logDeterminant = 0.0;
    double res;
    double[,] covb = new double[2, 2];
    double[,] q = new double[2, 2];
    double[,] r = { { 0.0 } };
    double[] b = { 0.0, 0.0 };
    double[,] z = { { 1.0, 0.0 } };
    double[,] t = {{0.0, 1.0},
                   {0.0, 0.0}};
    double[] y = new double[1];
    double[] ydata = {0.057466,-0.459067,1.236446,-1.864825,0.951507,
                      -1.489367,-0.864401,0.302984,-0.376017,0.610208,
                      -1.701583,2.271734,-0.917110,0.582576,1.018681,
                      0.107443,-0.557858,0.502818,0.858002,-1.417659,
                      0.839981,0.047021,0.404448,0.383749,-0.383227,
                      1.179607,-1.154339,1.927541,0.996344,1.019415,
                      -0.816815,-0.100467,-0.125334,-0.068065,-1.891685,
                      0.657241,-1.070823,1.023510,2.672653,-2.141434,
                      -1.232266,0.925311,-0.201665,-1.325580,-0.086926,
                      -0.647157,1.314749,-0.085708,1.430485,0.304040,
                      0.305559,1.669671,1.004800,-1.678350,0.631133,
                      0.502284,0.247378,-1.345484,0.994982,1.145546,
                      -1.248822,0.616784,-2.127822,2.264872,-1.590343,
                      0.365785,-1.056652,0.969750,1.028274,0.332050,
                      0.430686,0.364553,0.482446,-0.303248,1.581931,
                      -0.140942,-0.265280,-0.939284,0.464963,-0.778145,
                      0.583486,-0.113080,-0.009839,1.580189,-1.116377,
                      1.744513,-0.298106,0.332944,-0.228859,1.101747,
                      0.772369,-1.608111,2.671822,-0.504800,-1.647797,
                      -0.596313,0.845472,0.507869,0.833377,-0.460099,
                      0.416891,-1.139069,0.159028,-0.193971,-0.154656,
                      1.720997,-0.403189,0.400026,0.285921,-1.914338,
                      1.296864,1.426898,-0.426181,0.255961,-1.790193,
                      0.721048,1.150173,-0.980386,0.940958,0.313898,
                      0.505735,-1.058126,0.111918,0.185493,-0.296146,
                      -0.104457,1.151733,0.683025,-0.714269,-0.787972,
                      -1.277062,1.378816,-0.658115,-0.259860,-2.614008,
                      2.251646,-0.006773,-0.738467,-0.260685,1.896505,
                      -0.094919,0.089954,-1.627679,-1.675018,0.896835,
                      0.498690,-0.368775,0.131849,-2.060292,0.272666,
                      2.115804,-1.323451,0.557106,-0.602031,1.424524,
                      -0.107996,-1.580671,0.672012,-1.668931,2.474710,
                      -1.471518,1.780317,-0.419588,2.008474,-1.246855,
                      0.231161,0.706104,-0.474320,-0.705431,0.599358,
                      -2.469494,2.024374,0.849572,-2.410618,2.812321,
                      -1.066520,-0.539768,-0.067784,1.978078,0.592879,
                      -0.184623,-1.403912,-0.995537,1.727320,-0.313251,
                      0.472437,-0.241800,-0.875680,-0.159557,0.508238,
                      -0.116888,-0.981759,-0.472451,0.847273,-1.713030,
                      2.010192,-0.981891,1.190901,0.453348,-0.743333};
```

```
        if (Math.Abs(theta[0]) > 1.0)
        {
            res = 1.0e10;
        }
        else
        {
            q[0, 0] = 1.0;
            q[0, 1] = -theta[0];
            q[1, 0] = -theta[0];
            q[1, 1] = theta[0] * theta[0];

            covb[0, 0] = 1.0 + theta[0] * theta[0];
            covb[0, 1] = -theta[0];
            covb[1, 0] = -theta[0];
            covb[1, 1] = theta[0] * theta[0];

            KalmanFilter kalman = new KalmanFilter(b, covb, rank, ss, logDeterminant);

            for (int i = 0; i < nobs; i++)
            {
                y[0] = ydata[i];
                kalman.Update(y, z, r);
                kalman.SetTransitionMatrix(t);
                kalman.SetQ(q);
                kalman.Filter();
            }
            ss = kalman.SumOfSquares;
            logDeterminant = kalman.LogDeterminant;
            rank = kalman.Rank;
            res = rank * Math.Log(ss / rank) + logDeterminant;
        }
        return (res);
    }

    public static void Main(String[] args)
    {
        MinUnconMultiVar solver = new MinUnconMultiVar(1);
        double[] x = solver.ComputeMin(new KalmanFilterEx2());
        Console.Out.WriteLine("Maximum likelihood estimate, THETA = {0}", x[0]);
    }
}
```

Output

```
Maximum likelihood estimate, THETA = 0.452940213528753
```

Chapter 19: Multivariate Analysis

Types

Usage Notes

Cluster Analysis

`ClusterKMeans` performs a K-means cluster analysis. Basic K-means clustering attempts to find a clustering that minimizes the within-cluster sums-of-squares. In this method of clustering the data, matrix X is grouped so that each observation (row in X) is assigned to one of a fixed number, K, of clusters. The sum of the squared difference of each observation about its assigned cluster's mean is used as the criterion for assignment. In the basic algorithm, observations are transferred from one cluster or another when doing so decreases the within-cluster sums-of-squared differences. When no transfer occurs in a pass through the entire data set, the algorithm stops. `ClusterKMeans` is one implementation of the basic algorithm.

The usual course of events in K-means cluster analysis is to use `ClusterKMeans` to obtain the optimal clustering. The clustering is then evaluated by functions described in "Basic Statistics", and/or other chapters in this manual. Often, K-means clustering with more than one value of K is performed, and the

value of K that best fits the data is used.

Clustering can be performed either on observations or variables. The discussion of the function `ClusterKMeans` assumes the clustering is to be performed on the observations, which correspond to the rows of the input data matrix. If variables, rather than observations, are to be clustered, the data matrix should first be transposed. In the documentation for `ClusterKMeans`, the words "observation" and "variable" are interchangeable.

Principal Components

The idea in principal components is to find a small number of linear combinations of the original variables that maximize the variance accounted for in the original data. This amounts to an eigensystem analysis of the covariance (or correlation) matrix. In addition to the eigensystem analysis, when the principal component model is used, `FactorAnalysis` computes standard errors for the eigenvalues. Correlations of the original variables with the principal component scores also are computed.

Factor Analysis

Factor analysis and principal component analysis, while quite different in assumptions, often serve the same ends. Unlike principal components in which linear combinations yielding the highest possible variances are obtained, factor analysis generally obtains linear combinations of the observed variables according to a model relating the observed variable to hypothesized underlying factors, plus a random error term called the unique error or uniqueness. In factor analysis, the unique errors associated with each variable are usually assumed to be independent of the factors. Additionally, in the common factor model, the unique errors are assumed to be mutually independent. The factor analysis model is expressed in the following equation:

$$x - \mu = \Lambda f + e$$

where x is the p vector of observed values, μ is the p vector of variable means, Λ is the $p \times k$ matrix of factor loadings, f is the k vector of hypothesized underlying random factors, e is the p vector of hypothesized unique random errors, p is the number of variables in the observed variables, and k is the number of factors.

Because much of the computation in factor analysis was originally done by hand or was expensive on early computers, quick (but dirty) algorithms that made the calculations possible were developed. One result is the many factor extraction methods available today. Generally speaking, in the exploratory or model building phase of a factor analysis, a method of factor extraction that is not computationally intensive (such as principal components, principal factor, or image analysis) is used. If desired, a computationally intensive method is then used to obtain the final factors.

Discriminant Analysis

The class `DiscriminantAnalysis` allows linear or quadratic discrimination and the use of either reclassification, split sample, or the leaving-out-one methods in order to evaluate the rule. Moreover, `DiscriminantAnalysis` can be executed in an online mode, that is, one or more observations can be added to the rule during each invocation of `DiscriminantAnalysis`.

The mean vectors for each group of observations and an estimate of the common covariance matrix for all groups are input to `DiscriminantAnalysis`. Output from `DiscriminantAnalysis` are linear combinations of the observations, which at most separate the groups. These linear combinations may subsequently be used for discriminating between the groups. Their use in graphically displaying differences between the groups is possibly more important, however.

ClusterKMeans Class

`public class Imsl.Stat.ClusterKMeans`

Perform a *K*-means (centroid) cluster analysis.

`ClusterKMeans` is an implementation of Algorithm AS 136 by Hartigan and Wong (1979). It computes *K*-means (centroid) Euclidean metric clusters for an input matrix starting with initial estimates of the *K* cluster means. It allows for missing values (coded as NaN, *not a number*) and for weights and frequencies.

Let *p* denote the number of variables to be used in computing the Euclidean distance between observations. The idea in *K*-means cluster analysis is to find a clustering (or grouping) of the observations so as to minimize the total within-cluster sums of squares. In this case, the total sums of squares within each cluster is computed as the sum of the centered sum of squares over all nonmissing values of each variable. That is,

$$\phi = \sum_{i=1}^{K} \sum_{j=1}^{p} \sum_{m=1}^{n_i} f_{v_{im}} w_{v_{im}} \delta_{v_{im},j} \left(x_{v_{im},j} - \bar{x}_{ij}\right)^2$$

where v_{im} denotes the row index of the *m*-th observation in the *i*-th cluster in the matrix *X*; n_i is the number of rows of *X* assigned to group *i*; *f* denotes the frequency of the observation; w denotes its weight; *d* is zero if the *j*-th variable on observation v_{im} is missing, otherwise δ is one; and $\bar{x}_{ij}$ is the average of the nonmissing observations for variable *j* in group *i*. This method sequentially processes each observation and reassigns it to another cluster if doing so results in a decrease in the total within-cluster sums of squares. See Hartigan and Wong (1979) or Hartigan (1975) for details.

Property

MaxIterations

`public int MaxIterations {get; set; }`

Description

The maximum number of iterations.

Property Value

A `int` scalar specifying the maximum number of iterations.

Remarks

By default, `MaxIterations` = 30.

Constructor

ClusterKMeans

```
public ClusterKMeans(double[,] x, double[,] cs)
```

Description

Constructor for `ClusterKMeans`.

Parameters

`x` – A `double` matrix containing the observations to be clustered.

`cs` – A `double` matrix containing the cluster seeds, i.e. estimates for the cluster centers.

Exception

`System.ArgumentException` is thrown if `x.GetLength(0)`, `x.GetLength(1)` are equal 0, or `cs.GetLength(0)` is less than 1

Methods

Compute

```
public double[,] Compute()
```

Description

Computes the cluster means.

Returns

A `double` matrix containing computed result.

Exceptions

`Imsl.Stat.NoConvergenceException` is thrown if convergence did not occur within the maximum number of iterations

`Imsl.Stat.ClusterNoPointsException` is thrown if the cluster seed yields a cluster with no points

GetClusterCounts

```
public int[] GetClusterCounts()
```

Description

Returns the number of observations in each cluster. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

An `int` array containing the number of observations in each cluster.

GetClusterMembership

```
public int[] GetClusterMembership()
```

Description

Returns the cluster membership for each observation. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

An `int` array containing the cluster membership for each observation.

Remarks

Cluster membership 1 indicates the observation belongs to cluster 1, cluster membership 2 indicates the observation belongs to cluster 2, etc.

GetClusterSSQ

```
public double[] GetClusterSSQ()
```

Description

Returns the within sum of squares for each cluster. Note that the `Compute` method must be invoked first before invoking this method. Otherwise, the method throws a `NullReferenceException` exception.

Returns

A `double` array containing the within sum of squares for each cluster.

SetFrequencies

```
public void SetFrequencies(double[] frequencies)
```

Description

The frequency for each observation.

Parameter

`frequencies` – A `double` array of size `x.GetLength(0)` containing the frequency for each observation.

Remarks

By default, `Frequencies[]` = 1.

SetWeights

```
public void SetWeights(double[] weights)
```

Description

Sets the weight for each observation.

Parameter

`weights` – A `double` array of size `x.GetLength(0)` containing the weight for each observation.

Remarks

By default, `Weights[]` = 1.

Example: K-means Cluster Analysis

This example performs K-means cluster analysis on Fisher's iris data. The initial cluster seed for each iris type is an observation known to be in the iris type.

```
/*
* ------------------------------------------------------------------------
*    Copyright (c) 1999 Visual Numerics Inc. All Rights Reserved.
*
*    This software is confidential information which is proprietary to
*    and a trade secret of Visual Numerics, Inc.  Use, duplication or
*    disclosure is subject to the terms of an appropriate license
*    agreement.
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*    VISUAL NUMERICS MAKES NO REPRESENTATIONS OR WARRANTIES ABOUT THE
*    SUITABILITY OF THE SOFTWARE, EITHER EXPRESS OR IMPLIED, INCLUDING
*    BUT NOT LIMITED TO THE IMPLIED WARRANTIES OF MERCHANTABILITY,
*    FITNESS FOR A PARTICULAR PURPOSE, OR NON-INFRINGEMENT. VISUAL
*    NUMERICS SHALL NOT BE LIABLE FOR ANY DAMAGES SUFFERED BY LICENSEE
*    AS A RESULT OF USING, MODIFYING OR DISTRIBUTING THIS SOFTWARE OR
*    ITS DERIVATIVES.
*------------------------------------------------------------------------
*/
using System;
using Imsl.Stat;
using Imsl.Math;

public class ClusterKMeansEx1
{
    public static void  Main(String[] argv)
    {
        double[,] x = {{5.100, 3.500, 1.400, 0.200},
                       {4.900, 3.000, 1.400, 0.200},
                       {4.700, 3.200, 1.300, 0.200},
                       {4.600, 3.100, 1.500, 0.200},
                       {5.000, 3.600, 1.400, 0.200},
                       {5.400, 3.900, 1.700, 0.400},
                       {4.600, 3.400, 1.400, 0.300},
                       {5.000, 3.400, 1.500, 0.200},
                       {4.400, 2.900, 1.400, 0.200},
                       {4.900, 3.100, 1.500, 0.100},
                       {5.400, 3.700, 1.500, 0.200},
                       {4.800, 3.400, 1.600, 0.200},
                       {4.800, 3.000, 1.400, 0.100},
                       {4.300, 3.000, 1.100, 0.100},
                       {5.800, 4.000, 1.200, 0.200},
                       {5.700, 4.400, 1.500, 0.400},
                       {5.400, 3.900, 1.300, 0.400},
                       {5.100, 3.500, 1.400, 0.300},
                       {5.700, 3.800, 1.700, 0.300},
                       {5.100, 3.800, 1.500, 0.300},
                       {5.400, 3.400, 1.700, 0.200},
                       {5.100, 3.700, 1.500, 0.400},
                       {4.600, 3.600, 1.000, 0.200},
                       {5.100, 3.300, 1.700, 0.500},
```

```
{4.800, 3.400, 1.900, 0.200},
{5.000, 3.000, 1.600, 0.200},
{5.000, 3.400, 1.600, 0.400},
{5.200, 3.500, 1.500, 0.200},
{5.200, 3.400, 1.400, 0.200},
{4.700, 3.200, 1.600, 0.200},
{4.800, 3.100, 1.600, 0.200},
{5.400, 3.400, 1.500, 0.400},
{5.200, 4.100, 1.500, 0.100},
{5.500, 4.200, 1.400, 0.200},
{4.900, 3.100, 1.500, 0.200},
{5.000, 3.200, 1.200, 0.200},
{5.500, 3.500, 1.300, 0.200},
{4.900, 3.600, 1.400, 0.100},
{4.400, 3.000, 1.300, 0.200},
{5.100, 3.400, 1.500, 0.200},
{5.000, 3.500, 1.300, 0.300},
{4.500, 2.300, 1.300, 0.300},
{4.400, 3.200, 1.300, 0.200},
{5.000, 3.500, 1.600, 0.600},
{5.100, 3.800, 1.900, 0.400},
{4.800, 3.000, 1.400, 0.300},
{5.100, 3.800, 1.600, 0.200},
{4.600, 3.200, 1.400, 0.200},
{5.300, 3.700, 1.500, 0.200},
{5.000, 3.300, 1.400, 0.200},
{7.000, 3.200, 4.700, 1.400},
{6.400, 3.200, 4.500, 1.500},
{6.900, 3.100, 4.900, 1.500},
{5.500, 2.300, 4.000, 1.300},
{6.500, 2.800, 4.600, 1.500},
{5.700, 2.800, 4.500, 1.300},
{6.300, 3.300, 4.700, 1.600},
{4.900, 2.400, 3.300, 1.000},
{6.600, 2.900, 4.600, 1.300},
{5.200, 2.700, 3.900, 1.400},
{5.000, 2.000, 3.500, 1.000},
{5.900, 3.000, 4.200, 1.500},
{6.000, 2.200, 4.000, 1.000},
{6.100, 2.900, 4.700, 1.400},
{5.600, 2.900, 3.600, 1.300},
{6.700, 3.100, 4.400, 1.400},
{5.600, 3.000, 4.500, 1.500},
{5.800, 2.700, 4.100, 1.000},
{6.200, 2.200, 4.500, 1.500},
{5.600, 2.500, 3.900, 1.100},
{5.900, 3.200, 4.800, 1.800},
{6.100, 2.800, 4.000, 1.300},
{6.300, 2.500, 4.900, 1.500},
{6.100, 2.800, 4.700, 1.200},
{6.400, 2.900, 4.300, 1.300},
{6.600, 3.000, 4.400, 1.400},
{6.800, 2.800, 4.800, 1.400},
{6.700, 3.000, 5.000, 1.700},
{6.000, 2.900, 4.500, 1.500},
{5.700, 2.600, 3.500, 1.000},
```

```
{5.500, 2.400, 3.800, 1.100},
{5.500, 2.400, 3.700, 1.000},
{5.800, 2.700, 3.900, 1.200},
{6.000, 2.700, 5.100, 1.600},
{5.400, 3.000, 4.500, 1.500},
{6.000, 3.400, 4.500, 1.600},
{6.700, 3.100, 4.700, 1.500},
{6.300, 2.300, 4.400, 1.300},
{5.600, 3.000, 4.100, 1.300},
{5.500, 2.500, 4.000, 1.300},
{5.500, 2.600, 4.400, 1.200},
{6.100, 3.000, 4.600, 1.400},
{5.800, 2.600, 4.000, 1.200},
{5.000, 2.300, 3.300, 1.000},
{5.600, 2.700, 4.200, 1.300},
{5.700, 3.000, 4.200, 1.200},
{5.700, 2.900, 4.200, 1.300},
{6.200, 2.900, 4.300, 1.300},
{5.100, 2.500, 3.000, 1.100},
{5.700, 2.800, 4.100, 1.300},
{6.300, 3.300, 6.000, 2.500},
{5.800, 2.700, 5.100, 1.900},
{7.100, 3.000, 5.900, 2.100},
{6.300, 2.900, 5.600, 1.800},
{6.500, 3.000, 5.800, 2.200},
{7.600, 3.000, 6.600, 2.100},
{4.900, 2.500, 4.500, 1.700},
{7.300, 2.900, 6.300, 1.800},
{6.700, 2.500, 5.800, 1.800},
{7.200, 3.600, 6.100, 2.500},
{6.500, 3.200, 5.100, 2.000},
{6.400, 2.700, 5.300, 1.900},
{6.800, 3.000, 5.500, 2.100},
{5.700, 2.500, 5.000, 2.000},
{5.800, 2.800, 5.100, 2.400},
{6.400, 3.200, 5.300, 2.300},
{6.500, 3.000, 5.500, 1.800},
{7.700, 3.800, 6.700, 2.200},
{7.700, 2.600, 6.900, 2.300},
{6.000, 2.200, 5.000, 1.500},
{6.900, 3.200, 5.700, 2.300},
{5.600, 2.800, 4.900, 2.000},
{7.700, 2.800, 6.700, 2.000},
{6.300, 2.700, 4.900, 1.800},
{6.700, 3.300, 5.700, 2.100},
{7.200, 3.200, 6.000, 1.800},
{6.200, 2.800, 4.800, 1.800},
{6.100, 3.000, 4.900, 1.800},
{6.400, 2.800, 5.600, 2.100},
{7.200, 3.000, 5.800, 1.600},
{7.400, 2.800, 6.100, 1.900},
{7.900, 3.800, 6.400, 2.000},
{6.400, 2.800, 5.600, 2.200},
{6.300, 2.800, 5.100, 1.500},
{6.100, 2.600, 5.600, 1.400},
{7.700, 3.000, 6.100, 2.300},
```

```
                            {6.300, 3.400, 5.600, 2.400},
                            {6.400, 3.100, 5.500, 1.800},
                            {6.000, 3.000, 4.800, 1.800},
                            {6.900, 3.100, 5.400, 2.100},
                            {6.700, 3.100, 5.600, 2.400},
                            {6.900, 3.100, 5.100, 2.300},
                            {5.800, 2.700, 5.100, 1.900},
                            {6.800, 3.200, 5.900, 2.300},
                            {6.700, 3.300, 5.700, 2.500},
                            {6.700, 3.000, 5.200, 2.300},
                            {6.300, 2.500, 5.000, 1.900},
                            {6.500, 3.000, 5.200, 2.000},
                            {6.200, 3.400, 5.400, 2.300},
                            {5.900, 3.000, 5.100, 1.800}};


        double[,] cs = {{5.100, 3.500, 1.400, 0.200},
                        {7.000, 3.200, 4.700, 1.400},
                        {6.300, 3.300, 6.000, 2.500}};


        ClusterKMeans kmean = new ClusterKMeans(x, cs);

        double[,] cm = kmean.Compute();
        double[] wss = kmean.GetClusterSSQ();
        int[] ic = kmean.GetClusterMembership();
        int[] nc = kmean.GetClusterCounts();

        PrintMatrix pm = new PrintMatrix("Cluster Means");

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.0000";
        pm.Print(pmf, cm);

        new PrintMatrix("Cluster Membership").Print(ic);
        new PrintMatrix("Sum of Squares").Print(wss);
        new PrintMatrix("Number of observations").Print(nc);
    }
}
```

Output

```
           Cluster Means
     0       1       2       3
0  5.0060  3.4280  1.4620  0.2460
1  5.9016  2.7484  4.3935  1.4339
2  6.8500  3.0737  5.7421  2.0711

Cluster Membership
     0
  0  1
  1  1
  2  1
  3  1
  4  1
```

```
 5  1
 6  1
 7  1
 8  1
 9  1
10  1
11  1
12  1
13  1
14  1
15  1
16  1
17  1
18  1
19  1
20  1
21  1
22  1
23  1
24  1
25  1
26  1
27  1
28  1
29  1
30  1
31  1
32  1
33  1
34  1
35  1
36  1
37  1
38  1
39  1
40  1
41  1
42  1
43  1
44  1
45  1
46  1
47  1
48  1
49  1
50  2
51  2
52  3
53  2
54  2
55  2
56  2
57  2
58  2
59  2
60  2
```

61	2
62	2
63	2
64	2
65	2
66	2
67	2
68	2
69	2
70	2
71	2
72	2
73	2
74	2
75	2
76	2
77	3
78	2
79	2
80	2
81	2
82	2
83	2
84	2
85	2
86	2
87	2
88	2
89	2
90	2
91	2
92	2
93	2
94	2
95	2
96	2
97	2
98	2
99	2
100	3
101	2
102	3
103	3
104	3
105	3
106	2
107	3
108	3
109	3
110	3
111	3
112	3
113	2
114	2
115	3
116	3

```
117  3
118  3
119  2
120  3
121  2
122  3
123  2
124  3
125  3
126  2
127  2
128  3
129  3
130  3
131  3
132  3
133  2
134  3
135  3
136  3
137  3
138  2
139  3
140  3
141  3
142  2
143  3
144  3
145  3
146  2
147  3
148  3
149  2

   Sum of Squares
          0
0  15.151
1  39.8209677419355
2  23.8794736842105

Number of observations
   0
0  50
1  62
2  38
```

Dissimilarities Class

```
public class Imsl.Stat.Dissimilarities
```

Computes a matrix of dissimilarities (or similarities) between the columns (or rows) of a matrix.

Class `Dissimilarities` computes an upper triangular matrix (excluding the diagonal) of dissimilarities (or similarities) between the columns (or rows) of a matrix. Nine different distance measures can be computed. For the first three measures, three different scaling options can be employed. The distance matrix computed is generally used as input to clustering or multidimensional scaling functions.

The following discussion assumes that the distance measure is being computed between the columns of the matrix. If distances between the rows of the matrix are desired, use `Row = true`.

The distance method and scaling option used by Dissimilarities can be set via properties `DistanceMethod` and `ScalingOption`, respectively. For distance methods `L2Norm`, `L1Norm`, or `InfinityNorm`, each row of `x` is first scaled according to the value of `ScalingOption`. The scaling parameters are obtained from the values in the row scaled as either the standard deviation of the row or the row range; the standard deviation is computed from the unbiased estimate of the variance. If no scaling is performed, the parameters in the following discussion are all 1.0 (see `ScalingOption`). Once the scaling value (if any) has been computed, the distance between column i and column j is computed via the difference vector $z_k = \frac{(x_k - y_k)}{s_k}, i = 1, \ldots, ndstm$, where x_k denotes the k-th element in the i-th column, y_k denotes the corresponding element in the j-th column, and *ndstm* is the number of rows if differencing columns and the number of columns if differencing rows. For given z_i, the distance methods that allow scaling are defined as:

`DistanceMethod`	**Metric**
`L2Norm`	Euclidean distance (L_2 norm)
`L1Norm`	Sum of the absolute differences (L_1 norm)
`InfinityNorm`	Maximum difference (L_∞ norm)

The following distance measures do not allow for scaling.

`DistanceMethod`	**Metric**
`Mahalanobis`	Mahalanobis distance
`AbsCosine`	Absolute value of the cosine of the angle between the vectors
`AngleInRadians`	Angle in radians (0, π) between the lines through the origin defined by the vectors
`CorrelationCoefficient`	Correlation coefficient
`AbsCorrelationCoefficient`	Absolute value of the correlation coefficient
`ExactMatches`	Number of exact matches, where $x_i = y_i$.

For the Mahalanobis distance, any variable used in computing the distance measure that is (numerically) linearly dependent upon the previous variables in the `Index` property is omitted from the distance measure.

Properties

DistanceMatrix

```
virtual public double[,] DistanceMatrix {get; }
```

Description

The distance matrix.

DistanceMethod

```
virtual public Imsl.Stat.Dissimilarities.Method DistanceMethod {get; set; }
```

Description

The method in computing the dissimilarities or similarities.

Property Value

A `Dissimilarities.Method` identifying the method to use in computing the dissimilarities or similarities.

Remarks

Acceptable values of `DistanceMethod` are:

`DistanceMethod`	**Metric**
`L2Norm`	Euclidean distance (L_2 norm)
`L1Norm`	Sum of the absolute differences (L_1 norm)
`InfinityNorm`	Maximum difference (L_∞ norm)
`Mahalanobis`	Mahalanobis distance
`AbsCosine`	Absolute value of the cosine of the angle between the vectors
`AngleInRadians`	Angle in radians (0, π) between the lines through the origin defined by the vectors
`CorrelationCoefficient`	Correlation coefficient
`AbsCorrelationCoefficient`	Absolute value of the correlation coefficient
`ExactMatches`	Number of exact matches, where $x_i = y_i$.

See class description for more details. By default, `DistanceMethod` = `Method.L2Norm`.

Index

```
virtual public int[] Index {get; set; }
```

Description

The indices of the rows.

Property Value

An `int` array containing the indices of the columns (rows if `Row` = `false`) to use in computing the distance measure.

Remarks

By default, if `Row` = `true`, `Index` = `0, 1, ..., x.GetLength(1)-1`. if `Row` = `false`, `Index` = `0, 1, ..., x.GetLength(0)-1`, see `Row`.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Row

```
virtual public bool Row {get; set; }
```

Description

Identifies whether distances are computed between rows or columns of `x`.

Property Value

A `bool` identifying whether distances are computed between rows or columns of `x`.

Remarks

If `Row` = `true`, distances are computed between the rows of `x`. Otherwise, distances between the columns of `x` are computed. By default, `Row` = `true`.

ScalingOption

```
virtual public Imsl.Stat.Dissimilarities.Scaling ScalingOption {get; set; }
```

Description

The scaling option used if the `L2Norm`, `L1Norm`, or `InfinityNorm` distance methods are specified. See `DistanceMethod`.

Property Value

A `Dissimilarities.Scaling` containing the scaling option.

Remarks

By default, `ScalingOption` = `Scaling.None`.

ScalingOption	**Method**
None	No scaling is performed.
StdDev	If Row = false, scale each column by the standard deviation of the column. If Row = true, scale each row by the standard deviation of the row.
Range	If Row = false, scale each column by the range of the column. If Row = true, scale each row by the range of the row.

Constructor

Dissimilarities

```
public Dissimilarities(double[,] x)
```

Description

Constructor for Dissimilarities.

Parameter

x – A double matrix containing the data input matrix.

Method

Compute

```
public void Compute()
```

Description

Computes a matrix of dissimilarities (or similarities) between the columns (or rows) of a matrix.

Exceptions

Imsl.Stat.ScaleFactorZeroException is thrown when computations cannot continue because a scale factor is zero.

Imsl.Stat.NoPositiveVarianceException is thrown when no variable has positive variance

Imsl.Stat.ZeroNormException is thrown when the Euclidean norm of a column is equal to zero

Example: Dissimilarities

The following example illustrates the use of Dissimilarities for computing the Euclidean distance between the rows of a matrix:

```
using System;
using Imsl.Math;
```

```
using Imsl.Stat;

public class DissimilaritiesEx1
{
    public static void Main(String[] args)
    {
        double[,] x = { { 1.0, 1.0 }, { 1.0, 0.0 },
                        { 1.0, -1.0 }, { 1.0, 2.0 } };

        Dissimilarities dist = new Dissimilarities(x);
        dist.Compute();
        new PrintMatrix("Distance Matrix").Print(dist.DistanceMatrix);
    }
}
```

Output

```
Distance Matrix
   0  1  2  3
0  0  1  2  1
1  0  0  1  2
2  0  0  0  3
3  0  0  0  0
```

Dissimilarities.Method Enumeration

```
public enumeration Imsl.Stat.Dissimilarities.Method
```

Specifies the type of distance method.

Fields

AbsCorrelationCoefficient

```
public Imsl.Stat.Dissimilarities.Method AbsCorrelationCoefficient
```

Description

Indicates the absolute value of the correlation coefficient distance method.

AbsCosine

```
public Imsl.Stat.Dissimilarities.Method AbsCosine
```

Description

Indicates the absolute value of the cosine of the angle between the vectors distance method.

AngleInRadians

```
public Imsl.Stat.Dissimilarities.Method AngleInRadians
```

Description

Indicates the angle in radians $(0, \pi)$ between the lines through the origin defined by the vectors distance method.

CorrelationCoefficient

```
public Imsl.Stat.Dissimilarities.Method CorrelationCoefficient
```

Description

Indicates the correlation coefficient distance method.

ExactMatches

```
public Imsl.Stat.Dissimilarities.Method ExactMatches
```

Description

Indicates the number of exact matches distance method.

InfinityNorm

```
public Imsl.Stat.Dissimilarities.Method InfinityNorm
```

Description

Indicates the maximum difference (L_∞ norm) distance method.

L1Norm

```
public Imsl.Stat.Dissimilarities.Method L1Norm
```

Description

Indicates the sum of the absolute differences (L_1 norm) distance method.

L2Norm

```
public Imsl.Stat.Dissimilarities.Method L2Norm
```

Description

Indicates the Euclidean distance method (L_2 norm).

Mahalanobis

```
public Imsl.Stat.Dissimilarities.Method Mahalanobis
```

Description

Indicates the Mahalanobis distance method.

Dissimilarities.Scaling Enumeration

```
public enumeration Imsl.Stat.Dissimilarities.Scaling
```

Specifies the type of scaling.

Fields

None

```
public Imsl.Stat.Dissimilarities.Scaling None
```

Description

Indicates no scaling.

Range

```
public Imsl.Stat.Dissimilarities.Scaling Range
```

Description

Indicates scaling by the range.

StdDev

```
public Imsl.Stat.Dissimilarities.Scaling StdDev
```

Description

Indicates scaling by the standard deviation.

ClusterHierarchical Class

```
public class Imsl.Stat.ClusterHierarchical
```

Performs a hierarchical cluster analysis from a distance matrix.

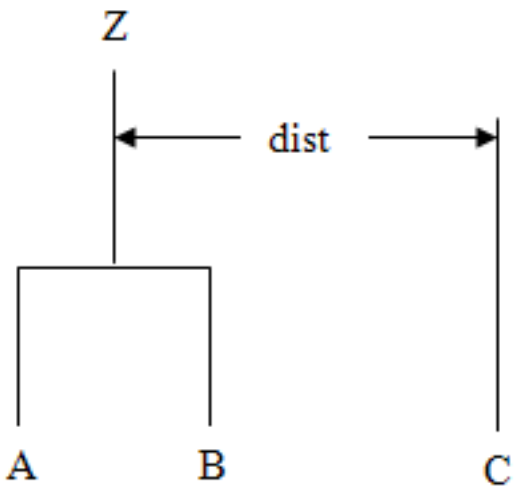
Class `ClusterHierarchical` conducts a hierarchical cluster analysis based upon a distance matrix, or, by appropriate use of the transformation specified in the `TransformType` property, based upon a similarity matrix. Only the upper triangular part of the input matrix is used.

Hierarchical clustering in `ClusterHierarchical` proceeds as follows:

Initially, each data point is considered to be a cluster, numbered 1 to `n`, where `n` is the number of rows in the input matrix, `dist`.

1. If the input matrix contains similarities, the matrix is transformed to a distance matrix using the transform type specified by the property `TransformType`. Set $k = 1$.
2. A search is made of the distance matrix to find the two closest clusters. These clusters are merged to form a new cluster, numbered `n` + k. The cluster numbers of the two clusters joined at this stage are saved as *Right Sons* and *Left Sons*, and the distance measure between the two clusters is stored as *Cluster Level* .
3. Based upon the method of clustering, updating of the distance measure in the row and column of `dist` corresponding to the new cluster is performed.
4. Set $k = k + 1$. If k is less than `n`, go to Step 2.

The five methods differ primarily in how the distance matrix is updated after two clusters have been joined. The `Method` property specifies how the distance of the cluster just merged with each of the remaining clusters will be updated. Class `ClusterHierarchical` allows five methods for computing the distances. To understand these measures, suppose in the following discussion that clusters *A* and *B* have just been joined to form cluster *Z*, and interest is in computing the distance of *Z* with another cluster called *C*.



Method	Description
Single	Single linkage (minimum distance). The distance from *Z* to *C* is the minimum of the distances (*A* to *C*, *B* to *C*).
Complete	Complete linkage (maximum distance). The distance from *Z* to *C* is the maximum of the distances (*A* to *C*, *B* to *C*).
AvgWithinClusters	Average-distance-within-clusters method. The distance from *Z* to *C* is the average distance of all objects that would be within the cluster formed by merging clusters *Z* and *C*. This average may be computed according to formulas given by Anderberg (1973, page 139).
AvgBetweenClusters	Average-distance-between-clusters method. The distance from *Z* to *C* is the average distance of objects within cluster *Z* to objects within cluster *C*. This average may be computed according to methods given by Anderberg (1973, page 140).
Wards	Ward's method: Clusters are formed so as to minimize the increase in the within-cluster sums of squares. The distance between two clusters is the increase in these sums of squares if the two clusters were merged. A method for computing this distance from a squared Euclidean distance matrix is given by Anderberg (1973, pages 142-145).

In general, single linkage will yield long thin clusters while complete linkage will yield clusters that are more spherical. Average linkage and Ward's linkage tend to yield clusters that are similar to those obtained with complete linkage.

Class `ClusterHierarchical` produces a unique representation of the binary cluster tree via the following three conventions; the fact that the tree is unique should aid in interpreting the clusters. First, when two clusters are joined and each cluster contains two or more data points, the cluster initially formed with the smallest level becomes the left son. Second, when a cluster containing more than one data point is joined with a cluster containing a single data point, the cluster with the single data point becomes the right son. Third, when two clusters containing only one object are joined, the cluster with the smallest cluster number becomes the right son.

Comments

1. The clusters corresponding to the original data points are numbered from 1 to `n`, where `n` is the number of rows in `dist`. The `n` - 1 clusters formed by merging clusters are numbered `n` + 1 to `n` + (`n` - 1).
2. Raw correlations, if used as similarities, should be made positive and transformed to a distance measure. One such transformation can be performed by setting `TransformType` = `ReciprocalAbs`.
3. The user may cluster either variables or observations with `ClusterHierarchical` since a dissimilarity matrix, not the original data, is used. Class Imsl.Stat.Dissimilarities (p. 1002) may be used to compute the matrix `dist` for either the variables or observations.

Properties

ClusterLeftSons

```
virtual public int[] ClusterLeftSons {get; }
```

Description

The left sons of each merged cluster.

Property Value

An `int` array containing the left sons of each merged cluster.

ClusterLevel

```
virtual public double[] ClusterLevel {get; }
```

Description

The level at which the clusters are joined.

Property Value

A `double` array containing the level at which the clusters are joined.

Remarks

Element [*k*-1] contains the distance (or similarity) level at which cluster `n` + *k* was formed. If the original data in `dist` was transformed, the inverse transformation is applied to the returned values.

ClusterRightSons

```
virtual public int[] ClusterRightSons {get; }
```

Description

The right sons of each merged cluster.

Property Value

An `int` array containing the right sons of each merged cluster.

Method

```
virtual public Imsl.Stat.ClusterHierarchical.Linkage Method {get; set; }
```

Description

The clustering method.

Property Value

A `ClusterHierarchical.Linkage` identifying the clustering method to use.

Remarks

By Default, `Method` = `Linkage.Single`.

Method	Description
Single	Single linkage (minimum distance).
Complete	Complete linkage (maximum distance).
AvgWithinClusters	Average distance within (average distance between objects within he merged cluster).
AvgBetweenClusters	Average distance between (average distance between objects in the two clusters).
Wards	Ward's method (minimize the within-cluster sums of squares). For Ward's method, the elements of dist are assumed to be Euclidean distances.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to NumberOfProcessors.

Property Value

An int indicating the maximum possible number of processors to use.

Remarks

By default, NumberOfProcessors = Environment.ProcessorCount. If NumberOfProcessors is set to a number less than 1 or greater than Environment.ProcessorCount, Environment.ProcessorCount will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

TransformType

```
virtual public Imsl.Stat.ClusterHierarchical.Transformation TransformType {get;
set; }
```

Description

The type of transformation.

Property Value

A ClusterHierarchical.Transformation identifying the type of transformation applied to the measures in dist.

Remarks

By Default, TransformType = Transformation.None.

TransformType	Description
None	No transformation is required. The elements of dist are distances.
Multiplication	Convert similarities to distances by multiplying by -1.0.
ReciprocalAbs	Convert similarities (usually correlations) to distances by taking the reciprocal of the absolute value.

Constructor

ClusterHierarchical

```
public ClusterHierarchical(double[,] dist)
```

Description

Constructor for `ClusterHierarchical`.

Parameter

`dist` – A `double` symmetric matrix containing the distance (or similarity) matrix.

Remarks

Only the upper triangular part of `dist` needs to be present.

Methods

Compute

```
public void Compute()
```

Description

Performs a hierarchical cluster analysis.

GetClusterMembership

```
public int[] GetClusterMembership(int nClusters)
```

Description

Returns the cluster membership of each observation.

Parameter

`nClusters` – An `int` which specifies the desired number of clusters.

Returns

An `int` array containing the cluster membership of each observation.

GetObsPerCluster

```
public int[] GetObsPerCluster(int nClusters)
```

Description

Returns the number of observations in each cluster.

Parameter

`nClusters` – An `int` which specifies the desired number of clusters.

Returns

An `int` array containing the number of observations in each cluster.

Example: ClusterHierarchical

This example illustrates a typical usage of `ClusterHierarchical`. The Fisher iris data is clustered. First the distance between irises is computed using the class `Dissimilarities`. The resulting distance matrix is then clustered using `ClusterHierarchical`, and cluster memberships for 5 clusters are computed.

```
using System;
using Imsl.Stat;

public class ClusterHierarchicalEx1
{
    public static void  Main(String[] args)
    {
        double[,] irisData = {
                    { 5.1, 3.5, 1.4, .2},
                    { 4.9, 3.0, 1.4, .2},
                    { 4.7, 3.2, 1.3, .2},
                    { 4.6, 3.1, 1.5, .2},
                    { 5.0, 3.6, 1.4, .2},
                    { 5.4, 3.9, 1.7, .4},
                    { 4.6, 3.4, 1.4, .3},
                    { 5.0, 3.4, 1.5, .2},
                    { 4.4, 2.9, 1.4, .2},
                    { 4.9, 3.1, 1.5, .1},
                    { 5.4, 3.7, 1.5, .2},
                    { 4.8, 3.4, 1.6, .2},
                    { 4.8, 3.0, 1.4, .1},
                    { 4.3, 3.0, 1.1, .1},
                    { 5.8, 4.0, 1.2, .2},
                    { 5.7, 4.4, 1.5, .4},
                    { 5.4, 3.9, 1.3, .4},
                    { 5.1, 3.5, 1.4, .3},
                    { 5.7, 3.8, 1.7, .3},
                    { 5.1, 3.8, 1.5, .3},
                    { 5.4, 3.4, 1.7, .2},
                    { 5.1, 3.7, 1.5, .4},
                    { 4.6, 3.6, 1.0, .2},
                    { 5.1, 3.3, 1.7, .5},
                    { 4.8, 3.4, 1.9, .2},
                    { 5.0, 3.0, 1.6, .2},
                    { 5.0, 3.4, 1.6, .4},
                    { 5.2, 3.5, 1.5, .2},
                    { 5.2, 3.4, 1.4, .2},
                    { 4.7, 3.2, 1.6, .2},
                    { 4.8, 3.1, 1.6, .2},
                    { 5.4, 3.4, 1.5, .4},
                    { 5.2, 4.1, 1.5, .1},
                    { 5.5, 4.2, 1.4, .2},
                    { 4.9, 3.1, 1.5, .2},
                    { 5.0, 3.2, 1.2, .2},
                    { 5.5, 3.5, 1.3, .2},
                    { 4.9, 3.6, 1.4, .1},
                    { 4.4, 3.0, 1.3, .2},
                    { 5.1, 3.4, 1.5, .2},
```

```
{ 5.0, 3.5, 1.3, .3},
{ 4.5, 2.3, 1.3, .3},
{ 4.4, 3.2, 1.3, .2},
{ 5.0, 3.5, 1.6, .6},
{ 5.1, 3.8, 1.9, .4},
{ 4.8, 3.0, 1.4, .3},
{ 5.1, 3.8, 1.6, .2},
{ 4.6, 3.2, 1.4, .2},
{ 5.3, 3.7, 1.5, .2},
{ 5.0, 3.3, 1.4, .2},
{ 7.0, 3.2, 4.7, 1.4},
{ 6.4, 3.2, 4.5, 1.5},
{ 6.9, 3.1, 4.9, 1.5},
{ 5.5, 2.3, 4.0, 1.3},
{ 6.5, 2.8, 4.6, 1.5},
{ 5.7, 2.8, 4.5, 1.3},
{ 6.3, 3.3, 4.7, 1.6},
{ 4.9, 2.4, 3.3, 1.0},
{ 6.6, 2.9, 4.6, 1.3},
{ 5.2, 2.7, 3.9, 1.4},
{ 5.0, 2.0, 3.5, 1.0},
{ 5.9, 3.0, 4.2, 1.5},
{ 6.0, 2.2, 4.0, 1.0},
{ 6.1, 2.9, 4.7, 1.4},
{ 5.6, 2.9, 3.6, 1.3},
{ 6.7, 3.1, 4.4, 1.4},
{ 5.6, 3.0, 4.5, 1.5},
{ 5.8, 2.7, 4.1, 1.0},
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{ 5.9, 3.2, 4.8, 1.8},
{ 6.1, 2.8, 4.0, 1.3},
{ 6.3, 2.5, 4.9, 1.5},
{ 6.1, 2.8, 4.7, 1.2},
{ 6.4, 2.9, 4.3, 1.3},
{ 6.6, 3.0, 4.4, 1.4},
{ 6.8, 2.8, 4.8, 1.4},
{ 6.7, 3.0, 5.0, 1.7},
{ 6.0, 2.9, 4.5, 1.5},
{ 5.7, 2.6, 3.5, 1.0},
{ 5.5, 2.4, 3.8, 1.1},
{ 5.5, 2.4, 3.7, 1.0},
{ 5.8, 2.7, 3.9, 1.2},
{ 6.0, 2.7, 5.1, 1.6},
{ 5.4, 3.0, 4.5, 1.5},
{ 6.0, 3.4, 4.5, 1.6},
{ 6.7, 3.1, 4.7, 1.5},
{ 6.3, 2.3, 4.4, 1.3},
{ 5.6, 3.0, 4.1, 1.3},
{ 5.5, 2.5, 4.0, 1.3},
{ 5.5, 2.6, 4.4, 1.2},
{ 6.1, 3.0, 4.6, 1.4},
{ 5.8, 2.6, 4.0, 1.2},
{ 5.0, 2.3, 3.3, 1.0},
{ 5.6, 2.7, 4.2, 1.3},
{ 5.7, 3.0, 4.2, 1.2},
```

```
            { 5.7, 2.9, 4.2, 1.3},
            { 6.2, 2.9, 4.3, 1.3},
            { 5.1, 2.5, 3.0, 1.1},
            { 5.7, 2.8, 4.1, 1.3},
            { 6.3, 3.3, 6.0, 2.5},
            { 5.8, 2.7, 5.1, 1.9},
            { 7.1, 3.0, 5.9, 2.1},
            { 6.3, 2.9, 5.6, 1.8},
            { 6.5, 3.0, 5.8, 2.2},
            { 7.6, 3.0, 6.6, 2.1},
            { 4.9, 2.5, 4.5, 1.7},
            { 7.3, 2.9, 6.3, 1.8},
            { 6.7, 2.5, 5.8, 1.8},
            { 7.2, 3.6, 6.1, 2.5},
            { 6.5, 3.2, 5.1, 2.0},
            { 6.4, 2.7, 5.3, 1.9},
            { 6.8, 3.0, 5.5, 2.1},
            { 5.7, 2.5, 5.0, 2.0},
            { 5.8, 2.8, 5.1, 2.4},
            { 6.4, 3.2, 5.3, 2.3},
            { 6.5, 3.0, 5.5, 1.8},
            { 7.7, 3.8, 6.7, 2.2},
            { 7.7, 2.6, 6.9, 2.3},
            { 6.0, 2.2, 5.0, 1.5},
            { 6.9, 3.2, 5.7, 2.3},
            { 5.6, 2.8, 4.9, 2.0},
            { 7.7, 2.8, 6.7, 2.0},
            { 6.3, 2.7, 4.9, 1.8},
            { 6.7, 3.3, 5.7, 2.1},
            { 7.2, 3.2, 6.0, 1.8},
            { 6.2, 2.8, 4.8, 1.8},
            { 6.1, 3.0, 4.9, 1.8},
            { 6.4, 2.8, 5.6, 2.1},
            { 7.2, 3.0, 5.8, 1.6},
            { 7.4, 2.8, 6.1, 1.9},
            { 7.9, 3.8, 6.4, 2.0},
            { 6.4, 2.8, 5.6, 2.2},
            { 6.3, 2.8, 5.1, 1.5},
            { 6.1, 2.6, 5.6, 1.4},
            { 7.7, 3.0, 6.1, 2.3},
            { 6.3, 3.4, 5.6, 2.4},
            { 6.4, 3.1, 5.5, 1.8},
            { 6.0, 3.0, 4.8, 1.8},
            { 6.9, 3.1, 5.4, 2.1},
            { 6.7, 3.1, 5.6, 2.4},
            { 6.9, 3.1, 5.1, 2.3},
            { 5.8, 2.7, 5.1, 1.9},
            { 6.8, 3.2, 5.9, 2.3},
            { 6.7, 3.3, 5.7, 2.5},
            { 6.7, 3.0, 5.2, 2.3},
            { 6.3, 2.5, 5.0, 1.9},
            { 6.5, 3.0, 5.2, 2.0},
            { 6.2, 3.4, 5.4, 2.3},
            { 5.9, 3.0, 5.1, 1.8}};

        Dissimilarities dist = new Dissimilarities(irisData);
```

```
        dist.ScalingOption = Dissimilarities.Scaling.StdDev;
        dist.Compute();
        ClusterHierarchical clink = new ClusterHierarchical(dist.DistanceMatrix);
        clink.Method = ClusterHierarchical.Linkage.AvgWithinClusters;
        clink.Compute();

        int nClusters = 5;
        int[] iclus = clink.GetClusterMembership(nClusters);
        int[] nclus = clink.GetObsPerCluster(nClusters);
        System.Console.Out.WriteLine("Cluster Membership");
        for (int i = 0; i < 15; i++)
        {
            for (int j = 0; j < 10; j++)
                Console.Out.Write(iclus[i * 10 + j] + " ");
            Console.Out.WriteLine();
        }

        System.Console.Out.WriteLine("\nObservations Per Cluster");
        for (int i = 0; i < nClusters; i++)
            System.Console.Out.Write(nclus[i] + " ");
        System.Console.Out.WriteLine();
    }
}
```

Output

```
Cluster Membership
5 5 5 5 5 5 5 5 5 5
5 5 5 5 5 5 5 5 5 5
5 5 5 5 5 5 5 5 5 5
5 5 5 5 5 5 5 5 5 5
5 5 5 5 5 5 5 5 5 5
3 3 3 4 3 4 3 4 3 4
4 3 4 3 4 3 4 4 4 4
3 3 3 3 3 3 3 3 3 4
4 4 4 3 4 3 3 4 4 4
4 3 4 4 4 4 4 3 4 4
2 3 2 3 2 1 4 1 3 2
2 3 2 3 3 2 3 2 1 4
2 3 1 3 2 1 3 3 3 1
1 2 3 3 3 1 2 3 3 2
2 2 3 2 2 2 3 3 2 3

Observations Per Cluster
8 19 44 29 50
```

ClusterHierarchical.Linkage Enumeration

```
public enumeration Imsl.Stat.ClusterHierarchical.Linkage
```

Specifies the type of linkage.

Fields

AvgBetweenClusters

```
public Imsl.Stat.ClusterHierarchical.Linkage AvgBetweenClusters
```

Description

Indicates the average distance between (average distance between objects in the two clusters) method.

AvgWithinClusters

```
public Imsl.Stat.ClusterHierarchical.Linkage AvgWithinClusters
```

Description

Indicates the average distance within (average distance between objects within the merged cluster) method.

Complete

```
public Imsl.Stat.ClusterHierarchical.Linkage Complete
```

Description

Indicates the complete linkage (maximum distance) method.

Single

```
public Imsl.Stat.ClusterHierarchical.Linkage Single
```

Description

Indicates the single linkage (minimum distance) method.

Wards

```
public Imsl.Stat.ClusterHierarchical.Linkage Wards
```

Description

Indicates the Ward's method.

ClusterHierarchical.Transformation Enumeration

```
public enumeration Imsl.Stat.ClusterHierarchical.Transformation
```

Specifies the type of transformation.

Fields

Multiplication

```
public Imsl.Stat.ClusterHierarchical.Transformation Multiplication
```

Description

Indicates transformation by multiplication by -1.0.

None

```
public Imsl.Stat.ClusterHierarchical.Transformation None
```

Description

Indicates no transformation.

ReciprocalAbs

```
public Imsl.Stat.ClusterHierarchical.Transformation ReciprocalAbs
```

Description

Indicates transformation by taking the reciprocal of the absolute value.

FactorAnalysis Class

```
public class Imsl.Stat.FactorAnalysis
```

Performs Principal Component Analysis or Factor Analysis on a covariance or correlation matrix.

Class `FactorAnalysis` computes principal components or initial factor loading estimates for a variance-covariance or correlation matrix using exploratory factor analysis models.

Models available are the principal component model for factor analysis and the common factor model with additions to the common factor model in alpha factor analysis and image analysis. Methods of estimation include principal components, principal factor, image analysis, unweighted least squares, generalized least squares, and maximum likelihood.

For the principal component model there are methods to compute the characteristic roots, characteristic vectors, standard errors for the characteristic roots, and the correlations of the principal component scores with the original variables. Principal components obtained from correlation matrices are the same as principal components obtained from standardized (to unit variance) variables.

The principal component scores are the elements of the vector $y = \Gamma^T x$ where Γ is the matrix whose columns are the characteristic vectors (eigenvectors) of the sample covariance (or correlation) matrix and x is the vector of observed (or standardized) random variables. The variances of the principal component scores are the characteristic roots (eigenvalues) of the covariance (correlation) matrix.

Asymptotic variances for the characteristic roots were first obtained by Girshick (1939) and are given more recently by Kendall, Stuart, and Ord (1983, page 331). These variances are computed either for variance-covariance matrices or for correlation matrices.

The correlations of the principal components with the observed (or standardized) variables are the same as the unrotated factor loadings obtained for the principal components model for factor analysis when a correlation matrix is input.

In the factor analysis model used for factor extraction, the basic model is given as $\Sigma = \Lambda\Lambda^T + \Psi$ where Σ is the $p \times p$ population covariance matrix. Λ is the $p \times k$ matrix of factor loadings relating the factors f to the observed variables x, and Ψ is the $p \times p$ matrix of covariances of the unique errors e. Here, p represents the number of variables and k is the number of factors. The relationship between the factors, the unique errors, and the observed variables is given as $x = \Lambda f + e$ where, in addition, it is assumed that the expected values of e, f, and x are zero. (The sample means can be subtracted from x if the expected value of x is not zero.) It is also assumed that each factor has unit variance, the factors are independent of each other, and that the factors and the unique errors are mutually independent. In the common factor model, the elements of the vector of unique errors e are also assumed to be independent of one another so that the matrix Ψ is diagonal. This is not the case in the principal component model in which the errors may be correlated.

Further differences between the various methods concern the criterion that is optimized and the amount of computer effort required to obtain estimates. Generally speaking, the least-squares and maximum likelihood methods, which use iterative algorithms, require the most computer time with the principal factor, principal component, and the image methods requiring much less time since the algorithms in these methods are not iterative. The algorithm in alpha factor analysis is also iterative, but the estimates in this method generally require somewhat less computer effort than the least-squares and maximum likelihood estimates. In all algorithms one eigensystem analysis is required on each iteration.

Properties

ConvergenceCriterion1

```
public double ConvergenceCriterion1 {get; set; }
```

Description

The convergence criterion used to terminate the iterations.

Property Value

A `double` used to terminate the iterations.

Remarks

For the least squares and and maximum likelihood methods convergence is assumed when the relative change in the criterion is less than `ConvergenceCriterion1`. For alpha factor analysis, convergence is assumed when the maximum change (relative to the variance) of a uniqueness is less than `ConvergenceCriterion1`. `ConvergenceCriterion1` is not referenced for the other estimation methods. By default, `ConvergenceCriterion1` is set to 0.0001.

ConvergenceCriterion2

```
public double ConvergenceCriterion2 {get; set; }
```

Description

The convergence criterion used to switch to exact second derivatives.

Property Value

A `double` used to switch to exact second derivatives.

Remarks

When the largest relative change in the unique standard deviation vector is less than `ConvergenceCriterion2`, exact second derivative vectors are used. By default, `ConvergenceCriterion2` is set to 0.1. Not referenced for principal component, principal factor, image factor, or alpha factor methods.

DegreesOfFreedom

```
public int DegreesOfFreedom {get; set; }
```

Description

The number of degrees of freedom.

Property Value

An `int` value specifying the number of degrees of freedom in the input matrix.

Remarks

If this property is not set, 100 degrees of freedom are assumed.

FactorLoadingEstimationMethod

```
public Imsl.Stat.FactorAnalysis.Model FactorLoadingEstimationMethod {get; set;
}
```

Description

The factor loading estimation method.

Property Value

Indicates the method to be applied for obtaining the factor loadings. Use `FactorAnalysis.Model` field `PrincipalComponent`, `PrincipalFactor`, `UnweightedLeastSquares`, `GeneralizedLeastSquares`, `MaximumLikelihood`, `ImageFactorAnalysis`, or `AlphaFactorAnalysis` for `FactorLoadingEstimationMethod`. By default, the `PrincipalComponent` is used.

Remarks

For the principal component and principal factor methods, the factor loading estimates are computed as

$$\hat{\Gamma}\hat{\Delta}^{-1/2}$$

where Γ and the diagonal matrix Δ are the eigenvalues and eigenvectors of a matrix. In the principal component model, the eigensystem analysis is performed on the sample covariance (correlation) matrix S while in the principal factor model the matrix $(S - \Psi)$ is used. If the unique error variances Ψ are not

known in the principal factor model, then they are estimated. This is achieved by setting the property `VarianceEstimationMethod` to 0. If the principal component model is used, the error variances in the `Variances` property are set to 0.0 automatically.

The basic idea in the principal component method is to find factors that maximize the variance in the original data that is explained by the factors. Because this method allows the unique errors to be correlated, some factor analysts insist that the principal component method is not a factor analytic method. Usually however, the estimates obtained via the principal component model and other models in factor analysis will be quite similar.

It should be noted that both the principal component and the principal factor methods give different results when the correlation matrix is used in place of the covariance matrix. Indeed, any rescaling of the sample covariance matrix can lead to different estimates with either of these methods. A further difficulty with the principal factor method is the problem of estimating the unique error variances. Theoretically, these must be known in advance and set using the the `Variances` property. In practice, the estimates of these parameters produced by setting the property `VarianceEstimationMethod` to 0 are often used. In either case, the resulting adjusted covariance (correlation) matrix

$$(S - \hat{\Psi})$$

may not yield the `nfactors` positive eigenvalues required for `nfactors` factors to be obtained. If this occurs, the user must either lower the number of factors to be estimated or give new unique error variance values.

For the least-squares and maximum likelihood methods an iterative algorithm is used to obtain the estimates (see joreskog 1977). As with the principal factor model, the user may either input the initial unique error variances or allow the algorithm to compute initial estimates. Unlike the principal factor method, the code then optimizes the criterion function with respect to both Ψ and Γ. (In the principal factor method, Ψ is assumed to be known. Given Ψ, estimates for Λ may be obtained.)

The major differences between the estimation methods described in this member function are in the criterion function that is optimized. Let S denote the sample covariance (correlation) matrix, and let Σ denote the covariance matrix that is to be estimated by the factor model. In the unweighted least-squares method, also called the iterated principal factor method or the minres method (see Harman 1976, page 177), the function minimized is the sum of the squared differences between S and Σ. This is written as $\Phi_u l = .5 trace((S-\Sigma)^2)$.

Generalized least-squares and maximum likelihood estimates are asymptotically equivalent methods. Maximum likelihood estimates maximize the (normal theory) likelihood $\{\Phi_m l = trace(\Sigma^{-1}S) - log(|\Sigma^{-1}S|)\}$. while generalized least squares optimizes the function $\Phi_g s = trace(\Sigma S^{-1} - I)^2$.

In all three methods, a two-stage optimization procedure is used. This proceeds by first solving the likelihood equations for Λ in terms of Ψ and substituting the solution into the likelihood. This gives a criterion $\Phi(\Psi, \Lambda(\Psi))$, which is optimized with respect to Ψ. In the second stage, the estimates

$$\hat{\Lambda}$$

are obtained from the estimates for Ψ.

The generalized least-squares and the maximum likelihood methods allow for the computation of a statistic for testing that `nfactors` common factors are adequate to fit the model. This is a chi-squared

test that all remaining parameters associated with additional factors are zero. If the probability of a larger chi-squared is small (see `stat[4]`) so that the null hypothesis is rejected, then additional factors are needed (although these factors may not be of any practical importance). Failure to reject does not legitimize the model. The statistic `stat[2]` is a likelihood ratio statistic in maximum likelihood estimates. As such, it asymptotically follows a chi-squared distribution with degrees of freedom given in `stat[3]`.

The Tucker and Lewis (1973) reliability coefficient, ρ, is returned in `stat[1]` when the maximum likelihood or generalized least-squares methods are used. This coefficient is an estimate of the ratio of explained to the total variation in the data. It is computed as follows:

$$\rho = \frac{mM_o - mM_k}{mM_o - 1}$$

$$m = d - \frac{2p+5}{6} - \frac{2k}{6}$$

$$M_o = \frac{-ln(|S|)}{p(p-1)/2}$$

$$M_k = \frac{\Phi}{((p-k)^2 - p - k)/2}$$

where $|S|$ is the determinant of `cov`, p is the number of variables, k is the number of factors, Φ is the optimized criterion, and d is the number of degrees of freedom.

The term “image analysis” is used here to denote the noniterative image method of Kaiser (1963). It is not the image factor analysis discussed by Harman (1976, page 226). The image method (as well as the alpha factor analysis method) begins with the notion that only a finite number from an infinite number of possible variables have been measured. The image factor pattern is calculated under the assumption that the ratio of the number of factors to the number of observed variables is near zero so that a very good estimate for the unique error variances (for standardized variables) is given as one minus the squared multiple correlation of the variable under consideration with all variables in the covariance matrix.

First, the matrix $D^2 = (diag(S^{-1}))^{-1}$ is computed where the operator “diag” results in a matrix consisting of the diagonal elements of its argument, and S is the sample covariance (correlation) matrix. Then, the eigenvalues Λ and eigenvectors Γ of the matrix $D^{-1}SD^{-1}$ are computed. Finally, the unrotated image factor pattern matrix is computed as $A = D\Gamma[(\Lambda - I)^2\Lambda^{-1}]^{1/2}$.

The alpha factor analysis method of Kaiser and Caffrey (1965) finds factor-loading estimates to maximize the correlation between the factors and the complete universe of variables of interest. The basic idea in this method is as follows: only a finite number of variables out of a much larger set of possible variables is observed. The population factors are linearly related to this larger set while the observed factors are linearly related to the observed variables. Let f denote the factors obtainable from a finite set of observed random variables, and let ξ denote the factors obtainable from the universe of observable variables. Then, the alpha method attempts to find factor-loading estimates so as to maximize the correlation between f and ξ. In order to obtain these estimates, the iterative algorithm of Kaiser and Caffrey (1965) is used.

MaxIterations

```
public int MaxIterations {get; set; }
```

Description

The maximum number of iterations in the iterative procedure.

Property Value

An `int` used as the maximum number of iterations allowed during the iterative portion of the algorithm.

Remarks

By default, `MaxIterations` is set to 60. `MaxIterations` is not referenced for factor loading methods PrincipalComponent, PrincipalFactor, or ImageFactorAnalysis.

MaxStep

```
public int MaxStep {get; set; }
```

Description

The maximum number of step halvings allowed during an iteration.

Property Value

An `int` used as the maximum number of step halvings allowed during an iteration.

Remarks

If this property is not set, `MaxStep` is set to 8. `MaxStep` is not referenced for `PrincipalComponent`, `PrincipalFactor`, `ImageFactorAnalysis`, or `AlphaFactorAnalysis` methods.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

VarianceEstimationMethod

```
public int VarianceEstimationMethod {get; set; }
```

Description

The variance estimation method.

Property Value

An `int` used to designate the method to be applied for obtaining the initial estimates of the unique variances.

Remarks

By default, `VarianceEstimationMethod` is set to 1.

init	Method
0	Initial estimates are taken as the constant 1-nfactors/(2*nvar) divided by the diagonal elements of the inverse of input matrix `cov`.
1	Initial estimates are input by the user in vector `uniq`.

Note that when the factor loading estimation method is PrincipalComponent, the initial estimates in `uniq` are reset to 0.0.

Constructor

FactorAnalysis

```
public FactorAnalysis(double[,] cov, Imsl.Stat.FactorAnalysis.MatrixType
matrixType, int nfactors)
```

Description

Constructor for `FactorAnalysis`.

Parameters

`cov` – A `double` matrix containing the covariance or correlation matrix.

`matrixType` – An `int` scalar indicating the type of matrix that is input.

`nfactors` – An `int` scalar indicating the number of factors in the model.

Remarks

`FactorAnalysis.matrixType` can specify a `VarianceCovariance` or `Correlation` matrix.

If `nfactors` is not known in advance, several different values of `nfactors` should be used, and the most reasonable value kept in the final solution. Since, in practice, the non-iterative methods often lead to solutions which differ little from the iterative methods, it is usually suggested that a non-iterative method be used in the initial tages of the factor analysis, and that the iterative methods be used once issues such as the number of factors have been resolved.

Methods

GetCorrelations

```
public double[,] GetCorrelations()
```

Description

Returns the correlations of the principal components.

Returns

A `double` matrix containing the correlations of the principal components with the observed/standardized variables.

Remarks

If a covariance matrix is input to the constructor, then the correlations are with the observed variables. Otherwise, the correlations are with the standardized (to a variance of 1.0) variables. Only valid for the Principal Components Model.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetFactorLoadings

```
public double[,] GetFactorLoadings()
```

Description

Returns the unrotated factor loadings.

Returns

A `double` matrix containing the unrotated factor loadings.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetParameterUpdates

```
public double[] GetParameterUpdates()
```

Description

Returns the parameter updates.

Returns

A `double` array containing the parameter updates when convergence was reached (or the iterations terminated).

Remarks

The parameter updates are only meaningful for the common factor model. The parameter updates are set to 0.0 for the principal component model.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetPercents

```
public double[] GetPercents()
```

Description

Returns the cumulative percent of the total variance explained by each principal component.

Returns

A `double` array containing the total variance explained by each principal component.

Remarks

Valid for the principal component model.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetStandardErrors

```
public double[] GetStandardErrors()
```

Description

Returns the estimated asymptotic standard errors of the eigenvalues.

Returns

A `double` array containing the estimated asymptotic standard errors of the eigenvalues.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetStatistics

```
public double[] GetStatistics()
```

Description

Returns statistics.

Returns

A `double` array containing output statistics.

Remarks

Statistics are not defined and set to `NaN` when the method used to obtain the estimates is the principal component method, principal factor method, image factor analysis method, or alpha analysis method.

i	Statistics[i]
0	Value of the function minimum.
1	Tucker reliability coefficient.
2	Chi-squared test statistic for testing that the number of factors in the model are adequate for the data.
3	Degrees of freedom in chi-squared. This is computed as $((nvar - nfactors)^2 - nvar - nfactors)/2$ where `nvar` is the number of variables and `nfactors` is the number of factors in the model.
4	Probability of a greater chi-squared statistic.
5	Number of iterations.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetValues

```
public double[] GetValues()
```

Description

Returns the eigenvalues.

Returns

A `double` array containing the eigenvalues of the matrix from which the factors were extracted ordered from largest to smallest.

Remarks

If Alpha Factor analysis is used, then the first `nfactors` positions of the array contain the Alpha coefficients. Here, `nfactors` is the number of factors in the model. If the algorithm fails to converge for a particular eigenvalue, that eigenvalue is set to `NaN`. Note that the eigenvalues are usually not the eigenvalues of the input matrix cov. They are the eigenvalues of the input matrix cov when the Principal Component method is used.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetVariances

```
public double[] GetVariances()
```

Description

Returns the unique variances.

Returns

A `double` array containing the unique variances.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

GetVectors

```
public double[,] GetVectors()
```

Description

Returns the eigenvectors.

Returns

A `double` matrix containing the eigenvectors of the matrix from which the factors were extracted.

Remarks

The j-th column of the eigenvector matrix corresponds to the j-th eigenvalue. The eigenvectors are normalized to each have Euclidean length equal to one. Also, the sign of each vector is set so that the largest component in magnitude (the first of the largest if there are ties) is made positive. Note that the eigenvectors are usually not the eigenvectors of the input matrix cov. They are the eigenvectors of the input matrix cov when the Principal Component method is used.

Exceptions

`Imsl.Stat.RankException` is thrown if the rank of the covariance matrix is less than the number of factors.

`Imsl.Stat.NotSemiDefiniteException` is thrown if the Hessian matrix not semi-definite.

`Imsl.Stat.NotPositiveSemiDefiniteException` is thrown if the covariance matrix is not positive semi-definite.

`Imsl.Stat.SingularException` is thrown if the covariance matrix is singular.

`Imsl.Stat.BadVarianceException` is thrown if the input variance is not in the allowed range.

`Imsl.Stat.EigenvalueException` is thrown if an error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

`Imsl.Stat.NonPositiveEigenvalueException` is thrown if in alpha factor analysis an eigenvalue is not positive. As all eigenvalues corresponding to the factors must be positive, either the number of factors must be reduced, or new initial estimates for the unique variances must be given.

SetVariances

```
public void SetVariances(double[] uniq)
```

Description

Sets the unique variances.

Parameter

`uniq` – A `double` array of length nvar containing the unique variances.

Remarks

If this member function is not called, the elements of `uniq`are set to 0.0. If the iterative methods fail for the unique variances used, new initial estimates should be tried. These may be obtained by use of another factoring method (use the final estimates from the new method as initial estimates in the old method). Another alternative is to call member function `VarianceEstimationMethod` and set the input argument to 0. This will cause the initial unique variances to be estimated by the code.

Example: Principal Components

This example illustrates the use of the FactorAnalysis class for a nine-variable matrix. `FactorAnalysis.Model.PrincipalComponent` and input matrix type `FactorAnalysis.MatrixType.Correlation` are selected.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class FactorAnalysisEx1
{
    public static void  Main(String[] args)
    {
        double[,] corr = {   {1.0, 0.523, 0.395, 0.471,
                                0.346, 0.426, 0.576, 0.434, 0.639},
                             {0.523, 1.0, 0.479, 0.506,
                                0.418, 0.462, 0.547, 0.283, 0.645},
                             {0.395, 0.479, 1.0, 0.355,
                                0.27, 0.254, 0.452, 0.219, 0.504},
                             {0.471, 0.506, 0.355, 1.0,
                                0.691, 0.791, 0.443, 0.285, 0.505},
                             {0.346, 0.418, 0.27, 0.691,
                                1.0, 0.679, 0.383, 0.149, 0.409},
                             {0.426, 0.462, 0.254, 0.791,
                                0.679, 1.0, 0.372, 0.314, 0.472},
                             {0.576, 0.547, 0.452, 0.443,
                                0.383, 0.372, 1.0, 0.385, 0.68},
                             {0.434, 0.283, 0.219, 0.285,
                                0.149, 0.314, 0.385, 1.0, 0.47},
                             {0.639, 0.645, 0.504, 0.505,
                                0.409, 0.472, 0.68, 0.47, 1.0}};

        FactorAnalysis pc = new FactorAnalysis(corr,
            FactorAnalysis.MatrixType.Correlation, 9);
        pc.FactorLoadingEstimationMethod =
            FactorAnalysis.Model.PrincipalComponent;
        pc.DegreesOfFreedom = 100;

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.0000";
        new PrintMatrix("Eigenvalues").Print(pmf, pc.GetValues());
        new PrintMatrix("Percents").Print(pmf, pc.GetPercents());
        new PrintMatrix
            ("Standard Errors").Print(pmf, pc.GetStandardErrors());
        new PrintMatrix("Eigenvectors").Print(pmf, pc.GetVectors());
        new PrintMatrix
            ("Unrotated Factor Loadings").Print(pmf, pc.GetFactorLoadings());
    }
}
```

Output

```
Eigenvalues
     0
0  4.6769
1  1.2640
2  0.8444
3  0.5550
4  0.4471
5  0.4291
6  0.3102
7  0.2770
8  0.1962

 Percents
     0
0  0.5197
1  0.6601
2  0.7539
3  0.8156
4  0.8653
5  0.9130
6  0.9474
7  0.9782
8  1.0000

Standard Errors
     0
0  0.6498
1  0.1771
2  0.0986
3  0.0879
4  0.0882
5  0.0890
6  0.0944
7  0.0994
8  0.1113

                                Eigenvectors
     0       1       2       3       4       5       6       7       8
0  0.3462  -0.2354   0.1386  -0.3317  -0.1088   0.7974   0.1735  -0.1240  -0.0488
1  0.3526  -0.1108  -0.2795  -0.2161   0.7664  -0.2002   0.1386  -0.3032  -0.0079
2  0.2754  -0.2697  -0.5585   0.6939  -0.1531   0.1511   0.0099  -0.0406  -0.0997
3  0.3664   0.4031   0.0406   0.1196   0.0017   0.1152  -0.4022  -0.1178   0.7060
4  0.3144   0.5022  -0.0733  -0.0207  -0.2804  -0.1796   0.7295   0.0075   0.0046
5  0.3455   0.4553   0.1825   0.1114   0.1202   0.0696  -0.3742   0.0925  -0.6780
6  0.3487  -0.2714  -0.0725  -0.3545  -0.5242  -0.4355  -0.2854  -0.3408  -0.1089
7  0.2407  -0.3159   0.7383   0.4329   0.0861  -0.1969   0.1862  -0.1623   0.0505
8  0.3847  -0.2533  -0.0078  -0.1468   0.0459  -0.1498  -0.0251   0.8521   0.1225

                          Unrotated Factor Loadings
     0       1       2       3       4       5       6       7       8
0  0.7487  -0.2646   0.1274  -0.2471  -0.0728   0.5224   0.0966  -0.0652  -0.0216
1  0.7625  -0.1245  -0.2568  -0.1610   0.5124  -0.1312   0.0772  -0.1596  -0.0035
2  0.5956  -0.3032  -0.5133   0.5170  -0.1024   0.0990   0.0055  -0.0214  -0.0442
3  0.7923   0.4532   0.0373   0.0891   0.0012   0.0755  -0.2240  -0.0620   0.3127
```

```
4  0.6799   0.5646  -0.0674  -0.0154  -0.1875  -0.1177   0.4063   0.0039   0.0021
5  0.7472   0.5119   0.1677   0.0830   0.0804   0.0456  -0.2084   0.0487  -0.3003
6  0.7542  -0.3051  -0.0666  -0.2641  -0.3505  -0.2853  -0.1589  -0.1794  -0.0482
7  0.5206  -0.3552   0.6784   0.3225   0.0576  -0.1290   0.1037  -0.0854   0.0224
8  0.8319  -0.2848  -0.0071  -0.1094   0.0307  -0.0981  -0.0140   0.4485   0.0543
```

Example: Factor Analysis

This example illustrates the use of the FactorAnalysis class. The following data were originally analyzed by Emmett(1949). There are 211 observations on 9 variables. Following Lawley and Maxwell (1971), three factors will be obtained by the method of maximum likelihood.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class FactorAnalysisEx2
{
    public static void  Main(String[] args)
    {
        double[,] cov = {
                            {1.0, 0.523, 0.395, 0.471,
                                   0.346, 0.426, 0.576, 0.434, 0.639},
                            {0.523, 1.0, 0.479, 0.506,
                                   0.418, 0.462, 0.547, 0.283, 0.645},
                            {0.395, 0.479, 1.0, 0.355,
                                   0.27, 0.254, 0.452, 0.219, 0.504},
                            {0.471, 0.506, 0.355, 1.0,
                                   0.691, 0.791, 0.443, 0.285, 0.505},
                            {0.346, 0.418, 0.27, 0.691,
                                   1.0, 0.679, 0.383, 0.149, 0.409},
                            {0.426, 0.462, 0.254, 0.791,
                                   0.679, 1.0, 0.372, 0.314, 0.472},
                            {0.576, 0.547, 0.452, 0.443,
                                   0.383, 0.372, 1.0, 0.385, 0.68},
                            {0.434, 0.283, 0.219, 0.285,
                                   0.149, 0.314, 0.385, 1.0, 0.47},
                            {0.639, 0.645, 0.504, 0.505,
                                   0.409, 0.472, 0.68, 0.47, 1.0}};
        FactorAnalysis fl = new FactorAnalysis(cov,
            FactorAnalysis.MatrixType.VarianceCovariance, 3);
        fl.ConvergenceCriterion1 = .000001;
        fl.ConvergenceCriterion2 = .01;
        fl.FactorLoadingEstimationMethod =
            FactorAnalysis.Model.MaximumLikelihood;
        fl.VarianceEstimationMethod = 0;
        fl.MaxStep = 10;
        fl.DegreesOfFreedom = 210;

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        pmf.NumberFormat = "0.0000";
        new PrintMatrix
            ("Unique Error Variances").Print(pmf, fl.GetVariances());
```

```
        new PrintMatrix
            ("Unrotated Factor Loadings").Print(pmf, fl.GetFactorLoadings());
        new PrintMatrix("Eigenvalues").Print(pmf, fl.GetValues());
        new PrintMatrix("Statistics").Print(pmf, fl.GetStatistics());
    }
}
```

Output

```
Unique Error Variances
     0
0  0.4505
1  0.4271
2  0.6166
3  0.2123
4  0.3805
5  0.1769
6  0.3995
7  0.4615
8  0.2309

  Unrotated Factor Loadings
     0        1        2
0  0.6642  -0.3209   0.0735
1  0.6888  -0.2471  -0.1933
2  0.4926  -0.3022  -0.2224
3  0.8372   0.2924  -0.0354
4  0.7050   0.3148  -0.1528
5  0.8187   0.3767   0.1045
6  0.6615  -0.3960  -0.0777
7  0.4579  -0.2955   0.4913
8  0.7657  -0.4274  -0.0117

Eigenvalues
     0
0  0.0626
1  0.2295
2  0.5413
3  0.8650
4  0.8937
5  0.9736
6  1.0802
7  1.1172
8  1.1401

 Statistics
      0
0  0.0350
1  1.0000
2  7.1494
3  12.0000
4  0.8476
5  5.0000
```

FactorAnalysis.MatrixType Enumeration

```
public enumeration Imsl.Stat.FactorAnalysis.MatrixType
```

Matrix type.

Fields

Correlation

```
public Imsl.Stat.FactorAnalysis.MatrixType Correlation
```

Description

Indicates correlation matrix.

VarianceCovariance

```
public Imsl.Stat.FactorAnalysis.MatrixType VarianceCovariance
```

Description

Indicates variance-covariance matrix.

FactorAnalysis.Model Enumeration

```
public enumeration Imsl.Stat.FactorAnalysis.Model
```

Model type.

Fields

AlphaFactorAnalysis

```
public Imsl.Stat.FactorAnalysis.Model AlphaFactorAnalysis
```

Description

Indicates alpha-factor analysis (common factor model) method used to obtain the estimates. Degrees of freedom is used for this estimation method.

GeneralizedLeastSquares

```
public Imsl.Stat.FactorAnalysis.Model GeneralizedLeastSquares
```

Description

Indicates generalized least-squares (common factor model) method used to obtain the estimates.

ImageFactorAnalysis

```
public Imsl.Stat.FactorAnalysis.Model ImageFactorAnalysis
```

Description

Indicates Image-factor analysis (common factor model) method used to obtain the estimates.

MaximumLikelihood

```
public Imsl.Stat.FactorAnalysis.Model MaximumLikelihood
```

Description

Indicates maximum likelihood method used to obtain the estimates. Degrees of freedom is used for this estimation method.

PrincipalComponent

```
public Imsl.Stat.FactorAnalysis.Model PrincipalComponent
```

Description

Indicates principal component (principal component model) used to obtain the estimates.

PrincipalFactor

```
public Imsl.Stat.FactorAnalysis.Model PrincipalFactor
```

Description

Indicates principal factor (common factor model) will be used to obtain the estimates.

UnweightedLeastSquares

```
public Imsl.Stat.FactorAnalysis.Model UnweightedLeastSquares
```

Description

Indicates unweighted least-squares (common factor model) method used to obtain the estimates. This option is the default.

DiscriminantAnalysis Class

```
public class Imsl.Stat.DiscriminantAnalysis
```

Performs a linear or a quadratic discriminant function analysis among several known groups.

`DiscriminantAnalysis` allows linear or a quadratic discrimination and the use of either reclassification, split sample, or the leaving-out-one methods in order to evaluate the rule. One or more observations can be added to the rule during each invocation of the `Update` method.

`DiscriminantAnalysis` results in the measure of distance between the groups, (see `GetMahalanobis` method), a table summarizing the classification results, (see `GetClassTable`), a matrix containing the posterior probabilities of group membership for each classified observation, (see `GetProbability`), the within-sample means, (see `GetMeans`) and covariance matrices computed from their LU factorizations, (see `GetCovariance`). The linear discriminant function coefficients are also computed, (see `GetCoefficients` method).

All observations can be input during one call to the `Update` method; this has the advantage of simplicity. Alternatively, one or more rows of observations can be input during separate calls to `Update`. This does not require all observations be memory resident, a significant advantage with large data sets. Note, however, to classify the same data set requires a second pass of the data to the `Classify` method. During the first pass to the `Update` method the discriminant functions are computed while in the second pass to the `Classify` method the observations are classified. When known groups are available the method `GetClassTable` is useful in comparing how well the alogorithm classifies. Multiple calls to the `Classify` method are also allowed. The class table, `GetClassTable`, is an accumulation of all observations classified. The class membership and probabilities, returned in `GetClassMembership` and `GetProbability`, will contain the membership for each observation from the most recent invocation of the `Classify` method.

Pooled only and pooled with group covariance computation cannot be mixed. By default, both pooled and group covariance matrices will be computed. An `InvalidOperationException` will be thrown if an attempt is made to change the covariance computation after the first call to the `Update` method. See the `CovarianceComputation` method for more details on specifying the covariance computation.

The within-group means are updated for all valid observations in `x`. Observations with invalid group numbers are ignored, as are observations with missing values (`Double.NaN`). The *LU* factorization of the covariance matrices are updated by adding (or deleting) observations via Givens rotations. See the `Downdate` method to delete observations.

During the algorithm's training process, or each invocation of the `Update` method, each observation in `x` is added to the means and the factorizations of the covariance matrices. Statistics of interest are computed: the linear discriminant functions, the prior probabilities, the log of the determinant of each of the covariance matrices, and a test statistic for testing that all of the within-group covariance matrices are equal. The matrix of Mahalanobis distances, which consists of the distances between the groups, is computed via the pooled covariance matrix when linear discrimination is specified. The row covariance matrix is used when the discrimination is quadratic. Covariance matrices are defined as follows. Let N_i denote the sum of the frequencies of the observations in group i, and let M_i denote the number of observations in group i. Then, if S_i denotes the within-group i covariance matrix,

$$S_i = \frac{1}{N_i - 1} \sum_{j=1}^{M_i} w_j f_j (x_j - \bar{x})(x_j - \bar{x})^T$$

where w_j is the weight of the j-th observation in group i, f_j is its frequency, x_j is the j-th observation column vector (in group i), and $\bar{x}$ denotes the mean vector of the observations in group i. The mean vectors are computed as

$$\bar{x} = \frac{1}{W_i} \sum_{j=1}^{M_i} w_j f_j x_j$$

where

$$W_i = \sum_{j=1}^{M_i} w_j f_j$$

Given the means and the covariance matrices, the linear discriminant function for group *i* is computed as:

$$z_i = \ln(p_i) - 0.5\overline{x_i}^T S_p^{-1}\overline{x_i} + x^T S_p^{-1}\overline{x_i}$$

where $\ln(p_i)$ is the natural log of the prior probability for the *i*-th group, *x* is the observation to be classified, and S_p denotes the pooled covariance matrix.

Let *S* denote either the pooled covariance matrix or one of the within-group covariance matrices S_i. (*S* will be the pooled covariance matrix in linear discrimination, and S_i otherwise.) The Mahalanobis distance between group *i* and group *j* is computed as:

$$D_{ij}^2 = (\overline{x_i} - \overline{x_j})^T S^{-1}(\overline{x_i} - \overline{x_j})$$

Finally, the asymptotic chi-squared test for the equality of covariance matrices is computed as follows (Morrison 1976, page 252):

$$\gamma = C^{-1}\sum_{i=1}^{k} n_i\{ln(|S_p|) - ln(|S_i|)\}$$

where n_i is the number of degrees of freedom in the *i*-th sample covariance matrix, *k* is the number of groups, and

$$C^{-1} = \frac{1-2p^2+3p-1}{6(p+1)(k-1)}\left(\sum_{i=1}^{k}\frac{1}{n_i} - \frac{1}{\sum_j n_j}\right)$$

where *p* is the number of variables.

The estimated posterior probability of each observation *x* belonging to group *i* is computed using the prior probabilities and the sample mean vectors and estimated covariance matrices under a multivariate normal assumption. Under quadratic discrimination, the within-group covariance matrices are used to compute the estimated posterior probabilities. The estimated posterior probability of an observation *x* belonging to group *i* is

$$\hat{q}_i(x) = \frac{e^{-\frac{1}{2}D_i^2(x)}}{\sum_{j=1}^{k} e^{-\frac{1}{2}D_j^2(x)}}$$

where

$$D_i^2(x) = \begin{cases} (x-\overline{x_i})^T S_i^{-1}(x-\overline{x_i}) + ln|S_i| - 2ln(p_i) & \text{linear or quadratic, pooled, group} \\ (x-\overline{x_i})^T S_p^{-1}(x-\overline{x_i}) - 2ln(p_i) & \text{linear, pooled} \end{cases}$$

For the leaving-out-one method of classification, the sample mean vector and sample covariance matrices in the formula for

$$D_i^2(x)$$

are adjusted so as to remove the observation *x* from their computation. For linear discrimination, the linear discriminant function coefficients are actually used to compute the same posterior probabilities.

Using the posterior probabilities, each observation in x is classified into a group; the result is tabulated in the matrix returned by `GetClassTable` and saved in the vector returned by `GetClassMembership`. If a group variable is provided and the group number is out of range, the classification table is not altered at this stage. If the reclassification method is specified, then all observations with no missing values are classified. When the leaving-out-one method is used, observations with invalid group numbers, weights, frequencies or classification variables are not classified. Regardless of the frequency, a 1 is added (or subtracted) from the classification table for each row of `x` that is classified and contains a valid group number. When the leaving-out-one method is used, adjustment is made to the posterior probabilities to remove the effect of the observation in the classification rule. In this adjustment, each observation is presumed to have a weight of w_j and a frequency of 1.0. See Lachenbruch (1975, page 36) for the required adjustment.

Properties

ClassificationMethod

```
public Imsl.Stat.DiscriminantAnalysis.Classification ClassificationMethod {get;
set; }
```

Description

The classification method.

Property Value

Indicates the method of classification.

Default: `Classification.Reclassification` is used.

Remarks

Use `Classification.Reclassification` or `Classification.LeaveOutOne`.

CovarianceComputation

```
public Imsl.Stat.DiscriminantAnalysis.CovarianceMatrix CovarianceComputation
{get; set; }
```

Description

The type of covariance matrices to be computed.

Property Value

Indicates the type of covariance matrices to be computed.

Default: `CovarianceMatrix.PooledGroup` is used.

Remarks

Use `CovarianceMatrix.Pooled` or `CovarianceMatrix.PooledGroup`.

DiscriminationMethod

```
public Imsl.Stat.DiscriminantAnalysis.Discrimination DiscriminationMethod {get;
set; }
```

Description

The discrimination method.

Property Value

Indicates the method of discrimination.

Default: `Discrimination.Linear` is used.

Remarks

Use `Discrimination.Linear` or `Discrimination.Quadratic`.

NumberOfRowsMissing

```
public int NumberOfRowsMissing {get; }
```

Description

The number of rows of data encountered containing missing values (`Double.NaN`).

Property Value

An `int` representing the number of rows of data encountered containing missing values (`Double.NaN`) for the classification, group, weight, and/or frequency variables.

Remarks

If a row of data contains a missing value (`Double.NaN`) for any of these variables, that row is excluded from the computations.

PriorType

```
public Imsl.Stat.DiscriminantAnalysis.PriorProbabilities PriorType {get; set; }
```

Description

The type of prior probabilities to be calculated.

Property Value

Indicates the type of prior probabilities to be calculated.

Default: PriorType = `PriorProbabilities.Equal`.

Remarks

Use `PriorProbabilities.Equal` to set equal prior probabilities, calculated as 1.0/`nGroups`. Or, use `PriorProbabilities.Proportional` to calculate the priors to be proportional to the sample size in each group. The sum of all prior probabilities is equal to 1.0. If the values calculated for the priors are less than 1.0e-20, they will be converted to the `Math.Log(1.0e-20)`. Prior probabilities are used in calculating statistics, coefficients, Mahalanobis, and classification probabilities.

Constructor

DiscriminantAnalysis

```
public DiscriminantAnalysis(int nVariables, int nGroups)
```

Description

Constructs a `DiscriminantAnalysis`.

Parameters

`nVariables` – An `int` representing the number of variables to be used in the discrimination.

`nGroups` – An `int` representing the number of groups in the data.

Methods

Classify

```
public void Classify(double[,] x)
```

Description

Classify a set of observations using the linear or quadratic discriminant functions generated during the training process.

Parameter

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The first `nVariables` columns correspond to the variables. Reclassification does not require group numbers be present. Any additional columns will be ignored.

Remarks

An `InvalidOperationException` is thrown if the leave-out-one classification method is chosen.

Exceptions

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

Classify

```
public void Classify(double[,] x, int[] varIndex)
```

Description

Classify a set of observations using the linear or quadratic discriminant functions generated during the training process.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables. Reclassification does not require group numbers be present. Additional columns will be ignored.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

Remarks

An `InvalidOperationException` is thrown if the leave-out-one classification method is chosen.

Exceptions

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

Classify

```
public void Classify(double[,] x, int[] frequencies, double[] weights)
```

Description

Classify a set of observations and associated frequencies and weights using the linear or quadratic discriminant functions generated during the training process.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The first `nVariables` columns correspond to the variables. Reclassification does not require group numbers be present. Any additional columns will be ignored.

`frequencies` – An `int` array containing the associated frequencies for each observation.

`weights` – A `double` array containing the associated weights for each observation

Remarks

An `InvalidOperationException` is thrown if the leave-out-one classification method is chosen.

Exceptions

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

Classify

```
public void Classify(double[,] x, int[] varIndex, int[] frequencies, double[]
weights)
```

Description

Classify a set of observations and associated frequencies and weights using the linear or quadratic discriminant functions generated during the training process.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables. Reclassification does not require group numbers be present. Additional columns in `x` will be ignored.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

`frequencies` – An `int` array containing the associated frequencies for each observation.

`weights` – A `double` array containing the associated weights for each observation.

Remarks

An `InvalidOperationException` is thrown if the leave-out-one classification method is chosen.

Exceptions

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

Classify

```
public void Classify(double[,] x, int[] group, int[] varIndex)
```

Description

Classify a set of observations and compare against known groups using the linear or quadratic discriminant functions generated during the training process.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables. Any additional columns will be ignored.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

Exceptions

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

Classify

```
public void Classify(double[,] x, int[] group, int[] varIndex, int[]
frequencies, double[] weights)
```

Description

Classify a set of observations, associated frequencies and weights, and compare against known groups using the linear or quadratic discriminant functions generated during the training process.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables. Additional columns are ignored.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

`frequencies` – An `int` array containing the associated frequencies for each observation.

`weights` – A `double` array containing the associated weights for each observation.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Downdate

```
public void Downdate(double[,] x, int[] group)
```

Description

Removes a set of observations from the discriminant functions.

Parameters

`x` – A `double` matrix containing the observations to be removed, with at least `nVariables` columns. The first `nVariables` columns correspond to the variables. Any additional columns will be ignored.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Downdate

```
public void Downdate(double[,] x, int[] group, int[] varIndex)
```

Description

Removes a set of observations from the discriminant functions.

Parameters

`x` – A `double` matrix containing the observations to be removed, with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Downdate

```
public void Downdate(double[,] x, int[] group, int[] frequencies, double[]
weights)
```

Description

Removes a set of observations and associated frequencies and weights from the discriminant functions.

Parameters

`x` – A `double` matrix containing the observations to be removed, with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

`frequencies` – An `int` array containing the associated frequencies for each observation.

`weights` – A `double` array containing the associated weights for each observation.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Downdate

```
public void Downdate(double[,] x, int[] group, int[] varIndex, int[]
frequencies, double[] weights)
```

Description

Removes a set of observations and associated frequencies and weights from the discriminant functions.

Parameters

`x` – A `double` matrix containing the observations to be removed, with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

`frequencies` – An `int` array containing the associated frequencies for each observation.

`weights` – A `double` array containing the associated weights for each observation.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

GetClassMembership

```
public int[] GetClassMembership()
```

Description

Returns the group number to which the observation was classified.

Returns

An `int` array containing the group to which the observation was classified. If an observation has an invalid group number, frequency, or weight when the leaving-out-one method has been specified, then the observation is not classified and the corresponding elements of the array are set to zero. Note this will return the class membership of the last set of observations classified.

Remarks

An `InvalidOperationException` is thrown if no data has been classified.

GetClassTable

```
public double[,] GetClassTable()
```

Description

Returns the classification table.

Returns

An `nGroups` by `nGroups` `double` matrix containing the classification table. The accumulation of each observation that is classified and has a group number equal to 1, 2, ..., `nGroups` is entered into the table. If a known group is provided, the rows of the table correspond to the known group membership. The columns refer to the group to which the observation was classified. If a known group is not provided, the table will only contain the accumulated classified groups in the column coresponding to the group to which the observation was classified.

Remarks

An `InvalidOperationException` is thrown if no data has been classified.

GetCoefficients

```
public double[,] GetCoefficients()
```

Description

Returns the linear discriminant function coefficients.

Returns

An `nGroups` by `nVariables` `double` matrix containing the linear discriminant function coefficients. The first column of the matrix contains the constant term, and the remaining columns contain the variable coefficients. The *i*-th row of the returned matrix corresponds to group *i*. The coefficients are always computed as linear discriminant function coefficients even when quadratic discrimination is specified.

Exceptions

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

GetCovariance

```
public double[,,] GetCovariance()
```

Description

Returns the array of covariances.

Returns

A *g* by `nVariables` by `nVariables` `double` array containing the covariances. Where, *g* = `nGroups`+1 if pooled, group covariance computation is specified or *g*=1 if pooled covariance computation is specified. When pooled only covariance matrices are computed, the within-group covariance matrices are not computed. The pooled covariance matrix is always computed and is returned as the *g*-th covariance matrix.

If this method is invoked before classification, the unscaled covariance matrix will be returned.

Exceptions

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

GetGroupCounts

```
public int[] GetGroupCounts()
```

Description

Returns the group counts.

Returns

An `int` array of length `nGroups` containing the number of observations in each group. If an update has not preceeded the invocation of this method, an array of all zeros will be returned.

GetMahalanobis

```
public double[,] GetMahalanobis()
```

Description

Returns the Mahalanobis distances between the group means.

Returns

An `nGroups` by `nGroups` `double` matrix containing the Mahalanobis distances between the group means. For linear discrimination, the Mahalanobis distance

$$D_{ij}^2(x)$$

between group means *i* and *j* is computed using the within covariance matrix for group *i* in place of the pooled covariance matrix.

Exceptions

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

GetMeans

```
public double[,] GetMeans()
```

Description

Returns the variable means.

Returns

An `nGroups` by `nVariables` `double` matrix containing the variable means. The *i*-th row contains the variable means for group *i*.

If this method is invoked before classification, the unscaled means will be returned.

Exceptions

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

GetPrior

```
public double[] GetPrior()
```

Description

Returns the prior probabilities.

Returns

A `double` array of length `nGroups` containing the prior probabilities for each group.

GetProbability

```
public double[,] GetProbability()
```

Description

Returns the posterior probabilities for each observation.

Returns

An `nGroups.GetLength(0)` by `nGroups` `double` matrix containing the posterior probabilities for each observation. Note this will return the probabilities of the last set of observations classified.

Remarks

An `InvalidOperationException` is thrown if no data has been classified.

GetStatistics

```
public double[] GetStatistics()
```

Description

Returns statistics.

Returns

A `double` array containing output statistics. The following table describes the statistics available:

index	Description
0	Sum of the degrees of freedom for the within-covariance matrices.
1	Chi-squared statistic.
2	The degrees of freedom in the chi-squared statistic.
3	Probability of a greater chi-squared, respectively, of a test of the homogeneity of the within-covariance matrices. (Not computed when the pooled only covariance matrix is computed).
4 thru (4+`nGroups`)	Log of the determinant of each group's covariance matrix (not computed when the pooled only covariance matrix is computed) and of the pooled covariance matrix.
Last (`nGroups` + 1) elements	Sum of the weights within each group.
Last element	Sum of the weights in all groups.

Exceptions

`Imsl.Stat.EmptyGroupException` is thrown when there are no observations in a group.

`Imsl.Stat.CovarianceSingularException` is thrown when the variance-covariance matrix is singular.

SetPrior

```
public void SetPrior(double[] prior)
```

Description

Specifies user supplied prior probabilities.

Parameter

`prior` – A `double` array of length `nGroups` containing the prior probabilities for each group.

Remarks

The elements of `prior` should sum to 1.0.

If the values of `prior` are less than 1.0e-20, they will be converted to `Math.Log(1.0e-20)`.

Default: The prior probablities are calculated to be equal, see property `PriorType`.

Update

```
public void Update(double[,] x, int[] group)
```

Description

Trains a set of observations and associated frequencies and weights by performing a linear or quadratic discriminant function analysis among several known groups.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The first `nVariables` correspond to the variables. Any additional columns will be ignored.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Update

```
public void Update(double[,] x, int[] group, int[] varIndex)
```

Description

Trains a set of observations and associated frequencies and weights by performing a linear or quadratic discriminant function analysis among several known groups.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables. Any additional columns will be ignored.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., .`nGroups` for each observation.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Update

```
public void Update(double[,] x, int[] group, int[] frequencies, double[]
weights)
```

Description

Trains a set of observations and associated frequencies and weights by performing a linear or quadratic discriminant function analysis among several known groups.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The first `nVariables` correspond to the variables. Any additional columns will be ignored.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

`frequencies` – An `int` array containing the associated frequencies for each observation.

`weights` – A `double` array containing the associated weights for each observation.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Update

```
public void Update(double[,] x, int[] group, int[] varIndex, int[] frequencies,
double[] weights)
```

Description

Trains a set of observations and associated frequencies and weights by performing a linear or quadratic discriminant function analysis among several known groups.

Parameters

`x` – A `double` matrix containing the observations with at least `nVariables` columns. The columns indicated in `varIndex` correspond to the variables.

`group` – An `int` array containing the group numbers. The groups must be numbered 1,2, ..., `nGroups` for each observation.

`varIndex` – An `int` array containing the column indices in `x` that correspond to the variables to be used in the analysis.

`frequencies` – An `int` array containing the associated frequencies for each observation.

`weights` – A `double` array containing the associated weights for each observation.

Exception

`Imsl.Stat.SumOfWeightsNegException` is thrown when the sum of the weights have become negative.

Example: Discriminant Analysis

This example uses linear discrimination with equal prior probabilities on Fisher's (1936) iris data. This example illustrates the use of the DiscriminantAnalysis class.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class DiscriminantAnalysisEx1
{
    public static void  Main(String[] args)
    {
        double[,] x = {
            {1.0, 5.1, 3.5, 1.4, .2},{1.0, 4.9, 3.0, 1.4, .2},
            {1.0, 4.7, 3.2, 1.3, .2},{1.0, 4.6, 3.1, 1.5, .2},
            {1.0, 5.0, 3.6, 1.4, .2},{1.0, 5.4, 3.9, 1.7, .4},
            {1.0, 4.6, 3.4, 1.4, .3},{1.0, 5.0, 3.4, 1.5, .2},
            {1.0, 4.4, 2.9, 1.4, .2},{1.0, 4.9, 3.1, 1.5, .1},
            {1.0, 5.4, 3.7, 1.5, .2},{1.0, 4.8, 3.4, 1.6, .2},
            {1.0, 4.8, 3.0, 1.4, .1},{1.0, 4.3, 3.0, 1.1, .1},
```

```
{1.0, 5.8, 4.0, 1.2, .2},{1.0, 5.7, 4.4, 1.5, .4},
{1.0, 5.4, 3.9, 1.3, .4},{1.0, 5.1, 3.5, 1.4, .3},
{1.0, 5.7, 3.8, 1.7, .3},{1.0, 5.1, 3.8, 1.5, .3},
{1.0, 5.4, 3.4, 1.7, .2},{1.0, 5.1, 3.7, 1.5, .4},
{1.0, 4.6, 3.6, 1.0, .2},{1.0, 5.1, 3.3, 1.7, .5},
{1.0, 4.8, 3.4, 1.9, .2},{1.0, 5.0, 3.0, 1.6, .2},
{1.0, 5.0, 3.4, 1.6, .4},{1.0, 5.2, 3.5, 1.5, .2},
{1.0, 5.2, 3.4, 1.4, .2},{1.0, 4.7, 3.2, 1.6, .2},
{1.0, 4.8, 3.1, 1.6, .2},{1.0, 5.4, 3.4, 1.5, .4},
{1.0, 5.2, 4.1, 1.5, .1},{1.0, 5.5, 4.2, 1.4, .2},
{1.0, 4.9, 3.1, 1.5, .2},{1.0, 5.0, 3.2, 1.2, .2},
{1.0, 5.5, 3.5, 1.3, .2},{1.0, 4.9, 3.6, 1.4, .1},
{1.0, 4.4, 3.0, 1.3, .2},{1.0, 5.1, 3.4, 1.5, .2},
{1.0, 5.0, 3.5, 1.3, .3},{1.0, 4.5, 2.3, 1.3, .3},
{1.0, 4.4, 3.2, 1.3, .2},{1.0, 5.0, 3.5, 1.6, .6},
{1.0, 5.1, 3.8, 1.9, .4},{1.0, 4.8, 3.0, 1.4, .3},
{1.0, 5.1, 3.8, 1.6, .2},{1.0, 4.6, 3.2, 1.4, .2},
{1.0, 5.3, 3.7, 1.5, .2},{1.0, 5.0, 3.3, 1.4, .2},
{2.0, 7.0, 3.2, 4.7, 1.4},{2.0, 6.4, 3.2, 4.5, 1.5},
{2.0, 6.9, 3.1, 4.9, 1.5},{2.0, 5.5, 2.3, 4.0, 1.3},
{2.0, 6.5, 2.8, 4.6, 1.5},{2.0, 5.7, 2.8, 4.5, 1.3},
{2.0, 6.3, 3.3, 4.7, 1.6},{2.0, 4.9, 2.4, 3.3, 1.0},
{2.0, 6.6, 2.9, 4.6, 1.3},{2.0, 5.2, 2.7, 3.9, 1.4},
{2.0, 5.0, 2.0, 3.5, 1.0},{2.0, 5.9, 3.0, 4.2, 1.5},
{2.0, 6.0, 2.2, 4.0, 1.0},{2.0, 6.1, 2.9, 4.7, 1.4},
{2.0, 5.6, 2.9, 3.6, 1.3},{2.0, 6.7, 3.1, 4.4, 1.4},
{2.0, 5.6, 3.0, 4.5, 1.5},{2.0, 5.8, 2.7, 4.1, 1.0},
{2.0, 6.2, 2.2, 4.5, 1.5},{2.0, 5.6, 2.5, 3.9, 1.1},
{2.0, 5.9, 3.2, 4.8, 1.8},{2.0, 6.1, 2.8, 4.0, 1.3},
{2.0, 6.3, 2.5, 4.9, 1.5},{2.0, 6.1, 2.8, 4.7, 1.2},
{2.0, 6.4, 2.9, 4.3, 1.3},{2.0, 6.6, 3.0, 4.4, 1.4},
{2.0, 6.8, 2.8, 4.8, 1.4},{2.0, 6.7, 3.0, 5.0, 1.7},
{2.0, 6.0, 2.9, 4.5, 1.5},{2.0, 5.7, 2.6, 3.5, 1.0},
{2.0, 5.5, 2.4, 3.8, 1.1},{2.0, 5.5, 2.4, 3.7, 1.0},
{2.0, 5.8, 2.7, 3.9, 1.2},{2.0, 6.0, 2.7, 5.1, 1.6},
{2.0, 5.4, 3.0, 4.5, 1.5},{2.0, 6.0, 3.4, 4.5, 1.6},
{2.0, 6.7, 3.1, 4.7, 1.5},{2.0, 6.3, 2.3, 4.4, 1.3},
{2.0, 5.6, 3.0, 4.1, 1.3},{2.0, 5.5, 2.5, 4.0, 1.3},
{2.0, 5.5, 2.6, 4.4, 1.2},{2.0, 6.1, 3.0, 4.6, 1.4},
{2.0, 5.8, 2.6, 4.0, 1.2},{2.0, 5.0, 2.3, 3.3, 1.0},
{2.0, 5.6, 2.7, 4.2, 1.3},{2.0, 5.7, 3.0, 4.2, 1.2},
{2.0, 5.7, 2.9, 4.2, 1.3},{2.0, 6.2, 2.9, 4.3, 1.3},
{2.0, 5.1, 2.5, 3.0, 1.1},{2.0, 5.7, 2.8, 4.1, 1.3},
{3.0, 6.3, 3.3, 6.0, 2.5},{3.0, 5.8, 2.7, 5.1, 1.9},
{3.0, 7.1, 3.0, 5.9, 2.1},{3.0, 6.3, 2.9, 5.6, 1.8},
{3.0, 6.5, 3.0, 5.8, 2.2},{3.0, 7.6, 3.0, 6.6, 2.1},
{3.0, 4.9, 2.5, 4.5, 1.7},{3.0, 7.3, 2.9, 6.3, 1.8},
{3.0, 6.7, 2.5, 5.8, 1.8},{3.0, 7.2, 3.6, 6.1, 2.5},
{3.0, 6.5, 3.2, 5.1, 2.0},{3.0, 6.4, 2.7, 5.3, 1.9},
{3.0, 6.8, 3.0, 5.5, 2.1},{3.0, 5.7, 2.5, 5.0, 2.0},
{3.0, 5.8, 2.8, 5.1, 2.4},{3.0, 6.4, 3.2, 5.3, 2.3},
{3.0, 6.5, 3.0, 5.5, 1.8},{3.0, 7.7, 3.8, 6.7, 2.2},
{3.0, 7.7, 2.6, 6.9, 2.3},{3.0, 6.0, 2.2, 5.0, 1.5},
{3.0, 6.9, 3.2, 5.7, 2.3},{3.0, 5.6, 2.8, 4.9, 2.0},
{3.0, 7.7, 2.8, 6.7, 2.0},{3.0, 6.3, 2.7, 4.9, 1.8},
{3.0, 6.7, 3.3, 5.7, 2.1},{3.0, 7.2, 3.2, 6.0, 1.8},
```

```
    {3.0, 6.2, 2.8, 4.8, 1.8},{3.0, 6.1, 3.0, 4.9, 1.8},
    {3.0, 6.4, 2.8, 5.6, 2.1},{3.0, 7.2, 3.0, 5.8, 1.6},
    {3.0, 7.4, 2.8, 6.1, 1.9},{3.0, 7.9, 3.8, 6.4, 2.0},
    {3.0, 6.4, 2.8, 5.6, 2.2},{3.0, 6.3, 2.8, 5.1, 1.5},
    {3.0, 6.1, 2.6, 5.6, 1.4},{3.0, 7.7, 3.0, 6.1, 2.3},
    {3.0, 6.3, 3.4, 5.6, 2.4},{3.0, 6.4, 3.1, 5.5, 1.8},
    {3.0, 6.0, 3.0, 4.8, 1.8},{3.0, 6.9, 3.1, 5.4, 2.1},
    {3.0, 6.7, 3.1, 5.6, 2.4},{3.0, 6.9, 3.1, 5.1, 2.3},
    {3.0, 5.8, 2.7, 5.1, 1.9},{3.0, 6.8, 3.2, 5.9, 2.3},
    {3.0, 6.7, 3.3, 5.7, 2.5},{3.0, 6.7, 3.0, 5.2, 2.3},
    {3.0, 6.3, 2.5, 5.0, 1.9},{3.0, 6.5, 3.0, 5.2, 2.0},
    {3.0, 6.2, 3.4, 5.4, 2.3},{3.0, 5.9, 3.0, 5.1, 1.8}
};

int[] group = new int[x.GetLength(0)];
int[] varIndex = { 1, 2, 3, 4 };

for (int i = 0; i < x.GetLength(0); i++)
{
    group[i] = (int)x[i,0];
}

int nvar = x.GetLength(1) - 1;

DiscriminantAnalysis da = new DiscriminantAnalysis(nvar, 3);
da.CovarianceComputation =
    DiscriminantAnalysis.CovarianceMatrix.Pooled;
da.ClassificationMethod =
DiscriminantAnalysis.Classification.Reclassification;

da.Update(x, group, varIndex);
da.Classify(x, group, varIndex);
PrintMatrixFormat pmf = new PrintMatrixFormat();
pmf.NumberFormat = "0.00";

new PrintMatrix("Xmean are:  ").Print(pmf, da.GetMeans());
new PrintMatrix("Coef:  ").Print(pmf, da.GetCoefficients());
new PrintMatrix("Counts: ").Print(da.GetGroupCounts());
new PrintMatrix("Stats: ").Print(pmf, da.GetStatistics());
int[] cm = da.GetClassMembership();
int[,] cMem = new int[1, cm.Length];
for (int i = 0; i < cm.Length; i++)
{
    cMem[0, i] = cm[i];
}
new PrintMatrix("ClassMembership").SetPageWidth(50).Print(cMem);
new PrintMatrix("ClassTable: ").Print(da.GetClassTable());
double[,,] cov = da.GetCovariance();
double[,] tmpCov = new double[cov.GetLength(1), cov.GetLength(2)];
for (int i = 0; i < cov.GetLength(0); i++)
{
    for (int j = 0; j < cov.GetLength(1); j++)
        for (int k = 0; k < cov.GetLength(2); k++)
            tmpCov[j, k] = cov[i, j, k];
    new PrintMatrix
        ("Covariance Matrix " + i + " : ").Print(pmf, tmpCov);
```

```
        }
        new PrintMatrix("Prior : ").Print(da.GetPrior());
        new PrintMatrix("PROB: ").Print(pmf, da.GetProbability());
        new PrintMatrix("MAHALANOBIS: ").Print(pmf, da.GetMahalanobis());
        Console.Out.WriteLine("nrmiss = " + da.NumberOfRowsMissing);
    }
}
```

Output

```
       Xmean are:
    0     1     2     3
0  5.01  3.43  1.46  0.25
1  5.94  2.77  4.26  1.33
2  6.59  2.97  5.55  2.03

                Coef:
      0       1      2      3       4
0  -86.31   23.54  23.59  -16.43  -17.40
1  -72.85   15.70  7.07    5.21    6.43
2  -104.37  12.45  3.69    12.77   21.08

Counts:
   0
0  50
1  50
2  50

   Stats:
       0
 0   147.00
 1   NaN
 2   NaN
 3   NaN
 4   NaN
 5   NaN
 6   NaN
 7  -9.96
 8   50.00
 9   50.00
10   50.00
11   150.00

                  ClassMembership
   0  1  2  3  4  5  6  7  8  9  10  11  12  13  14
0  1  1  1  1  1  1  1  1  1  1  1   1   1   1   1

   15  16  17  18  19  20  21  22  23  24  25  26  27  28  29
0  1   1   1   1   1   1   1   1   1   1   1   1   1   1   1

   30  31  32  33  34  35  36  37  38  39  40  41  42  43  44
0  1   1   1   1   1   1   1   1   1   1   1   1   1   1   1

   45  46  47  48  49  50  51  52  53  54  55  56  57  58  59
0  1   1   1   1   1   2   2   2   2   2   2   2   2   2   2
```

```
   60  61  62  63  64  65  66  67  68  69  70  71  72  73  74
0  2   2   2   2   2   2   2   2   2   2   3   2   2   2   2

   75  76  77  78  79  80  81  82  83  84  85  86  87  88  89
0  2   2   2   2   2   2   2   2   3   2   2   2   2   2   2

   90  91  92  93  94  95  96  97  98  99  100  101  102  103  104
0  2   2   2   2   2   2   2   2   2   2    3    3    3    3    3

   105  106  107  108  109  110  111  112  113  114  115  116  117  118  119
0   3    3    3    3    3    3    3    3    3    3    3    3    3    3    3

   120  121  122  123  124  125  126  127  128  129  130  131  132  133  134
0   3    3    3    3    3    3    3    3    3    3    3    3    3    2    3

   135  136  137  138  139  140  141  142  143  144  145  146  147  148  149
0   3    3    3    3    3    3    3    3    3    3    3    3    3    3    3

 ClassTable:
   0   1   2
0  50   0   0
1   0  48   2
2   0   1  49

  Covariance Matrix 0 :
     0     1     2     3
0  0.27  0.09  0.17  0.04
1  0.09  0.12  0.06  0.03
2  0.17  0.06  0.19  0.04
3  0.04  0.03  0.04  0.04

       Prior :
                0
0  0.333333333333333
1  0.333333333333333
2  0.333333333333333

        PROB:
      0     1     2
  0  1.00  0.00  0.00
  1  1.00  0.00  0.00
  2  1.00  0.00  0.00
  3  1.00  0.00  0.00
  4  1.00  0.00  0.00
  5  1.00  0.00  0.00
  6  1.00  0.00  0.00
  7  1.00  0.00  0.00
  8  1.00  0.00  0.00
  9  1.00  0.00  0.00
 10  1.00  0.00  0.00
 11  1.00  0.00  0.00
 12  1.00  0.00  0.00
 13  1.00  0.00  0.00
 14  1.00  0.00  0.00
 15  1.00  0.00  0.00
```

```
16  1.00  0.00  0.00
17  1.00  0.00  0.00
18  1.00  0.00  0.00
19  1.00  0.00  0.00
20  1.00  0.00  0.00
21  1.00  0.00  0.00
22  1.00  0.00  0.00
23  1.00  0.00  0.00
24  1.00  0.00  0.00
25  1.00  0.00  0.00
26  1.00  0.00  0.00
27  1.00  0.00  0.00
28  1.00  0.00  0.00
29  1.00  0.00  0.00
30  1.00  0.00  0.00
31  1.00  0.00  0.00
32  1.00  0.00  0.00
33  1.00  0.00  0.00
34  1.00  0.00  0.00
35  1.00  0.00  0.00
36  1.00  0.00  0.00
37  1.00  0.00  0.00
38  1.00  0.00  0.00
39  1.00  0.00  0.00
40  1.00  0.00  0.00
41  1.00  0.00  0.00
42  1.00  0.00  0.00
43  1.00  0.00  0.00
44  1.00  0.00  0.00
45  1.00  0.00  0.00
46  1.00  0.00  0.00
47  1.00  0.00  0.00
48  1.00  0.00  0.00
49  1.00  0.00  0.00
50  0.00  1.00  0.00
51  0.00  1.00  0.00
52  0.00  1.00  0.00
53  0.00  1.00  0.00
54  0.00  1.00  0.00
55  0.00  1.00  0.00
56  0.00  0.99  0.01
57  0.00  1.00  0.00
58  0.00  1.00  0.00
59  0.00  1.00  0.00
60  0.00  1.00  0.00
61  0.00  1.00  0.00
62  0.00  1.00  0.00
63  0.00  0.99  0.01
64  0.00  1.00  0.00
65  0.00  1.00  0.00
66  0.00  0.98  0.02
67  0.00  1.00  0.00
68  0.00  0.96  0.04
69  0.00  1.00  0.00
70  0.00  0.25  0.75
71  0.00  1.00  0.00
```

72	0.00	0.82	0.18
73	0.00	1.00	0.00
74	0.00	1.00	0.00
75	0.00	1.00	0.00
76	0.00	1.00	0.00
77	0.00	0.69	0.31
78	0.00	0.99	0.01
79	0.00	1.00	0.00
80	0.00	1.00	0.00
81	0.00	1.00	0.00
82	0.00	1.00	0.00
83	0.00	0.14	0.86
84	0.00	0.96	0.04
85	0.00	0.99	0.01
86	0.00	1.00	0.00
87	0.00	1.00	0.00
88	0.00	1.00	0.00
89	0.00	1.00	0.00
90	0.00	1.00	0.00
91	0.00	1.00	0.00
92	0.00	1.00	0.00
93	0.00	1.00	0.00
94	0.00	1.00	0.00
95	0.00	1.00	0.00
96	0.00	1.00	0.00
97	0.00	1.00	0.00
98	0.00	1.00	0.00
99	0.00	1.00	0.00
100	0.00	0.00	1.00
101	0.00	0.00	1.00
102	0.00	0.00	1.00
103	0.00	0.00	1.00
104	0.00	0.00	1.00
105	0.00	0.00	1.00
106	0.00	0.05	0.95
107	0.00	0.00	1.00
108	0.00	0.00	1.00
109	0.00	0.00	1.00
110	0.00	0.01	0.99
111	0.00	0.00	1.00
112	0.00	0.00	1.00
113	0.00	0.00	1.00
114	0.00	0.00	1.00
115	0.00	0.00	1.00
116	0.00	0.01	0.99
117	0.00	0.00	1.00
118	0.00	0.00	1.00
119	0.00	0.22	0.78
120	0.00	0.00	1.00
121	0.00	0.00	1.00
122	0.00	0.00	1.00
123	0.00	0.10	0.90
124	0.00	0.00	1.00
125	0.00	0.00	1.00
126	0.00	0.19	0.81
127	0.00	0.13	0.87

```
128  0.00  0.00  1.00
129  0.00  0.10  0.90
130  0.00  0.00  1.00
131  0.00  0.00  1.00
132  0.00  0.00  1.00
133  0.00  0.73  0.27
134  0.00  0.07  0.93
135  0.00  0.00  1.00
136  0.00  0.00  1.00
137  0.00  0.01  0.99
138  0.00  0.19  0.81
139  0.00  0.00  1.00
140  0.00  0.00  1.00
141  0.00  0.00  1.00
142  0.00  0.00  1.00
143  0.00  0.00  1.00
144  0.00  0.00  1.00
145  0.00  0.00  1.00
146  0.00  0.01  0.99
147  0.00  0.00  1.00
148  0.00  0.00  1.00
149  0.00  0.02  0.98

      MAHALANOBIS:
     0       1       2
0  0.00    89.86  179.38
1  89.86   0.00   17.20
2  179.38  17.20  0.00

nrmiss = 0
```

DiscriminantAnalysis.Discrimination Enumeration

```
public enumeration Imsl.Stat.DiscriminantAnalysis.Discrimination
```

Discrimination methods.

Fields

Linear

```
public Imsl.Stat.DiscriminantAnalysis.Discrimination Linear
```

Description

Indicates a linear discrimination method.

Quadratic

```
public Imsl.Stat.DiscriminantAnalysis.Discrimination Quadratic
```

Description

Indicates a quadratic discrimination method.

DiscriminantAnalysis.CovarianceMatrix Enumeration

```
public enumeration Imsl.Stat.DiscriminantAnalysis.CovarianceMatrix
```

Covariance matrix type.

Fields

Pooled

```
public Imsl.Stat.DiscriminantAnalysis.CovarianceMatrix Pooled
```

Description

Indicates pooled covariances computed.

PooledGroup

```
public Imsl.Stat.DiscriminantAnalysis.CovarianceMatrix PooledGroup
```

Description

Indicates pooled, group covariances computed.

DiscriminantAnalysis.Classification Enumeration

```
public enumeration Imsl.Stat.DiscriminantAnalysis.Classification
```

Classification method.

Fields

LeaveOutOne

```
public Imsl.Stat.DiscriminantAnalysis.Classification LeaveOutOne
```

Description

Indicates leave-out-one as the classification method.

Reclassification

```
public Imsl.Stat.DiscriminantAnalysis.Classification Reclassification
```

Description

Indicates reclassification as the classification method.

DiscriminantAnalysis.PriorProbabilities Enumeration

```
public enumeration Imsl.Stat.DiscriminantAnalysis.PriorProbabilities
```

Prior probabilities type.

Fields

Equal

```
public Imsl.Stat.DiscriminantAnalysis.PriorProbabilities Equal
```

Description

Indicates prior probability type is to be prior equal.

Proportional

```
public Imsl.Stat.DiscriminantAnalysis.PriorProbabilities Proportional
```

Description

Indicates prior probability type is to be prior proportional.

Chapter 20: Survival and Reliability Analysis

Types

Usage Notes

Survival Analysis

The functions described in this chapter have primary application in the areas of reliability and life testing, but they may find application in any situation in which analysis of binomial events over time is of interest. Kalbfleisch and Prentice (1980), Elandt-Johnson and Johnson (1980), Lee (1980), Gross and Clark (1975), Lawless (1982), and Chiang (1968) and Tanner and Wong (1984) are references for discussing the models and methods desribed in this chapter. Function KaplanMeierEstimates produces Kaplan-Meier (product-limit) estimates of the survival distribution in a single population. Function ProportionalHazards computes the parameter estimates in a proportional hazards model. Function SurvivalGLM fits any of several generalized linear models for survival data, and SurvivalEstimates computes estimates of survival probabilities based upon the same models. Function LifeTables computes and (optionally) prints an actuarial table based either upon a cohort followed over time or a cross-section of a population.

KaplanMeierECDF Class

```
public class Imsl.Stat.KaplanMeierECDF
```

Computes the Kaplan-Meier reliability function estimates or the CDF based on failure data that may be multi-censored.

The Kaplan-Meier (K-M) Product Limit procedure provides simple estimates of the reliability function or the CDF based on failure data that may be multi-censored. No underlying probability model is assumed; K-M estimation is an empirical (non-parametric) procedure. Exact times of failure are required.

Consider a situation in which we are reliability testing n (non-repairable) units taken randomly from a population. We are investigating the population to determine if its failure rate is acceptable. In the typical test scenario, we have a fixed time T to run the units to see if they survive or fail. The data obtained are called Censored Type I data.

During the T hours of test we observe r failures (where r can be any number from 0 to n). The failure times are $t_1, t_2, \ldots, t_r$, and there are $(n-r)$ units that survived the entire T-hour test without failing. Note that T is fixed in advance, and r is an output of the testing, since we don't know how many failures will occur until the test is run. Note that we assume the exact times of failure are recorded when they occur.

This type of data is also called "right censored" data since the times of failure to the right (i.e., larger than T) are missing. The steps for calculating K-M estimates are the following:

1. Order the actual failure times from t_1 through t_r, where there are r failures
2. Corresponding to each t_i, associate the number n_i with n_i = the number of operating units just before the ith failure occurred at time t_i
3. First estimate the survival $R(t_1) = (n_1 - 1)/n_1$
4. Estimate each ensuing survival $R(t_i) = R(t_{i-1})(n_i - 1)/n_1$, $i > 1$
5. Estimate the CDF $F(t_i) = 1 - R(t_i)$, $i = 1, 2, \ldots$

Note that non-failed units taken off testing (i.e., right-censored) only count up to the last actual failure time before they were removed. They are included in the n_i counts up to and including that failure time, but not after.

Property

NumberOfPoints

```
virtual public int NumberOfPoints {get; }
```

Description

The number of points in the empirical CDF.

Property Value

An `int` containing the number of points in the empirical CDF.

Exception

`System.InvalidOperationException` is thrown if the CDF has not been evaluated.

Constructor

KaplanMeierECDF

```
public KaplanMeierECDF(double[] t)
```

Description

Constructor for `KaplanMeierECDF`.

Parameter

`t` – A `double` array containing the failure times.

Methods

EvaluateCDF

```
virtual public double[] EvaluateCDF()
```

Description

Computes the empirical CDF and returns the CDF values up to, but not including the time values returned by `GetTimes`.

Returns

A `double` array of CDF values.

GetTimes

```
virtual public double[] GetTimes()
```

Description

Retrieves the time values where the step function CDF jumps to a greater value.

Returns

A `double` array of time values.

Remarks

The returned array has right-censored values of `t` removed.

Exception

`System.InvalidOperationException` is thrown if the CDF has not been evaluated.

SetCensor

```
virtual public void SetCensor(int[] censor)
```

Description

Set flags to note right-censoring.

Parameter

`censor` – An `int` array of 0 or 1 flags to note right-censoring. Values of 0 = continue to use datum and 1 = remove datum.

Default: No data is right-censored.

SetFrequency

```
virtual public void SetFrequency(int[] freq)
```

Description

Sets the frequency for each entry in `t`.

Parameter

`freq` – An `int` array containing the repeat count for each entry in `t`.

Default: A frequency of 1 is used for each entry in `t` .

Example: Kaplan Meier Empirical CDF

This example illustrates the K-M procedure. Assume 20 units are on life test and 6 failures occur at the following times: 10, 32, 56, 98, 122, and 181 hours. There were 4 working units removed from the test for other experiments at the following times: 50, 100, 125, and 150 hours. The remaining 10 working units were removed from the test at 200 hours. The K-M estimates for this life test are:

R(10) = 19/20
R(32) = 19/20 x 18/19
R(56) = 19/20 x 18/19 x 16/17
R(98) = 19/20 x 18/19 x 16/17 x 15/16
R(122) = 19/20 x 18/19 x 16/17 x 15/16 x 13/14
R(181) = 19/20 x 18/19 x 16/17 x 15/16 x 13/14 x 10/11

```
using System;
using Imsl.Stat;
using Imsl.Math;

public class KaplanMeierECDFEx1
{

    public static void  Main(String[] args)
    {
        double[] y = {10.0, 32.0, 56.0, 98.0,
                      122.0, 181.0, 50.0, 100.0,
                      125.0, 150.0, 200.0
                    };
        int[] freq = {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 10};
        int[] censor = {0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1};

        KaplanMeierECDF km = new KaplanMeierECDF(y);
        km.SetFrequency(freq);
        km.SetCensor(censor);
        double[] fx = km.EvaluateCDF();
```

```
        int ntimes = km.NumberOfPoints;
        Console.Out.WriteLine("Number of points = " + ntimes);

        double[] x = km.GetTimes();
        PrintMatrix p =
         new PrintMatrix("CDF = 1 - survival of life test subjects");
        p.Print(fx);
        p.SetTitle("Times of change in CDF");
        p.Print(x);
    }
}
```

Output

```
Number of points = 6
CDF = 1 - survival of life test subjects
           0
0  0.05
1  0.1
2  0.152941176470588
3  0.205882352941177
4  0.262605042016807
5  0.329640947288006

Times of change in CDF
    0
0   10
1   32
2   56
3   98
4  122
5  181
```

KaplanMeierEstimates Class

`public class Imsl.Stat.KaplanMeierEstimates`

Computes Kaplan-Meier (or product-limit) estimates of survival probabilities for a sample of failure times that possibly contain right consoring.

Class `KaplanMeierEstimates` computes Kaplan-Meier (or product-limit) estimates of survival probabilities for a sample of failure times that can be right censored or exact times. A survival probability $S(t)$ is defined as 1 - $F(t)$, where $F(t)$ is the cumulative distribution function of the failure times t. Greenwood's estimate of the standard errors of the survival probability estimates are also computed. (See Kalbfleisch and Prentice, 1980, pages 13 and 14.)

Let (t_i, δ_i), for $i = 1,..., n$ denote the failure censoring times and the censoring codes for the n observations in a single sample. Here, $t_i = x_{i-1,responseIndex}$ is a failure time if δ_i is 0, where

$\delta_i = x_{i-1,censorIndex}$. Also, t_i is a right censoring time if δ_i is 1. Rows in `x` containing values other than 0 or 1 for δ_i are ignored. Let the number of observations in the sample that have not failed by time $s_{(t)}$ be denoted by $n_{(t)}$, where $s_{(t)}$ is an ordered (from smallest to largest) listing of the distinct failure times (censoring times are omitted). Then the Kaplan-Meier estimate of the survival probabilities is a step function, which in the interval from $s_{(i)}$ to $s_{(i+1)}$ (including the lower endpoint) is given by

$$\hat{S}(t) = \prod_{j=1}^{i} \left(\frac{n_{(j)} - d_{(j)}}{n_{(j)}} \right)$$

where $d_{(j)}$ denotes the number of failures occurring at time $s_{(j)}$, and $n_{(j)}$ is the number of observations that have not failed prior to $s_{(j)}$.

Note that one row of `x` may correspond to more than one failed (or censored) observation when the frequency option is in effect (see `FrequencyColumn`). The Kaplan-Meier estimate of the survival probability prior to time $s_{(1)}$ is 1.0, while the Kaplan-Meier estimate of the survival probability after the last failure time is not defined.

Greenwood's estimate of the variance of

$$\hat{S}(t)$$

in the interval from $s_{(i)}$ to $s_{(i+1)}$ is given as

$$\text{est.var}(\hat{S}(t)) = \hat{S}^2(t) \sum_{j=1}^{i} \frac{d_{(j)}}{n_{(j)}(n_{(j)} - d_{(j)})}$$

`KaplanMeierEstimates` computes the single sample estimates of the survival probabilities for all samples of data included in `x` during a single call. This is accomplished through the stratum column of `x`, which if present, must contain a distinct code for each sample of observations (see `StratumColumn`). If a stratum column is not specified, there is no grouping , and all observations are assumed to come from the same sample.

When failures and right-censored observations are tied and the data are to be sorted by `KaplanMeierEstimates` (`Sorted=true` is not used), `KaplanMeierEstimates` assumes that the time of censoring for the tied-censored observations is immediately after the tied failure (within the same sample). When `Sorted=true` is used, the data are assumed to be sorted from smallest to largest according to the response time column of `x` within each stratum (see `ResponseColumn`). Furthermore, a small increment of time is assumed (theoretically) to elapse between the failed and censored observations that are tied (in the same sample). Thus, when `Sorted=true` is used, the user must sort all of the data in `x` from smallest to largest according to the response time column (and the stratum column, if set). By appropriate sorting of the observations, the user can handle censored and failed observations that are tied in any manner desired.

Properties

CensorColumn

```
virtual public int CensorColumn {get; set; }
```

Description

The column index of `x` containing the optional censoring code for each observation.

Property Value

An `int` specifying the column index of `x` containing the optional censoring code for each observation.

Default: It is assumed that there is no censor code column in `x`. All observations are assumed to be exact failure times.

Remarks

If `x[i,CensorColumn]` equals 0, the failure time `x[i,ResponseColumn]` is treated as an exact time of failure. Otherwise, it is treated as right-censored time.

FrequencyColumn

```
virtual public int FrequencyColumn {get; set; }
```

Description

The column index of `x` containing the frequency of response for each observation.

Property Value

An `int` specifying the column index of `x` containing the frequency of response for each observation.

Default: It is assumed that there is no frequency response column recorded in `x`. Each observation in the data array is assumed to be for a single failure.

NumberOfRowsMissing

```
virtual public int NumberOfRowsMissing {get; }
```

Description

The number of rows of data in `x` that contain missing values in one or more specific columns of `x`.

Property Value

An `int` scalar representing the number of rows of data in `x` that contain missing values in one or more specific columns of `x`.

ResponseColumn

```
virtual public int ResponseColumn {get; set; }
```

Description

The column index of `x` containing the response time for each observation.

Property Value

An `int` specifying the column index of `x` containing the response time for each observation.

Default: `ResponseColumn` = 0.

Remarks

The interpretation of these times as either right-consored or exact failure times depends on the setting of the censor codes in the censor code column. See property `CensorColumn`.

Sorted

```
virtual public bool Sorted {get; set; }
```

Description

The `boolean` which indicates that the column of response times in `x` are already sorted.

Property Value

A `boolean` indicating whether or not column `ResponseColumn` of `x` is already sorted.

Default: It is assumed that column `ResponseColumn` of `x` is not sorted, so a detached sort is performed.

Remarks

`Sorted = true` indicates that column `ResponseColumn` of `x` is already sorted. Otherwise, a detached sort is performed prior to analysis. If sorting is performed, all censored individuals are assumed to follow tied failures.

StratumColumn

```
virtual public int StratumColumn {get; set; }
```

Description

The column index of `x` containing the stratum number for each observation.

Property Value

An `int` specifying the column index of `x` containing the stratum number for each observation.

Default: It is assumed that there is no stratum number column recorded in `x`. The data is assumed to come from one statum.

Remarks

Column `StratumColumn` of `x` contains a unique value for each stratum in the data. Kaplan-Meier estimates are computed within each stratum.

Constructor

KaplanMeierEstimates

```
public KaplanMeierEstimates(double[,] x)
```

Description

Constructor for `KaplanMeierEstimates`.

Parameter

`x` – A `double` matrix containing the data, including optional data.

Default: It is assumed the response times are in column 0.

Methods

GetGroupTotal

```
virtual public int GetGroupTotal(double groupValue)
```

Description

Returns the total number in the group for the specified group value.

Parameter

`groupValue` – A `double` specifying the group value.

Returns

An `int` representing the total number in the group which has value `groupValue`.

GetLogLikelihood

```
virtual public double GetLogLikelihood(double groupValue)
```

Description

Returns the Kaplan-Meier log-likelihood of the group with the specified group value.

Parameter

`groupValue` – A `double` specifying the group value.

Returns

A `double` representing the Kaplan-Meier log-likelihood of the group which has value `groupValue`.

Remarks

The Kaplan-Meier log-likelihood is computed as:

$$\ell = \sum_j d_{(j)} \ln d_{(j)} + (n_{(j)} - d_{(j)}) \ln(n_{(j)} - d_{(j)}) - n_{(j)} \ln n_{(j)}$$

where the sum is with respect to the distinct failure times $s_{(j)}$.

GetNumberAtRisk

```
virtual public int[] GetNumberAtRisk()
```

Description

Returns the number of individuals at risk at each failure point.

Returns

An `int` array containing the number of individuals at risk at each failure point.

GetNumberOfFailures

```
virtual public int[] GetNumberOfFailures()
```

Description

Returns the number of failures which occurred at each failure point.

Returns

An `int` array containing the number of failures which occurred at each failure point.

GetStandardErrors

```
virtual public double[] GetStandardErrors()
```

Description

Returns Greenwood's estimated standard errors.

Returns

A `double` array containing Greenwood's estimate of the standard errors for the survival probabilities.

GetSurvivalProbabilities

```
public double[] GetSurvivalProbabilities()
```

Description

Returns the estimated survival probabilities.

Returns

A `double` array containing the estimated survival probabilities.

GetTotalNumberOfFailures

```
virtual public int GetTotalNumberOfFailures(double groupValue)
```

Description

Returns the total number failing in the group for the specified group value.

Parameter

`groupValue` – A `double` specifying the group value.

Returns

An `int` representing the total number failing in the group which has value `groupValue`.

Example : KaplanMeierEstimates

The following example is taken from Kalbfleisch and Prentice (1980, page 1). The first column in `x` contains the death/censoring times for rats suffering from vaginal cancer. The second column contains information as to which of two forms of treatment were provided, while the third column contains the censoring code. Finally, the fourth column contains the frequency of each observation. The product-limit estimates of the survival probabilities are computed for both groups along with their standard error estimates. Tables containing these values along with other statistics are printed.

```
using System;
using Imsl.Stat;

public class KaplanMeierEstimatesEx1
{
    public static void  Main(String[] args)
    {
```

```
double[,] x = new double[,]{
    {143, 5, 0, 1}, {164, 5, 0, 1}, {188, 5, 0, 2}, {190, 5, 0, 1},
    {192, 5, 0, 1}, {206, 5, 0, 1}, {209, 5, 0, 1}, {213, 5, 0, 1},
    {216, 5, 0, 1}, {220, 5, 0, 1}, {227, 5, 0, 1}, {230, 5, 0, 1},
    {234, 5, 0, 1}, {246, 5, 0, 1}, {265, 5, 0, 1}, {304, 5, 0, 1},
    {216, 5, 1, 1}, {244, 5, 1, 1}, {142, 7, 0, 1}, {156, 7, 0, 1},
    {163, 7, 0, 1}, {198, 7, 0, 1}, {205, 7, 0, 1}, {232, 7, 0, 2},
    {233, 7, 0, 4}, {239, 7, 0, 1}, {240, 7, 0, 1}, {261, 7, 0, 1},
    {280, 7, 0, 2}, {296, 7, 0, 2}, {323, 7, 0, 1}, {204, 7, 1, 1},
    {344, 7, 1, 1}};
int nobs = x.GetLength(0), censorIndex = 2, frequencyIndex = 3;
int stratumIndex = 1, responseIndex = 0;
int i, groupValue;
int[] atRisk = new int[nobs];
int[] nFailing = new int[nobs];
double[] prob = new double[nobs];
double[] se = new double[nobs];

// Get Kaplan-Meir Estimates

KaplanMeierEstimates km = new KaplanMeierEstimates(x);
km.CensorColumn = censorIndex;
km.FrequencyColumn = frequencyIndex;
km.StratumColumn = stratumIndex;
atRisk = km.GetNumberAtRisk();
nFailing = km.GetNumberOfFailures();
prob = km.GetSurvivalProbabilities();
se = km.GetStandardErrors();

// Print Results
i = 0;
while (i < nobs)
{
    groupValue = (int) x[i,stratumIndex];
    Console.Out.WriteLine("\n       Kaplan-Meier Survival "+
        "Probabilities");
    Console.Out.WriteLine("               For Group Value = "+
        "{0,5:0.####}", groupValue);
    Console.Out.WriteLine("\nNumber     Number" +
             "                Survival     Estimated");
    Console.Out.WriteLine("at risk    Failing     Time" +
        "     Probability   Std. Error");
    while (i < nobs && ((int) x[i,stratumIndex] == groupValue))
    {
        if ((int) x[i,censorIndex] != 1)
        {
            Console.Out.WriteLine("{0,5:0.####}    "+
                "{1,5:0.####}        {2,5:0.####}       {3,5:0.####}"+
                "       {4,5:0.####}",atRisk[i],nFailing[i],
                ((int) x[i,responseIndex]),(prob[i]),(se[i]));
        }
        i++;
    }
    Console.Out.WriteLine("\nTotal number in group = "+
        "{0,5:0.####}",km.GetGroupTotal(groupValue));
    Console.Out.WriteLine("Total number failing = "+
```

```
                "{0,5:0.####}",km.GetTotalNumberOfFailures(groupValue));
            Console.Out.WriteLine("Product Limit likelihood = "+
                "{0,5:0.####}",km.GetLogLikelihood(groupValue));
        }
        Console.Out.WriteLine(
         "\n\n\nThe number of rows of x with missing values is {0,5:0.####}",
         km.NumberOfRowsMissing);
    }
}
```

Output

```
       Kaplan-Meier Survival Probabilities
              For Group Value =     5

Number     Number                Survival     Estimated
at risk    Failing     Time     Probability   Std. Error
   19        1          143       0.9474       0.0512
   18        1          164       0.8947       0.0704
   17        2          188       0.7895       0.0935
   15        1          190       0.7368       0.101
   14        1          192       0.6842       0.1066
   13        1          206       0.6316       0.1107
   12        1          209       0.5789       0.1133
   11        1          213       0.5263       0.1145
   10        1          216       0.4737       0.1145
    8        1          220       0.4145       0.1145
    7        1          227       0.3553       0.1124
    6        1          230       0.2961       0.1082
    5        1          234       0.2368       0.1015
    3        1          246       0.1579       0.0934
    2        1          265       0.0789       0.0728
    1        1          304           0         NaN

Total number in group =    19
Total number failing =    17
Product Limit likelihood = -49.1692

       Kaplan-Meier Survival Probabilities
              For Group Value =     7

Number     Number                Survival     Estimated
at risk    Failing     Time     Probability   Std. Error
   21        1          142       0.9524       0.0465
   20        1          156       0.9048       0.0641
   19        1          163       0.8571       0.0764
   18        1          198       0.8095       0.0857
   16        1          205       0.7589       0.0941
   15        2          232       0.6577       0.1053
   13        4          233       0.4554       0.1114
    9        1          239       0.4048       0.1099
    8        1          240       0.3542       0.1072
    7        1          261       0.3036       0.1031
    6        2          280       0.2024       0.0902
    4        2          296       0.1012       0.0678
```

```
    2        1          323       0.0506       0.0493

Total number in group =    21
Total number failing =    19
Product Limit likelihood = -50.4277


The number of rows of x with missing values is     0
```

ProportionalHazards Class

`public class Imsl.Stat.ProportionalHazards`

Analyzes survival and reliability data using Cox's proportional hazards model.

Class `ProportionalHazards` computes parameter estimates and other statistics in Proportional Hazards Generalized Linear Models. These models were first proposed by Cox (1972). Two methods for handling ties are allowed. Time-dependent covariates are not allowed. The user is referred to Cox and Oakes (1984), Kalbfleisch and Prentice (1980), Elandt-Johnson and Johnson (1980), Lee (1980), or Lawless (1982), among other texts, for a thorough discussion of the Cox proportional hazards model.

Let $\lambda(t, z_i)$ represent the hazard rate at time t for observation number i with covariables contained as elements of row vector z_i. The basic assumption in the proportional hazards model (the proportionality assumption) is that the hazard rate can be written as a product of a time varying function $\lambda_0(t)$, which depends only on time, and a function $f(z_i)$, which depends only on the covariable values. The function $f(z_i)$ used in `ProportionalHazards` is given as $f(z_i) = \exp(w_i + \beta z_i)$ where w_i is a fixed constant assigned to the observation, and b is a vector of coefficients to be estimated. With this function one obtains a hazard rate $\lambda(t, z_i) = \lambda_0(t)\exp(w_i + \beta z_i)$. The form of $\lambda_0(t)$ is not important in proportional hazards models.

The constants w_i may be known theoretically. For example, the hazard rate may be proportional to a known length or area, and the w_i can then be determined from this known length or area. Alternatively, the w_i may be used to fix a subset of the coefficients β (say, β_1) at specified values. When w_i is used in this way, constants $w_i = \beta_1 z_{1i}$ are used, while the remaining coefficients in β are free to vary in the optimization algorithm. Constants are defined as 0.0 by default. If user-specified constants are desired, use the `ConstantColumn` property to specify which column contains the constant.

With this definition of $\lambda(t, z_i)$, the usual partial (or marginal, see Kalbfleisch and Prentice (1980)) likelihood becomes

$$L = \prod_{i=1}^{n_d} \frac{\exp(w_i + \beta z_i)}{\sum_{j \in R(t_i)} \exp(w_j + \beta z_j)}$$

where $R(t_i)$ denotes the set of indices of observations that have not yet failed at time t_i (the risk set), t_i denotes the time of failure for the i-th observation, n_d is the total number of observations that fail.

Right-censored observations (i.e., observations that are known to have survived to time t_i, but for which no time of failure is known) are incorporated into the likelihood through the risk set $R(t_i)$. Such observations never appear in the numerator of the likelihood. When `TiesOption` is set to `BreslowsApproximate` (the default), all observations that are censored at time t_i are not included in $R(t_i)$, while all observations that fail at time t_i are included in $R(t_i)$.

If it can be assumed that the dependence of the hazard rate upon the covariate values remains the same from stratum to stratum, while the time-dependent term, $\lambda_0(t)$, may be different in different strata, then `ProportionalHazards` allows the incorporation of strata into the likelihood as follows. Let k index the m strata (set with `StratumColumn`). Then, the likelihood is given by

$$L_S = \prod_{k=1}^{m} \left[\prod_{i=1}^{n_k} \frac{\exp(w_{ki} + \beta z_{ki})}{\sum_{j \in R(t_{ki})} \exp(w_{kj} + \beta z_{kj})} \right]$$

In `ProportionalHazards`, the log of the likelihood is maximized with respect to the coefficients β. A quasi-Newton algorithm approximating the Hessian via the matrix of sums of squares and cross products of the first partial derivatives is used in the initial iterations. When the change in the log-likelihood from one iteration to the next is less than 100 times the convergence tolerance, Newton-Raphson iteration is used. If, during any iteration, the initial step does not lead to an increase in the log-likelihood, then step halving is employed to find a step that will increase the log-likelihood.

Once the maximum likelihood estimates have been computed, the algorithm computes estimates of a probability associated with each failure. Within stratum k, an estimate of the probability that the i-th observation fails at time t_i given the risk set $R(t_{ki})$ is given by

$$p_{ki} = \frac{\exp(w_{ki} + \beta z_{ki})}{\sum_{j \in R(t_{ki})} \exp(w_{kj} + \beta z_{kj})}$$

A diagnostic "influence" or "leverage" statistic is computed for each noncensored observation as:

$$l_{ki} = -g'_{ki} H_s^{-1} g'_{ki}$$

where H_s is the matrix of second partial derivatives of the log-likelihood, and

$$g'_{ki}$$

is computed as:

$$g'_{ki} = z_{ki} - \frac{z_{ki} \exp(w_{ki} + \beta z_{ki})}{\sum_{j \in R(t_{ki})} \exp(w_{kj} + \beta z_{kj})}$$

Influence statistics are not computed for censored observations.

A "residual" is computed for each of the input observations according to methods given in Cox and Oakes (1984, page 108). Residuals are computed as

$$r_{ki} = \exp(w_{ki} + \hat{\beta} z_{ki}) \sum_{j \in R(t_{ki})} \frac{d_{kj}}{\sum_{l \in R(t_{kj})} \exp(w_{kl} + \hat{\beta} z_{kl})}$$

where d_{kj} is the number of tied failures in group k at time t_{kj}. Assuming that the proportional hazards assumption holds, the residuals should approximate a random sample (with censoring) from the unit exponential distribution. By subtracting the expected values, centered residuals can be obtained. (The j-th expected order statistic from the unit exponential with censoring is given as

$$e_j = \sum_{l \leq j} \frac{1}{h - l + 1}$$

where h is the sample size, and censored observations are not included in the summation.)

An estimate of the cumulative baseline hazard within group k is given as

$$\hat{H}_{k0}(t_{ik}) = \sum_{t_{kj} \leq t_{ki}} \frac{d_{kj}}{\sum_{l \in R(t_{kj})} \exp(w_{kl} + \hat{\beta} z_{kl})}$$

The observation proportionality constant is computed as

$$\exp(w_{ki} + \hat{\beta} z_{ki})$$

Note that one can use logging to generate intermediate output for this class. Accumulated levels of detail correspond to Fine, Finer, and Finest logging levels with Fine yielding the smallest amount of information and Finest yielding the most. The levels of output yield the following:

Level	Output
Fine	Logging is enabled, but observational statistics are not printed.
Finer	All output statistics are printed.
Finest	Tracks progress through internal methods.

Properties

CensorColumn

```
virtual public int CensorColumn {get; set; }
```

Description

The column index of `x` containing the optional censoring code for each observation.

Property Value

An `int` specifying the column index of `x` containing the optional censoring code for each observation.

Default: It is assumed that there is no censor code column in `x` and all observations are assumed to be exact failure times.

Remarks

If `x[i][CensorColumn]` equals 0, the failure time `x[i][CensorColumn]` is treated as an exact time of failure. Otherwise, it is treated as right-censored time.

ConstantColumn

```
virtual public int ConstantColumn {get; set; }
```

Description

The column index of `x` containing the constant to be added to the linear response.

Property Value

An `int` specifying the column index of `x` containing the constant to be added to the linear response.

Default: It is assumed that $w_i = 0$ for all observations.

Remarks

The linear response is taken to be $w_i + z_i\hat{\beta}$ where w_i is the observation constant, z_i is the observation design row vector, and $\hat{\beta}$ is the vector of estimated parameters. The "fixed" constant allows one to test hypotheses about parameters via the log-likelihoods.

ConvergenceTol

```
virtual public double ConvergenceTol {get; set; }
```

Description

The convergence tolerance.

Property Value

A `double` specifying the convergence tolerance.

Default: `ConvergenceTol` is 0.0001.

Remarks

Convergence is assumed when the relative change in the maximum likelihood from one iteration to the next is less than `ConvergenceTol`. If `ConvergenceTol` is zero, `ConvergenceTol` = 0.0001 is assumed.

FrequencyColumn

```
virtual public int FrequencyColumn {get; set; }
```

Description

The column index of `x` containing the frequency of response for each observation.

Property Value

An `int` specifying the column index of `x` containing the frequency of response for each observation.

Default: It is assumed that there is no frequency response column recorded in `x`. Each observation in the data array is assumed to be for a single failure; that is, the frequency of response for each observation is 1.

HessianOption

```
virtual public bool HessianOption {get; set; }
```

Description

The `boolean` used to indicate whether or not to compute the Hessian and gradient at the initial estimates.

Property Value

A `boolean` specifying whether or not the Hessian and gradient are to be computed at the initial estimates.

Default: The Hessian and gradient are not computed at the initial estimates.

Remarks

`HessianOption` equal to `true` indicates that the Hessian and gradient are to be computed. If `HessianOption` is `true` the user must set the initial estimates via the `SetInitialEstimates` method.

Logger

```
virtual public Imsl.Logger Logger {get; set; }
```

Description

Returns the logger object and enables logging.

Property Value

A `Imsl.Logger` object, if present, or `null`.

MaxClass

```
virtual public int MaxClass {get; set; }
```

Description

The upper bound used on the sum of the number of distinct values found among the classification variables in `x`.

Property Value

An `int` representing the upper bound used on the sum of the number of distinct values found among the classification variables in `x`.

Default: `MaxClass` is the number of observations in `x`.

Remarks

Consider a model consisting of two class variables, one with the values {1, 2, 3, 4} and a second with the values {0, 1}, then the total number of different classification values is 4 + 2 = 6, and `MaxClass` $>=$ 6.

MaximumLikelihood

```
virtual public double MaximumLikelihood {get; }
```

Description

Returns the maximized log-likelihood.

Property Value

A `double` representing the maximized log-likelihood.

Remarks

The log-likelihood is fully described in the `ProportionalHazards` class description.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through `MaxClass` has been exceeded.

MaxIterations

```
virtual public int MaxIterations {get; set; }
```

Description

The maximum number of iterations allowed.

Property Value

An `int` specifying the maximum number of iterations allowed.

Default: `MaxIterations` is 30.

NumberOfCoefficients

```
virtual public int NumberOfCoefficients {get; }
```

Description

Returns the number of estimated coefficients in the model.

Property Value

An `int` scalar representing the number of estimated coefficients in the model.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

NumberRowsMissing

```
virtual public int NumberRowsMissing {get; }
```

Description

The number of rows of data in `x` that contain missing values in one or more columns of `x`.

Property Value

An `int` scalar representing the number of rows of data in `x` that contain missing values in one or more columns of `x`.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

ResponseColumn

```
virtual public int ResponseColumn {get; set; }
```

Description

The column index of `x` containing the response time for each observation.

Property Value

An `int` specifying the column index of `x` containing the response time for each observation.

Default: `ResponseColumn = 0`.

Remarks

For point observations, `x[i][ResponseColumn]` contains the time of the *i*-th event. For right-censored observations, `x[i][ResponseColumn]` contains the right-censoring time. Note that because `ProportionalHazards` only uses the order of the events, negative "times" are allowed.

StratumColumn

```
virtual public int StratumColumn {get; set; }
```

Description

The column index of `x` containing the stratum number for each observation.

Property Value

An `int` specifying the column index of `x` containing the stratum number for each observation.

Default: It is assumed that all obvservations are from one stratum.

Remarks

Column `StratumColumn` of `x` contains a unique value for each stratum in the data. The risk set for an observation is determined by its stratum.

StratumRatio

```
virtual public double StratumRatio {get; set; }
```

Description

The ratio at which a stratum is split into two strata.

Property Value

A `double` specifying the ratio at which a stratum is split into two strata.

Default: `StratumRatio` = 1000.

Remarks

Let

$$r_k = \exp(z_k\hat{\beta} + w_k)$$

be the observation proportionality constant, where z_k is the design row vector for the k-th observation and w_k is the optional fixed parameter specified by $x_{k,\text{ConstantColumn}}$. Let r_{min} be the minimum value r_k in a stratum, where, for failed observations, the minimum is over all times less than or equal to the time of occurrence of the k-th observation. Let r_{max} be the maximum value of r_k for the remaining observations in the group. Then, if $r_{min} > \texttt{value} * r_{max}$, the observations in the group are divided into two groups at k. Set `StratumRatio = -1` if no division into strata is to be made.

TiesOption

```
virtual public Imsl.Stat.ProportionalHazards.TieHandling TiesOption {get; set;
}
```

Description

The method used for handling ties.

Property Value

A `ProportionalHazards.TieHandling` specifying the method to be used in handling ties.

Default: `TiesOption = ProportionalHazards.TieHandling.BreslowsApproximate`.

Remarks

`TiesOption` is one of the values in the following table:

value	Method
`BreslowsApproximate`	Breslow's approximate method. This is the default.
`SortedAsPerObservations`	Failures are assumed to occur in the same order as the observations input in `x`. The observations in `x` must be sorted from largest to smallest failure time within each stratum, and grouped by stratum. All observations are treated as if their failure/censoring times were distinct when computing the log-likelihood.

Constructor

ProportionalHazards

```
public ProportionalHazards(double[,] x, int[] nVarEffects, int[] indEffects)
```

Description

Constructor for `ProportionalHazards`.

Parameters

`x` – A `double` matrix containing the data, including optional data.

`nVarEffects` – An `int` array containing the number of variables associated with each effect in the model.

`indEffects` – An `int` array containing the column numbers of `x` associated with each effect. The first `nVarEffects[0]` elements of `indEffects` contain the column numbers of `x` for the variables in the first effect. The next `nVarEffects[1]` elements of `indEffects` contain the column numbers of `x` for the variables in the second effect, etc.

Methods

GetCaseStatistics

```
virtual public double[,] GetCaseStatistics()
```

Description

Returns the case statistics for each observation.

Returns

A `double` matrix containing the case statistics.

Remarks

There is one row for each observation, and the columns of the returned matrix contain the following:

Column	Statistic
0	Estimated survival probability at the observation time.
1	Estimated observation influence or leverage.
2	A residual estimate.
3	Estimated cumulative baseline hazard rate.
4	Observation proportionality constant.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetClassValueCounts

```
virtual public int[] GetClassValueCounts()
```

Description

Returns the number of values taken by each classification variable.

Returns

An `int` array containing the number of values taken by each classification variable.

Remarks

The *i*-th element of the returned array is the number of distinct values taken by the *i*-th classification variable.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetClassValues

```
virtual public double[] GetClassValues()
```

Description

Returns the class values taken by each classification variable.

Returns

A `double` array containing the values taken by each classification variable.

Remarks

For description purposes, let `nclval = GetClassValueCounts()`. Then the first `nclval[0]` elements contain the values for the first classification variable, the next `nclval[1]` elements contain the values for the second classification variable, etc.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetGradient

```
virtual public double[] GetGradient()
```

Description

Returns the inverse of the Hessian times the gradient vector, computed at the initial estimates.

Returns

A `double` array containing the inverse of the Hessian times the gradient vector, computed at the initial estimates.

Remarks

Note that the `HessianOption` property must be set to `true` and the `SetInitialEstimates` method must be invoked prior to invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetHessian

```
virtual public double[,] GetHessian()
```

Description

Returns the inverse of the Hessian of the negative of the log-likelihood, computed at the initial estimates.

Returns

A `double` matrix containing the inverse of the Hessian of the negative of the log-likelihood, computed at the initial estimates.

Remarks

Note that the `HessianOption` property must set to `true` and the `SetInitialEstimates` method must be invoked prior to invoking this method. Otherwise, the method throws an `InvalidOperationException` exception.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetInitialEstimates

```
virtual public double[] GetInitialEstimates()
```

Description

Gets the initial parameter estimates.

Returns

A `double` array containing the initial parameter estimates.

GetLastUpdates

```
virtual public double[] GetLastUpdates()
```

Description

Gets the last parameter updates.

Returns

A `double` array containing the last parameter updates (excluding step halvings).

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetMeans

```
virtual public double[] GetMeans()
```

Description

Returns the means of the design variables.

Returns

A `double` array containing the means of the design variables.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetParameterStatistics

```
virtual public double[,] GetParameterStatistics()
```

Description

Returns the parameter estimates and associated statistics.

Returns

A `double` matrix containing the parameter estimates and associated statistics.

Remarks

There is one row for each coefficient, and the columns of the returned matrix contain the following:

Column	Statistic
0	The coefficient estimate, $\hat{\beta}$
1	Estimated standard deviation of the estimated coefficient
2	Asymptotic normal score for testing that the coefficient is zero against the two-sided alternative
3	p-value associated with the normal score in column 2

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetStratumNumbers

```
virtual public int[] GetStratumNumbers()
```

Description

Returns the stratum number used for each observation.

Returns

An `int` array containing the stratum number used for each observation.

Remarks

If property `StratumRatio` is not -1.0, additional “strata” (other than those specified by the column of `x` set via the `StratumColumn` property) may be generated. The array also contains a record of the generated strata. See the `ProportionalHazards` class description for more detail.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

GetVarianceCovarianceMatrix

```
virtual public double[,] GetVarianceCovarianceMatrix()
```

Description

Returns the estimated asymptotic variance-covariance matrix of the parameters.

Returns

A `double` matrix containing the estimated asymptotic variance-covariance matrix of the parameters.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

SetClassVarColumns

```
virtual public void SetClassVarColumns(int[] classVarIndices)
```

Description

Sets the column indices of `x` that are the classification variables.

Parameter

`classVarIndices` – An `int` array containing the column numbers of `x` that are the classification variables.

Remarks

Default: It is assumed there are no classification variables.

SetInitialEstimates

```
virtual public void SetInitialEstimates(double[] initialCoef)
```

Description

Sets the initial parameter estimates.

Parameter

`initialCoef` – A `double` array containing the initial parameter estimates.

Remarks

Care should be taken to ensure that the supplied estimates for the model coefficients β correspond to the generated covariate vector z_{ki}.

Default: The initial parameter estimates are all 0.0.

Exception

`Imsl.Stat.ClassificationVariableLimitException` is thrown if the classification variable limit set by the user through the `MaxClass` property has been exceeded.

Example: ProportionalHazards

The following example is taken from Lawless (1982, page 287) and involves the survival of lung cancer patients based upon their initial tumor types and treatment type. In the example, the likelihood is maximized with no strata present in the data. This corresponds to Example 7.2.3 in Lawless (1982, page 367). The model is given as:

$$\ln(\lambda) = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \alpha_i + \gamma_j$$

where α_i and γ_j correspond to dummy variables generated from classification variables in columns 5 and 6 of x. Respectively, x_1 corresponds to column index 2, x_2 corresponds to column index 3, and x_3 corresponds to column index 4 of x. Column 0 of x contains the response and column 1 of x contains the censoring code. Logging is used to print output statistics.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using Imsl;

public class ProportionalHazardsEx1
{
    public static void  Main(string[] args)
    {
        double[,] x =
            {{411, 0, 7, 64, 5, 1, 0}, {126, 0, 6, 63, 9, 1, 0},
            {118, 0, 7, 65, 11, 1, 0}, {92, 0, 4, 69, 10, 1, 0},
            {8, 0, 4, 63, 58, 1, 0}, {25, 1, 7, 48, 9, 1, 0},
            {11, 0, 7, 48, 11, 1, 0}, {54, 0, 8, 63, 4, 2, 0},
            {153, 0, 6, 63, 14, 2, 0}, {16, 0, 3, 53, 4, 2, 0},
            {56, 0, 8, 43, 12, 2, 0}, {21, 0, 4, 55, 2, 2, 0},
            {287, 0, 6, 66, 25, 2, 0}, {10, 0, 4, 67, 23, 2, 0},
            {8, 0, 2, 61, 19, 3, 0}, {12, 0, 5, 63, 4, 3, 0},
            {177, 0, 5, 66, 16, 4, 0}, {12, 0, 4, 68, 12, 4, 0},
            {200, 0, 8, 41, 12, 4, 0}, {250, 0, 7, 53, 8, 4, 0},
            {100, 0, 6, 37, 13, 4, 0}, {999, 0, 9, 54, 12, 1, 1},
            {231, 1, 5, 52, 8, 1, 1}, {991, 0, 7, 50, 7, 1, 1},
            {1, 0, 2, 65, 21, 1, 1}, {201, 0, 8, 52, 28, 1, 1},
            {44, 0, 6, 70, 13, 1, 1}, {15, 0, 5, 40, 13, 1, 1},
            {103, 1, 7, 36, 22, 2, 1}, {2, 0, 4, 44, 36, 2, 1},
            {20, 0, 3, 54, 9, 2, 1}, {51, 0, 3, 59, 87, 2, 1},
            {18, 0, 4, 69, 5, 3, 1}, {90, 0, 6, 50, 22, 3, 1},
            {84, 0, 8, 62, 4, 3, 1}, {164, 0, 7, 68, 15, 4, 1},
            {19, 0, 3, 39, 4, 4, 1}, {43, 0, 6, 49, 11, 4, 1},
            {340, 0, 8, 64, 10, 4, 1}, {231, 0, 7, 67, 18, 4, 1}};
        int[] indef = {2, 3, 4, 5, 6};
        int[] nvef = {1, 1, 1, 1, 1};
        int[] indcl = {5, 6};
        int maxcl = 6, icen = 1;
        double ratio = 10000.0;

        ProportionalHazards ph = new ProportionalHazards(x, nvef, indef);
        ph.MaxClass = maxcl;
        ph.CensorColumn = icen;
        ph.SetClassVarColumns(indcl);
        ph.StratumRatio = ratio;

        // Level.FINER prints most output statistics

        Logger logger = ph.Logger;
        logger.LogLevel = Logger.Level.Finer;
```

```
        double[,] coef = ph.GetParameterStatistics();
        new PrintMatrix("\nFinal Coefficient Matrix").Print(coef);
    }
}
```

Output

```
ProportionalHazards: Initial Estimates
   0
0  0
1  0
2  0
3  0
4  0
5  0
6  0


ProportionalHazards: Method  Iteration  Step       Maximum scaled     Log
ProportionalHazards:                    Size        coef. update      likelihood
ProportionalHazards:   Q-N        0                              -102.400565515673
ProportionalHazards:   Q-N        1      1  0.503384047154466 -91.0439507780637
ProportionalHazards:   Q-N        2      1  0.578199557315753 -88.0680849909176
ProportionalHazards:   N-R        3      1  0.113104528935929 -87.9223344687152
ProportionalHazards:   N-R        4      1  0.0695795111007414 -87.887787543875
ProportionalHazards:   N-R        5      1  0.000814904865501336 -87.8877800993911
ProportionalHazards: Log-Likelihood = -87.8877800993911
ProportionalHazards:                        Coefficient Statistics
   Coefficient  Std. error  Asymptotic z-stat  Asymptotic p-value
0    -0.5846      0.1368         -4.2721             0
1    -0.0131      0.0206         -0.6342             0.526
2     0.0008      0.0118          0.0645             0.9486
3    -0.367       0.4848         -0.7572             0.449
4    -0.0077      0.5068         -0.0152             0.9878
5     1.1129      0.6331          1.758              0.0787
6     0.3797      0.4058          0.9357             0.3494


ProportionalHazards:                  Asymptotic Coefficient Covariance
     0        1        2        3        4        5        6
0  0.0187   0.0003   0.0003   0.0057   0.0097   0.0043   0.0021
1           0.0004   0       -0.0017  -0.0008  -0.0031  -0.0029
2                    0.0001   0.0008  -0.0018   0.0006   0.0017
3                             0.235    0.098    0.1184   0.0373
4                                      0.2568   0.1253  -0.0194
5                                               0.4008   0.0629
6                                                        0.1647


ProportionalHazards:                                Case Analysis
    Survival Prob.  Influence  Residual  Cumulative hazard  Prop. constant
 0      0.0022       0.0414     2.0531        6.1032           0.3364
 1      0.2988       0.1088     0.7409        1.2078           0.6134
 2      0.3424       0.1184     0.3576        1.0719           0.3336
 3      0.4336       0.1554     1.5272        0.8357           1.8274
```

```
 4      0.9555       0.5567     0.0933       0.0455          2.0499
 5      0.7365       NaN        0.1272       0.3058          0.4158
 6      0.9204       0.3729     0.0346       0.083           0.4164
 7      0.5876       0.2637     0.1446       0.5317          0.2719
 8      0.2577       0.1173     1.196        1.3561          0.882
 9      0.8457       0.1486     0.9656       0.1676          5.7608
10      0.5481       0.3133     0.2135       0.6012          0.3551
11      0.7365       0.2108     0.9551       0.3058          3.1232
12      0.0293       0.0602     3.018        3.5289          0.8552
13      0.9382       0.0935     0.173        0.0638          2.7135
14      0.9555       0.1595     1.3142       0.0455          28.8855
15      0.8854       0.2322     0.5864       0.1217          4.8164
16      0.1814       0.0918     2.6217       1.707           1.5358
17      0.8854       0.1869     0.3258       0.1217          2.6765
18      0.1414       0.2303     0.7187       1.9565          0.3673
19      0.0522       0.0943     1.6591       2.9529          0.5618
20      0.3899       0.2212     1.1745       0.9419          1.2469
21      0            0          1.7281       21.1049         0.0819
22      0.0806       NaN        2.1865       2.5177          0.8684
23      0.0001       0.0049     2.4603       8.8921          0.2767
24      0.9892       0.3072     0.0462       0.0108          4.2758
25      0.1074       0.1724     0.3406       2.2311          0.1527
26      0.664        0.2513     0.1573       0.4095          0.3841
27      0.8655       0.2215     0.1472       0.1444          1.0196
28      0.3899       NaN        0.4533       0.9419          0.4812
29      0.9781       0.2495     0.0561       0.0222          2.531
30      0.769        0.2556     1.0257       0.2627          3.9045
31      0.6291       0.3509     1.7991       0.4635          3.8817
32      0.8233       0.2598     1.0635       0.1944          5.4705
33      0.4739       0.26       1.6474       0.7468          2.2058
34      0.5104       0.3191     0.3886       0.6725          0.5779
35      0.2173       0.183      0.485        1.5267          0.3177
36      0.7979       0.2642     1.0764       0.2258          4.7675
37      0.7          0.16       0.2598       0.3567          0.7282
38      0.0094       0.2267     0.8668       4.6642          0.1858
39      0.0806       0.2045     0.8122       2.5177          0.3226


ProportionalHazards: Last Coefficient Update

0  -5.835E-8
1   1.401E-9
2  -8.597E-9
3  -2.822E-7
4  -4.566E-8
5   1.256E-7
6   1.058E-8


ProportionalHazards: Covariate Means

0  5.65
1  56.575
2  15.65
3  0.35
4  0.275
```

```
5  0.125
6  0.525


ProportionalHazards: Distinct Values For Each Class Variable
ProportionalHazards: Variable 0: 1 2 3 4
ProportionalHazards: Variable 1: 0 1
ProportionalHazards: Stratum Numbers For Each Observation

 0  1
 1  1
 2  1
 3  1
 4  1
 5  1
 6  1
 7  1
 8  1
 9  1
10  1
11  1
12  1
13  1
14  1
15  1
16  1
17  1
18  1
19  1
20  1
21  1
22  1
23  1
24  1
25  1
26  1
27  1
28  1
29  1
30  1
31  1
32  1
33  1
34  1
35  1
36  1
37  1
38  1
39  1


ProportionalHazards: Number of Missing Values = 0

Final Coefficient Matrix
             0                   1                   2                   3
0  -0.584594437935942     0.136840243992857   -4.27209438450326    1.93645575283785E-05
```

```
1  -0.0130519374088305    0.0205802738950756  -0.634196487149451   0.525952599824556
2   0.000761767774242064  0.0118179341006006   0.0644586243041712  0.948605051586236
3  -0.36704523412352      0.484770317924572   -0.757152863019617   0.448958286523091
4  -0.00772085680226222   0.506751409428954   -0.0152359848608269  0.987843913222082
5   1.11293966441439      0.633058877893231    1.75803500002743    0.0787415542394656
6   0.379709386883123     0.405801960576063    0.935701213331005   0.34942704574484
```

ProportionalHazards.TieHandling Enumeration

```
public enumeration Imsl.Stat.ProportionalHazards.TieHandling
```

Tie handling options.

Fields

BreslowsApproximate

```
public Imsl.Stat.ProportionalHazards.TieHandling BreslowsApproximate
```

Description

Indicates Breslows approximate method of handling ties.

SortedAsPerObservations

```
public Imsl.Stat.ProportionalHazards.TieHandling SortedAsPerObservations
```

Description

Indicates failures are assumed to occur in the same order as the observations input in `x`.

Remarks

The observations in `x` must be sorted from largest to smallest failure time within each stratum, and grouped by stratum. All observations are treated as if their failure/censoring times were distinct when computing the log-likelihood.

LifeTables Class

```
public class Imsl.Stat.LifeTables
```

Computes population (current) or cohort life tables based upon the observed population sizes at the middle (for population table) or the beginning (for cohort table) of some user specified age intervals. The number of deaths in each of these intervals must also be observed.

The probability of dying prior to the middle of the interval, given that death occurs somewhere in the interval, may also be specified. Often, however, this probability is taken to be 0.5. For a discussion of the probability models underlying the life table here, see the references.

Let t_i, for i = 0, 1, ..., t_n denote the time grid defining the n age intervals, and note that the length of the age intervals may vary. Following Gross and Clark (1975, page 24), let d_i denote the number of individuals dying in age interval i, where age interval i ends at time t_i. For population table, the death rate at the middle of the interval is given by $r_i = d_i/(M_i h_i)$, where M_i is the number of individuals alive at the middle of the interval, and $h_i = t_i - t_{i-1}$, $t_0 = 0$. The number of individuals alive at the beginning of the interval may be estimated by $P_i = M_i + (1 - a_i)d_i$ where a_i is the probability that an individual dying in the interval dies prior to the interval midpoint. For cohort table, P_i is input directly while the death rate in the interval, r_i, is not needed.

The probability that an individual dies during the age interval from t_{i-1} to t_i is given by $q_i = d_i/P_i$. It is assumed that all individuals alive at the beginning of the last interval die during the last interval. Thus, q_n = 1.0. The asymptotic variance of q_i can be estimated by

$$\sigma_i^2 = q_i(1 - q_i)/P_i$$

For a population table, the number of individuals alive in the middle of the time interval (input in `nCohort[i]`) must be adjusted to correspond to the number of deaths observed in the interval. The algorithm assumes that the number of deaths observed in interval h_i occur over a time period equal to h_i. If d_i is measured over a period u_i, where $u_i \neq d_i$, then `nCohort[i]` must be adjusted to correspond to d_i by multiplication by u_i/h_i, i.e., the value M_i input as `nCohort[i]` is computed as

$$M_i^* = M_i u_i / h_i$$

Let S_i denote the number of survivors at time t_i from a hypothetical (for population table) or observed (for cohort table) population. Then, S_0 `=initialPopulation` for population table, and S_0 = `nCohort[0]` for cohort table, and S_i is given by $S_i = S_{i-1} - \delta_{i-1}$ where $\delta_i = S_i q_i$ is the number of individuals who die in the ith interval. The proportion of survivors in the interval is given by $V_i = S_i/S_0$ while the asymptotic variance of V_i can be estimated as follows:

$$\text{var}(V_i) = V_i^2 \sum_{j=1}^{i-1} \frac{\sigma_j^2}{(1 - q_j)^2}$$

The expected lifetime at the beginning of the interval is calculated as the total lifetime remaining for all survivors alive at the beginning of the interval divided by the number of survivors at the beginning of the interval. If e_i denotes this average expected lifetime, then the variance of e_i can be estimated as (see Chiang 1968):

$$\text{var}(e_i) = \frac{\sum_{j=i}^{n-1} P_j^2 \sigma_j^2 \left[e_{j+1} + h_{j+1}(1 - a_j)\right]^2}{P_j^2}$$

where var(e_n) = 0.0.

Finally, the total number of time units lived by all survivors in the time interval can be estimated as:

$$U_i = h_i[S_i - \delta_i(1 - a_i)]$$

Property

PopulationSize

```
virtual public int PopulationSize {get; set; }
```

Description

Sets the population size at the beginning of the first age interval in requesting a population table.

Property Value

An `int` scalar specifying the initial population.

Default: A value of 10,000 is used to allow easy entry of `nCohorts` and `nDeaths` when numbers are available as percentages.

Constructor

LifeTables

```
public LifeTables(int[] nCohort, double[] age, double[] a)
```

Description

Constructs a new `LifeTables` instance. The number of classes, *nClasses* is equal to the length of the input array `nCohort`.

Parameters

`nCohort` – An `int` array of length *nClasses* containing the cohort sizes during each interval. If the Population Table option is used, then `nCohort[i]` contains the size of the population at the midpoint of interval `i`. Otherwise, `nCohort[i]` contains the size of the cohort at the beginning of interval `i`. When requesting a population table, the population sizes in `nCohort` may need to be adjusted to correspond to the number of deaths in `nDeaths`. See the class description section for more information.

`age` – A `double` array of length `nClasses + 1` containing the lowest age in each age interval, and in `age[nClasses]`, the endpoint of the last age interval. Negative `age[0]` indicates that the age intervals are all of length |`age[0]`| and that the initial age interval is from 0.0 to |`age[0]`|. In this case, all other elements of `age` need not be specified. `age[nClasses]` need not be specified when getting a cohort table.

`a` – A `double` array of length *nClasses* containing the fraction of those dying within each interval who die before the interval midpoint. A common choice for all `a[i]` is 0.5. This choice may also be specified by setting `a[0]` to any negative value. In this case, the remaining values of `a` need not be specified.

Methods

GetLifeTable

```
virtual public double[,] GetLifeTable()
```

Description

Compute a cohort table.

Returns

A `double` matrix of dimensions *nClasses* by 12 containing the population table. Entries in the *i*th row are for the age interval defined by `age[i]`. Column definitions are described in the following table.

Column	*Description*
0	Lowest age in the age interval.
1	Fraction of those dying within the interval who die before the interval midpoint.
2	Number surviving to the beginning of the interval.
3	Number of deaths in the interval.
4	Death rate in the interval. For cohort table, this column is set to NaN (not a number).
5	Proportion dying in the interval.
6	Standard error of the proportion dying in the interval.
7	Proportion of survivors at the beginning of the interval.
8	Standard error of the proportion of survivors at the beginning of the interval.
9	Expected lifetime at the beginning of the interval.
10	Standard error of the expected life at the beginning of the interval.
11	Total number of time units lived by all of the population in the interval.

GetPopulationTable

```
virtual public double[,] GetPopulationTable(int[] nDeaths)
```

Description

Compute a population table.

Parameter

`nDeaths` – An `int` array of `nClasses` containing the number of deaths in each age interval.

Returns

A `double` matrix of dimensions `nClasses` by 12 containing the population table. Entries in the *i*th row are for the age interval defined by `age[i]`. Column definitions are the same as in `GetLifeTable`.

Example: Life Tables

This example is taken from Chiang (1968). The cohort life table has thirteen equally spaced intervals, so age[0] is set to -5.0. Similarly, the probabilities of death prior to the middle of the interval are all taken to be 0.5, so a[0] is set to -1.0.

```
using System;
using Imsl.Stat;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

public class LifeTablesEx1
{
    public static void  Main(String[] args)
    {
        int[] nCohort = {270, 268, 264, 261, 254, 251, 248, 232,
            166, 130, 76, 34, 13};
        double[] age = new double[nCohort.Length + 1];
        double[] a = new double[nCohort.Length];

        age[0] = - 5.0;
        a[0] = - 1.0;

        LifeTables lt = new LifeTables(nCohort, age, a);
        double[,] table = lt.GetLifeTable();

        PrintMatrixFormat pmf = new PrintMatrixFormat();
        String[] cols = {"Age", "PDHALF", "Alive",
            "Deaths", "Death Rate", "P(D)", "Std(P(D))", "P(S)", "Std(P(S))",
            "Lifetime", "Std(Life)", "Time Units"};
        pmf.SetColumnLabels(cols);
        pmf.NumberFormat = "0.0####";
        PrintMatrix pm = new PrintMatrix("Life Table");
        pm.SetPageWidth(40);
        pm.Print(pmf, table);
    }
}
```

Output

```
          Life Table
    Age   PDHALF  Alive  Deaths
 0  0.0    0.5    270.0   2.0
 1  5.0    0.5    268.0   4.0
 2  10.0   0.5    264.0   3.0
 3  15.0   0.5    261.0   7.0
 4  20.0   0.5    254.0   3.0
 5  25.0   0.5    251.0   3.0
 6  30.0   0.5    248.0   16.0
 7  35.0   0.5    232.0   66.0
 8  40.0   0.5    166.0   36.0
 9  45.0   0.5    130.0   54.0
10  50.0   0.5    76.0    42.0
11  55.0   0.5    34.0    21.0
```

```
12  60.0   0.5    13.0    13.0

    Death Rate   P(D)    Std(P(D))   P(S)
 0     NaN      0.00741   0.00522   1.0
 1     NaN      0.01493   0.00741   0.99259
 2     NaN      0.01136   0.00652   0.97778
 3     NaN      0.02682   0.01      0.96667
 4     NaN      0.01181   0.00678   0.94074
 5     NaN      0.01195   0.00686   0.92963
 6     NaN      0.06452   0.0156    0.91852
 7     NaN      0.28448   0.02962   0.85926
 8     NaN      0.21687   0.03199   0.61481
 9     NaN      0.41538   0.04322   0.48148
10     NaN      0.55263   0.05704   0.28148
11     NaN      0.61765   0.08334   0.12593
12     NaN      1.0       0.0       0.04815

    Std(P(S))  Lifetime  Std(Life)  Time Units
 0   0.0       43.18519   0.69929     1345.0
 1   0.00522   38.48881   0.67074     1330.0
 2   0.00897   34.03409   0.62303     1312.5
 3   0.01092   29.39655   0.59401     1287.5
 4   0.01437   25.1378    0.54028     1262.5
 5   0.01557   20.40837   0.52367     1247.5
 6   0.01665   15.625     0.51485     1200.0
 7   0.02116   11.53017   0.49815     995.0
 8   0.02962   10.12048   0.46017     740.0
 9   0.03041   7.23077    0.4328      515.0
10   0.02737   5.59211    0.43607     275.0
11   0.02019   4.41176    0.41671     117.5
12   0.01303   2.5        0.0         32.5
```

Chapter 21: Probability Distribution Functions and Inverses

Types

Usage Notes

Definitions and discussions of the terms basic to this chapter can be found in Johnson and Kotz (1969, 1970a, 1970b). These are also good references for the specific distributions.

In order to keep the calling sequences simple, whenever possible, the methods/classes described in this chapter are written for standard forms of statistical distributions. Hence, the number of parameters for any given distribution may be fewer than the number often associated with the distribution. Also, the methods relating to the normal distribution, `Cdf.Normal`, `InvCdf.Normal`, and `Pdf.Normal` are for a normal distribution with mean equal to zero and variance equal to one. For other means and variances, it is very easy for the user to standardize the variables by subtracting the mean and dividing by the square root of the variance.

The *distribution function* for the (real, single-valued) random variable X is the function F defined for all

real x by

$$F(x) = \text{Prob}(X \leq x)$$

where $\text{Prob}(\cdot)$ denotes the probability of an event. The distribution function is often called the *cumulative distribution function* (CDF).

For distributions with finite ranges, such as the beta distribution, the CDF is 0 for values less than the left endpoint and 1 for values greater than the right endpoint. The methods in the `Cdf`, `InvCdf`, and `Pdf` classes described in this chapter return the correct values for the distribution functions when values outside of the range of the random variable are input, but warning error conditions are set in these cases.

Discrete Random Variables

For discrete distributions, the function giving the probability that the random variable takes on specific values is called the *probability function*, defined by

$$p(x) = \text{Prob}(X = x)$$

The CDF for a discrete random variable is

$$F(x) = \sum_{A} p(k)$$

where A is set such that $k \leq x$. Since the distribution function is a step function, its inverse does not exist uniquely.

Continuous Distributions

For continuous distributions, a probability function, as defined above, would not be useful because the probability of any given point is 0. For such distributions, the useful analog is the *probability density function* (PDF). The integral of the PDF is the probability over the interval, if the continuous random variable X has PDF f, then

$$\text{Prob}(a \leq X \leq b) = \int_a^b f(x)\,dx$$

The relationship between the CDF and the PDF is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

For (absolutely) continuous distributions, the value of F(`x`) uniquely determines `x` within the support of the distribution. The methods in the `InvCdf` class compute the inverses of the distribution functions. That is, given F(`x`), a method such as, `Beta` in the `InvCdf` class computes `x`. The inverses are defined only over the open interval (0,1).

Additional Comments

Whenever a probability close to 1.0 results from a call to a distribution function or is to be input to an inverse function, it is often impossible to achieve good accuracy because of the nature of the representation of numeric values. In this case, it may be better to work with the complementary distribution function (one minus the distribution function). If the distribution is symmetric about some point (as the normal distribution, for example) or is reflective about some point (as the beta distribution, for example), the complementary distribution function has a simple relationship with the distribution function. For example, to evaluate the standard normal distribution at 4.0, using the `Normal` method in the `Cdf` class directly, the result to six places is 0.999968. Only two of those digits are really useful, however. A more useful result may be 1.000000 minus this value, which can be obtained to six places as 3.16712e-05 by evaluating `Normal` at -4.0. For the normal distribution, the two values are related by $\Phi(x) = 1 - \Phi(-x)$, where $\Phi(\cdot)$ is the normal distribution function. Another example is the beta distribution with parameters 2 and 10. This distribution is skewed to the right, so evaluating `Cdf.Beta` at 0.7, 0.999953 is obtained. A more precise result is obtained by evaluating `Cdf.Beta` with parameters 10 and 2 at 0.3. This yields 4.72392e-5.

Many of the algorithms used by the classes in this chapter are discussed by Abramowitz and Stegun (1964). The algorithms make use of various expansions and recursive relationships and often use different methods in different regions.

Cumulative distribution functions are defined for all real arguments. However, if the input to one of the distribution functions in this chapter is outside the range of the random variable, an error is issued.

Cdf Class

```
public class Imsl.Stat.Cdf
```

Cumulative probability distribution functions.

See Also

Probability density function (p. 1149), Inverse Cumulative Distribution Function (p. 1137)

Methods

Beta

```
static public double Beta(double x, double pin, double qin)
```

Description

Evaluates the beta cumulative probability distribution function.

Parameters

`x` – A `double`, the argument at which the function is to be evaluated.

`pin` – A `double`, the first beta distribution parameter.

`qin` – A `double`, the second beta distribution parameter.

Returns

A `double`, the probability that a beta random variable takes on a value less than or equal to `x`.

Remarks

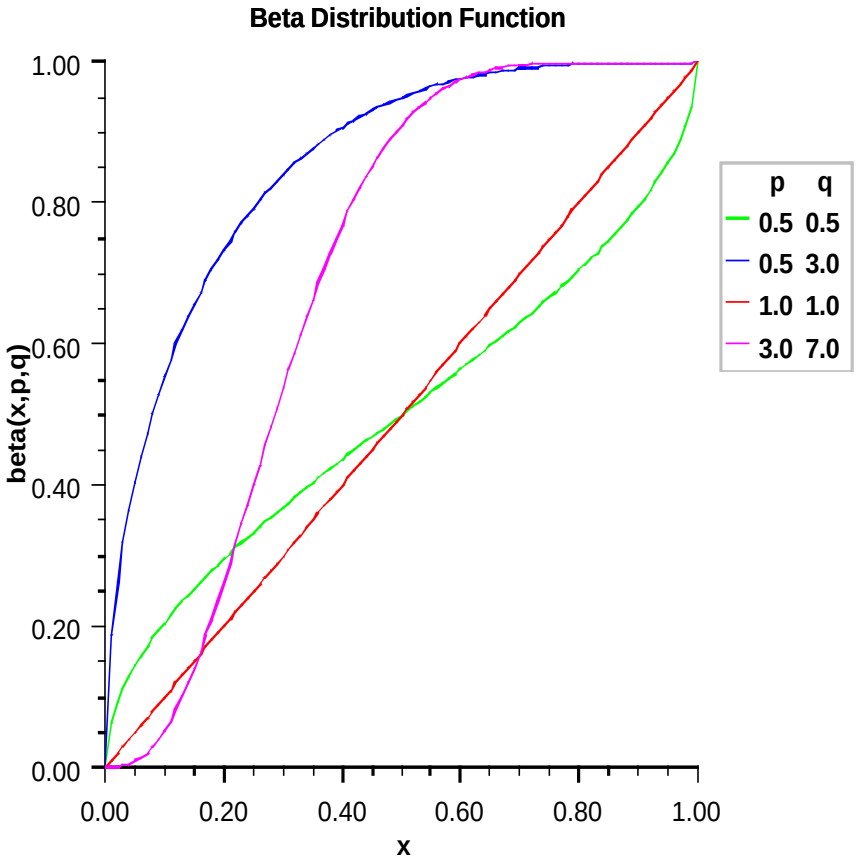
Method `Cdf.Beta` evaluates the distribution function of a beta random variable with parameters `pin` and `qin`. This function is sometimes called the *incomplete beta ratio* and, with p = `pin` and q = `qin`, is denoted by $I_x(p,q)$. It is given by

$$I_x(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_0^x t^{p-1}(1-t)^{q-1}\,dt$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function $I_x(p,q)$ is the probability that the random variable takes a value less than or equal to `x`.

The integral in the expression above is called the *incomplete beta function* and is denoted by $\beta_x(p,q)$. The constant in the expression is the reciprocal of the *beta function* (the incomplete function evaluated at one) and is denoted by $\beta_1(p,q)$.

`Cdf.Beta` uses the method of Bosten and Battiste (1974).



BetaMean

```
static public double BetaMean(double pin, double qin)
```

Description

Evaluates the mean of the beta cumulative probability distribution function.

Parameters

`pin` – A `double`, the first beta distribution parameter.

`qin` – A `double`, the second beta distribution parameter.

Returns

A `double`, the mean of the beta distribution function.

BetaVariance

```
static public double BetaVariance(double pin, double qin)
```

Description

Evaluates the variance of the beta cumulative probability distribution function.

Parameters

`pin` – A `double`, the first beta distribution parameter.

`qin` – A `double`, the second beta distribution parameter.

Returns

A `double`, the variance of the beta distribution function.

Binomial

```
static public double Binomial(int k, int n, double pin)
```

Description

Evaluates the binomial cumulative probability distribution function.

Parameters

`k` – The `int` argument for which the binomial distribution function is to be evaluated.

`n` – The `int` number of Bernoulli trials.

`pin` – A `double` scalar value representing the probability of success on each independent trial.

Returns

A `double` scalar value representing the probability that a binomial random variable takes a value less than or equal to `k`. This value is the probability that `k` or fewer successes occur in `n` independent Bernoulli trials, each of which has a `pin` probability of success.

Remarks

Method `Cdf.Binomial` evaluates the distribution function of a binomial random variable with parameters `n` and p with p=`pin`. It does this by summing probabilities of the random variable taking on the specific values in its range. These probabilities are computed by the recursive relationship

$$\Pr(X = j) = \frac{(n+1-j)\,p}{j\,(1-p)} \Pr(X = j-1)$$

To avoid the possibility of underflow, the probabilities are computed forward from 0, if `k` is not greater than `n` times p, and are computed backward from `n`, otherwise. The smallest positive machine number, ε, is used as the starting value for summing the probabilities, which are rescaled by $(1-p)^n\varepsilon$ if forward computation is performed and by $p^n\varepsilon$ if backward computation is done. For the special case of p = 0, `Cdf.Binomial` is set to 1; and for the case p = 1, `Cdf.Binomial` is set to 1 if `k = n` and to 0 otherwise.

BivariateNormal

```
static public double BivariateNormal(double x, double y, double rho)
```

Description

Evaluates the bivariate normal cumulative probability distribution function.

Parameters

`x` – The *x*-coordinate of the point for which the bivariate normal distribution function is to be evaluated.

`y` – The *y*-coordinate of the point for which the bivariate normal distribution function is to be evaluated.

`rho` – The correlation coefficient.

Returns

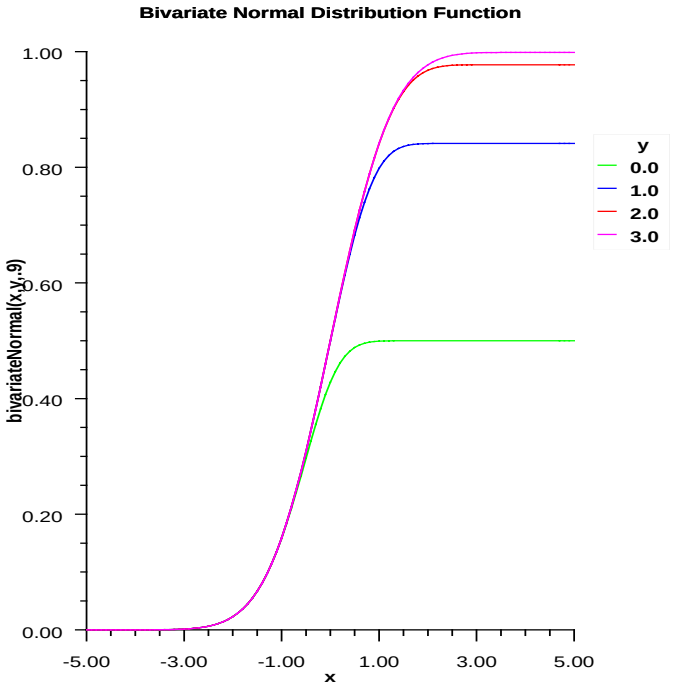
The probability that a bivariate normal random variable (X,Y) with correlation `rho` satisfies $X \leq$ `x` and $Y \leq$ `y`.

Remarks

Let (X,Y) be a bivariate normal variable with mean $(0,0)$ and variance-covariance matrix

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

This method computes the probability that $X \leq$ `x` and $Y \leq$ `y`.



Chi

```
static public double Chi(double chsq, double df)
```

Description

Evaluates the chi-squared cumulative probability distribution function.

Parameters

`chsq` – A `double` specifying the argument at which the function is to be evaluated.

`df` – A `double` specifying the number of degrees of freedom. This must be at least 0.5.

Returns

A `double` specifying the probability that a chi-squared random variable takes a values less than or equal to `chsq`.

Remarks

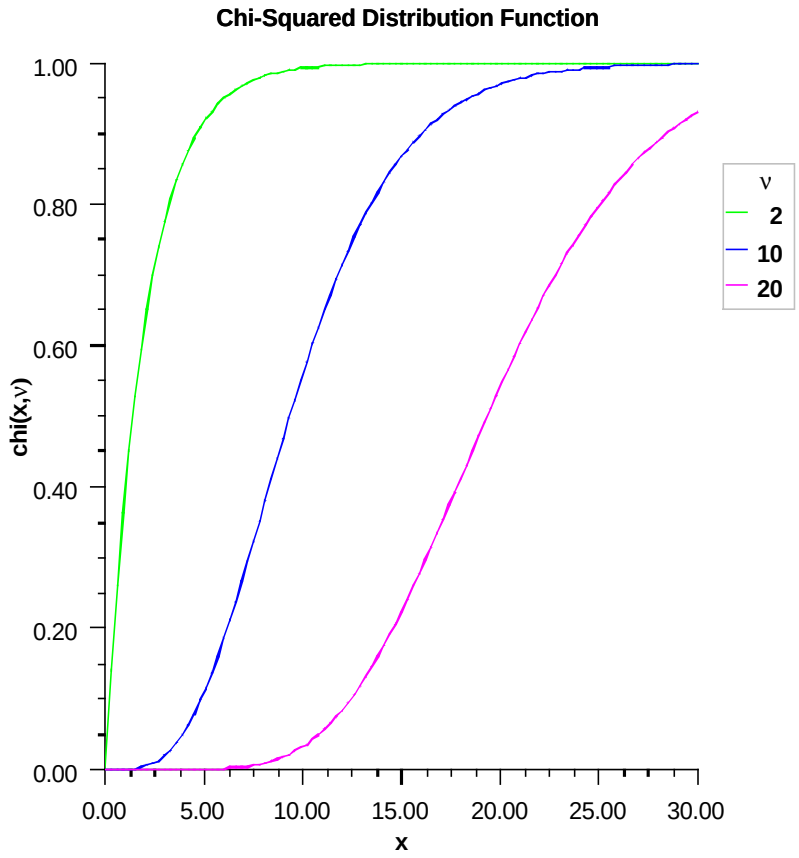
Method `Cdf.Chi` evaluates the distribution function, F, of a chi-squared random variable with `df` degrees of freedom, that is, with $v = \text{df}$, and $x = \text{chsq}$,

$$F(x) = \frac{1}{2^{v/2}\Gamma(v/2)} \int_0^x e^{-t/2} t^{v/2-1} dt$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x.

For $v > 343$, `Cdf.Chi` uses the Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.17) to the normal distribution, and method `Cdf.Normal` is used to evaluate the normal distribution function.

For $v \leq 343$, `Cdf.Chi` uses series expansions to evaluate the distribution function. If $x < \max(v/2, 26)$, `Cdf.Chi` uses the series 6.5.29 in Abramowitz and Stegun (1964), otherwise, it uses the asymptotic expansion 6.5.32 in Abramowitz and Stegun.



For greater right tail accuracy, see Cdf.ComplementaryChi (p. 1108).

ChiMean

```
static public double ChiMean(double df)
```

Description

Evaluates the mean of the chi-squared cumulative probability distribution function.

Parameter

`df` – A `double` scalar value representing the number of degrees of freedom. This must be at least 0.5.

Returns

A `double`, the mean of the chi-squared distribution function.

ChiVariance

```
static public double ChiVariance(double df)
```

Description

Evaluates the variance of the chi-squared cumulative probability distribution function.

Parameter

`df` – A `double` scalar value representing the number of degrees of freedom. This must be at least 0.5.

Returns

A `double`, the variance of the chi-squared distribution function.

ComplementaryChi

```
static public double ComplementaryChi(double chsq, double df)
```

Description

Calculates the complement of the chi-squared distribution.

Parameters

`chsq` – A `double` scalar value at which $Pr\left(x > \chi^2\right)$ is to be evaluated.

`df` – A `double` specifying the number of degrees of freedom. This must be at least 0.5.

Returns

A `double` specifying the probability that a chi-squared random variable takes a value greater than `chsq`.

Remarks

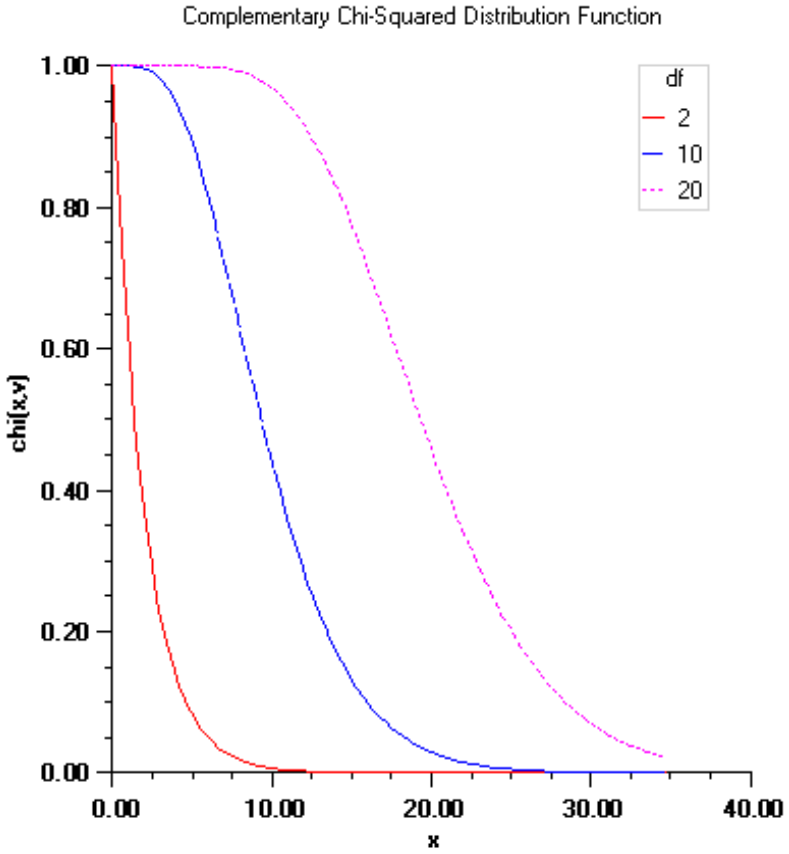
Method `Cdf.ComplementaryChi` evaluates the distribution function, $1 - F$, of a chi-squared random variable with `df` degrees of freedom, that is, with $\nu =$ df, and $x =$ chsq,

$$F(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} \int_0^x e^{-t/2} t^{\nu/2-1} dt$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function at the point x is the probability that the random variable takes a value greater than x.

For $v > 343$, `Cdf.ComplementaryChi` uses the Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.17) to the one minus the normal distribution, and method `Cdf.Normal` is used to evaluate the normal distribution function.

For $v \leq 343$, `Cdf.ComplementaryChi` uses series expansions to evaluate the distribution function. If $x < \max(v/2, 26)$, `Cdf.ComplementaryChi` uses the series 6.5.29 in Abramowitz and Stegun (1964), otherwise, it uses the asymptotic expansion 6.5.32 in Abramowitz and Stegun. This function provides higher right tail accuracy for the Chi-squared distribution.



ComplementaryF

```
static public double ComplementaryF(double x, double dfn, double dfd)
```

Description

Calculates the complement of the F distribution function.

Parameters

`x` – A `double`, the argument at which $Pr(x > F)$ is to be evaluated.

`dfn` – A `double`, the numerator degrees of freedom. It must be positive.

`dfd` – A `double`, the denominator degrees of freedom. It must be positive.

Returns

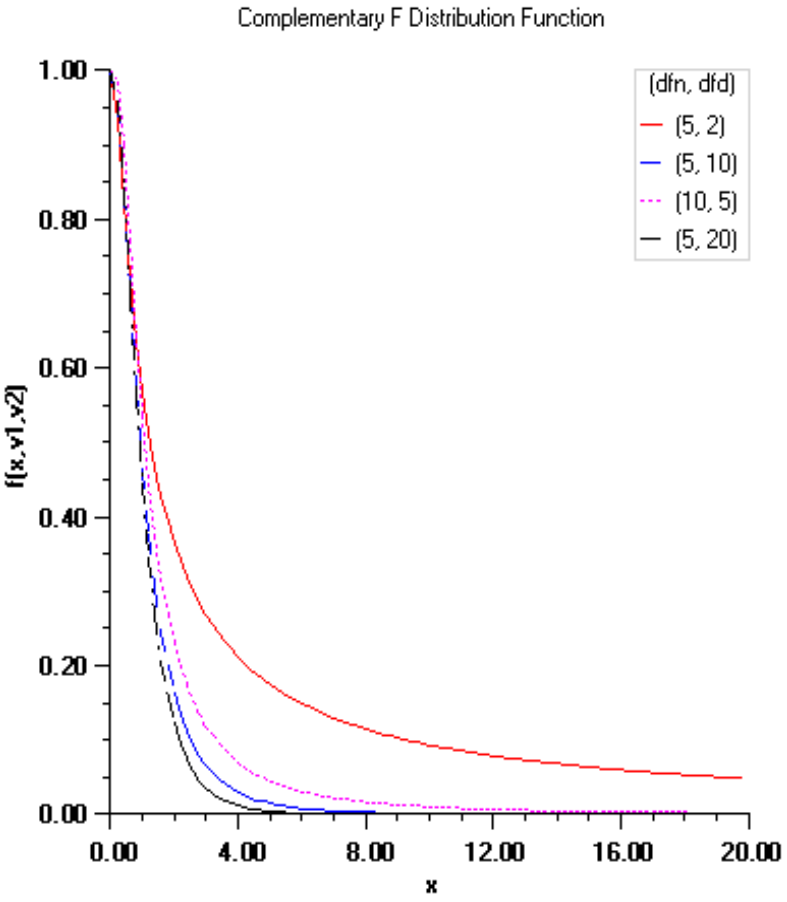
A `double`, the probability that an F random variable takes on a value greater than `x`.

Remarks

`Cdf.ComplementaryF` evaluates one minus the distribution function of a Snedecor's F random variable with `dfn` numerator degrees of freedom and `dfd` denominator degrees of freedom. The function is evaluated by making a transformation to a beta random variable and then using the function `Cdf.Beta`. If X is an F variate with v_1 and v_2 degrees of freedom and $Y = v_1X/(v_2 + v_1X)$, then Y is a beta variate with parameters $p = v_1/2$ and $q = v_2/2$. `Cdf.ComplementaryF` also uses a relationship between F random variables that can be expressed as follows:

$$\mathrm{F}(X, dfn, dfd) = \mathrm{F}(1/X, dfd, dfn)$$

This function provides higher right tail accuracy for the F distribution.



ComplementaryStudentsT

```
static public double ComplementaryStudentsT(double t, double df)
```

Description

Calculates the complement of the Student's *t* distribution.

Parameters

`t` – A `double` scalar value for which $Pr(x > t)$ is to be evaluated.

`df` – A `double` scalar value representing the number of degrees of freedom. This must be at least one.

Returns

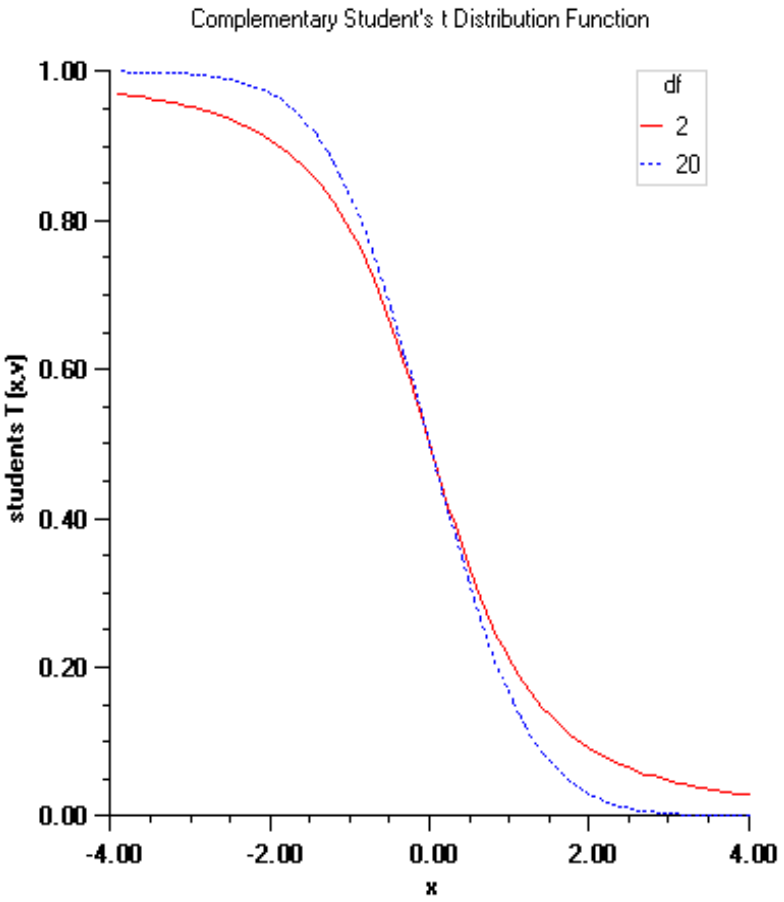
A `double` scalar value representing the probability that a Student's *t* random variable takes a value

greater than `t`.

Remarks

Method `Cdf.ComplementaryStudentsT` evaluates one minus the distribution function of a Student's *t* random variable with `df` degrees of freedom. If the square of `t` is greater than or equal to `df`, the relationship of a *t* to an *f* random variable (and subsequently, to a beta random variable) is exploited, and routine `Cdf.Beta` is used. Otherwise, the method described by Hill (1970) is used. If `df` is not an integer, if `df` is greater than 19, or if `df` is greater than 200, a Cornish-Fisher expansion is used to evaluate the distribution function. If `df` is less than 20 and |t| is less than 2.0, a trigonometric series (see Abramowitz and Stegun 1964, equations 26.7.3 and 26.7.4, with some rearrangement) is used. For the remaining cases, a series given by Hill (1970) that converges well for large values of `t` is used.

This function provides higher right tail accuracy for the Student's *t* distribution.



DiscreteUniform

```
static public double DiscreteUniform(int x, int n)
```

Description

Evaluates the discrete uniform cumulative probability distribution function.

Parameters

`x` – An `int` scalar value representing the argument at which the function is to be evaluated.

`n` – An `int` scalar value representing the upper limit of the discrete uniform distribution.

Returns

A `double` scalar value representing the probability that a discrete uniform random variable takes a value less than or equal to `x`.

Exponential

```
static public double Exponential(double x, double scale)
```

Description

Evaluates the exponential cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`scale` – A `double` scalar value representing the scale parameter, *b*.

Returns

A `double` scalar value representing the probability that an exponential random variable takes on a value less than or equal to `x`.

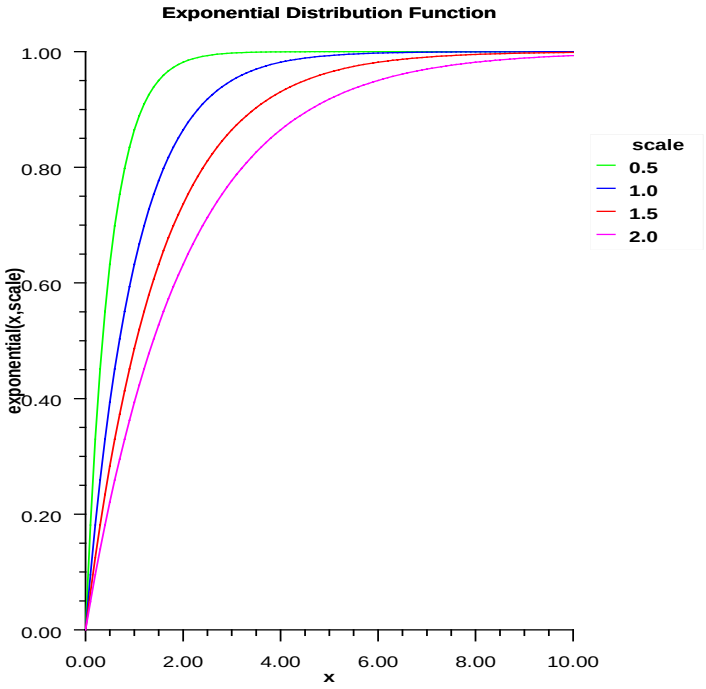
Remarks

Method `Cdf.Exponential` is a special case of the gamma distribution function, which evaluates the distribution function, *F*, with scale parameter *b* and shape parameter *a*, used in the gamma distribution function equal to 1.0. That is,

$$F(x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t/b} dt$$

where $\Gamma(\cdot)$ is the gamma function. (The gamma function is the integral from 0 to ∞ of the same integrand as above). The value of the distribution function at the point `x` is the probability that the random variable takes a value less than or equal to `x`.

If `x` is less than or equal to 1.0, `Cdf.Gamma` uses a series expansion. Otherwise, a continued fraction expansion is used. (See Abramowitz and Stegun, 1964.)



ExtremeValue

```
static public double ExtremeValue(double x, double mu, double beta)
```

Description

Evaluates the extreme value cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter, μ.

`beta` – A `double` scalar value representing the scale parameter, β.

Returns

A `double` scalar value representing the probability that an extreme value random variable takes on a value less than or equal to `x`.

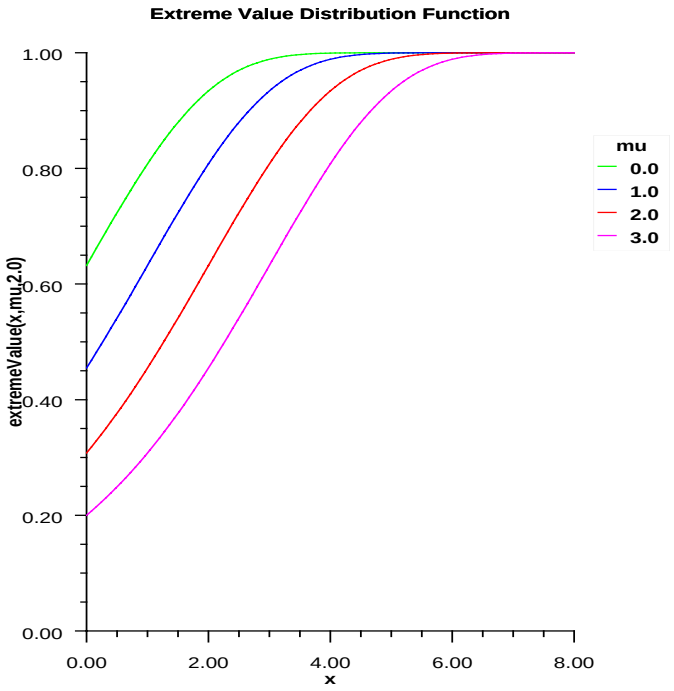
Remarks

Method `Cdf.ExtremeValue`, also known as the Gumbel minimum distribution, evaluates the extreme value distribution function, F, of a uniform random variable with location parameter μ and shape parameter β; that is,

$$F(x) = \int_0^x 1 - e^{-e^{\frac{x-\mu}{\beta}}}\, dt$$

The case where $\mu = 0$ and $\beta = 1$ is called the standard Gumbel distribution.

Random numbers are generated by evaluating uniform variates u_i, equating the continuous distribution function, and then solving for x_i by first computing $\frac{x_i - \mu}{\beta} = log(-log(1 - u_i))$.



F

```
static public double F(double x, double dfn, double dfd)
```

Description

Evaluates the F cumulative probability distribution function.

Parameters

`x` – A `double`, the argument at which the function is to be evaluated.

`dfn` – A `double`, the numerator degrees of freedom. It must be positive.

`dfd` – A `double`, the denominator degrees of freedom. It must be positive.

Returns

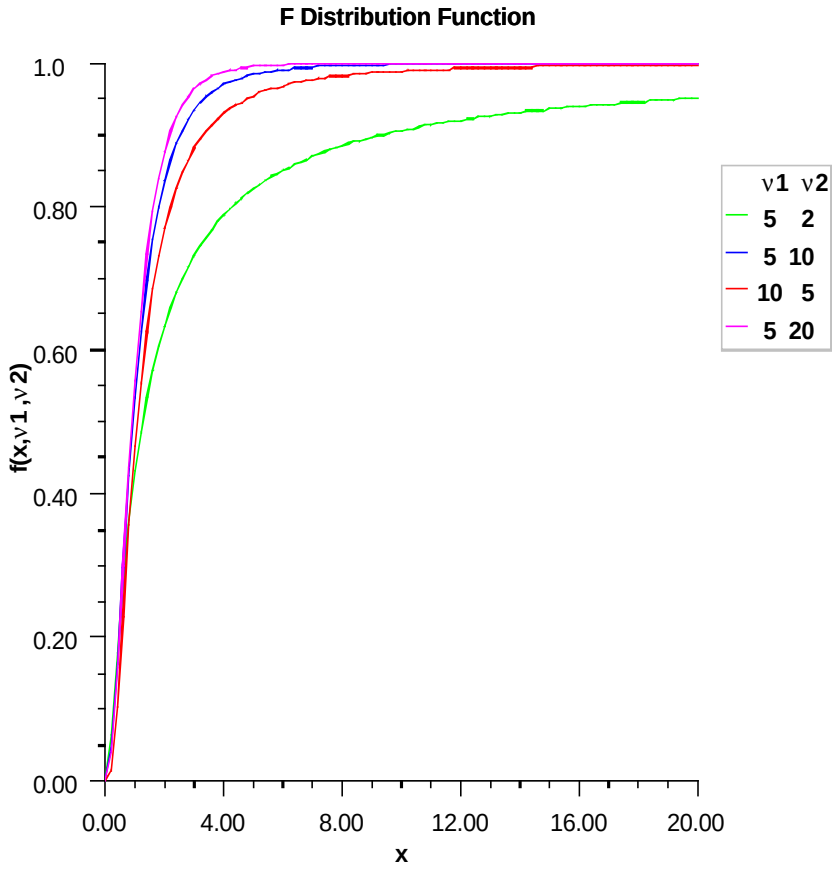
A `double`, the probability that an F random variable takes on a value less than or equal to `x`.

Remarks

`Cdf.F` evaluates the distribution function of a Snedecor's F random variable with `dfn` numerator degrees of freedom and `dfd` denominator degrees of freedom. The function is evaluated by making a transformation to a beta random variable and then using the function `Cdf.Beta`. If X is an F variate with v_1 and v_2 degrees of freedom and $Y = v_1 X/(v_2 + v_1 X)$, then Y is a beta variate with parameters $p = v_1/2$ and $q = v_2/2$. `F` also uses a relationship between F random variables that can be expressed as follows:

$$\mathrm{F}(X, dfn, dfd) = \mathrm{F}(1/X, dfd, dfn)$$

For greater right tail accuracy, see Cdf.ComplementaryF (p. 1109).



Gamma

```
static public double Gamma(double x, double a)
```

Description

Evaluates the gamma cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`a` – A `double` scalar value representing the shape parameter. This must be positive.

Returns

A `double` scalar value representing the probability that a gamma random variable takes on a value less than or equal to `x`.

Remarks

Method `Cdf.Gamma` evaluates the distribution function, F, of a gamma random variable with shape parameter a; that is,

$$F(x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt$$

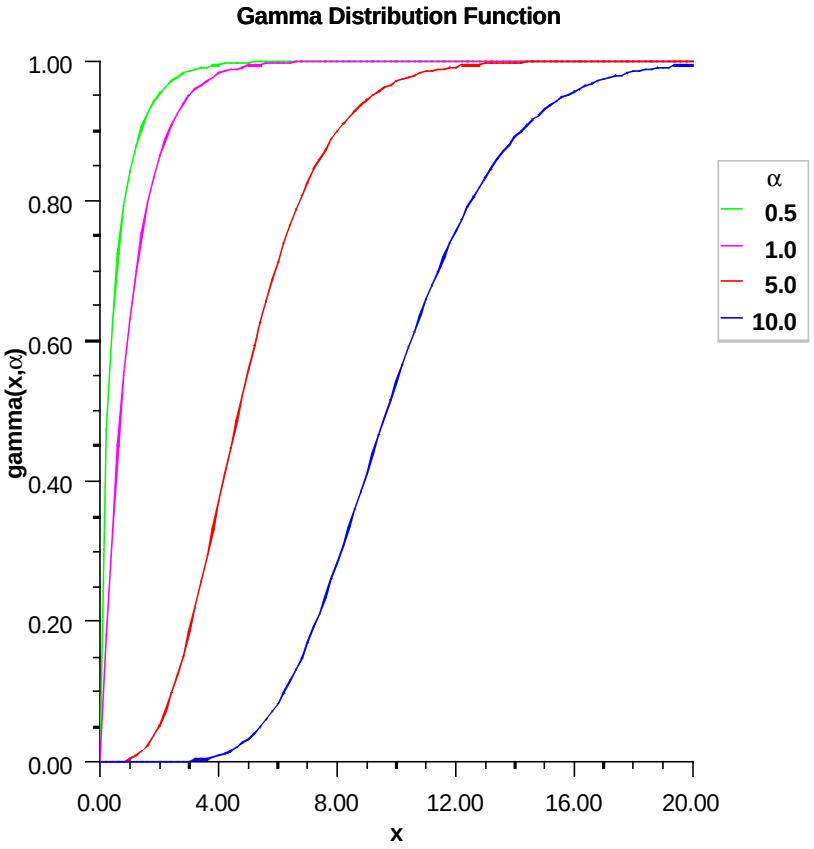
where $\Gamma(\cdot)$ is the gamma function. (The gamma function is the integral from 0 to ∞ of the same integrand as above). The value of the distribution function at the point `x` is the probability that the random variable takes a value less than or equal to `x`.

The gamma distribution is often defined as a two-parameter distribution with a scale parameter b (which must be positive), or even as a three-parameter distribution in which the third parameter c is a location parameter. In the most general case, the probability density function over (c, ∞) is

$$f(t) = \frac{1}{b^a \Gamma(a)} e^{-(t-c)/b} (x-c)^{a-1}$$

If T is such a random variable with parameters a, b, and c, the probability that $T \leq t_0$ can be obtained from `Cdf.Gamma` by setting $X = (t_0 - c)/b$.

If X is less than `a` or if X is less than or equal to 1.0, `Cdf.Gamma` uses a series expansion. Otherwise, a continued fraction expansion is used. (See Abramowitz and Stegun, 1964.)



Geometric

```
static public double Geometric(int x, double pin)
```

Description

Evaluates the discrete geometric cumulative probability distribution function.

Parameters

`x` – An `int` scalar value representing the argument at which the function is to be evaluated.

`pin` – A `double` scalar value representing the probability parameter for each independent trial (the probability of success for each independent trial).

Returns

A `double` scalar value representing the probability that a geometric random variable takes a value less than or equal to `x`. The return value is the probability that up to `x` trials would be observed before observing a success.

Hypergeometric

```
static public double Hypergeometric(int k, int sampleSize, int defectivesInLot,
int lotSize)
```

Description

Evaluates the hypergeometric cumulative probability distribution function.

Parameters

`k` – An `int`, the argument at which the function is to be evaluated.

`sampleSize` – An `int`, the sample size, `n`.

`defectivesInLot` – An `int`, the number of defectives in the lot, `m`.

`lotSize` – An `int`, the lot size, `l`.

Returns

A `double`, the probability that a hypergeometric random variable takes a value less than or equal to `k`.

Remarks

Method `Cdf.Hypergeometric` evaluates the distribution function of a hypergeometric random variable with parameters *n*, *l*, and *m*. The hypergeometric random variable *X* can be thought of as the number of items of a given type in a random sample of size *n* that is drawn without replacement from a population of size *l* containing *m* items of this type. The probability function is

$$\Pr(X = j) = \frac{\binom{m}{j}\binom{l-m}{n-j}}{\binom{l}{n}} \text{for } j = i,\ i+1,\ i+2,\ \ldots,\ \min(n,m)$$

where $i = \max(0, n-l+m)$.

If `k` is greater than or equal to *i* and less than or equal to $\min(n,m)$, `Cdf.Hypergeometric` sums the terms in this expression for *j* going from *i* up to `k`. Otherwise, `Cdf.Hypergeometric` returns 0 or 1, as appropriate. So, as to avoid rounding in the accumulation, `Cdf.Hypergeometric` performs the summation differently depending on whether or not `k` is greater than the mode of the distribution, which is the greatest integer less than or equal to $(m+1)(n+1)/(l+2)$.

Logistic

```
static public double Logistic(double x, double mu, double s)
```

Description

Evaluates the logistic cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter, μ.

`s` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value representing the probability that a logistic random variable takes a value less than or equal to `x`.

Remarks

Method `Cdf.Logistic` evaluates the distribution function, *F*, of a logistic random variable with location parameter μ and scale parameter *s*. It is given by

$$F(x,\mu,s) = \frac{1}{1+e^{-(x-\mu)/s}}$$

where $s > 0$.

LogNormal

```
static public double LogNormal(double x, double mu, double sigma)
```

Description

Evaluates the standard lognormal cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter.

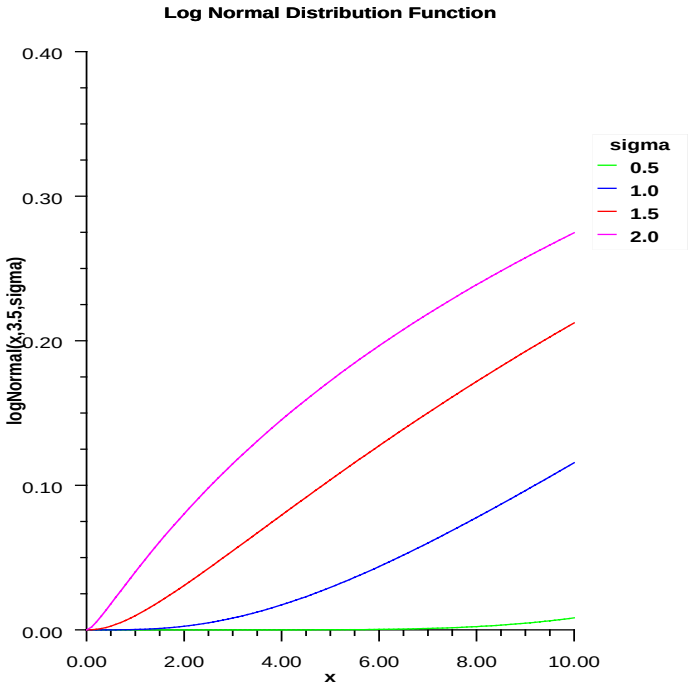
`sigma` – A `double` scalar value representing the shape parameter. `sigma` must be a positive.

Returns

A `double` scalar value representing the probability that a standard lognormal random variable takes a value less than or equal to `x`.

Remarks

$$F(x) = \frac{1}{x^{\sigma}\sqrt{2\pi}} \int \frac{1}{t} e^{-\frac{\ln t-\mu^2}{2\sigma^2}}$$



NoncentralBeta

```
static public double NoncentralBeta(double x, double shape1, double shape2,
double lambda)
```

Description

Evaluates the noncentral beta cumulative probability distribution function (*CDF*).

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated. `x` must be nonnegative and less than or equal to 1.

`shape1` – A `double` scalar value representing the first shape parameter. `shape1` must be positive.

`shape2` – A `double` scalar value representing the second shape parameter. `shape2` must be positive.

`lambda` – A `double` scalar value representing the noncentrality parameter. `lambda` must nonnegative.

Returns

A `double` scalar value representing the probability that a noncentral beta random variable takes a value less than or equal to `x`.

Remarks

The noncentral beta distribution is a generalization of the beta distribution. If Z is a noncentral chi-square random variable with noncentrality parameter λ and $2\alpha_1$ degrees of freedom, and Y is a chi-square random variable with $2\alpha_2$ degrees of freedom which is statistically independent of Z, then

$$X = \frac{Z}{Z + Y} = \frac{\alpha_1 F}{\alpha_1 F + \alpha_2}$$

is a noncentral beta-distributed random variable and

$$F = \frac{\alpha_2 Z}{\alpha_1 Y} = \frac{\alpha_2 X}{\alpha_1 (1 - X)}$$

is a noncentral F-distributed random variable. The CDF for noncentral beta variable X can thus be simply defined in terms of the noncentral F CDF:

$$CDF_{nc\beta}(x, \alpha_1, \alpha_2, \lambda) = CDF_{ncF}(f, 2\alpha_1, 2\alpha_2, \lambda)$$

where $CDF_{nc\beta}(x, \alpha_1, \alpha_2, \lambda)$ is the noncentral beta CDF with x = `x`, α_1 = `shape1`, α_2 = `shape2`, and noncentrality parameter λ = `lambda`; $CDF_{ncF}(f, 2\alpha_1, 2\alpha_2, \lambda)$ is the noncentral F CDF with argument f, numerator and denominator degrees of freedom $2\alpha_1$ and $2\alpha_2$ respectively, and noncentrality parameter λ; and:

$$f = \frac{\alpha_2 x}{\alpha_1 (1 - x)}; \; x = \frac{\alpha_1 f}{\alpha_1 f + \alpha_2}$$

(See documentation for class `Cdf` method `NoncentralF` for a discussion of how the noncentral F CDF is defined and calculated.)

With a noncentrality parameter of zero, the noncentral beta distribution is the same as the beta distribution.

Noncentralchi

```
static public double Noncentralchi(double chsq, double df, double alam)
```

Description

Evaluates the noncentral chi-squared cumulative probability distribution function.

Parameters

`chsq` – A `double` scalar value representing the argument at which the function is to be evaluated.

`df` – A `double` scalar value representing the number of degrees of freedom. `df` must be positive.

`alam` – A `double` scalar value representing the noncentrality parameter. This must be nonnegative, and `alam + df` must be less than or equal to 200,000.

Returns

A `double` scalar value representing the probability that a chi-squared random variable takes a value less than or equal to `chsq`.

Remarks

Method `Cdf.Noncentralchi` evaluates the distribution function, *F*, of a noncentral chi-squared random variable with `df` degrees of freedom and noncentrality parameter `alam`, that is, with $\nu = \text{df}$, $\lambda = \text{alam}$, and $\chi = \text{chsq}$,

$$F(x) = \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} \int_0^x \frac{t^{(\nu+2i)/2-1}e^{-t/2}}{2^{(\nu+2i)/2}\Gamma\left(\frac{\nu+2i}{2}\right)} dt$$

where $\Gamma(\cdot)$ is the gamma function. This is a series of central chi-squared distribution functions with Poisson weights. The value of the distribution function at the point *x* is the probability that the random variable takes a value less than or equal to *x*.

The noncentral chi-squared random variable can be defined by the distribution function above, or alternatively and equivalently, as the sum of squares of independent normal random variables. If the Y_i have independent normal distributions with means μ_i and variances equal to one and

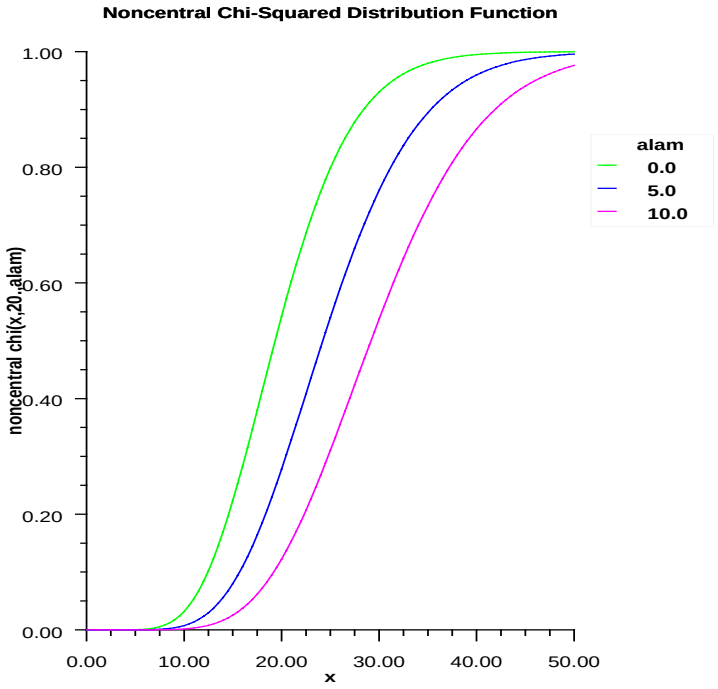
$$\chi = \sum_{i=1}^{n} Y_i^2$$

then χ has a noncentral chi-squared distribution with *n* degrees of freedom and noncentrality parameter equal to

$$\sum_{i=1}^{n} \mu_i^2$$

With a noncentrality parameter of zero, the noncentral chi-squared distribution is the same as the chi-squared distribution.

`Cdf.Noncentralchi` determines the point at which the Poisson weight is greatest, and then sums forward and backward from that point, terminating when the additional terms are sufficiently small or when a maximum of 1000 terms have been accumulated. The recurrence relation 26.4.8 of Abramowitz and Stegun (1964) is used to speed the evaluation of the central chi-squared distribution functions.



NoncentralF

```
static public double NoncentralF(double f, double df1, double df2, double
lambda)
```

Description

Evaluates the noncentral *F* cumulative distribution function (*CDF*).

Parameters

`f` – A `double` value representing the argument at which the function is to be evaluated. `f` must be nonnegative.

`df1` – A `double` value representing the number of numerator degrees of freedom. `df1` must be positive.

`df2` – A `double` value representing the number of denominator degrees of freedom. `df2` must be positive.

`lambda` – A `double` value representing the noncentrality parameter. `lambda` must be nonnegative.

Returns

A `double` scalar value representing the probability that a noncentral *F* random variable takes a value less than or equal to `f`.

Remarks

The noncentral *F* distribution is a generalization of the *F* distribution. If X is a noncentral chi-square random variable with noncentrality parameter λ and ν_1 degrees of freedom, and Y is a chi-square random variable with ν_2 degrees of freedom which is statistically independent of X, then

$$F = \frac{(X/\nu_1)}{(Y/\nu_2)}$$

is a noncentral *F*-distributed random variable whose *CDF* is given by:

$$CDF(f, \nu_1, \nu_2, \lambda) = \sum_{j=0}^{\infty} c_j$$

where:

$$c_j = \omega_j \, I_x(\frac{\nu_1}{2} + j, \frac{\nu_1}{2})$$

$$\omega_j = e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} = \frac{\lambda}{2j} \omega_{j-1}$$

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}$$

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt = x^a \sum_{j=0}^{\infty} \frac{\Gamma(j+1-b)}{(a+j)\,\Gamma(1-b)\,j!}) \, x^j$$

$$B(a,b) = B_1(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

$$I_x(a+1,b) = I_x(a,b) - T_x(a,b)$$

$$T_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\,\Gamma(b)} x^a(1-x)^b = T_x(a-1,b) \, \frac{a-1+b}{a} \, x$$

$$x = \frac{\nu_1 f}{\nu_2 + \nu_1 f}$$

and $\Gamma(\cdot)$ is the gamma function, ν_1 = `df1`, ν_2 = `df2`, λ = `lambda`, and f = `f`.

With a noncentrality parameter of zero, the noncentral *F* distribution is the same as the *F* distribution.

NoncentralstudentsT

```
static public double NoncentralstudentsT(double t, int idf, double delta)
```

Description

Evaluates the noncentral Student's *t* cumulative probability distribution function.

Parameters

`t` – A `double` scalar value representing the argument at which the function is to be evaluated.

`idf` – An `int` scalar value representing the number of degrees of freedom. This must be positive.

`delta` – A `double` scalar value representing the noncentrality parameter.

Returns

A `double` scalar value representing the probability that a noncentral Student's *t* random variable takes a value less than or equal to `t`.

Remarks

Method `Cdf.NoncentralstudentsT` evaluates the distribution function `F` of a noncentral *t* random variable with `idf` degrees of freedom and noncentrality parameter `delta`; that is, with $\nu = \text{idf}$, $\delta = \text{delta}$, and $t_0 = \text{t}$,

$$F(t_0) = \int_{-\infty}^{t_0} \frac{\nu^{\nu/2} e^{-\delta^2/2}}{\sqrt{\pi}\Gamma(\nu/2)(\nu+x^2)^{(\nu+1)/2}} \sum_{i=0}^{\infty} \Gamma((\nu+i+1)/2) \left(\frac{\delta^i}{i!}\right) \left(\frac{2x^2}{\nu+x^2}\right)^{i/2} dx$$

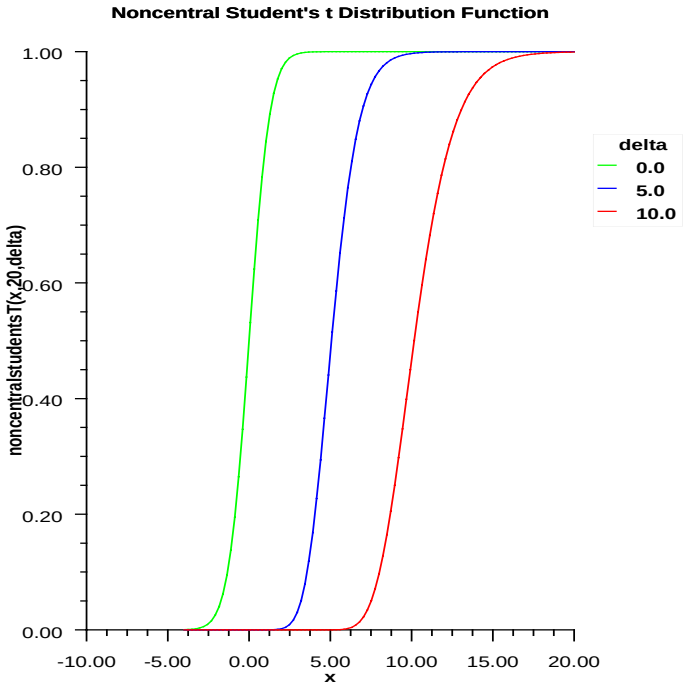
where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function at the point t_0 is the probability that the random variable takes a value less than or equal to t_0.

The noncentral *t* random variable can be defined by the distribution function above, or alternatively and equivalently, as the ratio of a normal random variable and an independent chi-squared random variable. If *w* has a normal distribution with mean δ and variance equal to one, *u* has an independent chi-squared distribution with ν degrees of freedom, and

$$x = w/\sqrt{u/\nu}$$

then *x* has a noncentral *t* distribution with ν degrees of freedom and noncentrality parameter δ.

The distribution function of the noncentral *t* can also be expressed as a double integral involving a normal density function (see, for example, Owen 1962, page 108). The method `Cdf.NoncentralstudentsT` uses the method of Owen (1962, 1965), which uses repeated integration by parts on that alternate expression for the distribution function.



Normal

```
static public double Normal(double x)
```

Description

Evaluates the normal (Gaussian) cumulative probability distribution function.

Parameter

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

Returns

A `double` scalar value representing the probability that a normal variable takes a value less than or equal to `x`.

Remarks

Method `Cdf.Normal` evaluates the distribution function, Φ, of a standard normal (Gaussian) random variable, that is,

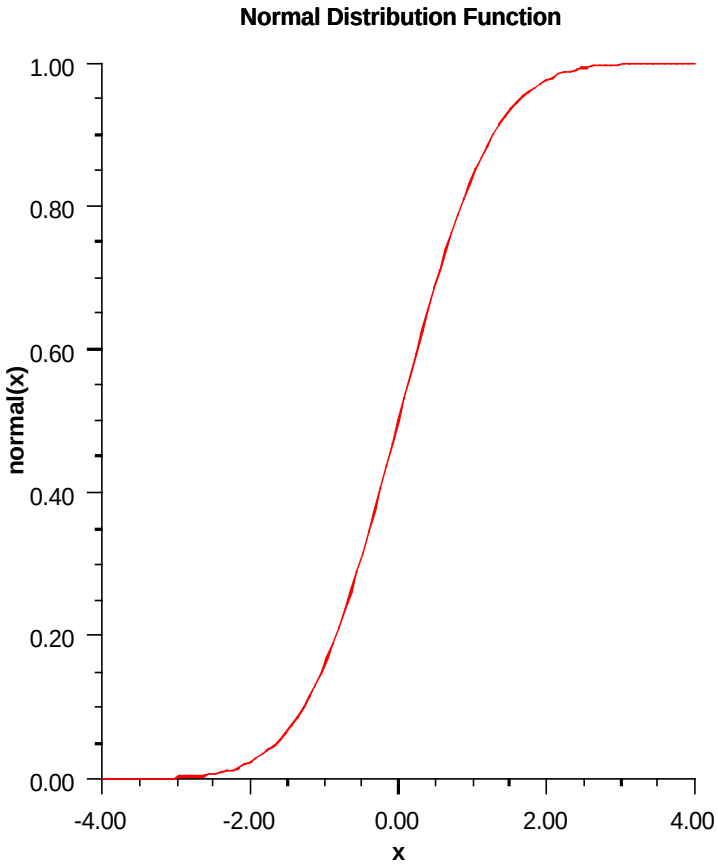
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

The value of the distribution function at the point `x` is the probability that the random variable takes a value less than or equal to `x`.

The standard normal distribution (for which `Cdf.Normal` is the distribution function) has mean of 0 and variance of 1. The probability that a normal random variable with mean μ and variance σ^2 is less than y is given by `Cdf.Normal` evaluated at $(y-\mu)/\sigma$.

$\Phi(x)$ is evaluated by use of the complementary error function, erfc. The relationship is:

$$\Phi(x) = \text{erfc}(-x/\sqrt{2.0})/2$$



Pareto

```
static public double Pareto(double x, double xm, double k)
```

Description

Evaluates the Pareto cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`xm` – A `double` scalar value representing the scale parameter, x_m.

`k` – A `double` scalar value representing the shape parameter, k.

Returns

A `double` scalar value representing the probability that a Pareto random variable takes a value less than or equal to `x`.

Remarks

Method `Cdf.Pareto` evaluates the distribution function, F, of a Pareto random variable with scale parameter x_m and shape parameter `k`. It is given by

$$F(x, x_m, k) = 1 - \left(\frac{x_m}{x}\right)^k$$

where $x_m > 0$ and $k > 0$. The function is only defined for $x \geq x_m$.

Poisson

```
static public double Poisson(int k, double theta)
```

Description

Evaluates the Poisson cumulative probability distribution function.

Parameters

`k` – The `int` argument for which the Poisson distribution function is to be evaluated.

`theta` – A `double` scalar value representing the mean of the Poisson distribution.

Returns

A `double` scalar value representing the probability that a Poisson random variable takes a value less than or equal to `k`.

Remarks

`Cdf.Poisson` evaluates the distribution function of a Poisson random variable with parameter `theta`. `theta`, which is the mean of the Poisson random variable, must be positive. The probability function (with $\theta = theta$) is

$$f(x) = e^{-\theta}\theta^x/x! \text{ for } x = 0, 1, 2, \ldots$$

The individual terms are calculated from the tails of the distribution to the mode of the distribution and summed. `Cdf.Poisson` uses the recursive relationship

$$f(x+1) = f(x)(\theta/(x+1)), \quad \text{for } x = 0, 1, 2, \ldots k-1$$

with $f(0) = e^{-\theta}$.

Rayleigh

```
static public double Rayleigh(double x, double alpha)
```

Description

Evaluates the Rayleigh cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated. It must be non-negative.

`alpha` – A `double` scalar value representing the scale parameter.

Returns

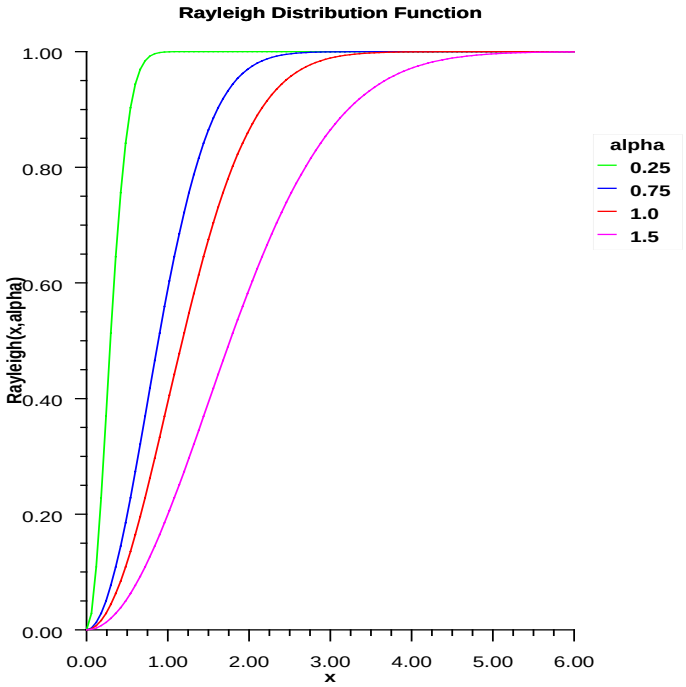
A `double` scalar value representing the probability that a Rayleigh random variable takes a value less than or equal to `x`.

Remarks

Method `Cdf.Rayleigh` is a special case of Weibull distribution function where the shape parameter `gamma` is 2.0; that is,

$$F(x) = 1 - e^{-\frac{x^2}{2\alpha^2}}$$

where `alpha` is the scale parameter.



StudentsT

```
static public double StudentsT(double t, double df)
```

Description

Evaluates the Student's t cumulative probability distribution function.

Parameters

`t` – A `double` scalar value representing the argument at which the function is to be evaluated.

`df` – A `double` scalar value representing the number of degrees of freedom. This must be at least one.

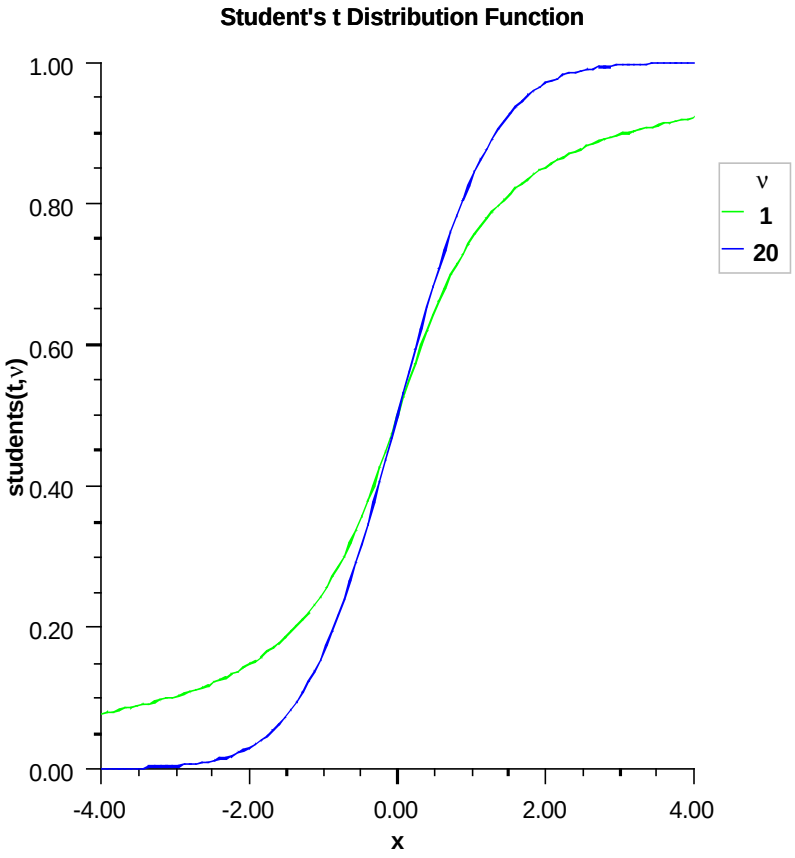
Returns

A `double` scalar value representing the probability that a Student's t random variable takes a value less than or equal to `t`.

Remarks

Method `Cdf.StudentsT` evaluates the distribution function of a Student's *t* random variable with `df` degrees of freedom. If the square of `t` is greater than or equal to `df`, the relationship of a `t` to an *f* random variable (and subsequently, to a beta random variable) is exploited, and routine `Cdf.Beta` is used. Otherwise, the method described by Hill (1970) is used. If `df` is not an integer, if `df` is greater than 19, or if `df` is greater than 200, a Cornish-Fisher expansion is used to evaluate the distribution function. If `df` is less than 20 and |t| is less than 2.0, a trigonometric series (see Abramowitz and Stegun 1964, equations 26.7.3 and 26.7.4, with some rearrangement) is used. For the remaining cases, a series given by Hill (1970) that converges well for large values of `t` is used.

For greater right tail accuracy, see Cdf.ComplementaryStudentsT (p. 1111).



Uniform

```
static public double Uniform(double x, double aa, double bb)
```

Description

Evaluates the uniform cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`aa` – A `double` scalar value representing the location parameter.

`bb` – A `double` scalar value representing the scale parameter.

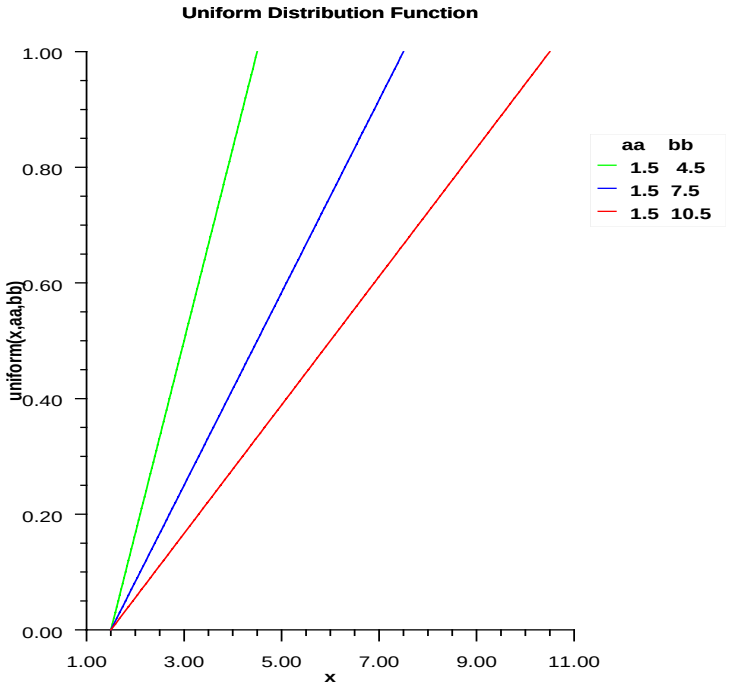
Returns

A `double` scalar value representing the probability that a Uniform random variable takes a value less than or equal to `x`.

Remarks

Method `Cdf.Uniform` evaluates the distribution function, *F*, of a uniform random variable with location parameter `aa` and scale parameter `bb`; that is,

$$f(x) = \begin{cases} 0 & \text{if } x < aa \\ \frac{x-aa}{bb-aa} & \text{if } aa \leq x \leq bb \\ 1 & \text{if } x > bb \end{cases}$$



Weibull

```
static public double Weibull(double x, double gamma, double alpha)
```

Description

Evaluates the Weibull cumulative probability distribution function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated. It must be non-negative.

`gamma` – A `double` scalar value representing the shape parameter, γ.

`alpha` – A `double` scalar value representing the scale parameter, α.

Returns

A `double` scalar value representing the probability that a Weibull random variable takes a value less than

or equal to x.

Remarks

Method Cdf.Weibull evaluates the distribution function given by

$$F(x, \gamma, \alpha) = 1 - e^{-(x/\alpha)^{\gamma}}$$

Example: The Cumulative Distribution Functions

Various cumulative distribution functions are exercised. Their use in this example typifies the manner in which other functions in the Cdf class would be used.

```
using System;
using Imsl.Stat;

public class CdfEx1
{
    public static void  Main(String[] args)
    {
        double x, prob, result;
        int p, q, k, n;
        // Beta
        x = .5;
        p = 12;
        q = 12;
        result = Cdf.Beta(x, p, q);
        Console.Out.WriteLine("Cdf.Beta(.5, 12, 12) is " + result);


        // binomial
        k = 3;
        n = 5;
        prob = .95;
        result = Cdf.Binomial(k, n, prob);
        Console.Out.WriteLine("Cdf.Binomial(3, 5, .95) is " + result);

        // Chi
        x = .15;
        n = 2;
        result = Cdf.Chi(x, n);
        Console.Out.WriteLine("Cdf.Chi(.15, 2) is " + result);

    }
}
```

Output

```
Cdf.Beta(.5, 12, 12) is 0.500000000000002
Cdf.Binomial(3, 5, .95) is 0.0225925
Cdf.Chi(.15, 2) is 0.0722565136714471
```

InvCdf Class

```
public class Imsl.Stat.InvCdf
```

Inverse cumulative probability distribution functions.

See Also

Cumulative Distribution Function (p. 1137), Probability density function (p. 1149)

Constructor

InvCdf

```
public InvCdf()
```

Description

Initializes a new instance of the Imsl.Stat.InvCdf (p. 1137) class.

Methods

Beta

```
static public double Beta(double p, double pin, double qin)
```

Description

Evaluates the inverse of the beta cumulative probability distribution function.

Parameters

`p` – A `double`, the probability for which the inverse of the beta CDF is to be evaluated.

`pin` – A `double`, the first beta distribution parameter.

`qin` – A `double`, the second beta distribution parameter.

Returns

A `double`, the probability that a beta random variable takes a value less than or equal to this returned value is `p`.

Remarks

Method `InvCdf.Beta` evaluates the inverse distribution function of a beta random variable with parameters `pin` and `qin`, that is, with P = `p`, p = `pin`, and q = `qin`, it determines x (equal to `InvCdf.Beta(p, pin, qin)`), such that

$$P = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_0^x t^{p-1}(1-t)^{q-1}\,dt$$

where $\Gamma(\cdot)$ is the Gamma function. The probability that the random variable takes a value less than or equal to x is P.

Chi

```
static public double Chi(double p, double df)
```

Description

Evaluates the inverse of the chi-squared cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse chi-squared function is to be evaluated.

`df` – A `double` scalar value representing the number of degrees of freedom. This must be at least 0.5.

Returns

A `double` scalar value. The probability that a chi-squared random variable takes a value less than or equal to this returned value is `p`.

Remarks

Method `InvCdf.Chi` evaluates the inverse distribution function of a chi-squared random variable with `df` degrees of freedom, that is, with P = `p` and v = `df`, it determines x (equal to `InvCdf.Chi(p, df)`), such that

$$P = \frac{1}{2^{v/2}\Gamma(v/2)} \int_0^x e^{-t/2} t^{v/2-1}\,dt$$

where $\Gamma(\cdot)$ is the Gamma function. The probability that the random variable takes a value less than or equal to x is P.

For $v < 40$, `InvCdf.Chi` uses bisection, if $v \geq 2$ or $P > 0.98$, or regula falsi to find the point at which the chi-squared distribution function is equal to P. The distribution function is evaluated using Cdf.Chi (p. 1106). For $40 \leq v < 100$, a modified Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.18) to the normal distribution is used, and `InvCdf.Normal` is used to evaluate the inverse of the normal distribution function. For $v \geq 100$, the ordinary Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.17) is used.

DiscreteUniform

```
static public int DiscreteUniform(double p, int n)
```

Description

Returns the inverse of the discrete uniform cumulative probability distribution function.

Parameters

p – A `double` scalar value representing the probability for which the inverse discrete uniform function is to be evaluated.

n – An `int` scalar value representing the upper limit of the discrete uniform distribution.

Returns

An `int` scalar value. The probability that a discrete uniform random variable takes a value less than or equal to this returned value is p.

Exponential

```
static public double Exponential(double p, double scale)
```

Description

Evaluates the inverse of the exponential cumulative probability distribution function.

Parameters

p – A `double` scalar value representing the probability at which the function is to be evaluated.

`scale` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value. The probability that an exponential random variable takes a value less than or equal to this returned value is p.

Remarks

Method `InvCdf.Exponential` evaluates the inverse distribution function of a Gamma random variable with scale parameter $=b$ and shape parameter a=1.0, that is, it determines $x = \text{exponential}(p, 1.0))$, such that

$$P = \frac{1}{\Gamma(a)} \int_o^x e^{-t/b} dt$$

where $\Gamma(\cdot)$ is the Gamma function. The probability that the random variable takes a value less than or equal to x is P. See the Cdf.Gamma (p. 1117) remarks for further discussion of the Gamma distribution.

`InvCdf.Exponential` uses bisection and modified regula falsi to invert the distribution function, which is evaluated using method `Cdf.Gamma`.

ExtremeValue

```
static public double ExtremeValue(double p, double mu, double beta)
```

Description

Returns the inverse of the extreme value cumulative probability distribution function.

Parameters

p – A `double` scalar value representing the probability for which the inverse extreme value function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter.

`beta` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value. The probability that an extreme value random variable takes a value less than or equal to this returned value is `p`.

F

```
static public double F(double p, double dfn, double dfd)
```

Description

Returns the inverse of the F cumulative probability distribution function.

Parameters

`p` – A `double`, the probability for which the inverse of the F distribution function is to be evaluated. Argument `p` must be in the open interval (0.0, 1.0).

`dfn` – A `double`, the numerator degrees of freedom. It must be positive.

`dfd` – A `double`, the denominator degrees of freedom. It must be positive.

Returns

A `double`, the probability that an F random variable takes a value less than or equal to this returned value is `p`.

Remarks

Method `InvCdf.F` evaluates the inverse distribution function of a Snedecor's F random variable with `dfn` numerator degrees of freedom and `dfd` denominator degrees of freedom. The function is evaluated by making a transformation to a beta random variable and then using InvCdf.Beta (p. 1137). If X is an F variate with v_1 and v_2 degrees of freedom and $Y = v_1X/(v_2 + v_1X)$, then Y is a beta variate with parameters $p = v_1/2$ and $q = v_2/2$. If $P \leq 0.5$, `F` uses this relationship directly, otherwise, it also uses a relationship between X random variables that can be expressed as follows, using `f`, which is the F cumulative distribution function:

$$\mathrm{F}(X, dfn, dfd) = 1 - \mathrm{F}(1/X, dfd, dfn)$$

Gamma

```
static public double Gamma(double p, double a)
```

Description

Evaluates the inverse of the Gamma cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability at which the function is to be evaluated.

`a` – A `double` scalar value representing the shape parameter. This must be positive.

Returns

A `double` scalar value. The probability that a Gamma random variable takes a value less than or equal to this returned value is `p`.

Remarks

Method `InvCdf.Gamma` evaluates the inverse distribution function of a Gamma random variable with shape parameter `a`, that is, it determines $x = \text{Gamma}(p, a))$, such that

$$P = \frac{1}{\Gamma(a)} \int_o^x e^{-t} t^{a-1} dt$$

where $\Gamma(\cdot)$ is the Gamma function. The probability that the random variable takes a value less than or equal to x is P. See the Cdf.Gamma (p. 1117) remarks for further discussion of the `Gamma` distribution.

`InvCdf.Gamma` uses bisection and modified regula falsi to invert the distribution function, which is evaluated using method `Cdf.Gamma`.

Geometric

```
static public double Geometric(double r, double pin)
```

Description

Returns the inverse of the discrete geometric cumulative probability distribution function.

Parameters

`r` – A `double` scalar value representing the probability for which the inverse geometric function is to be evaluated.

`pin` – An `int` scalar value representing the probability parameter for each independent trial (the probability of success for each independent trial).

Returns

A `double` scalar value. The probability that a geometric random variable takes a value less than or equal to this returned value is the input probability, `r`.

Logistic

```
static public double Logistic(double p, double mu, double s)
```

Description

Returns the inverse of the logistic cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse logistic function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter, μ.

`s` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value. The probability that a logistic random variable takes a value less than or equal to this returned value is `p`.

LogNormal

```
static public double LogNormal(double p, double mu, double sigma)
```

Description

Returns the inverse of the standard LogNormal cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse LogNormal function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter.

`sigma` – A `double` scalar value representing the shape parameter. `sigma` must be a positive.

Returns

A `double` scalar value. The probability that a standard LogNormal random variable takes a value less than or equal to this returned value is `p`.

NoncentralBeta

```
static public double NoncentralBeta(double p, double shape1, double shape2,
double lambda)
```

Description

Evaluates the inverse of the noncentral beta cumulative distribution function (*CDF*).

Parameters

`p` – A `double` scalar value representing the probability for which the inverse of the noncentral beta cumulative distribution function is to be evaluated. `p` must be non-negative and less than or equal to one.

`shape1` – A `double` scalar value representing the first shape parameter. `shape1` must be positive.

`shape2` – A `double` scalar value representing the second shape parameter. `shape2` must be positive.

`lambda` – A `double` scalar value representing the noncentrality parameter. `lambda` must be nonnegative.

Returns

A `double` scalar value representing the inverse of the noncentral beta distribution function evaluated at `p`. The probability that a noncentral beta random variable takes a value less than or equal to `NoncentralBeta` is `p`.

Remarks

If Z is a noncentral chi-square random variable with noncentrality parameter λ and $2\alpha_1$ degrees of freedom, and Y is a chi-square random variable with $2\alpha_2$ degrees of freedom which is statistically independent of Z, then

$$X = \frac{Z}{Z + Y} = \frac{\alpha_1 F}{\alpha_1 F + \alpha_2}$$

is a noncentral beta-distributed random variable and

$$F = \frac{\alpha_2 Z}{\alpha_1 Y} = \frac{\alpha_2 X}{\alpha_1 (1 - X)}$$

is a noncentral *F*-distributed random variable. The CDF for noncentral beta variable *X* can thus be simply defined in terms of the noncentral *F* CDF:

$$CDF_{nc\beta}(x,\ \alpha_1,\ \alpha_2,\ \lambda)\ =\ CDF_{ncF}(f,\ 2\alpha_1,\ 2\alpha_2,\ \lambda)$$

where $CDF_{nc\beta}(x,\ \alpha_1,\ \alpha_2,\ \lambda)$ is the noncentral beta CDF with x = `x`, α_1 = `shape1`, α_2 = `shape2`, and noncentrality parameter λ = `lambda`; $CDF_{ncF}(f,\ 2\alpha_1,\ 2\alpha_2,\ \lambda)$ is the noncentral F CDF with argument *f*, numerator and denominator degrees of freedom $2\alpha_1$ and $2\alpha_2$ respectively, and noncentrality parameter λ; and:

$$f\ =\ \frac{\alpha_2 x}{\alpha_1(1-x)};\ x\ =\ \frac{\alpha_1 f}{\alpha_1 f\ +\ \alpha_2}$$

(See documentation for class `Cdf` method `NoncentralF` for a discussion of how the noncentral F CDF is defined and calculated.)

Method `InvCdf.NoncentralBeta` evaluates

$$x\ =\ CDF^{-1}_{nc\beta}(p,\ \alpha_1,\ \alpha_2,\ \lambda)$$

by first evaluating:

$$f\ =\ CDF^{-1}_{ncF}(p,\ 2\alpha_1,\ 2\alpha_2,\ \lambda)$$

and then solving for x using $x\ =\ \frac{\alpha_1 f}{\alpha_1 f+\alpha_2}$. (See documentation for InvCdf.NoncentralF (p. 1144) for a discussion of how the inverse noncentral *F* CDF is calculated.)

Noncentralchi

```
static public double Noncentralchi(double p, double df, double alam)
```

Description

Evaluates the inverse of the noncentral chi-squared cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse noncentral chi-squared distribution function is to be evaluated. `p` must be in the open interval (0.0, 1.0).

`df` – A `double` scalar value representing the number of degrees of freedom. This must be at least 0.5. but less than or equal to 200,000.

`alam` – A `double` scalar value representing the noncentrality parameter. This must be nonnegative, and `alam` + `df` must be less than or equal to 200,000.

Returns

A `double` scalar value. The probability that a noncentral chi-squared random variable takes a value less than or equal to this returned value is `p`.

Remarks

Method `InvCdf.Noncentralchi` evaluates the inverse distribution function of a noncentral chi-squared random variable with `df` degrees of freedom and noncentrality parameter `alam`, that is, with P = `p`, v = df, and λ = alam, it determines c_0 = `InvCdf.Noncentralchi(p, df, alam)`, such that

$$P=\sum_{i=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^i}{i!}\int_0^{c_0}\frac{x^{(v+2i)/2-1}e^{-x/2}}{2^{(v+2i)/2}\Gamma\left(\frac{v+2i}{2}\right)}dx$$

where $\Gamma(\cdot)$ is the Gamma function. The probability that the random variable takes a value less than or equal to c_0 is P.

Method `InvCdf.Noncentralchi` uses bisection and modified regula falsi to invert the distribution function, which is evaluated using `Cdf.Noncentralchi`. See Cdf.Noncentralchi (p. 1123) for an alternative definition of the noncentral chi-squared random variable in terms of normal random variables.

NoncentralF

```
static public double NoncentralF(double p, double dfn, double dfd, double
lambda)
```

Description

Evaluates the inverse of the noncentral F cumulative distribution function (*CDF*).

Parameters

`p` – A `double` scalar value representing the probability for which the inverse of the noncentral *F* cumulative distribution function is to be evaluated. `p` must be non-negative and less than one.

`dfn` – A `double` scalar value representing the number of numerator degrees of freedom. `dfn` must be positive.

`dfd` – A `double` scalar value representing the number of denominator degrees of freedom. `dfd` must be positive.

`lambda` – A `double` scalar value representing the noncentrality parameter. `lambda` must nonnegative.

Returns

A `double` scalar value representing the inverse of the noncentral *F* distribution function evaluated at `p`. The probability that a noncentral *F* random variable takes a value less than or equal to `InvCdf.NoncentralF(p, dfn, dfd, lambda)` is `p`.

Remarks

If X is a noncentral chi-square random variable with noncentrality parameter λ and ν_1 degrees of freedom, and Y is a chi-square random variable with ν_2 degrees of freedom which is statistically independent of X, then

$$F = (X/\nu_1)/(Y/\nu_2)$$

is a noncentral F-distributed random variable whose CDF is given by:

$$CDF(f, \nu_1, \nu_2, \lambda) = \int_0^f PDF(x, \nu_1, \nu_2, \lambda)dx$$

where the probability density function $PDF(x, \nu_1, \nu_2, \lambda)$ is given by:

$$PDF(x, \nu_1, \nu_2, \lambda) = \Psi \sum_{k=0}^{\infty} \Phi_k$$

$$\Psi = \frac{e^{-\lambda/2}(\nu_1 x)^{\nu_1/2}(\nu_2)^{\nu_2/2}}{x\,(\nu_1 x + \nu_2)^{(\nu_1+\nu_2)/2}\,\Gamma(\nu_2/2)}$$

$$\Phi_k = \frac{R^k\,\Gamma(\frac{\nu_1+\nu_2}{2}+k)}{k!\,\Gamma(\frac{\nu_1}{2}+k)}$$

$$R = \frac{\lambda\nu_1 x}{2(\nu_1 x+\nu_2)}$$

where $\Gamma(\cdot)$ is the Gamma function, ν_1 = `dfn`, ν_2 = `dfd`, λ = `lambda`, and $p = CDF(f,\nu_1,\nu_2,\lambda)$ is the probability that $F \le f$.

Method `InvCdf.NoncentralF` evaluates

$$f = CDF^{-1}(p,\nu_1,\nu_2,\lambda)$$

Method `InvCdf.NoncentralF` uses bisection and modified regula falsi search algorithms to invert the distribution function $CDF(f,\nu_1,\nu_2,\lambda)$, which is evaluated using method Cdf.NoncentralF (p. 1125). For sufficiently small p, an accurate approximation of $CDF^{-1}(p,\nu_1,\nu_2,\lambda)$ can be used which requires no such inverse search algorithms.

NoncentralstudentsT

```
static public double NoncentralstudentsT(double p, int idf, double delta)
```

Description

Evaluates the inverse of the noncentral Student's *t* cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the function is to be evaluated.

`idf` – An `int` scalar value representing the number of degrees of freedom. This must be positive.

`delta` – A `double` scalar value representing the noncentrality parameter.

Returns

A `double` scalar value. The probability that a noncentral Student's t random variable takes a value less than or equal to this returned value is `p`.

Remarks

Method `InvCdf.NoncentralstudentsT` evaluates the inverse distribution function of a noncentral *t* random variable with `idf` degrees of freedom and noncentrality parameter `delta`; that is, with $P = p$, $\nu = idf$, $\delta = delta$, it determines t_0 = `InvCdf.NoncentralstudentsT(p, idf, delta)`, such that

$$P = \int_{-\infty}^{t_0} \frac{\nu^{\nu/2}e^{-\delta^2/2}}{\sqrt{\pi}\Gamma(\nu/2)(\nu+x^2)^{(\nu+1)/2}} \sum_{i=0}^{\infty} \Gamma((\nu+i+1)/2)\left(\frac{\delta^i}{i!}\right)\left(\frac{2x^2}{\nu+x^2}\right)^{i/2} dx$$

where $\Gamma(\cdot)$ is the Gamma function. The probability that the random variable takes a value less than or equal to t_0 is `P`. See Cdf.NoncentralstudentsT (p. 1126) for an alternative definition in terms of normal and chi-squared random variables. The method `InvCdf.NoncentralstudentsT` uses bisection and modified regula falsi to invert the distribution function, which is evaluated using `Cdf.NoncentralstudentsT`.

Normal

```
static public double Normal(double p)
```

Description

Evaluates the inverse of the normal (Gaussian) cumulative probability distribution function.

Parameter

`p` – A `double` scalar value representing the probability at which the function is to be evaluated.

Returns

A `double` scalar value. The probability that a standard normal random variable takes a value less than or equal to this returned value is `p`.

Remarks

Method `InvCdf.Normal` evaluates the inverse of the distribution function, Φ, of a standard normal (Gaussian) random variable, that is, `InvCdf.Normal` $(\mathrm{p}) = \Phi - 1(p)$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x. The standard normal distribution has a mean of 0 and a variance of 1.

Pareto

```
static public double Pareto(double p, double xm, double k)
```

Description

Returns the inverse of the Pareto cumulative probability density function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse Pareto function is to be evaluated.

`xm` – A `double` scalar value representing the scale parameter, x_m.

`k` – A `double` scalar value representing the shape parameter.

Returns

A `double` scalar value. The probability that a Pareto random variable takes on a value less than or equal to this returned value is `p`.

Rayleigh

```
static public double Rayleigh(double p, double alpha)
```

Description

Returns the inverse of the Rayleigh cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse Rayleigh function is to be evaluated.

`alpha` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value. The probability that a Rayleigh random variable takes a value less than or equal to this returned value is `p`.

StudentsT

```
static public double StudentsT(double p, double df)
```

Description

Returns the inverse of the Student's t cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse Student's *t* function is to be evaluated.

`df` – A `double` scalar value representing the number of degrees of freedom. This must be at least one.

Returns

A `double` scalar value. The probability that a Student's *t* random variable takes a value less than or equal to this returned value is `p`.

Remarks

`InvCdf.StudentsT` evaluates the inverse distribution function of a Student's *t* random variable with `df` degrees of freedom. Let *v* = `df`. If *v* equals 1 or 2, the inverse can be obtained in closed form, if *v* is between 1 and 2, the relationship of a *t* to a beta random variable is exploited and InvCdf.Beta (p. 1137) is used to evaluate the inverse; otherwise the algorithm of Hill (1970) is used. For small values of *v* greater than 2, Hill's algorithm inverts an integrated expansion in $1/(1+t^2/v)$ of the *t* density. For larger values, an asymptotic inverse Cornish-Fisher type expansion about normal deviates is used.

Uniform

```
static public double Uniform(double p, double aa, double bb)
```

Description

Returns the inverse of the uniform cumulative probability distribution function.

Parameters

`p` – A `double` scalar value representing the probability for which the inverse uniform function is to be evaluated.

`aa` – A `double` scalar value representing the minimum value.

`bb` – A `double` scalar value representing the maximum value.

Returns

A `double` scalar value. The probability that a uniform random variable takes a value less than or equal to this returned value is `p`.

Weibull

```
static public double Weibull(double p, double gamma, double alpha)
```

Description

Returns the inverse of the Weibull cumulative probability distribution function.

Parameters

p – A `double` scalar value representing the probability for which the inverse Weibull function is to be evaluated.

gamma – A `double` scalar value representing the shape parameter.

alpha – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value. The probability that a Weibull random variable takes a value less than or equal to this returned value is p.

Example: The Inverse Cumulative Distribution Functions

Examples of the inverse cumulative distribution functions are exercised. Their use in this example typifies the manner in which other functions in the InvCdf class would be used.

```
using System;
using InvCdf = Imsl.Stat.InvCdf;

public class InvCdfEx1
{

    public static void  Main(String[] args)
    {

        // Inverse Beta
        double x = .5;
        double pin = 12.0;
        double qin = 12.0;
        double result = InvCdf.Beta(x, pin, qin);
      Console.Out.WriteLine("InvCdf.Beta(.5, 12., 12.) is {0,5:0.0000}", result);

        // Inverse Chi
      double prob = .99;
        int n = 2;
        result = InvCdf.Chi(prob, n);
      Console.Out.WriteLine("InvCdf.Chi(.99, 2) is {0,5:0.0000}", result);
    }
}
```

Output

```
InvCdf.Beta(.5, 12., 12.) is 0.5000
InvCdf.Chi(.99, 2) is 9.2103
```

Pdf Class

`public class Imsl.Stat.Pdf`

Probability density functions.

See Also

Cumulative distribution function (p. 1101), Inverse Cumulative Distribution Function (p. 1137)

Methods

Beta

`static public double Beta(double x, double pin, double qin)`

Description

Evaluates the beta probability density function.

Parameters

`x` – A `double`, the argument at which the function is to be evaluated.

`pin` – A `double`, the first beta distribution parameter.

`qin` – A `double`, the second beta distribution parameter.

Returns

A `double`, the value of the probability density function at `x`.

Binomial

`static public double Binomial(int k, int n, double pin)`

Description

Evaluates the binomial probability density function.

Parameters

`k` – The `int` argument for which the binomial distribution function is to be evaluated.

`n` – The `int` number of Bernoulli trials.

`pin` – A `double` scalar value representing the probability of success on each independent trial.

Returns

A `double` scalar value representing the probability that a binomial random variable takes a value equal to `k`.

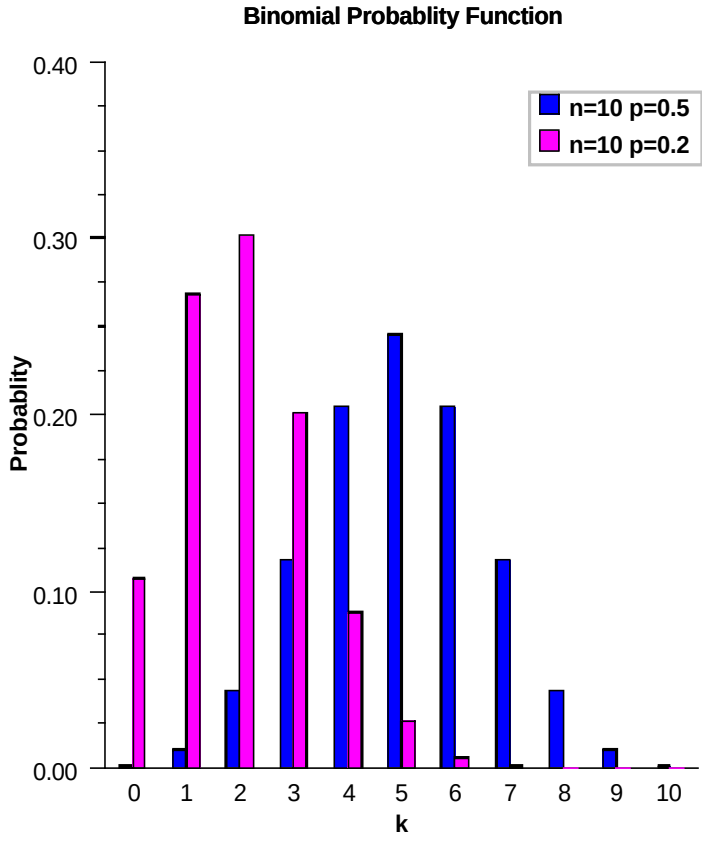
Remarks

Method `Pdf.Binomial` evaluates the probability that a binomial random variable with parameters `n` and p with p=`pin` takes on the value `k`. It does this by computing probabilities of the random variable taking on the values in its range less than (or the values greater than) `k`. These probabilities are computed by the recursive relationship

$$\Pr(X = j) = \frac{(n+1-j)\,p}{j\,(1-p)} \Pr(X = j-1)$$

To avoid the possibility of underflow, the probabilities are computed forward from 0, if `k` is not greater than $n \times p$, and are computed backward from `n`, otherwise. The smallest positive machine number, ε, is used as the starting value for computing the probabilities, which are rescaled by $(1-p)^n\varepsilon$ if forward computation is performed and by $p^n\varepsilon$ if backward computation is done.

For the special case of p = 0, `Pdf.Binomial` is set to 0 if `k` is greater than 0 and to 1 otherwise; and for the case p = 1, `Pdf.Binomial` is set to 0 if `k` is less than `n` and to 1 otherwise.



Chi

```
static public double Chi(double chsq, double df)
```

Description

Evaluates the chi-squared probability density function

Parameters

`chsq` – A `double` scalar value representing the argument at which the function is to be evaluated.

`df` – A `double` scalar value representing the number of degrees of freedom. `df` must be positive.

Returns

A `double` scalar value, the value of the probability density function at `chsq`.

DiscreteUniform

```
static public double DiscreteUniform(int x, int n)
```

Description

Evaluates the discrete uniform probability density function.

Parameters

`x` – An `int` argument for which the discrete uniform probability density function is to be evaluated. `x` should be a value between the lower limit 0 and upper limit `n`.

`n` – An `int` scalar value representing the upper limit of the discrete uniform distribution.

Returns

A `double` scalar value representing the probability that a discrete uniform random variable takes a value equal to `x`.

Exponential

```
static public double Exponential(double x, double scale)
```

Description

Evaluates the exponential probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`scale` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value, the value of the probability density function at `x`.

ExtremeValue

```
static public double ExtremeValue(double x, double mu, double beta)
```

Description

Evaluates the extreme value probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter.

`beta` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value representing the probability density function at `x`.

F

```
static public double F(double x, double dfn, double dfd)
```

Description

Evaluates the F probability density function.

Parameters

`x` – A `double`, the argument at which the function is to be evaluated.

`dfn` – A `double`, the numerator degrees of freedom. It must be positive.

`dfd` – A `double`, the denominator degrees of freedom. It must be positive.

Returns

A `double`, the value of the probability density function at `x`.

Remarks

The probability density function of the F distribution is

$$f(x, dfn, dfd) = \frac{\Gamma(\frac{v_1+v_2}{2})(\frac{v_1}{v_2})^{\frac{v_1}{2}} x^{\frac{v_1}{2}}}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})(1+\frac{v_1 x}{v_2})^{\frac{v_1+v_2}{2}}}$$

where v_1 and v_2 are the shape parameters `dfn` and `dfd` and Γ is the gamma function,

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$

Gamma

```
static public double Gamma(double x, double a, double b)
```

Description

Evaluates the gamma probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`a` – A `double` scalar value representing the shape parameter. This must be positive.

`b` – A `double` scalar value representing the scale parameter. This must be positive.

Returns

A `double` scalar value, the probability density function at `x`.

Remarks

The probability density function of the gamma distribution is

$$f(x; a, b) = x^{a-1} \frac{1}{b^a \Gamma(a)} e^{-x/b}$$

where `a` is the shape parameter and `b` is the scale parameter.

Geometric

```
static public double Geometric(int x, double pin)
```

Description

Evaluates the discrete geometric probability density function.

Parameters

`x` – The `int` argument for which the geometric probability function is to be evaluated.

`pin` – A `double` scalar value representing the probability parameter of the geometric distribution (the probability of success for each independent trial.

Returns

A `double` scalar value representing the probability that a geometric random variable takes a value equal to `x`.

Remarks

Method `Pdf.Geometric` evaluates the geometric distribution for the number of trials before the first success.

Hypergeometric

```
static public double Hypergeometric(int k, int sampleSize, int defectivesInLot,
int lotSize)
```

Description

Evaluates the hypergeometric probability density function.

Parameters

`k` – An `int`, the argument at which the function is to be evaluated.

`sampleSize` – An `int`, the sample size, `n`.

`defectivesInLot` – An `int`, the number of defectives in the lot, `m`.

`lotSize` – An `int`, the lot size, `l`.

Returns

A `double`, the probability that a hypergeometric random variable takes on a value equal to `k`.

Remarks

Method `Pdf.Hypergeometric` evaluates the probability density function of a hypergeometric random variable with parameters *n*, *l*, and *m*. The hypergeometric random variable *X* can be thought of as the number of items of a given type in a random sample of size *n* that is drawn without replacement from a population of size *l* containing *m* items of this type. The probability density function is:

$$\Pr(X = k) = \frac{\binom{m}{k}\binom{l-m}{n-k}}{\binom{l}{n}} \text{for } k = i,\ i+1, i+2\ \ldots,\ \min(n, m)$$

where *i = max(0, n - l + m)*. `Pdf.Hypergeometric` evaluates the expression using log gamma functions.

Logistic

```
static public double Logistic(double x, double mu, double s)
```

Description

Evaluates the logistic probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter.

`s` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value representing the probability density function at `x`.

Remarks

The probability density function of the logistic distribution is

$$f(x,\mu,s) = \frac{e^{-(x-\mu)/s}}{s\left(1+e^{-(x-\mu)/s}\right)^2}$$

where μ is the location parameter and the scale parameter $s > 0$.

LogNormal

```
static public double LogNormal(double x, double mu, double sigma)
```

Description

Evaluates the standard lognormal probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`mu` – A `double` scalar value representing the location parameter.

`sigma` – A `double` scalar value representing the shape parameter. `sigma` must be a positive.

Returns

A `double` scalar value representing the probability density function at `x`.

Remarks

$$F\left(x\right) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$$

NoncentralBeta

```
static public double NoncentralBeta(double x, double shape1, double shape2,
double lambda)
```

Description

Evaluates the noncentral beta probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated. `x` must be nonnegative and less than or equal to 1.

`shape1` – A `double` scalar value representing the first shape parameter. `shape1` must be positive.

`shape2` – A `double` scalar value representing the second shape parameter. `shape2` must be positive.

`lambda` – A `double` scalar value representing the noncentrality parameter. `lambda` must nonnegative.

Returns

A `double` scalar value representing the probability density associated with a noncentral beta random variable with value `x`.

Remarks

The noncentral beta distribution is a generalization of the beta distribution. If Z is a noncentral chi-square random variable with noncentrality parameter λ and $2\alpha_1$ degrees of freedom, and Y is a chi-square random variable with $2\alpha_2$ degrees of freedom which is statistically independent of Z, then

$$X = \frac{Z}{Z + Y} = \frac{\alpha_1 F}{\alpha_1 F + \alpha_2}$$

is a noncentral beta-distributed random variable and

$$F = \frac{\alpha_2 Z}{\alpha_1 Y} = \frac{\alpha_2 X}{\alpha_1 (1 - X)}$$

is a noncentral F-distributed random variable. The PDF for noncentral beta variable X can thus be simply defined in terms of the noncentral F PDF:

$$PDF_{nc\beta}(x,\ \alpha_1,\ \alpha_2,\ \lambda) = PDF_{ncF}(f,\ 2\alpha_1,\ 2\alpha_2,\ \lambda)\ \frac{df}{dx}$$

where $PDF_{nc\beta}(x,\ \alpha_1,\ \alpha_2,\ \lambda)$ is the noncentral beta PDF with x = `x`, α_1 = `shape1`, α_2 = `shape2`, and noncentrality parameter λ = `lambda`; $PDF_{ncF}(f,\ 2\alpha_1,\ 2\alpha_2,\ \lambda)$ is the noncentral F PDF with argument f, numerator and denominator degrees of freedom $2\alpha_1$ and $2\alpha_2$ respectively, noncentrality parameter λ,

$$f = \frac{\alpha_2 x}{\alpha_1 (1 - x)},$$

$$x = \frac{\alpha_1 f}{\alpha_1 f + \alpha_2},$$

and

$$\frac{df}{dx} = \frac{(\alpha_1 f + \alpha_2)^2}{\alpha_1 \alpha_2} = \frac{\alpha_2}{\alpha_1 (1 - x)^2}.$$

(See documentation for class `Pdf` method `NoncentralF` for a discussion of how the noncentral F PDF is defined and calculated.)

With a noncentrality parameter of zero, the noncentral beta distribution is the same as the beta distribution.

NoncentralChi

```
static public double NoncentralChi(double chsq, double df, double alam)
```

Description

Evaluates the noncentral chi-squared probability density function.

Parameters

`chsq` – A `double` scalar value at which the function is to be evaluated. `chsq` must be nonnegative.

`df` – A `double` scalar value representing the number of degrees of freedom. `df` must be positive.

`alam` – A `double` scalar value representing the noncentrality parameter. `alam` must be nonnegative.

Returns

A `double` scalar value representing the probability density associated with a noncentral chi-squared random variable with value `chsq`.

Remarks

The noncentral chi-squared distribution is a generalization of the chi-squared distribution. If $\{X_i\}$ are k independent, normally distributed random variables with means μ_i and variances σ_i^2, then the random variable

$$X = \sum_{i=1}^{k} \left(\frac{X_i}{\sigma_i} \right)^2$$

is distributed according to the noncentral chi-squared distribution. The noncentral chi-squared distribution has two parameters, k which specifies the number of degrees of freedom (i.e. the number of X_i), and λ which is related to the mean of the random variables X_i by

$$\lambda = \sum_{i=1}^{k} \left(\frac{\mu_i}{\sigma_i} \right)^2$$

The noncentral chi-squared distribution is equivalent to a (central) chi-squared distribution with $k+2i$ degrees of freedom, where i is the value of a Poisson distributed random variable with parameter $\lambda/2$. Thus, the probability density function is given by:

$$F(x,k,\lambda) = \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} f(x,k+2i)$$

where the (central) chi-squared Pdf $f(x,k)$ is given by:

$$f(x,k) \;=\; \frac{(x/2)^{k/2}\, e^{-x/2}}{x\,\Gamma(k/2)} \quad for\ x > 0,\ else\ 0$$

where $\Gamma(\cdot)$ is the gamma function. The above representation of $F(x,k,\lambda)$ can be shown to be equivalent to the representation:

$$F(x,k,\lambda) \;=\; \frac{e^{-(\lambda+x)/2}\,(x/2)^{k/2}}{x} \sum_{i=0}^{\infty} \phi_i$$

$$\phi_i \;=\; \frac{(\lambda x/4)^i}{i!\,\Gamma(k/2 \;+\; i)}$$

Method `Pdf.NoncentralChi` evaluates the probability density function, $F(x,k,\lambda)$, of a noncentral chi-squared random variable with `df` degrees of freedom and noncentrality parameter `alam`, corresponding to k = `df`, λ = `alam`, and x = `chsq`.

With a noncentrality parameter of zero, the noncentral chi-squared distribution is the same as the central chi-squared distribution.

NoncentralF

```
static public double NoncentralF(double f, double df1, double df2, double
lambda)
```

Description

Evaluates the noncentral F probability density function.

Parameters

`f` – A `double` value representing the argument at which the function is to be evaluated. `f` must be nonnegative.

`df1` – A `double` value representing the number of numerator degrees of freedom. `df1` must be positive.

`df2` – A `double` value representing the number of denominator degrees of freedom. `df2` must be positive.

`lambda` – A `double` value representing the noncentrality parameter. `lambda` must be nonnegative.

Returns

A `double` value representing the probability density associated with a noncentral F random variable with value `f`.

Remarks

The noncentral F distribution is a generalization of the F distribution. If x is a noncentral chi-square random variable with noncentrality parameter λ and ν_1 degrees of freedom, and y is a chi-square random variable with ν_2 degrees of freedom which is statistically independent of X, then

$$F = (x/\nu_1)/(y/\nu_2)$$

is a noncentral F-distributed random variable whose PDF is given by:

$$\text{PDF}(f,\nu_1,\nu_2,\lambda) = \Psi \sum_{k=0}^{\infty} \Phi_k$$

where

$$\Psi = \frac{e^{-\lambda/2}(\nu_1 f)^{\nu_1/2}(\nu_2)^{\nu_2/2}}{f\,(\nu_1 f + \nu_2)^{(\nu_1+\nu_2)/2}\,\Gamma(\nu_2/2)}$$

$$\Phi_k = \frac{R^k\,\Gamma(\frac{\nu_1+\nu_2}{2} + k)}{k!\,\Gamma(\frac{\nu_1}{2} + k)}$$

$$R = \frac{\lambda\,\nu_1 f}{2(\nu_1 f + \nu_2)}$$

where $\Gamma(\cdot)$ is the gamma function, ν_1 = `df1`, ν_2 = `df2`, λ = `lambda`, and f = `f`.

With a noncentrality parameter of zero, the noncentral F distribution is the same as the F distribution.

The efficiency of the calculation of the above series is enhanced by:

1. calculating each term Φ_k in the series recursively in terms of either the term Φ_{k-1} preceding it or the term Φ_{k+1} following it, and
2. initializing the sum with the largest series term and adding the subsequent terms in order of decreasing magnitude.

Special cases:

For $R = \lambda \quad f = 0$:

$$\text{PDF}(f,\nu_1,\nu_2,\lambda) = \Psi\,\Phi_0 = \Psi\,\frac{\Gamma([\nu_1+\nu_2]/2)}{\Gamma(\nu_1/2)}$$

For $\lambda = 0$:

$$\text{PDF}(f,\nu_1,\nu_2,\lambda) = \frac{(\nu_1 f)^{\nu_1/2}\,(\nu_2)^{\nu_2/2}\,\Gamma([\nu_1+\nu_2]/2)}{f\,(\nu_1 f + \nu_2)^{(\nu_1+\nu_2)/2}\,\Gamma(\nu_1/2)\,\Gamma(\nu_2/2)}$$

For $f = 0$:

$$\text{PDF}(f,\nu_1,\nu_2,\lambda) = \frac{e^{-\lambda/2}\,f^{\nu_1/2\,-\,1}\,(\nu_1/\nu_2)^{\nu_1/2}\,\Gamma([\nu_1 + \nu_2]/2)}{\Gamma(\nu_1/2)\,\Gamma(\nu_2/2)}$$

$$\text{PDF}(f,\nu_1,\nu_2,\lambda) = \begin{cases} 0 & \text{if } \nu_1 > 2; \\ e^{-\lambda/2} & \text{if } \nu_1 = 2; \\ \infty & \text{if } \nu_1 < 2 \end{cases}$$

NoncentralStudentsT

```
static public double NoncentralStudentsT(double t, double df, double delta)
```

Description

Evaluates the noncentral Student's *t* probability density function.

Parameters

`t` – A `double` value representing the argument at which the function is to be evaluated.

`df` – A `double` value representing the number of degrees of freedom. `df` must be positive.

`delta` – A `double` value representing the noncentrality parameter.

Returns

A `double` value representing the probability density associated with a noncentral Student's *t* random variable with value `t`.

Remarks

The noncentral Student's *t*-distribution is a generalization of the Student's *t*-distribution. If *w* is a normally distributed random variable with unit variance and mean δ and *u* is a chi-square random variable with ν degrees of freedom that is statistically independent of *w*, then

$$T = w/\sqrt{u/\nu}$$

is a noncentral *t*-distributed random variable with ν degrees of freedom and noncentrality parameter δ, that is, with ν = `df`, and δ = `delta`. The probability density function for the noncentral *t*-distribution is:

$$f(t,\nu,\delta) = \frac{\nu^{\nu/2}\, e^{-\delta^2/2}}{\sqrt{\pi}\,\Gamma(\nu/2)\,(\nu+t^2)^{(\nu+1)/2}} \sum_{i=0}^{\infty} \Phi_i$$

where

$$\Phi_i = \frac{\Gamma((\nu+i+1)/2)}{i!}\,[\delta t]^i \left(\frac{2}{\nu+t^2}\right)^{i/2}$$

and t = `t`.

For noncentrality parameter $\delta = 0$, the PDF reduces to the (central) Student's *t* PDF:

$$f(t,\nu,0) = \frac{\Gamma((\nu+1)/2)\,\left(1 + (t^2/\nu)\right)^{-(\nu+1)/2}}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}$$

and, for $t = 0$, the Pdf becomes:

$$f(0,\nu,\delta) = \frac{\Gamma((\nu+1)/2)\, e^{-\delta^2/2}}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}$$

Normal

```
static public double Normal(double x, double mean, double stdev)
```

Description

Evaluates the normal (Gaussian) probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`mean` – A `double` scalar value containing the mean.

`stdev` – A `double` scalar value containing the standard deviation.

Returns

A `double` containing the value of the probability density function at `x`

Remarks

The probability density function for a normal distribution is given by

$$\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are the conditional mean and standard deviation.

Pareto

```
static public double Pareto(double x, double xm, double k)
```

Description

Evaluates the Pareto probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated.

`xm` – A `double` scalar value representing the scale parameter, x_m.

`k` – A `double` scalar value representing the shape parameter.

Returns

A `double` scalar value representing the probability density function at `x`.

Remarks

The probability density function of the Pareto distribution is

$$f(x, x_m, k) = 1 - \frac{k x_m^k}{x^{k+1}}$$

where the scale parameter $x_m > 0$ and the shape parameter $k > 0$. The function is only defined for $x \geq x_m$

Poisson

```
static public double Poisson(int k, double theta)
```

Description

Evaluates the Poisson probability density function.

Parameters

`k` – An `int` scalar for which the Poisson probability function is to be evaluated.

`theta` – A `double` scalar value representing the mean of the Poisson distribution.

Returns

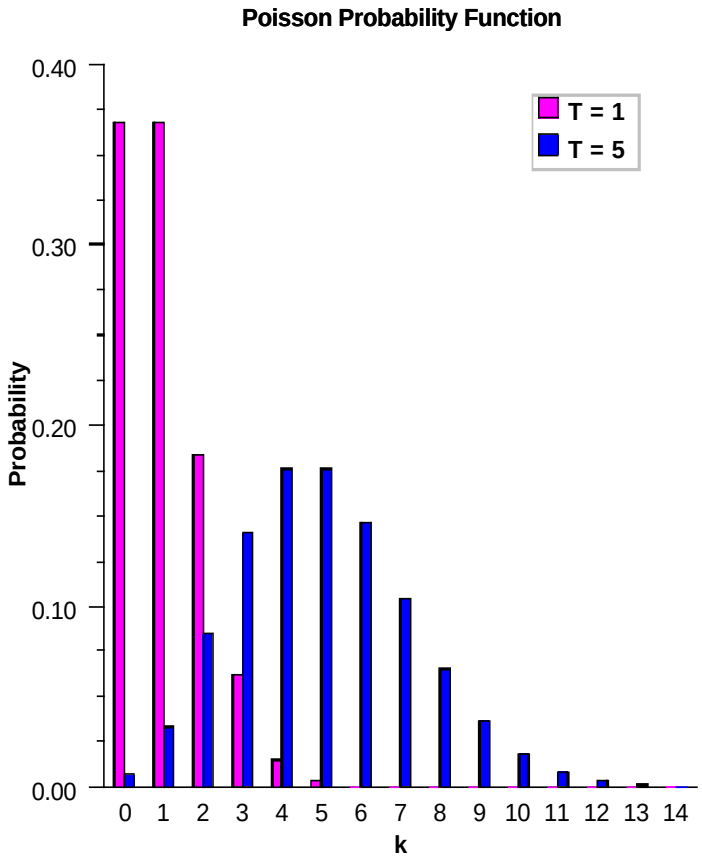
A `double` scalar value representing the probability that a Poisson random variable takes a value equal to *k*.

Remarks

Method `Pdf.Poisson` evaluates the probability density function of a Poisson random variable with parameter `theta`. `theta`, which is the mean of the Poisson random variable, must be positive. The probability function (with $\theta = theta$) is

$$f(x) = e^{-\theta}\ \theta^k / k!, \quad for\ \ k = 0,\ 1,\ 2, \ldots$$

`Pdf.Poisson` evaluates this function directly, taking logarithms and using the log gamma function.



Rayleigh

```
static public double Rayleigh(double x, double alpha)
```

Description

Evaluates the Rayleigh probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated. It must be non-negative.

`alpha` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value representing the probability density function at `x`.

Weibull

```
static public double Weibull(double x, double gamma, double alpha)
```

Description

Evaluates the Weibull probability density function.

Parameters

`x` – A `double` scalar value representing the argument at which the function is to be evaluated. It must be non-negative.

`gamma` – A `double` scalar value representing the shape parameter.

`alpha` – A `double` scalar value representing the scale parameter.

Returns

A `double` scalar value, the probability density function at `x`.

Example: The Probability Distribution Functions

Examples of the probability distribution functions are exercised. Their use in this example typifies the manner in which other functions in the Pdf class would be used.

```
using System;
using Pdf = Imsl.Stat.Pdf;

public class PdfEx1
{

    public static void  Main(String[] args)
    {

        // Beta
        double x = .5;
        double pin = 12.0;
        double qin = 12.0;
        double result = Pdf.Beta(x, pin, qin);
      Console.Out.WriteLine("Pdf.Beta(.5, 12., 12.) is {0,5:0.0000}", result);

      // binomial
      int k = 3;
      int n = 5;
      double prob = .95;
      result = Pdf.Binomial(k, n, prob);
      Console.Out.WriteLine("Pdf.Binomial(3, 5, .95) is " + result);


        // Chi
        prob = .99;
        n = 2;
        result = Pdf.Chi(prob, n);
      Console.Out.WriteLine("Pdf.Chi(.99, 2) is {0,5:0.0000}", result);
    }
}
```

Output

```
Pdf.Beta(.5, 12., 12.) is 3.8683
Pdf.Binomial(3, 5, .95) is 0.021434375
Pdf.Chi(.99, 2) is 0.3048
```

ICdfFunction Interface

`public interface Imsl.Stat.ICdfFunction`

Interface for the user-supplied cumulative distribution function to be used by `InverseCdf` and `ChiSquaredTest`.

Method

CdfFunction

`abstract public double CdfFunction(double p)`

Description

User-supplied cumulative distribution function to be used by `InverseCdf`.

Parameter

`p` – A `double` scalar value representing the point at which the inverse CDF is desired.

Returns

A `double` scalar value representing the probability that a random variable for this CDF takes a value less than or equal to this value is p.

InverseCdf Class

`public class Imsl.Stat.InverseCdf`

Inverse of user-supplied cumulative distribution function.

Class `InverseCdf` evaluates the inverse of a continuous, strictly monotone function. Its most obvious use is in evaluating inverses of continuous distribution functions that can be defined by a user-supplied function, which implements the `ICdfFunction` interface. The inverse is computed using regula falsi and/or bisection, possibly with the Illinois modification (see Dahlquist and Bjorck 1974). A maximum of 100 iterations are performed.

Property

Tolerance

```
public double Tolerance {get; set; }
```

Description

The tolerance to be used as the convergence criterion.

Property Value

A `double` scalar value representing the convergence criterion.

Remarks

When the relative change from one iteration to the next is less than tolerance, convergence is assumed. The default value for tolerance is 0.0001.

Constructor

InverseCdf

```
public InverseCdf(Imsl.Stat.ICdfFunction cdf)
```

Description

Constructor for the inverse of a user-supplied cummulative distribution function.

Parameter

`cdf` – A `ICdfFunction` object that contains the user-supplied function to be inverted.

Remarks

The cdf function must be continuous and strictly monotone.

Method

Eval

```
public double Eval(double p, double guess)
```

Description

Evaluates the inverse CDF function.

Parameters

`p` – A `double` scalar value representing the point at which the inverse CDF is desired.

`guess` – A `double` scalar value representing an initial estimate of the inverse at `p`.

Returns

A `double` scalar value representing the inverse of the CDF at the point `p`.

Remarks

Cdf(`InverseCdf`) is “close” to `p`.

Exception

`Imsl.Stat.DidNotConvergeException` is thrown if the interation to find the inverse of the CDF did not converge.

Example: Inverse of a User-Supplied Cumulative Distribution Function

In this example, InverseCdf is used to compute the point such that the probability is 0.9 that a standard normal random variable is less than or equal to the computed point.

```
using System;
using Imsl.Stat;

public class InverseCdfEx1 : ICdfFunction
{
    public double CdfFunction(double x)
    {
        return Cdf.Normal(x);
    }

    public static void  Main(String[] args)
    {
        double p = 0.9; ;
        ICdfFunction normal = new InverseCdfEx1();
        InverseCdf inv = new InverseCdf(normal);
        inv.Tolerance = 1.0e-10;
        double x1 = inv.Eval(p, 0.0);
        Console.Out.WriteLine
            ("The 90th percentile of a standard normal is " + x1);
    }
}
```

Output

```
The 90th percentile of a standard normal is 1.2815515655446
```

IDistribution Interface

`public interface Imsl.Stat.IDistribution`

Public interface for the user-supplied distribution function.

The purpose of this interface is to fit the probability distribution to a given set of data and return the probability density at each value of the given set of data.

Method

Eval

```
abstract public double[] Eval(double[] xData)
```

Description

Evaluation method to fit the user-supplied probability density function to input data.

Parameter

`xData` – A `double` array representing the points at which the probability density function is to be evaluated.

Returns

A `double` array representing the probability density at each value of `xData`.

IProbabilityDistribution Interface

```
public interface Imsl.Stat.IProbabilityDistribution :  Imsl.Stat.IDistribution
```

Public interface for a user-supplied probability distribution.

The purpose of this interface is to evaluate the probability density of a given set of data by either fitting the probability density function to the data or by evaluating the probability density function with supplied parameters. Both `Eval` methods return the probability density at each value of the given set of data. After the probability distribution is fitted to the data, the `GetParameters` method can be used to return the distribution parameters used to fit the probability density function to the data.

The `DataMining` package class `NaiveBayesClassifier` uses an implementation of `IProbabilityDistribution` to train continuous data.

See Also

Naive Bayes Example 3

Methods

Eval

```
abstract public double[] Eval(double[] xData, object[] parameters)
```

Description

Evaluates the user-supplied probability density of each value in `xData` using the supplied probability distribution parameters.

Parameters

`xData` – A `double` array containing the points at which the probability density function is to be evaluated.

`parameters` – An `Object` array containing the probability distribution parameters to be used in evaluating `xData`. See method `GetParameters`.

Returns

A `double` array representing the probability density of each value of `xData`.

Eval

```
abstract public double Eval(double xData, object[] parameters)
```

Description

Evaluation method for the user-supplied distribution function and parameters. Evaluates the user-supplied probability density at `xData` using the supplied probability distribution parameters.

Parameters

`xData` – A `double` scalar value containing the point the distribution function is to evaluate.

`parameters` – An `Object` array containing the probability distribution parameters to be used in evaluating `xData`. See method `GetParameters`.

Returns

A `double` scalar value representing the probability density at `xData`.

GetParameters

```
abstract public object[] GetParameters()
```

Description

Returns the current parameters of the probability density function.

Returns

An `Object` array containing the parameters resulting from the last invocation of the (`IDistribution`) `Eval` method with the following signature, `double[] Eval(double[] xData)`. This `Object` array can be used as input to the `Eval` methods that require an `Object` array of distribution parameters as input.

NormalDistribution Class

```
public class Imsl.Stat.NormalDistribution :
Imsl.Stat.IProbabilityDistribution, Imsl.Stat.IDistribution
```

Evaluates the normal (Gaussian) probability density for a given set of data.

`NormalDistribution` evaluates the normal probability density of a given set of data, `xData`. If parameters are not supplied, the `Eval` method fits the normal probability density function to the data by first calculating the mean and standard deviation of `xData`. The normal probability density function is defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are the mean and standard deviation.

The `DataMining` package class `NaiveBayesClassifier` uses `NormalDistribution` as the default method to train continuous data.

See Also

Naive Bayes Example 1

Properties

Mean

```
virtual public double Mean {get; }
```

Description

Returns the population mean of `xData`.

Property Value

A `double` representing the population mean of `xData`.

StandardDeviation

```
virtual public double StandardDeviation {get; }
```

Description

Returns the population standard deviation.

Property Value

A `double` representing the population standard deviation of `xData`.

Constructor

NormalDistribution

```
public NormalDistribution()
```

Description

Initializes a new instance of the Imsl.Stat.NormalDistribution (p. 1170) class.

Methods

Eval

```
virtual public double[] Eval(double[] xData)
```

Description

Fits a normal (Gaussian) probability distribution to `xData` and returns the probability density at each value.

Parameter

`xData` – A `double` array representing the points at which the normal probability distribution function is to be evaluated.

Returns

A `double` array representing the normal probability density at each value in `xData`.

Eval

```
virtual public double[] Eval(double[] xData, object[] parameters)
```

Description

Evaluates a normal (Gaussian) probability distribution with the given parameters at each point in `xData` and returns the probability density at each value.

Parameters

`xData` – A `double` array representing the points at which the normal probability distribution function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the normal probability density function, see method `GetParameters`.

Returns

A `double` array representing the normal probability density of each value in `xData`.

Eval

```
virtual public double Eval(double xData, object[] parameters)
```

Description

Evaluates a normal (Gaussian) probability density at a given point `xData`.

Parameters

`xData` – A `double` containing the point at which the normal probability density function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the normal probability density, see method `GetParameters`.

Returns

A `double` representing the normal probability density at `xData`.

GetParameters

```
virtual public object[] GetParameters()
```

Description

Returns the current parameters of the normal probability density function.

Returns

An `Object` array containing the parameters resulting from the last invocation of the (`IDistribution`) `Eval` method with the following signature, `double[] Eval(double[] xData)`. This `Object` array can be used as input to the `Eval` methods that require an `Object` array of distribution parameters as input.

GammaDistribution Class

```
public class Imsl.Stat.GammaDistribution :  Imsl.Stat.IProbabilityDistribution,
Imsl.Stat.IDistribution
```

Evaluates a gamma probability density for a given set of data.

`GammaDistribution` evaluates the gamma density of a given set of data, `xData`. If parameters are not supplied, the `Eval` method fits the gamma probability density function to the data by first calculating the shape and scale parameters using an MLE technique for a best fit. The gamma probability density function is defined as:

$$f(x) = x^{a-1}\frac{e^{-\frac{x}{b}}}{b^a\Gamma(a)},\ x > 0, a > 0 \text{ and } b > 0,$$

where a and b are the scale and shape parameters.

The `DataMining` package class `NaiveBayesClassifier` uses `GammaDistribution` as a method to train continuous data.

See Also

Naive Bayes Example 1

Properties

ScaleParameter

```
virtual public double ScaleParameter {get; }
```

Description

The maximum-likelihood estimate found for the gamma scale parameter.

Property Value

A `double` representing the maximum-likelihood estimate found for the gamma scale parameter.

ShapeParameter

```
virtual public double ShapeParameter {get; }
```

Description

The maximum-likelihood estimate found for the gamma shape parameter.

Property Value

A `double` representing the maximum-likelihood estimate found for the gamma shape parameter.

Constructor

GammaDistribution

```
public GammaDistribution()
```

Description

Initializes a new instance of the Imsl.Stat.GammaDistribution (p. 1172) class.

Methods

Eval

```
virtual public double[] Eval(double[] xData)
```

Description

Fits a gamma probability distribution to `xData` and returns the probability density at each value.

Parameter

`xData` – A `double` array representing the points at which the gamma probability distribution function is to be evaluated.

Returns

A `double` array representing the gamma probability density at each value of `xData`.

Eval

```
virtual public double[] Eval(double[] xData, object[] parameters)
```

Description

Evaluates a gamma probability distribution with a given set of parameters at each point in `xData` and returns the probability density at each value.

Parameters

`xData` – A `double` array representing the points at which the gamma probability distribution function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the gamma distribution, see method `GetParameters`.

Returns

A `double` array representing the gamma probability density at each value of `xData`.

Eval

```
virtual public double Eval(double xData, object[] parameters)
```

Description

Evaluates a gamma probability density at a given point `xData`.

Parameters

`xData` – A `double` representing the point at which the gamma probability distribution function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the gamma distribution, see method `GetParameters`.

Returns

A `double` representing the gamma probability density at `xData`.

GetParameters

```
virtual public object[] GetParameters()
```

Description

Returns the current parameters of the gamma probability density function.

Returns

An `Object` array containing the parameters resulting from the last invocation of the (`IDistribution`) `Eval` method with the following signature, `double[] Eval(double[] xData)`. This `Object` array can be used as input to the `Eval` methods that require an `Object` array of distribution parameters as input.

LogNormalDistribution Class

```
public class Imsl.Stat.LogNormalDistribution :
Imsl.Stat.IProbabilityDistribution, Imsl.Stat.IDistribution
```

Evaluates a lognormal probability density for a given set of data.

`LogNormalDistribution` evaluates the lognormal probability density of a given set of data, `xData`. If parameters are not supplied, the `Eval` method fits the lognormal probability density function to the data by first calculating the mean and standard deviation. The lognormal probability density function is defined as:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

where μ and σ are the mean and standard deviation.

The `DataMining` package class `NaiveBayesClassifier` uses `LognormalDistribution` as a method to train continuous data.

See Also

Naive Bayes Example 1

Properties

Mean

```
virtual public double Mean {get; }
```

Description

The lognormal probability distribution mean parameter.

Property Value

A `double` representing the mean parameter.

StandardDeviation

```
virtual public double StandardDeviation {get; }
```

Description

The lognormal probability distribution standard deviation parameter.

Property Value

A `double` representing the standard deviation parameter.

Constructor

LogNormalDistribution

```
public LogNormalDistribution()
```

Description

Initializes a new instance of the Imsl.Stat.LogNormalDistribution (p. 1175) class.

Methods

Eval

```
virtual public double[] Eval(double[] xData)
```

Description

Fits a lognormal probability distribution to `xData` and returns the probability density at each value.

Parameter

`xData` – A `double` array representing the points at which the lognormal probability distribution function is to be evaluated.

Returns

A `double` array representing the lognormal probability density at each value of `xData`.

Eval

```
virtual public double[] Eval(double[] xData, object[] parameters)
```

Description

Evaluates a lognormal probability distribution with a given set of parameters at each point in `xData` and returns the probability density at each value.

Parameters

`xData` – A `double` array representing the points at which the lognormal probability distribution function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the lognormal distribution, see method `GetParameters`.

Returns

A `double` array representing the lognormal probability density at each value of `xData`.

Eval

```
virtual public double Eval(double xData, object[] parameters)
```

Description

Evaluates a lognormal probability density function at a given point `xData`.

Parameters

`xData` – A `double` representing the point at which the lognormal probability distribution function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the lognormal distribution, see method `GetParameters`.

Returns

A `double` representing the lognormal probability density at `xData`.

GetParameters

```
virtual public object[] GetParameters()
```

Description

Returns the current parameters of the lognormal probability density function.

Returns

An `Object` array containing the parameters resulting from the last invocation of the (`IDistribution`) `Eval` method with the following signature, `double[] Eval(double[] xData)`. This `Object` array can be used as input to the `Eval` methods that require an `Object` array of distribution parameters as input.

PoissonDistribution Class

```
public class Imsl.Stat.PoissonDistribution :
Imsl.Stat.IProbabilityDistribution, Imsl.Stat.IDistribution
```

Evaluates a Poisson probability density of a given set of data.

`PoissonDistribution` evaluates the Poisson probability density of a given set of data, `xData`. If parameters are not supplied, the `Eval` method fits the Poisson probability density function by first calculating *theta*, θ. The Poisson probability density function is defined as:

$$f(x) = \frac{\theta^x e^{-\theta}}{x!}, x \geq 0 \text{ and } \theta > 0.$$

The DataMining package class NaiveBayesClassifier uses PoissonDistribution as a method to train continuous data.

See Also

Naive Bayes Example 1

Property

Theta

```
virtual public double Theta {get; }
```

Description

The mean number of successes in a given time period of the Poisson probability distribution.

Property Value

A double representing the mean number of successes in a given time period of the Poisson probability distribution.

Constructor

PoissonDistribution

```
public PoissonDistribution()
```

Description

Initializes a new instance of the Imsl.Stat.PoissonDistribution (p. 1177) class.

Methods

Eval

```
virtual public double[] Eval(double[] xData)
```

Description

Fits a Poisson probability distribution to xData and returns the probability density at each value.

Parameter

xData – A double array representing the points at which the Poisson probability distribution function is to be evaluated.

Returns

A `double` array representing the Poisson probability density at each value of `xData`.

Eval

```
virtual public double[] Eval(double[] xData, object[] parameters)
```

Description

Evaluates a Poisson probability distribution with a given set of parameters at each point in `xData` and returns the probability density at each value.

Parameters

`xData` – A `double` array representing the points at which the Poisson probability distribution function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the Poisson distribution, see method `GetParameters`.

Returns

A `double` array representing the Poisson probability density at each value of `xData`.

Eval

```
virtual public double Eval(double xData, object[] parameters)
```

Description

Evaluates a Poisson probability density function at a given point `xData`.

Parameters

`xData` – A `double` representing the point at which the Poisson probability distribution function is to be evaluated.

`parameters` – An `Object` array representing the parameters used to evaluate the Poisson distribution, see method `GetParameters`.

Returns

A `double` representing the Poisson probability density at `xData`.

GetParameters

```
virtual public object[] GetParameters()
```

Description

Returns the current parameters of the Poisson probability density function.

Returns

An `Object` array containing the parameters resulting from the last invocation of the (`IDistribution`) `Eval` method with the following signature, `double[] Eval(double[] xData)`. This `Object` array can be used as input to the `Eval` methods that require an `Object` array of distribution parameters as input.

Chapter 22: Random Number Generation

Types

Usage Notes

Overview of Random Number Generation

This chapter describes functions for the generation of random numbers that are useful for applications in simulation studies.

In the following discussions, the phrases *random numbers*, *random deviates*, *deviates*, and *variates* are used interchangeably. The phrase *pseudorandom* is sometimes used to emphasize that the numbers generated are really not *random* since they result from a deterministic process. The usefulness of pseudorandom numbers is derived from the similarity, in a statistical sense, of samples of the pseudorandom numbers to samples of observations from the specified distributions. In short, while the pseudorandom numbers are completely deterministic and repeatable, they simulate the realizations of independent and identically distributed random variables.

Class `Imsl.Stat.Random` extends `System.Random`. It adds the ability to generate random numbers with a number of different non-uniform distributions. The non-uniform random number generators use a uniform random number generator. The uniform random number generator in `System.Random` can be used. `Imsl.Stat.Random` adds a linear congruential generator. It is also possible to use aother uniform generator as long as it implements the `Random.BaseGenerator` interface.

The `MersenneTwister` and `MersenneTwister64` uniform random number generators generate random number series with very long periods. They both implement the `Random.BaseGenerator` interface and

so can be used as the base uniform generator in `Imsl.Stat.Random`.

The class `FaureSequence` generates the *low-discrepancy* Faure sequence. This is also called a *quasi-random* generator. The sequence is a series of points in *n*-dimensions, which are close to being as equally spaced as possible. Low-discrepancy refers to the difference from actually being as equally spaced as possible.

The `FaureSequence` implements the `IRandomSequence` interface. This interface defines a sequence of points in *n*-dimensions. It is used by the class `Imsl.Math.HyperRectangleQuadrature` to evaluate *n*-dimensional integrals.

Random Class

```
public class Imsl.Stat.Random :  Random
```

Generate uniform and non-uniform random number distributions.

The non-uniform distributions are generated from a uniform distribution. By default, this class uses the uniform distribution generated by the base class System.Random . If the multiplier is set in this class then a multiplicative congruential method is used. The form of the generator is

$$x_i \equiv cx_{i-1} \mathrm{mod}(2^{31} - 1)$$

Each x_i is then scaled into the unit interval (0,1). If the multiplier, c, is a primitive root modulo $2^{31} - 1$ (which is a prime), then the generator will have a maximal period of $2^{31} - 2$. There are several other considerations, however. See Knuth (1981) for a good general discussion. Possible values for c are 16807, 397204094, and 950706376. The selection is made by the property Multiplier (p. 1184). Evidence suggests that the performance of 950706376 is best among these three choices (Fishman and Moore 1982).

Alternatively, one can select a 32-bit or 64-bit Mersenne Twister generator by first instantiating Imsl.Stat.MersenneTwister (p. 1207) or Imsl.Stat.MersenneTwister64 (p. 1211). These generators have a period of $2^{19937} - 1$ and a 623-dimensional equidistribution property. See Matsumoto et al. 1998 for details.

The generation of uniform (0,1) numbers is done by the method NextFloat (p. 1191).

Nonuniform random numbers are generated using a variety of transformation procedures. All of the transformations used are exact (mathematically). The most straightforward transformation is the *inverse CDF technique*, but it is often less efficient than others involving *acceptance/rejection* and mixtures. See Kennedy and Gentle(1980) for discussion of these and other techniques.

Many of the nonuniform generators use different algorithms depending on the values of the parameters of the distributions. This is particularly true of the generators for discrete distributions. Schmeiser (1983) gives an overview of techniques for generating deviates from discrete distributions.

Extensive empirical tests of some of the uniform random number generators available in the `Random` class are reported by Fishman and Moore (1982 and 1986). Results of tests on the generator using the

multiplier 16807 are reported by Learmonth and Lewis (1973). If the user wishes to perform additional tests, the routines in Chapter 17, *Tests of Goodness of Fit*, may be of use. Often in Monte Carlo applications, it is appropriate to construct an ad hoc test that is sensitive to departures that are important in the given application. For example, in using Monte Carlo methods to evaluate a one-dimensional integral, autocorrelations of order one may not be harmful, but they may be disastrous in evaluating a two-dimensional integral. Although generally the routines in this chapter for generating random deviates from nonuniform distributions use exact methods, and, hence, their quality depends almost solely on the quality of the underlying uniform generator, it is often advisable to employ an ad hoc test of goodness of fit for the transformations that are to be applied to the deviates from the nonuniform generator.

Three methods are associated with copulas. A *copula* is a multivariate cumulative probability distribution (CDF) whose arguments are random variables uniformly distributed on the interval [0,1] corresponding to the probabilities (variates) associated with arbitrarily distributed marginal deviates. The copula structure allows the multivariate CDF to be partitioned into the copula, which has associated with it information characterizing the dependence among the marginal variables, and the set of separate marginal deviates, each of which has its own distribution structure.

Two methods, `NextGaussianCopula` and `NextStudentsTCopula`, allow the user to specify a correlation structure (in the form of a Cholesky matrix) which can be used to imprint correlation information on a sequence of multivariate random vectors. Each call to one of these methods returns a random vector whose elements (variates) are each uniformly distributed on the interval [0,1] and correlated according to a user-specified Cholesky matrix. These variate vector sequences may then be inverted to marginal deviate sequences whose distributions and imprinted correlations are user-specified. Method `NextGaussianCopula` generates a random Gaussian copula sequence by inverting uniform [0,1] random numbers to N(0,1) deviates vectors, imprinting each vector with the correlation information by multiplying it with the Cholesky matrix, and then using the N(0,1) CDF to map the imprinted deviates back to uniform [0,1] variates. Method `NextStudentsTCopula` inverts a vector of uniform [0,1] random numbers to a Student's t deviate vector with mean 0 and user specified degrees of freedom `df` which is then imprinted with the Cholesky matrix and mapped back to uniform [0,1] variates.

The third copula method, `CanonicalCorrelation`, extracts a correlation matrix from a sequence of multivariate deviate vectors whose component marginals are arbitrarily distributed. This is accomplished by first extracting the empirical CDF from each of the marginal deviates and then using this CDF to map the deviates to uniform [0,1] variates which are then inverted to Normal (0,1) deviates. Each element C_{ij} of the correlation matrix can then be extracted by averaging the products $z_{it}z_{jt}$ of deviates i and j over the t-indexed sequence. The utility of method `CanonicalCorrelation` is that because the correlation matrix is derived from N(0,1) deviates, the correlation is unbiased, i.e. undistorted by the arbitrary marginal distribution structures of the original deviate vector sequences. This is important in such financial applications as portfolio optimization, where correlation is used to estimate and minimize risk.

The use of these copula methods is illustrated with RandomEx2.cs, which first uses method `NextGaussianCopula` to create a correlation imprinted sequence of random deviate vectors and then uses method `CanonicalCorrelation` to extract the correlation matrix from the imprinted sequence of vectors.

Properties

Multiplier

```
public long Multiplier {get; set; }
```

Description

The multiplier for a linear congruential random number generator.

Property Value

A `long` which represents the random number generator multiplier.

Remarks

If not set, the multiplier has the value zero. If a multiplier is set then the linear congruential generator, defined in the base class System.Random, is replaced by the generator

seed = (multiplier*seed) mod $(2^{31} - 1)$

See Donald Knuth, The Art of Computer Programming, Volume 2, for guidelines in choosing a multiplier. Some possible values are 16807, 397204094, 950706376.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Constructors

Random

```
public Random()
```

Description

Constructor for the Random number generator class.

Random

```
public Random(int seed)
```

Description

Constructor for the Random number generator class with supplied seed.

Parameter

`seed` – An `int` which represents the random number generator seed in the range of -2,147,483,647 to +2,147,483,647.

Random

```
public Random(Imsl.Stat.Random.BaseGenerator baseGenerator)
```

Description

Constructor for the Random number generator class with an alternate basic number genrator.

Parameter

`baseGenerator` – A `BaseGenerator` used to override the method `Next`.

Methods

CanonicalCorrelation

```
public double[][] CanonicalCorrelation(double[][] deviate)
```

Description

Method `CanonicalCorrelation` generates a canonical correlation matrix from an arbitrarily distributed multivariate deviate sequence with *nvar* deviate variables, *nseq* steps in the sequence, and a Gaussian Copula dependence structure.

Parameter

`deviate` – A double *nseq* by *nvar* array of input deviate values.

Remarks

Method `CanonicalCorrelation` first maps each of the `j=1..`*nvar* input deviate sequences `deviate[k=1..`*nseq*`][j]` into a corresponding sequence of variates, say *variate*`[k][j]` (where variates are values of the empirical cumulative probability function, $CDF(x)$, defined as the probability that random deviate variable $X \leq x$). The variate matrix *variate*`[k][j]` is then mapped into Normal(0,1) distributed deviates z_{kj} using the method `InvCdf.Normal(`*variate*`[k][j])` and then the standard covariance estimator

$$C_{ij} = \frac{1}{n_{seq}} \sum_{k=1}^{n_{seq}} z_{ki}\, z_{kj}$$

is used to calculate the canonical correlation matrix `correlation = CanonicalCorrelation(deviate)`, where C_{ij} = *correlation*`[i][j]` and n_{seq} = *nseq*.

If a multivariate distribution has Gaussian marginal distributions, then the standard "empirical" correlation matrix given above is "unbiased", i.e. an accurate measure of dependence among the variables. But when the marginal distributions depart significantly from Gaussian, i.e. are skewed or flattened, then the empirical correlation may become biased. One way to remove such bias from

dependence measures is to map the non-Gaussian-distributed marginal deviates to Gaussian N(0,1) deviates (by mapping the non-Gaussian marginal deviates to empirically derived marginal CDF variate values, then inverting the variates to N(0,1) deviates as described above), and calculating the standard empirical correlation matrix from these N(0,1) deviates as in the equation above. The resulting "(Gaussian) canonical correlation" matrix thereby avoids the bias that would occur if the empirical correlation matrix were extracted from the non-Gaussian marginal distributions directly.

The canonical correlation matrix may be of value in such applications as Markowitz portfolio optimization, where an unbiased measure of dependence is required to evaluate portfolio risk, defined in terms of the portfolio variance which is in turn defined in terms of the correlation among the component portfolio instruments.

The utility of the canonical correlation derives from the observation that a "copula" multivariate distribution with uniformly-distributed deviates (corresponding to the CDF probabilities associated with the marginal deviates) may be mapped to arbitrarily distributed marginals, so that an unbiased dependence estimator derived from one set of marginals (N(0,1) distributed marginals) can be used to represent the dependence associated with arbitrarily-distributed marginals. The "Gaussian Copula" (whose variate arguments are derived from N(0,1) marginal deviates) is a particularly useful structure for representing multivariate dependence.

This is demonstrated in Example 2 where method `Random.NextGaussianCopula(chol)` (where `chol` is a Cholesky object derived from a user-specified covariance matrix) is used to imprint correlation information on otherwise arbitrarily distributed and independent random sequences. Method `Random.CanonicalCorrelation` is then used to extract an unbiased correlation matrix from these imprinted deviate sequences.

Next

```
override public int Next(int maxValue)
```

Description

Returns a nonnegative pseudorandom `int`.

Parameter

`maxValue` – An `int` which specifies the upper bound of the random number to be generated. `maxValue` must be greater than or equal to zero.

Returns

An `int` greater than or equal to zero and less than `maxValue`.

Next

```
override public int Next(int minValue, int maxValue)
```

Description

Returns a nonnegative pseudorandom `int` in the specified range.

Parameters

`minValue` – An `int` which specifies the lower bound of the random number returned.

`maxValue` – An `int` which specifies the upper bound of the random number to be generated. `maxValue` must be greater than or equal to zero.

Returns

An `int` greater than or equal to `minValue` and less than `maxValue`; that is, the range of return values includes `minValue` but not `maxValue`. If `minValue` equals `maxValue`, `minValue` is returned.

NextBeta

```
virtual public double NextBeta(double p, double q)
```

Description

Generate a pseudorandom number from a beta distribution.

Parameters

`p` – A `double` which specifies the first beta distribution parameter, $p > 0$.

`q` – A `double` which specifies the second beta distribution parameter, $q > 0$.

Returns

A `double` which specifies a pseudorandom number from a beta distribution.

Remarks

Method `NextBeta` generates pseudorandom numbers from a beta distribution with parameters p and q, both of which must be positive. The probability density function is

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} \quad for\, 0 \le x \le 1$$

where $\Gamma(\cdot)$ is the gamma function.

The algorithm used depends on the values of p and q. Except for the trivial cases of $p = 1$ or $q = 1$, in which the inverse CDF method is used, all of the methods use acceptance/rejection. If p and q are both less than 1, the method of Johnk (1964) is used; if either p or q is less than 1 and the other is greater than 1, the method of Atkinson (1979) is used; if both p and q are greater than 1, algorithm BB of Cheng (1978), which requires very little setup time, is used.

The value returned is less than 1.0 and greater than ε, where ε is the smallest positive number such that $1.0 - \varepsilon$ is less than 1.0.

NextBinomial

```
virtual public int NextBinomial(int n, double p)
```

Description

Generate a pseudorandom number from a Binomial distribution.

Parameters

`n` – A `int` which specifies the number of Bernoulli trials.

`p` – A `double` which specifies the probability of success on each trial, $0 < p < 1$.

Returns

A `int` which specifies the pseudorandom number from a Binomial distribution.

Remarks

`NextBinomial` generates pseudorandom numbers from a Binomial distribution with parameters *n* and *p*. *n* and *p* must be positive, and *p* must be less than 1. The probability function (with *n* = *n* and *p* = *p*) is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$.

The algorithm used depends on the values of *n* and *p*. If $np < 10$ or if *p* is less than a machine epsilon, the inverse CDF technique is used; otherwise, the BTPE algorithm of Kachitvichyanukul and Schmeiser (see Kachitvichyanukul 1982) is used. This is an acceptance/rejection method using a composition of four regions. (TPE equals Triangle, Parallelogram, Exponential, left and right.)

NextCauchy

```
virtual public double NextCauchy()
```

Description

Generates a pseudorandom number from a Cauchy distribution.

Returns

A `double` which specifies a pseudorandom number from a Cauchy distribution.

Remarks

The probability density function is

$$f(x) = \frac{1}{\pi(1+x^2)}$$

Use of the inverse CDF technique would yield a Cauchy deviate from a uniform (0, 1) deviate, *u*, as $\tan[\pi(u-.5)]$. Rather than evaluating a tangent directly, however, `NextCauchy` generates two uniform (-1, 1) deviates, x_1 and x_2. These values can be thought of as sine and cosine values. If

$$x_1^2 + x_2^2$$

is less than or equal to 1, then x_1/x_2 is delivered as the Cauchy deviate; otherwise, x_1 and x_2 are rejected and two new uniform (-1, 1) deviates are generated. This method is also equivalent to taking the ratio of two independent normal deviates.

Deviates from the Cauchy distribution with median *t* and first quartile *t* - *s*, that is, with density

$$f(x) = \frac{s}{\pi\left[s^2 + (x-t)^2\right]}$$

can be obtained by scaling the output from `NextCauchy`. To do this, first scale the output from `NextCauchy` by *S* and then add *T* to the result.

NextChiSquared

```
virtual public double NextChiSquared(double df)
```

Description

Generates a pseudorandom number from a Chi-squared distribution.

Parameter

`df` – A `double` which specifies the number of degrees of freedom. It must be positive.

Returns

A `double` which specifies a pseudorandom number from a Chi-squared distribution.

Remarks

`NextChiSquared` generates pseudorandom numbers from a chi-squared distribution with `df` degrees of freedom. If `df` is an even integer less than 17, the chi-squared deviate r is generated as

$$r = -2\ln\left(\prod_{i=1}^{n} u_i\right)$$

where $n = \mathrm{df}/2$ and the u_i are independent random deviates from a uniform (0, 1) distribution. If `df` is an odd integer less than 17, the chi-squared deviate is generated in the same way, except the square of a normal deviate is added to the expression above. If `df` is greater than 16 or is not an integer, and if it is not too large to cause overflow in the gamma random number generator, the chi-squared deviate is generated as a special case of a gamma deviate, using `NextGamma`. If overflow would occur in `NextGamma`, the chi-squared deviate is generated in the manner described above, using the logarithm of the product of uniforms, but scaling the quantities to prevent underflow and overflow.

NextDouble

```
override public double NextDouble()
```

Description

Generates the next pseudorandom number.

Returns

A `double` which specifies the next pseudorandom value from this random number generator's sequence.

Remarks

If the `multiplier` is set then the multiplicative congruential method is used. Otherwise, `super.Next(bits)` is used. Where bits is the number of random bits required.

NextExponential

```
virtual public double NextExponential()
```

Description

Generates a pseudorandom number from a standard exponential distribution.

Returns

A `double` which specifies a pseudorandom number from a standard exponential distribution.

Remarks

The probability density function is $f(x) = e^{-x}$; for $x > 0$.

`NextExponential` uses an antithetic inverse CDF technique; that is, a uniform random deviate *U* is generated and the inverse of the exponential cumulative distribution function is evaluated at *1.0 - U* to yield the exponential deviate.

Deviates from the exponential distribution with mean `THETA` can be generated by using `NextExponential` and then multiplying the result by `THETA`.

NextExponentialMix

```
virtual public double NextExponentialMix(double theta1, double theta2, double
p)
```

Description

Generate a pseudorandom number from a mixture of two exponential distributions.

Parameters

`theta1` – A `double` which specifies the mean of the exponential distribution that has the larger mean.

`theta2` – A `double` which specifies the mean of the exponential distribution that has the smaller mean. `theta2` must be positive and less than or equal to `theta1`.

`p` – A `double` which specifies the mixing parameter. It must satisfy $0 \leq p \leq \text{theta1}/(\text{theta1} - \text{theta2})$.

Returns

A `double` which specifies a pseudorandom number from a mixture of the two exponential distributions.

Remarks

The probability density function is

$$f(x) = \frac{p}{\theta} e^{-x/\theta_1} + \frac{1-p}{\theta_2} e^{-x/\theta_2} \quad for\, x > 0$$

where $p = \text{p}$, $\theta_1 = theta1$, and $\theta_2 = theta2$.

In the case of a convex mixture, that is, the case $0 < p < 1$, the mixing parameter *p* is interpretable as a probability; and `NextExponentialMix` with probability *p* generates an exponential deviate with mean θ_1, and with probability *1 - p* generates an exponential with mean θ_2. When *p* is greater than 1, but less than $\theta_1/(\theta_1 - \theta_2)$, then either an exponential deviate with mean θ_2 or the sum of two exponentials with means θ_1 and θ_2 is generated. The probabilities are $q = p - (p-1)\theta_1/\theta_2$ and *1 - q*, respectively, for the single exponential and the sum of the two exponentials.

NextExtremeValue

```
virtual public double NextExtremeValue(double mu, double beta)
```

Description

Generate a pseudorandom number from an extreme value distribution.

Parameters

`mu` – A `double` scalar value representing the location parameter.

`beta` – A `double` scalar value representing the scale parameter.

Returns

A `double` pseudorandom number from an extreme value distribution

NextF

```
virtual public double NextF(double dfn, double dfd)
```

Description

Generate a pseudorandom number from the F distribution.

Parameters

`dfn` – A `double`, the numerator degrees of freedom. It must be positive.

`dfd` – A `double`, the denominator degrees of freedom. It must be positive.

Returns

A `double`, a pseudorandom number from an F distribution

NextFloat

```
public float NextFloat()
```

Description

Generates the next pseudorandom number.

Returns

A `float` which specifies the next pseudorandom value from this random number generator's sequence.

Remarks

If the `multiplier` is set then the multiplicative congruential method is used. Otherwise, `super.Next(bits)` is used. Where bits is the number of random bits required.

NextGamma

```
virtual public double NextGamma(double a)
```

Description

Generates a pseudorandom number from a standard gamma distribution.

Parameter

`a` – A `double` which specifies the shape parameter of the gamma distribution. It must be positive.

Returns

A `double` which specifies a pseudorandom number from a standard gamma distribution.

Remarks

Method `NextGamma` generates pseudorandom numbers from a gamma distribution with shape parameter *a*. The probability density function is

$$P = \frac{1}{\Gamma(a)} \int_o^x e^{-t} t^{a-1} dt$$

Various computational algorithms are used depending on the value of the shape parameter *a*. For the special case of *a = 0.5*, squared and halved normal deviates are used; and for the special case of *a = 1.0*, exponential deviates (from method `NextExponential`) are used. Otherwise, if *a* is less than 1.0, an acceptance-rejection method due to Ahrens, described in Ahrens and Dieter (1974), is used; if *a* is greater than 1.0, a ten-region rejection procedure developed by Schmeiser and Lal (1980) is used.

The Erlang distribution is a standard gamma distribution with the shape parameter having a value equal to a positive integer; hence, `NextGamma` generates pseudorandom deviates from an Erlang distribution with no modifications required.

NextGaussianCopula

```
virtual public double[] NextGaussianCopula(Imsl.Math.Cholesky chol)
```

Description

Generate pseudorandom numbers from a Gaussian Copula distribution.

Parameter

`chol` – A `Cholesky` object containing the Cholesky factorization of the correlation matrix of order k.

Returns

A `double` array which contains the pseudorandom numbers from a multivariate Gaussian Copula distribution.

Remarks

`NextGaussianCopula` generates pseudorandom numbers from a multivariate Gaussian Copula distribution which are uniformly distributed on the interval (0,1) representing the probabilities associated with N(0,1) deviates imprinted with correlation information from input Cholesky object `chol`. Cholesky matrix `R` is defined as the "square root" of a user-defined correlation matrix, that is `R` is an upper triangular matrix such that the transpose of `R` times `R` is the correlation matrix.

First, a length *k* vector of independent random normal deviates with mean 0 and variance 1 is generated, and then this deviate vector is post-multiplied by cholesky matrix `R`. Finally, the Cholesky-imprinted random N(0,1) deviates are mapped to output probabilities using the N(0,1) cumulative distribution function (CDF).

Random deviates from arbitrary marginal distributions which are imprinted with the correlation information contained in Cholesky matrix `R` can then be generated by inverting the output probabilities using user-specified inverse CDF functions.

NextGeometric

```
virtual public int NextGeometric(double p)
```

Description

Generate a pseudorandom number from a geometric distribution.

Parameter

`p` – A `double` which specifies the probability of success on each trial, $0 < p \leq 1$.

Returns

A `int` which specifies a pseudorandom number from a geometric distribution.

Remarks

`NextGeometric` generates pseudorandom numbers from a geometric distribution with parameter *p*, where *P =p* is the probability of getting a success on any trial. A geometric deviate can be interpreted as the number of trials until the first success (including the trial in which the first success is obtained). The probability function is

$$f(x) = P(1-P)^{x-1}$$

for $x = 1, 2, \ldots$ and $0 < P < 1$.

The geometric distribution as defined above has mean *1/P*.

The *i*-th geometric deviate is generated as the smallest integer not less than $log(U_i)/log(1-P)$, where the U_i are independent uniform (0, 1) random numbers (see Knuth, 1981).

The geometric distribution is often defined on 0, 1, 2, ..., with mean *(1 - P)/P*. Such deviates can be obtained by subtracting 1 from each element returned value.

NextHypergeometric

```
virtual public int NextHypergeometric(int n, int m, int l)
```

Description

Generate a pseudorandom number from a hypergeometric distribution.

Parameters

`n` – A `int` which specifies the number of items in the sample, $n > 0$.

`m` – A `int` which specifies the number of special items in the population, or lot, $m > 0$.

`l` – A `int` which specifies the number of items in the lot, $l > max(n, m)$.

Returns

A `int` which specifies the number of special items in a sample of size n drawn without replacement from a population of size l that contains m such special items.

Remarks

Method `NextHypergeometric` generates pseudorandom numbers from a hypergeometric distribution with parameters *n*, *m*, and *l*. The hypergeometric random variable *x* can be thought of as the number of items of a given type in a random sample of size *n* that is drawn without replacement from a population of size *l* containing *m* items of this type. The probability function is

$$f(x) = \frac{\binom{m}{x}\binom{l-m}{n-x}}{\binom{l}{n}}$$

for $x = \max(0, n-l+m), 1, 2, \ldots, \min(n, m)$.

If the hypergeometric probability function with parameters *n*, *m*, and *l* evaluated at *n* - *l* + *m* (or at 0 if this is negative) is greater than the machine epsilon, and less than 1.0 minus the machine epsilon, then `NextHypergeometric` uses the inverse CDF technique. The method recursively computes the hypergeometric probabilities, starting at $x = \max(0, n-l+m)$ and using the ratio *f (x = x + 1)/f(x = x)* (see Fishman 1978, page 457).

If the hypergeometric probability function is too small or too close to 1.0, then `NextHypergeometric` generates integer deviates uniformly in the interval $[1, l-i]$, for $i = 0, 1, \ldots$; and at the *I*-th step, if the generated deviate is less than or equal to the number of special items remaining in the lot, the occurrence of one special item is tallied and the number of remaining special items is decreased by one. This process continues until the sample size or the number of special items in the lot is reached, whichever comes first. This method can be much slower than the inverse CDF technique. The timing depends on *n*. If *n* is more than half of *l* (which in practical examples is rarely the case), the user may wish to modify the problem, replacing *n* by *l* - *n*, and to consider the deviates to be the number of special items *not* included in the sample.

NextLogarithmic

```
virtual public int NextLogarithmic(double a)
```

Description

Generate a pseudorandom number from a logarithmic distribution.

Parameter

`a` – A `double` which specifies the parameter of the logarithmic distribution, $0 < a < 1$.

Returns

A `int` which specifies a pseudorandom number from a logarithmic distribution.

Remarks

Method `NextLogarithmic` generates pseudorandom numbers from a logarithmic distribution with parameter *a*. The probability function is

$$f(x) = -\frac{a^x}{x \ln(1-a)}$$

for $x = 1, 2, 3, \ldots$, and $0 < a < 1$.

The methods used are described by Kemp (1981) and depend on the value of *a*. If *a* is less than 0.95, Kemp's algorithm LS, which is a "chop-down" variant of an inverse CDF technique, is used. Otherwise, Kemp's algorithm LK, which gives special treatment to the highly probable values of 1 and 2, is used.

NextLogNormal

```
virtual public double NextLogNormal(double mean, double stdev)
```

Description

Generate a pseudorandom number from a lognormal distribution.

Parameters

`mean` – A `double` which specifies the mean of the underlying normal distribution.

`stdev` – A `double` which specifies the standard deviation of the underlying normal distribution. It must be positive.

Returns

A `double` which specifies a pseudorandom number from a lognormal distribution.

Remarks

Method `NextLogNormal` generates pseudorandom numbers from a lognormal distribution with parameters `mean` and `stdev`. The scale parameter in the underlying normal distribution, `stdev`, must be positive. The method is to generate normal deviates with mean `mean` and standard deviation `stdev` and then to exponentiate the normal deviates.

With $\mu = mean$ and $\sigma = stdev$, the probability density function for the lognormal distribution is

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(\ln x - \mu)^2\right] \quad for\, x > 0$$

The mean and variance of the lognormal distribution are $\exp(\mu + \sigma 2/2)$ and $\exp(2\mu + 2\sigma 2) - \exp(2\mu + \sigma 2)$, respectively.

NextMultivariateNormal

```
virtual public double[] NextMultivariateNormal(Imsl.Math.Cholesky matrix)
```

Description

Generate pseudorandom numbers from a multivariate normal distribution.

Parameter

`matrix` – The `Cholesky` factorization of the variance-covariance matrix of order k.

Returns

A `double` array which contains the pseudorandom numbers from a multivariate normal distribution.

Remarks

`NextMultivariateNormal` generates pseudorandom numbers from a multivariate normal distribution with mean vector consisting of all zeroes and variance-covariance matrix whose Cholesky factor (or "square root") is `matrix`; that is, `matrix` is an upper triangular matrix such that the transpose of `matrix` times `matrix` is the variance-covariance matrix. First, independent random normal deviates with mean 0 and variance 1 are generated, and then the matrix containing these deviates is post-multiplied by `matrix`.

Deviates from a multivariate normal distribution with means other than zero can be generated by using `NextMultivariateNormal` and then by adding the means to the deviates.

NextNegativeBinomial

```
virtual public int NextNegativeBinomial(double rk, double p)
```

Description

Generate a pseudorandom number from a negative Binomial distribution.

Parameters

`rk` – A `double` which specifies the negative binomial parameter, $rk > 0$.

`p` – A `double` which specifies the probability of success on each trial. It must be greater than machine precision and less than one.

Returns

A `int` which specifies the pseudorandom number from a negative binomial distribution. If rk is an integer, the deviate can be thought of as the number of failures in a sequence of Bernoulli trials before rk successes occur.

Remarks

Method `NextNegativeBinomial` generates pseudorandom numbers from a negative Binomial distribution with parameters rk and p. rk and p must be positive and p must be less than 1. The probability function with ($r =$ rk and $p =$ p) is

$$f(x) = \begin{pmatrix} r+x-1 \\ x \end{pmatrix} (1-p)^r p^x$$

for $x = 0, 1, 2, \ldots$.

If r is an integer, the distribution is often called the Pascal distribution and can be thought of as modeling the length of a sequence of Bernoulli trials until r successes are obtained, where p is the probability of getting a success on any trial. In this form, the random variable takes values r, $r + 1$, $r+2, \ldots$ and can be obtained from the negative binomial random variable defined above by adding r to the negative binomial variable. This latter form is also equivalent to the sum of r geometric random variables defined as taking values $1, 2, 3, \ldots$.

If *rp/(1 - p)* is less than 100 and $(1-p)^r$ is greater than the machine epsilon, `NextNegativeBinomial` uses the inverse CDF technique; otherwise, for each negative binomial deviate, `NextNegativeBinomial` generates a gamma *(r, p/(1 - p))* deviate y and then generates a Poisson deviate with parameter y.

NextNormal

```
virtual public double NextNormal()
```

Description

Generate a pseudorandom number from a standard normal distribution using an inverse CDF method.

Returns

A `double` which represents a pseudorandom number from a standard normal distribution.

Remarks

In this method, a uniform (0,1) random deviate is generated, then the inverse of the normal distribution function is evaluated at that point using `InverseNormal`. This method is slower than the acceptance/rejection technique used in `NextNormalAR` to generate standard normal deviates. Deviates from the normal distribution with mean x_m and standard deviation x_{std} can be obtained by scaling the output from `NextNormal`. To do this first scale the output of `NextNormal` by x_{std} and then add x_m to the result.

NextNormalAR

```
virtual public double NextNormalAR()
```

Description

Generate a pseudorandom number from a standard normal distribution using an acceptance/rejection method.

Returns

A `double` which represents a pseudorandom number from a standard normal distribution.

Remarks

`NextNormalAR` generates pseudorandom numbers from a standard normal (Gaussian) distribution using an acceptance/rejection technique due to Kinderman and Ramage (1976). In this method, the normal density is represented as a mixture of densities over which a variety of acceptance/rejection methods due to Marsaglia (1964), Marsaglia and Bray (1964), and Marsaglia, MacLaren, and Bray (1964) are applied. This method is faster than the inverse CDF technique used in `NextNormal` to generate standard normal deviates.

Deviates from the normal distribution with mean x_m and standard deviation x_{std} can be obtained by scaling the output from `NextNormalAR`. To do this first scale the output of `NextNormalAR` by x_{std} and then add x_m to the result.

NextPoisson

```
virtual public int NextPoisson(double theta)
```

Description

Generate a pseudorandom number from a Poisson distribution.

Parameter

`theta` – A `double` which specifies the mean of the Poisson distribution, $theta > 0$.

Returns

A `int` which specifies a pseudorandom number from a Poisson distribution.

Remarks

Method `NextPoisson` generates pseudorandom numbers from a Poisson distribution with parameter `theta`. `theta`, which is the mean of the Poisson random variable, must be positive. The probability function (with $\theta =$ theta) is

$$f(x) = e^{-\theta}\,\theta^x/x!$$

for $x = 0, 1, 2, \ldots$

If `theta` is less than 15, `NextPoisson` uses an inverse CDF method; otherwise the `PTPE` method of Schmeiser and Kachitvichyanukul (1981) (see also Schmeiser 1983) is used.

The `PTPE` method uses a composition of four regions, a triangle, a parallelogram, and two negative exponentials. In each region except the triangle, acceptance/rejection is used. The execution time of the method is essentially insensitive to the mean of the Poisson.

NextRayleigh

```
virtual public double NextRayleigh(double alpha)
```

Description

Generate a pseudorandom number from a Rayleigh distribution.

Parameter

`alpha` – A `double` which specifies the scale parameter of the Rayleigh distribution

Returns

A `double`, a pseudorandom number from a Rayleigh distribution

Remarks

Method `NextRayleigh` generates pseudorandom numbers from a Rayleigh distribution with scale parameter *alpha*.

NextStudentsT

```
virtual public double NextStudentsT(double df)
```

Description

Generate a pseudorandom number from a Student's t distribution.

Parameter

`df` – A `double` which specifies the number of degrees of freedom. It must be positive.

Returns

A `double` which specifies a pseudorandom number from a Student's t distribution.

Remarks

`NextStudentsT` generates pseudo-random numbers from a Student's *t* distribution with `df` degrees of freedom, using a method suggested by Kinderman, Monahan, and Ramage (1977). The method ("TMX" in the reference) involves a representation of the *t* density as the sum of a triangular density over (-2, 2) and the difference of this and the *t* density. The mixing probabilities depend on the degrees of freedom of the *t* distribution. If the triangular density is chosen, the variate is generated as the sum of two uniforms; otherwise, an acceptance/rejection method is used to generate a variate from the difference density.

For degrees of freedom less than 100, `NextStudentsT` requires approximately twice the execution time as `NextNormalAR`, which generates pseudorandom normal deviates. The execution time of `NextStudentsT` increases very slowly as the degrees of freedom increase. Since for very large degrees of freedom the normal distribution and the *t* distribution are very similar, the user may find that the difference in the normal and the *t* does not warrant the additional generation time required to use `NextStudentsT` instead of `NextNormalAR`.

NextStudentsTCopula

```
virtual public double[] NextStudentsTCopula(double df, Imsl.Math.Cholesky chol)
```

Description

Generate pseudorandom numbers from a Student's t Copula distribution.

Parameters

`df` – A `double` which specifies the degrees of freedom parameter.

`chol` – A `Cholesky` object containing the Cholesky factorization of the correlation matrix of order *k*.

Returns

A `double` array which contains the pseudorandom numbers from a multivariate Student's t Copula distribution with `df` degrees of freedom.

Remarks

`NextStudentsTCopula` generates pseudorandom numbers from a multivariate Student's t Copula distribution which are uniformly distributed on the interval (0,1) representing the probabilities associated with deviates from Student's t distributions with `df` degrees of freedom imprinted with correlation information from the input Cholesky object `chol`. Cholesky matrix `R` is defined as the "square root" of a user-defined correlation matrix, that is `R` is an upper triangular matrix such that the transpose of `R` times `R` is the correlation matrix. First, a length *k* vector of independent random Student's t deviates with mean 0 and `df` degrees of freedom is generated, and then this deviate vector is post-multiplied by Cholesky matrix `R`. Finally, the Cholesky-imprinted random Student's t deviates are mapped to output probabilities using the Student's t cumulative distribution function (CDF) with `df` degrees of freedom.

Random deviates from arbitrary marginal distributions which are imprinted with the correlation information contained in Cholesky matrix `R` can then be generated by inverting the output probabilities using user-specified inverse CDF functions.

NextTriangular

```
virtual public double NextTriangular()
```

Description

Generate a pseudorandom number from a triangular distribution on the interval (0,1).

Returns

A `double` which specifies a pseudorandom number from a triangular distribution on the interval (0,1).

Remarks

The probability density function is $f(x) = 4x$, for $0 \leq x \leq .5$, and $f(x) = 4(1-x)$, for $.5 < x \leq 1$. `NextTriangular` uses an inverse CDF technique.

NextVonMises

```
virtual public double NextVonMises(double c)
```

Description

Generate a pseudorandom number from a von Mises distribution.

Parameter

`c` – A `double` which specifies the parameter of the von Mises distribution, $p > 7.4e-9$.

Returns

A `double` which specifies a pseudorandom number from a von Mises distribution.

Remarks

Method `NextVonMises` generates pseudorandom numbers from a von Mises distribution with parameter *c*, which must be positive. With $c = C$, the probability density function is

$$f(x) = \frac{1}{2\pi I_0(c)} \exp[c\cos(x)] \; for -\pi < x < \pi$$

where $I_0(c)$ is the modified Bessel function of the first kind of order 0. The probability density equals 0 outside the interval $(-\pi, \pi)$.

The algorithm is an acceptance/rejection method using a wrapped Cauchy distribution as the majorizing distribution. It is due to Best and Fisher (1979).

NextWeibull

```
virtual public double NextWeibull(double a)
```

Description

Generate a pseudorandom number from a Weibull distribution.

Parameter

`a` – A `double` which specifies the shape parameter of the Weibull distribution, $a > 0$.

Returns

A `double` which specifies a pseudorandom number from a Weibull distribution.

Remarks

Method `NextWeibull` generates pseudorandom numbers from a Weibull distribution with shape parameter *a*. The probability density function is

$$f(x) = Ax^{A-1}e^{-x^A} \; for x \geq 0$$

`NextWeibull` uses an antithetic inverse CDF technique to generate a Weibull variate; that is, a uniform random deviate *U* is generated and the inverse of the Weibull cumulative distribution function is evaluated at *1.0 - u* to yield the Weibull deviate.

Deviates from the two-parameter Weibull distribution with shape parameter *a* can be generated by using `NextWeibull` and then multiplying the result by *b*.

The Rayleigh distribution with probability density function,

$$r(x) = \frac{1}{\alpha^2} x e^{\left(-x^2/2\alpha^2\right)} \; for x \geq 0$$

is the same as a Weibull distribution with shape parameter *a* equal to 2 and scale parameter *b* equal to

$$\sqrt{2\alpha}$$

hence, `NextWeibull` and simple multiplication can be used to generate Rayleigh deviates.

NextZigguratNormalAR

```
virtual public double NextZigguratNormalAR()
```

Description

Generates pseudorandom numbers using the Ziggurat method.

Returns

A `double` containing the random normal deviate.

Remarks

The `NextZigguratNormalAR` method cuts the density into many small pieces. For each random number generated, an interval is chosen at random and a random normal is generated from the chosen interval. In this implementation, the density is cut into 256 pieces, but symmetry is used so that only 128 pieces are needed by the computation. Following Doornik (2005), different uniform random deviates are used to determine which slice to use and to determine the normal deviate from the slice.

Skip

```
virtual public void Skip(int n)
```

Description

Resets the seed to skip ahead in the base linear congruential generator.

Parameter

n – An `int` which specifies the number of random deviates to skip.

Remarks

This method can be used only if a linear congruential multiplier is explicitly defined by a call to Multiplier (p. 1184).

The method skips ahead in the deviates returned by the protected method `Random.Next`. The public methods use `Next(int)` as their source of uniform random deviates. Some methods call it more than once. For instance, each call to NextDouble (p. 1189) calls it twice.

Example: Random Number Generation

In this example, a discrete normal random sample of size 1000 is generated via `NextNormal`. After the `ChiSquaredTest` constructor is called, the random observations are added to the test one at a time to simulate streaming data. The Chi-squared test is performed using `Cdf.Normal` as the cumulative distribution function object to see how well the random numbers fit the normal distribution.

```
using System;
using Imsl.Stat;

public class RandomEx1 : ICdfFunction
{
    public double CdfFunction(double x)
    {
        return Cdf.Normal(x);
    }

    public static void  Main(String[] args)
    {
        int nObservations = 1000;
        Imsl.Stat.Random r = new Imsl.Stat.Random(123457);
        ICdfFunction normal = new RandomEx1();
        ChiSquaredTest test = new ChiSquaredTest(normal, 10, 0);
        for (int k = 0; k < nObservations; k++)
        {
            test.Update(r.NextNormal(), 1.0);
        }

        double p = test.P;
        Console.Out.WriteLine("The P-value is " + p);
    }
}
```

Output

```
The P-value is 0.496307043723263
```

Example 2: Using Copulas to imprint and extract correlation information

This example uses method `Random.NextGaussianCopula` to generate a multivariate sequence `GCdevt[k=0..nseq-1][j=0..nvar-1]` whose marginal distributions are user-defined and imprinted with a user-specified correlation matrix `CorrMtrxIn[i=0..nvar-1][j=0..nvar-1]` and then uses method `Random.CanonicalCorrelation` to extract from this multivariate random sequence a canonical correlation matrix `CorrMtrx[i=0..nvar-1][j=0..nvar-1]`.

This example illustrates two useful copula related procedures. The first procedure generates a random multivariate sequence with arbitrary user-defined marginal deviates whose dependence is specified by a user-defined correlation matrix. The second procedue is the inverse of the first: an arbitary multivariate deviate input sequence is first mapped to a corresponding sequence of empirically derived variates, i.e. cumulative distribution function values representing the probability that each random variable has a value less than or equal to the input deviate. The variates are then inverted, using the inverse Normal(0,1) function, to N(0,1) deviates; and finally, a canonical covariance matrix is extracted from the multivariate N(0,1) sequence using the standard sum of products.

This example demonstrates that the `nextGaussianCopula` method correctly imbeds the user-defined correlation information into an arbitrary marginal distribution sequence by extracting the canonical correlation from these sequences and showing that they differ from the original correlation matrix by a small relative error, which generally decreases as the number of multivariate sequence vectors increases.

```
using System;
using Imsl.Math;
using Imsl.Stat;

public class RandomEx2
{

    internal static Imsl.Stat.Random IMSLRandom()
    {
        Imsl.Stat.Random r = new Imsl.Stat.Random(123457);
        r.Multiplier = 16807;
        return r;
    }

    public static void Main(string[] args)
    {
        double[,] CorrMtrxIn = new double[,]{
            {1.0, - 0.9486832980505138, 0.8164965809277261},
            {- 0.9486832980505138, 1.0, -0.6454972243679028},
            {0.8164965809277261, -0.6454972243679028, 1.0}};

        int nvar = 3;

        Console.WriteLine("Random Example 2:");
        Console.WriteLine();

        for (int i = 0; i < nvar; i++)
        {
            for (int j = 0; j < i; j++)
            {
```

```
        Console.WriteLine("CorrMtrxIn[" + i + "," + j + "] = " + CorrMtrxIn[i, j]);
    }
}

PrintMatrixFormat pmf = new PrintMatrixFormat();
pmf.NumberFormat = "0.000000000";

new PrintMatrix("Input Correlation Matrix: ").Print(pmf, CorrMtrxIn);
Console.WriteLine("Correlation Matrices calculated from");
Console.WriteLine(" Gaussian Copula imprinted multivariate sequence:");
Console.WriteLine();

// Compute the Cholesky factorization of CorrMtrxIn
Cholesky CholMtrx = new Cholesky(CorrMtrxIn);

for (int kmax = 500; kmax < 1000000; kmax *= 10)
{

    Console.WriteLine("# vectors in multivariate sequence: " + kmax);

    double[][] GCvart = new double[kmax][];
    double[][] GCdevt = new double[kmax][];
    for (int i2 = 0; i2 < kmax; i2++)
    {
        GCdevt[i2] = new double[nvar];
    }
    Imsl.Stat.Random r = IMSLRandom();
    for (int k = 0; k < kmax; k++)
    {
        GCvart[k] = r.NextGaussianCopula(CholMtrx); //probs
        for (int j = 0; j < nvar; j++)
        {
            /*
            *  invert Gaussian Copula probabilities to deviates using
            *  variable-specific inversions: j = 0: Chi Square; 1: F;
            *  2: Normal(0,1); will end up with deviate sequences ready
            *  for mapping to canonical correlation matrix:
            */
            if (j == 0)
            {
                //convert probs into ChiSquare(df=10) deviates:
                GCdevt[k][j] = InvCdf.Chi(GCvart[k][j], 10.0);
            }
            else if (j == 1)
            {
                //convert probs into F(dfn=15,dfd=10) deviates:
                GCdevt[k][j] = InvCdf.F(GCvart[k][j], 15.0, 10.0);
            }
            else
            {
                //convert probs into Normal(mean=0,variance=1) deviates:
                GCdevt[k][j] = InvCdf.Normal(GCvart[k][j]);
            }
        }
    }
    /*
```

```
            *  extract Canonical Correlation matrix from arbitrarily distributed
            *  deviate sequences GCdevt[k=0..kmax-1][j=0..nvar-1] which have been
            *  imprinted with CorrMtrxIn[i=1..nvar][j=1..nvar] above:
            */
            double[][] CorrMtrx = r.CanonicalCorrelation(GCdevt);
            double relerr;
            for (int i = 0; i < nvar; i++)
            {
                for (int j = 0; j < i; j++)
                {
                    relerr = Math.Abs(1.0 -
                        (CorrMtrx[i][j] / CorrMtrxIn[i, j]));
                    Console.WriteLine("CorrMtrx[" + i + "][" + j + "] = " +
                        CorrMtrx[i][j] + "; relerr = " + relerr);
                }
            }
            new PrintMatrix("Correlation Matrix: ").Print(pmf, CorrMtrx);
        }
    }
}
```

Output

```
Random Example 2:

CorrMtrxIn[1,0] = -0.948683298050514
CorrMtrxIn[2,0] = 0.816496580927726
CorrMtrxIn[2,1] = -0.645497224367903
         Input Correlation Matrix:
         0             1             2
0   1.000000000  -0.948683298   0.816496581
1  -0.948683298   1.000000000  -0.645497224
2   0.816496581  -0.645497224   1.000000000

Correlation Matrices calculated from
 Gaussian Copula imprinted multivariate sequence:

# vectors in multivariate sequence: 500
CorrMtrx[1][0] = -0.95029565568146; relerr = 0.00169957417218125
CorrMtrx[2][0] = 0.805260514673251; relerr = 0.0137613145197848
CorrMtrx[2][1] = -0.640202740166643; relerr = 0.00820217965529457
            Correlation Matrix:
         0             1             2
0   1.000000000  -0.950295656   0.805260515
1  -0.950295656   1.000000000  -0.640202740
2   0.805260515  -0.640202740   1.000000000

# vectors in multivariate sequence: 5000
CorrMtrx[1][0] = -0.948611734364931; relerr = 7.54347480657058E-05
CorrMtrx[2][0] = 0.815532446740145; relerr = 0.00118081840157325
CorrMtrx[2][1] = -0.646255361981671; relerr = 0.00117450174090306
            Correlation Matrix:
         0             1             2
0   1.000000000  -0.948611734   0.815532447
1  -0.948611734   1.000000000  -0.646255362
```

```
2   0.815532447  -0.646255362   1.000000000

# vectors in multivariate sequence: 50000
CorrMtrx[1][0] = -0.948314953115713; relerr = 0.00038826965285188
CorrMtrx[2][0] = 0.817827046711005; relerr = 0.00162948114463246
CorrMtrx[2][1] = -0.646669093465958; relerr = 0.00181545180028175
            Correlation Matrix:
        0             1             2
0   1.000000000  -0.948314953   0.817827047
1  -0.948314953   1.000000000  -0.646669093
2   0.817827047  -0.646669093   1.000000000

# vectors in multivariate sequence: 500000
CorrMtrx[1][0] = -0.94873295551488; relerr = 5.23435634081082E-05
CorrMtrx[2][0] = 0.81728795761655; relerr = 0.000969234540976194
CorrMtrx[2][1] = -0.646929884520875; relerr = 0.00221946756529401
            Correlation Matrix:
        0             1             2
0   1.000000000  -0.948732956   0.817287958
1  -0.948732956   1.000000000  -0.646929885
2   0.817287958  -0.646929885   1.000000000
```

Random.BaseGenerator Interface

`public interface Imsl.Stat.Random.BaseGenerator`

Base pseudorandom number.

Methods

Next

`abstract public int Next()`

Description

Generates the next pseudorandom number.

Returns

The next pseudorandom value from this random number generator’s sequence.

NextDouble

`abstract public double NextDouble()`

Description

Generates the next pseudorandom `double`.

Returns

A `double` which is the next pseudorandom value from this random number generator's sequence.

NextFloat

```
abstract public float NextFloat()
```

Description

Generates the next pseudorandom `float`.

Returns

A `float` which is the next pseudorandom value from this random number generator's sequence.

MersenneTwister Class

```
public class Imsl.Stat.MersenneTwister :  Imsl.Stat.Random.BaseGenerator,
ICloneable
```

A 32-bit Mersenne Twister generator.

By default, the class Imsl.Stat.Random (p. 1182) uses the uniform distribution generated by the base class System.Random . Alternatively, one can instantiate Imsl.Stat.MersenneTwister (p. 1207) or Imsl.Stat.MersenneTwister64 (p. 1211) to generate uniform psuedorandom numbers via the Mersenne Twister algorithm. These generators have a period of $2^{19937} - 1$ and a 623-dimensional equidistribution property. See Matsumoto et al. 1998 for details. The series of random numbers can be generated using a seed for initialization or by using an array of type `int` or This generator can be used to generate non-uniform distributions by creating an Imsl.Stat.Random (p. 1182) object using an instance of this class as an argument to the constructor. One can also save the state of the generator at initialization to be re-used later.

This C# code was translated from the the following C program.

A C-program for MT19937, with initialization improved 2002/1/26. Coded by Takuji Nishimura and Makoto Matsumoto.

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mailto: m-mat@math.sci.hiroshima-u.ac.jp

Constructors

MersenneTwister

```
public MersenneTwister(int s)
```

Description

Initializes the 32-bit Mersenne Twister generator using a seed.

Parameter

`s` – An `int` which contains the seed that is used to initialize the 32-bit Mersenne Twister generator.

MersenneTwister

```
public MersenneTwister(uint s)
```

Description

Initializes the 32-bit Mersenne Twister generator using a seed.

Parameter

`s` – A `uint` which contains the seed that is used to initialize the 32-bit Mersenne Twister generator.

MersenneTwister

```
public MersenneTwister(int[] key)
```

Description

Initializes the 32-bit Mersenne Twister generator using an array.

Parameter

`key` – An `int` array used to initialize the 32-bit Mersenne Twister generator.

MersenneTwister

```
public MersenneTwister(uint[] key)
```

Description

Initializes the 32-bit Mersenne Twister generator using an array.

Parameter

`key` – A `uint` array used to initialize the 32-bit Mersenne Twister generator.

Methods

Clone

```
Final public object Clone()
```

Description

Returns a clone of this object.

Returns

An `Object` which is a clone of this `MersenneTwister` object.

Next

```
virtual public int Next()
```

Description

Returns a nonnegative pseudorandom `int`.

Returns

An `int` greater than or equal to zero and less than `System.Int32.MaxValue`.

NextDouble

```
virtual public double NextDouble()
```

Description

Returns a random number between 0.0 and 1.0.

Returns

A `double` greater than or equal to 0.0, and less than 1.0.

Remarks

Only the first 32 bits of the `double` are pseudorandom.

NextFloat

```
virtual public float NextFloat()
```

Description

Returns a random number between 0.0 and 1.0.

Returns

A `float` greater than or equal to 0.0, and less than 1.0.

Example: Mersenne Twister Random Number Generation

In this example, four simulation streams are generated. The first series is generated with the seed used for initialization. The second series is generated using an array for initialization. The third series is obtained by resetting the generator back to the state it had at the beginning of the second stream. Therefore, the second and third streams are identical. The fourth stream is obtained by resetting the generator back to its original, uninitialized state, and having it reinitialize using the seed. The first and fourth streams are therefore the same.

```
using System;
using System.IO;
using System.Runtime.Serialization;
using System.Runtime.Serialization.Formatters.Binary;
using Imsl.Stat;

public class MersenneTwisterEx1
{
   /// <summary>
   /// The main entry point for the application.
   /// </summary>


   static void Main(string[] args)
   {
      int nr = 4;
      double[] r = new double[nr];
      int s = 123457;

      /* Initialize MersenneTwister with a seed */
      MersenneTwister mt1 = new MersenneTwister(s);
      MersenneTwister mt2 = (MersenneTwister) mt1.Clone();

      /* Save the state of MersenneTwister */
      Stream stm = new FileStream("mt", FileMode.Create);
      IFormatter fmt = new BinaryFormatter();
      fmt.Serialize(stm,mt1);
      stm.Flush();
      stm.Close();

      Imsl.Stat.Random rndm = new Imsl.Stat.Random(mt1);

      /* Get the next five random numbers */
      for (int k=0; k < nr; k++)
      {
         r[k] = rndm.NextDouble();
      }
```

```
      Console.WriteLine("              First Stream Output");
      Console.WriteLine(r[0]+"     "+r[1]+"     "+r[2]+"     "+r[3]);

      /* Check the cloned copy against the original */
      Imsl.Stat.Random rndm2 = new Imsl.Stat.Random(mt2);
      for (int k=0; k < nr; k++)
      {
         r[k] = rndm2.NextDouble();
      }

      Console.WriteLine("\n              Clone Stream Output");
      Console.WriteLine(r[0]+"     "+r[1]+"     "+r[2]+"     "+r[3]);

      /* Check the serialized copy against the original */
      System.IO.Stream stm2 = new FileStream("mt", FileMode.Open);
      IFormatter fmt2 = new BinaryFormatter();
      mt2 = (MersenneTwister)fmt2.Deserialize(stm2);
      stm2.Close();

      Imsl.Stat.Random rndm3 = new Imsl.Stat.Random(mt2);
      for (int k=0; k < nr; k++)
      {
         r[k] = rndm3.NextDouble();
      }
      Console.WriteLine("\n              Serialized Stream Output");
      Console.WriteLine(r[0]+"     "+r[1]+"     "+r[2]+"     "+r[3]);
   }
}
```

Output

```
              First Stream Output
0.434745062375441     0.352208853699267     0.0138511140830815     0.20914130914025

              Clone Stream Output
0.434745062375441     0.352208853699267     0.0138511140830815     0.20914130914025

              Serialized Stream Output
0.434745062375441     0.352208853699267     0.0138511140830815     0.20914130914025
```

MersenneTwister64 Class

```
public class Imsl.Stat.MersenneTwister64 :  Imsl.Stat.Random.BaseGenerator,
ICloneable
```

A 64-bit Mersenne Twister generator.

`MersenneTwister64` generates uniform pseudorandom 64-bit numbers with a period of $2^{19937} - 1$ and a 623-dimensional equidistribution property. See Matsumoto et al. 1998 for details.

Since 64-bit numbers are generated, all of the bits of both `nextFloat` and `nextDouble` are pseudorandom.

The series of random numbers can be generated using a seed for initialization or by using an array of type `int`. One can also save the state of the generator at initialization to be re-used later.

This C# code was translated from the the following C program.

A C-program for MT19937, with initialization improved 2002/1/26. Coded by Takuji Nishimura and Makoto Matsumoto.

Before using, initialize the state by using init_genrand(seed) or init_by_array(init_key, key_length).

Any feedback is very welcome.

http://www.math.sci.hiroshima-u.ac.jp/m̃-mat/MT/emt.html

email: m-mat@math.sci.hiroshima-u.ac.jp

Constructors

MersenneTwister64

```
public MersenneTwister64(int seed)
```

Description

Initializes the 64-bit Mersenne Twister generator using a seed.

Parameter

`seed` – An `int` which contains the seed that is used to initialize the 64-bit Mersenne Twister generator.

MersenneTwister64

```
public MersenneTwister64(ulong seed)
```

Description

Initializes the 64-bit Mersenne Twister generator using a seed.

Parameter

`seed` – A `ulong` which represents the seed used to initialize the 64-bit Mersenne Twister generator.

MersenneTwister64

```
public MersenneTwister64(int[] key)
```

Description

Initializes the 64-bit Mersenne Twister generator with supplied array.

Parameter

`key` – A `int` array used to initialize the 64-bit Mersenne Twister generator.

MersenneTwister64

```
public MersenneTwister64(ulong[] key)
```

Description

Initializes the 64-bit Mersenne Twister generator with supplied array.

Parameter

`key` – A `ulong` array used to initialize the 64-bit Mersenne Twister generator.

Methods

Clone

```
Final public object Clone()
```

Description

Returns a clone of this object.

Returns

An `Object` which is a clone of this `MersenneTwister64` object.

Next

```
virtual public int Next()
```

Description

Returns a nonnegative random number.

Returns

A 32-bit signed integer greater than or equal to zero.

NextDouble

```
virtual public double NextDouble()
```

Description

Returns a random number between 0.0 and 1.0.

Returns

A `double` greater than or equal to 0.0, and less than 1.0.

NextFloat

```
virtual public float NextFloat()
```

Description

Returns a random number between 0.0 and 1.0.

Returns

A `float` greater than or equal to 0.0, and less than 1.0.

NextLong

```
virtual public long NextLong()
```

Description

Generates the next pseudorandom, uniformly distributed `long` value from this random number generator's sequence.

Returns

A `long` from this random number generator's sequence.

Example: Mersenne Twister Random Number Generation

In this example, four simulation streams are generated. The first series is generated with the seed used for initialization. The second series is generated using an array for initialization. The third series is obtained by resetting the generator back to the state it had at the beginning of the second stream. Therefore, the second and third streams are identical. The fourth stream is obtained by resetting the generator back to its original, uninitialized state, and having it reinitialize using the seed. The first and fourth streams are therefore the same.

```
using System;
using System.IO;
using System.Runtime.Serialization;
using System.Runtime.Serialization.Formatters.Binary;
using Imsl.Stat;
```

```
public class MersenneTwister64Ex1
{
   /// <summary>
   /// The main entry point for the application.
   /// </summary>


   static void Main(string[] args)
   {
      int nr = 4;
      double[] r = new double[nr];
      int s = 123457;

      /* Initialize MersenneTwister64 with a seed */
      MersenneTwister64 mt1 = new MersenneTwister64(s);
      MersenneTwister64 mt2 = (MersenneTwister64) mt1.Clone();

      /* Save the state of MersenneTwister64 */
      Stream stm = new FileStream("mt", FileMode.Create);
      IFormatter fmt = new BinaryFormatter();
      fmt.Serialize(stm,mt1);
      stm.Flush();
      stm.Close();

      Imsl.Stat.Random rndm = new Imsl.Stat.Random(mt1);

      /* Get the next five random numbers */
      for (int k=0; k < nr; k++)
      {
         r[k] = rndm.NextDouble();
      }

      Console.WriteLine("              First Stream Output");
      Console.WriteLine(r[0]+"     "+r[1]+"     "+r[2]+"     "+r[3]);

      /* Check the cloned copy against the original */
      Imsl.Stat.Random rndm2 = new Imsl.Stat.Random(mt2);
      for (int k=0; k < nr; k++)
      {
         r[k] = rndm2.NextDouble();
      }

      Console.WriteLine("\n              Clone Stream Output");
      Console.WriteLine(r[0]+"     "+r[1]+"     "+r[2]+"     "+r[3]);

      /* Check the serialized copy against the original */
      System.IO.Stream stm2 = new FileStream("mt", FileMode.Open);
      IFormatter fmt2 = new BinaryFormatter();
      mt2 = (MersenneTwister64)fmt2.Deserialize(stm2);
      stm2.Close();

      Imsl.Stat.Random rndm3 = new Imsl.Stat.Random(mt2);
      for (int k=0; k < nr; k++)
      {
         r[k] = rndm3.NextDouble();
      }
```

```
      Console.WriteLine("\n              Serialized Stream Output");
      Console.WriteLine(r[0]+"     "+r[1]+"     "+r[2]+"     "+r[3]);
   }
}
```

Output

```
              First Stream Output
0.579916541818503     0.940114746325065     0.710159376724905     0.163995293979278

              Clone Stream Output
0.579916541818503     0.940114746325065     0.710159376724905     0.163995293979278

              Serialized Stream Output
0.579916541818503     0.940114746325065     0.710159376724905     0.163995293979278
```

FaureSequence Class

`public class Imsl.Stat.FaureSequence :  Imsl.Stat.IRandomSequence`

Generates the low-discrepancy Faure sequence.

Discrepancy measures the deviation from uniformity of a point set.

The discrepancy of the point set $x_1, \ldots, x_n \in [0,1]^d$, $d \geq 1$, is

$$D_n^{(d)} = \sup_E \left| \frac{A(E;n)}{n} - \lambda(E) \right|,$$

where the supremum is over all subsets of $[0,1]^d$ of the form

$$E = [0,t_1) \times \cdots \times [0,t_d), \; 0 \leq t_j \leq 1, 1 \leq j \leq d,$$

λ is the Lebesque measure, and *A(E;n)* is the number of the x_j contained in E.

The sequence $x_1, x_2, \ldots$ of points in $[0,1]^d$ is a low-discrepancy sequence if there exists a constant *c(d)*, depending only on *d*, such that

$$D_n^{(d)} \leq c(d) \frac{(\log n)^d}{n}$$

for all $n > 1$.

Generalized Faure sequences can be defined for any prime base $b \geq d$. The lowest bound for the discrepancy is obtained for the smallest prime $b \geq d$, so the base defaults to the smallest prime greater than or equal to the dimension.

The generalized Faure sequence $x_1, x_2, \ldots$, is computed as follows:

Write the positive integer n in its b-ary expansion,

$$n = \sum_{i=0}^{\infty} a_i(n) b^i$$

where $a_i(n)$ are integers, $0 \leq a_j(n) < b$.

The j-th coordinate of x_n is

$$x_n^{(j)} = \sum_{k=0}^{\infty} \sum_{d=0}^{\infty} c_{kd}^{(j)} a_d(n) b^{-k-1},\ 1 \leq j \leq d$$

The generator matrix for the series, $c_{kd}^{(j)}$, is defined to be

$$c_{kd}^{(j)} = j^{d-k} c_{kd}$$

and c_{kd} is an element of the Pascal matrix,

$$c_{kd} = \begin{cases} \frac{d!}{c!(d-c)!} & k \leq d \\ 0 & k > d \end{cases}$$

It is faster to compute a shuffled Faure sequence than to compute the Faure sequence itself. It can be shown that this shuffling preserves the low-discrepancy property.

The shuffling used is the b-ary Gray code. The function *G(n)* maps the positive integer n into the integer given by its b-ary expansion. The sequence computed by this function is $\vec{x}(G(n))$, where $\vec{x}$ is the generalized Faure sequence.

Properties

Base

```
public int Base {get; }
```

Description

The base.

Property Value

A `int` which specifies the base.

Dimension

```
Final public int Dimension {get; }
```

Description

Returns the dimension of the sequence.

Property Value

A `int` which specifies the dimension.

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

Skip

```
public int Skip {get; }
```

Description

Returns the number of points skipped at the beginning of the sequence.

Property Value

A `int` which specifies the number of points skipped.

Constructors

FaureSequence

```
public FaureSequence(int dimension)
```

Description

Creates a Faure sequence with the default base.

Parameter

`dimension` – An `int` which specifies the dimension of the sequence.

Remarks

The base defaults to the smallest prime equal to or greater than `dimension`.

FaureSequence

```
public FaureSequence(int dimension, int baseSequence, int nSkip)
```

Description

Creates a Faure sequence.

Parameters

`dimension` – An `int` which specifies the dimension of the sequence.

`baseSequence` – A `int` which specifies the smallest prime number greater than or equal to `dimension`.

`nSkip` – An `int` which specifies the number of initial points to skip.

Remarks

If `nSkip` is negative then $base^{m/2-1}$, where m is the number of digits needed to represent the largest `Int32` in the base, points are skipped.

Methods

ComputeParameters

```
void ComputeParameters()
```

Description

Compute needed parameters.

NextDouble

```
public double NextDouble()
```

Description

Returns the first value of the next point in the sequence.

Returns

A `double` array which specifies the next sequence value.

Remarks

This method is intended for use when `dimension` is 1.

NextPoint

```
Final public double[] NextPoint()
```

Description

Returns the next point in the sequence.

Returns

A `double` array which specifies the next point in the sequence.

NextPrime

```
static public int NextPrime(int n)
```

Description

Returns the smallest prime greater than or equal to n.

Parameter

n – An int which specifies the first number to try as a prime.

Returns

An int which specifies a prime greater than or equal to n.

Remarks

If n is less than or equal to 2 then 2 is returned.

Example: FaureSequence

In this example, ten points of the Faure sequence are computed. The points are in a four-dimensional cube.

```
using System;
using FaureSequence = Imsl.Stat.FaureSequence;
using PrintMatrix = Imsl.Math.PrintMatrix;

public class FaureSequenceEx1
{
    public static void  Main(String[] args)
    {
        FaureSequence seq = new FaureSequence(4);
        double[][] x = new double[10][];
        for (int k = 0; k < 10; k++)
        {
            x[k] = seq.NextPoint();
        }
        new PrintMatrix("Faure Sequence").Print(x);
    }
}
```

Output

```
              Faure Sequence
      0         1         2         3
0  0.201344  0.274944  0.532544  0.694144
1  0.401344  0.474944  0.732544  0.894144
2  0.601344  0.674944  0.932544  0.094144
3  0.801344  0.874944  0.132544  0.294144
4  0.841344  0.114944  0.572544  0.934144
5  0.041344  0.314944  0.772544  0.134144
6  0.241344  0.514944  0.972544  0.334144
7  0.441344  0.714944  0.172544  0.534144
8  0.641344  0.914944  0.372544  0.734144
9  0.681344  0.154944  0.612544  0.374144
```

IRandomSequence Interface

```
public interface Imsl.Stat.IRandomSequence
```

Interface implemented by generators of random or quasi-random multidimension sequences.

Property

Dimension

```
abstract public int Dimension {get; }
```

Description

Returns the dimension of the sequence.

Property Value

A `int` which specifies the dimension.

Method

NextPoint

```
abstract public double[] NextPoint()
```

Description

Returns the next multidimensional point in the sequence.

Returns

A `double` array of length *dimension*.

Chapter 23: Finance

Types

Usage Notes

Users can perform financial computations by using pre-defined data types. Most of the financial functions require one or more of the following:

- Date
- Number of payments per year
- A variable to indicate when payments are due
- Day count basis

The `Bond.Frequency` field indicates the number of payments for each year.

`Bond.Frequency`	**Meaning**
`Bond.Annual`	One payment per year (Annual payment)
`Bond.SemiAnnual`	Two payments per year (Semi-annual payment)
`Bond.Quarterly`	Four payments per year (Quarterly payment)

The `Finance.Period` field indicates when payments are due.

`Finance.Period`	**Meaning**
`Finance.At_End_of_Period`	Payments are due at the end of the period
`Finance.AT_Beginning_of_Period`	Payments are due at the beginning of the period

The `DayCountBasis` class provides fields to indicate the type of day count basis. Day count basis is the method for computing the number of days between two dates.

Class Field	Day count basis
`DayCountBasis.BasisNASD`	US (NASD) 30/360
`DayCountBasis.BasisActualActual`	Actual/Actual
`DayCountBasis.BasisActual360`	Actual/360
`DayCountBasis.BasisActual365`	Actual/365
`DayCountBasis.Basis30e360`	European 30/360

Additional Information

In preparing the finance and bond functions we incorporated standards used by *SIA Standard Securities Calculation Methods*.

More detailed information on finance and bond functionality can be found in the following manuals:

- *SIA Standard Securities Calculation Methods* 1993, vols. 1 and 2, Third Edition
- *Microsoft Excel 5, Worksheet Function Reference.*

Finance Class

```
public class Imsl.Finance.Finance
```

Collection of finance functions.

Constructor

Finance

```
public Finance()
```

Description

Initializes a new instance of the Imsl.Finance.Finance (p. 1224) class.

Methods

Cumipmt

```
static public double Cumipmt(double rate, int nper, double pv, int firstPeriod,
int lastPeriod, Imsl.Finance.Finance.Period period)
```

Description

Returns the cumulative interest paid between two periods.

Parameters

`rate` – A `double` which specifies the interest rate.

`nper` – A `int` which specifies the total number of payment periods.

`pv` – A `double` which specifies the present value.

`firstPeriod` – A `int` containing the first period in the caclulation. Periods are numbered starting with one.

`lastPeriod` – A `int` which specifies the last period in the calculation.

`period` – A `int` which specifies the time in each period when the payment is made, either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)

Returns

A `double` which specifies the cumulative interest paid between the first period and the last period.

Remarks

It is computed using the following:

$$\sum_{i=firstPeriod}^{lastPeriod} interest_i$$

where $interest_i$ is computed from `Ipmt` for the i-th period.

Cumprinc

```
static public double Cumprinc(double rate, int nper, double pv, int
firstPeriod, int lastPeriod, Imsl.Finance.Finance.Period time)
```

Description

Returns the cumulative principal paid between two periods.

Parameters

`rate` – A `double` which specifies the interest rate.

`nper` – A `int` which specifies the total number of payment periods.

`pv` – A `double` which specifies the present value.

`firstPeriod` – A `int` which specifies the first period in the calculation. Periods are numbered starting with one.

`lastPeriod` – A `int` which specifies the last period in the calculation.

`time` – The time of a `Period` when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `double` which specifies the cumulative principal paid between the first period and the last period.

Remarks

It is computed using the following:

$$\sum_{i=firstPeriod}^{lastPeriod} principal_i$$

where $principal_i$ is computed from `Ppmt` for the i-th period.

Db

```
static public double Db(double cost, double salvage, int life, int period, int
month)
```

Description

Returns the depreciation of an asset using the fixed-declining balance method.

Parameters

`cost` – A `double` which specifies the initial cost of the asset.

`salvage` – A `double` which specifies the salvage value of the asset.

`life` – A `int` which specifies the number of periods over which the asset is being depreciated.

`period` – A `int` which specifies the period for which the depreciation is to be computed.

`month` – A `int` which specifies the number of months in the first year.

Returns

A `double` which specifies the depreciation of an asset for a specified period using the fixed-declining balance method.

Remarks

Method `Db` varies depending on the specified value for the argument period, see table below.

If period = 1,

$$\text{cost} \times \text{rate} \times \frac{\text{month}}{12}$$

If period = life,

$$(\text{cost} - \text{total depreciation from periods}) \times \text{rate} \times \frac{12 - \text{month}}{12}$$

If period other than 1 or life,

$$(\text{cost} - \text{total depreciation from priorperiods}) \times rate$$

where

$$rate = 1 - \left(\frac{\text{salvage}}{\text{cost}}\right)^{\left(\frac{1}{life}\right)}$$

NOTE: *rate* is rounded to three decimal places.

Ddb

```
static public double Ddb(double cost, double salvage, int life, int period,
double factor)
```

Description

Returns the depreciation of an asset using the double-declining balance method.

Parameters

`cost` – A `double` which specifies the initial cost of the asset.

`salvage` – A `double` which specifies the salvage value of the asset.

`life` – A `int` which specifies the number of periods over which the asset is being depreciated.

`period` – A `int` which specifies the period.

`factor` – A `double` which specifies the rate at which the balance declines.

Returns

A `double` which specifies the depreciation of an asset for a specified period.

Remarks

It is computed using the following:

$$[cost - salvage\,(total\,depreciation\,from\,prior\,periods)]\,\frac{factor}{life}$$

Dollarde

```
static public double Dollarde(double fractionalDollar, int fraction)
```

Description

Converts a fractional price to a decimal price.

Parameters

`fractionalDollar` – A `double` which specifies a fractional number.

`fraction` – A `int` which specifies the denominator.

Returns

A `double` which specifies the dollar price expressed as a decimal number.

Remarks

It is computed using the following:

$$idollar + (fractionalDollar - idollar) \times \frac{10^{(ifrac+1)}}{fraction}$$

where *idollar* is the integer part of *fractionalDollar*, and *ifrac* is the integer part of $\log(fraction)$.

Dollarfr

```
static public double Dollarfr(double decimalDollar, int fraction)
```

Description

Converts a decimal price to a fractional price.

Parameters

`decimalDollar` – A `double` which specifies a decimal number.

`fraction` – A `int` which specifies the denominator.

Returns

A `double` which specifies a dollar price expressed as a fraction.

Remarks

It is computed using the following:

$$idollar + \frac{decimalDollar - idollar}{10^{(ifrac+1)}/fraction}$$

where *idollar* is the integer part of the *decimalDollar*, and *ifrac* is the integer part of $log(fraction)$.

Effect

```
static public double Effect(double nominalRate, int nper)
```

Description

Returns the effective annual interest rate.

Parameters

`nominalRate` – A `double` which specifies the nominal interest rate.

`nper` – A `int` which specifies the number of compounding periods per year.

Returns

A `double` which specifies the effective annual interest rate.

Remarks

The nominal interest rate is the periodically-compounded interest rate as stated on the face of a security. The effective annual interest rate is computed using the following:

$$\left(1 + \frac{nominalRate}{nper}\right)^{nper} - 1$$

Fv

```
static public double Fv(double rate, int nper, double pmt, double pv,
Imsl.Finance.Finance.Period period)
```

Description

Returns the future value of an investment.

Parameters

`rate` – A `double` which specifies the interest rate.

`nper` – A `int` which specifies the total number of payment periods.

`pmt` – A `double` which specifies the payment made in each period.

`pv` – A `double` which specifies the present value.

`period` – A `int` which specifies the time in each period when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `double` which specifies the future value of an investment.

Remarks

The future value is the value, at some time in the future, of a current amount and a stream of payments. It can be found by solving the following:

If $rate = 0$,

$$pv + pmt \times nper + fv = 0$$

If $rate \neq 0$,

$$pv(1+rate)^{nper} + pmt\,[1 + rate\,(period)]\,\frac{(1+rate)^{nper} - 1}{rate} + fv = 0$$

Fvschedule

```
static public double Fvschedule(double principal, double[] schedule)
```

Description

Returns the future value of an initial principal taking into consideration a schedule of compound interest rates.

Parameters

`principal` – A `double` which specifies the present value.

`schedule` – A `double` array of interest rates to apply.

Returns

A `double` which specifies the future value of an initial principal

Remarks

It is computed using the following:

$$\sum_{i=1}^{count} (principal \times schedule_i)$$

where $schedule_i$ = interest rate at the i-th period.

Ipmt

```
static public double Ipmt(double rate, int period, int nper, double pv, double
fv, Imsl.Finance.Finance.Period time)
```

Description

Returns the interest payment for an investment for a given period.

Parameters

`rate` – A `double` which specifies the interest rate.

`period` – A `int` which specifies the payment period.

`nper` – A `int` which specifies the total number of periods.

`pv` – A `double` which specifies the present value.

`fv` – A `double` which specifies the future value.

`time` – The time of a `Period` when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `double` which specifies the interest payment for a given period for an investment.

Remarks

It is computed using the following:

$$\left\{ pv(1+rate)^{nper-1} + pmt(1+rate \times period)\frac{(1+rate)^{nper-1}}{rate} \right\} rate$$

Irr

```
static public double Irr(double[] pmt)
```

Description

Returns the internal rate of return for a schedule of cash flows.

Parameter

`pmt` – A `double` array which contains cash flow values which occur at regular intervals.

Returns

A `double` which specifies the internal rate of return.

Remarks

It is found by solving the following:

$$0 = \sum_{i=1}^{count} \frac{value_i}{(1+rate)^i}$$

where $value_i$ = the ith cash flow, $rate$ is the internal rate of return.

Irr

```
static public double Irr(double[] pmt, double guess)
```

Description

Returns the internal rate of return for a schedule of cash flows.

Parameters

`pmt` – A `double` array which contains cash flow values which occur at regular intervals.

`guess` – A `double` value which represents an initial guess at the return value from this function.

Returns

A `double` which specifies the internal rate of return.

Remarks

It is found by solving the following:

$$0 = \sum_{i=1}^{count} \frac{value_i}{(1+rate)^i}$$

where $value_i$ = the *ith* cash flow, *rate* is the internal rate of return.

Mirr

```
static public double Mirr(double[] cashFlow, double financeRate, double
reinvestRate)
```

Description

Returns the modified internal rate of return for a schedule of periodic cash flows.

Parameters

`cashFlow` – A `double` array of cash flows.

`financeRate` – A `double` which specifies the interest you pay on the money you borrow.

`reinvestRate` – A `double` which specifies the interest rate you receive on the cash flows.

Returns

A `double` which specifies the modified internal rate of return.

Remarks

The modified internal rate of return differs from the ordinary internal rate of return in assuming that the cash flows are reinvested at the cost of capital, not at the internal rate of return. It also eliminates the multiple rates of return problem. It is computed using the following:

$$\left\{ \frac{-(pnpv)(1+reinvestRate)^{n\_per}}{(nnpv)(1+financeRate)} \right\}^{\frac{1}{nper-1}} - 1$$

where *pnpv* is calculated from `Npv` for positive values in values using `reinvestRate`, and where *nnpv* is calculated from `Npv` for negative values in `values` using `financeRate`.

Nominal

```
static public double Nominal(double effectiveRate, int nper)
```

Description

Returns the nominal annual interest rate.

Parameters

`effectiveRate` – A `double` which specifies the effective interest rate.

`nper` – A `int` which specifies the number of compounding periods per year.

Returns

A `double` which specifies the nominal annual interest rate.

Remarks

The nominal interest rate is the interest rate as stated on the face of a security. It is computed using the following:

$$\left[(1+effectiveRate)^{\frac{1}{nper}} - 1\right] \times nper$$

Nper

```
static public double Nper(double rate, double pmt, double pv, double fv,
Imsl.Finance.Finance.Period period)
```

Description

Returns the number of periods for an investment for which periodic, and constant payments are made and the interest rate is constant.

Parameters

`rate` – A `double` which specifies the interest rate.

`pmt` – A `double` which specifies the payment.

`pv` – A `double` which specifies the present value.

`fv` – A `double` which specifies the future value.

`period` – A `int` which specifies the time in each period when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `int` which specifies the number of periods for an investment.

Remarks

It can be found by solving the following:

If $rate = 0$,

$$pv + pmt \times nper + fv = 0$$

If $rate \neq 0$,

$$pv(1+rate)^{nper} + pmt\,[1 + rate\,(period)]\,\frac{(1+rate)^{nper} - 1}{rate} + fv = 0$$

Npv

```
static public double Npv(double rate, double[] eqCashFlow)
```

Description

Returns the net present value of a stream of equal periodic cash flows, which are subject to a given discount rate.

Parameters

`rate` – A `double` which specifies the interest rate per period.

`eqCashFlow` – A `double` array of equally-spaced cash flows.

Returns

A `double` which specifies the net present value of the investment.

Remarks

It is found by solving the following:

$$\sum_{i=1}^{count} \frac{value_i}{(1+rate)^i}$$

where $value_i$ = the ith cash flow.

PeriodicPayment

```
static public double PeriodicPayment(double rate, int nper, double pv, double
fv, Imsl.Finance.Finance.Period period)
```

Description

Returns the periodic payment for an investment.

Parameters

`rate` – A `double` which specifies the interest rate.

`nper` – A `int` which specifies the total number of periods.

`pv` – A `double` which specifies the present value.

`fv` – A `double` which specifies the future value.

`period` – A `int` which specifies the time in each period when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `double` which specifies the interest payment for a given period for an investment.

Remarks

It can be found by solving the following:

If $rate = 0$,

$$pv + pmt \times nper + fv = 0$$

If $rate \neq 0$,

$$pv(1+rate)^{nper} + pmt\,[1+rate\,(period)]\,\frac{(1+rate)^{nper}-1}{rate} + fv = 0$$

Ppmt

```
static public double Ppmt(double rate, int period, int nper, double pv, double
fv, Imsl.Finance.Finance.Period time)
```

Description

Returns the payment on the principal for a specified period.

Parameters

`rate` – A `double` which specifies the interest rate.

`period` – A `int` which specifies the payment period.

`nper` – A `int` which specifies the total number of periods.

`pv` – A `double` which specifies the present value.

`fv` – A `double` which specifies the future value.

`time` – The time of a `Period` when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `double` which specifies the payment on the principal for a given period.

Remarks

It is computed using the following:

$$payment_i - interest_i$$

where $payment_i$ is computed from `pmt` for the i-th period, $interest_i$ is calculated from `Ipmt` for the i-th period.

Pv

```
static public double Pv(double rate, int nper, double pmt, double fv,
Imsl.Finance.Finance.Period time)
```

Description

Returns the net present value of a stream of equal periodic cash flows, which are subject to a given discount rate.

Parameters

`rate` – A `double` which specifies the interest rate per period.

`nper` – A `int` which specifies the number of periods.

`pmt` – A `double` which specifies the payment made each period.

`fv` – A `double` which specifies the annuity's value after the last payment.

`time` – The time in a `Period` when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `double` which specifies the present value of the investment.

Remarks

It can be found by solving the following:

If $rate = 0$,

$$pv + pmt \times nper + fv = 0$$

If $rate \neq 0$,

$$pv(1+rate)^{nper} + pmt\,[1 + rate\,(period)]\,\frac{(1+rate)^{nper} - 1}{rate} + fv = 0$$

Rate

```
static public double Rate(int nper, double pmt, double pv, double fv,
Imsl.Finance.Finance.Period time)
```

Description

Returns the interest rate per period of an annuity.

Parameters

`nper` – A `int` which specifies the number of periods.

`pmt` – A `double` which specifies the payment made each period.

`pv` – A `double` which specifies the present value.

`fv` – A `double` which specifies the annuity's value after the last payment.

`time` – The time in a `Period` when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

Returns

A `double` which specifies the interest rate per period of an annuity.

Remarks

Rate is calculated by iteration and can have zero or more solutions. It can be found by solving the following:

If $rate = 0$,

$$pv + pmt \times nper + fv = 0$$

If $rate \neq 0$,

$$pv(1+rate)^{nper} + pmt\,[1 + rate\,(period)]\,\frac{(1+rate)^{nper} - 1}{rate} + fv = 0$$

Rate

```
static public double Rate(int nper, double pmt, double pv, double fv,
Imsl.Finance.Finance.Period time, double guess)
```

Description

Returns the interest rate per period of an annuity with an initial guess.

Parameters

`nper` – A `int` which specifies the number of periods.

`pmt` – A `double` which specifies the payment made each period.

`pv` – A `double` which specifies the present value.

`fv` – A `double` which specifies the annuity's value after the last payment.

`time` – The time in a `Period` when the payment is made (either Imsl.Finance.Finance.Period.AtEnd (p. 1254) or Imsl.Finance.Finance.Period.AtBeginning (p. 1254)).

`guess` – A `double` value which represents an initial guess at the interest rate per period of an annuity.

Returns

A `double` which specifies the interest rate per period of an annuity.

Remarks

Rate is calculated by iteration and can have zero or more solutions. It can be found by solving the following:

If $rate = 0$,

$$pv + pmt \times nper + fv = 0$$

If $rate \neq 0$,

$$pv(1+rate)^{nper} + pmt\,[1 + rate\,(period)]\,\frac{(1+rate)^{nper} - 1}{rate} + fv = 0$$

Sln

```
static public double Sln(double cost, double salvage, int life)
```

Description

Returns the depreciation of an asset using the straight line method.

Parameters

`cost` – A `double` which specifies the initial cost of the asset.

`salvage` – A `double` which specifies the salvage value of the asset.

`life` – A `int` which specifies the number of periods over which the asset is being depreciated.

Returns

A `double` which specifies the straight line depreciation of an asset for one period.

Remarks

It is computed using the following:

$$cost - salvage/life$$

Syd

```
static public double Syd(double cost, double salvage, int life, int per)
```

Description

Returns the depreciation of an asset using the sum-of-years digits method.

Parameters

`cost` – A `double` which specifies the initial cost of the asset.

`salvage` – A `double` which specifies the salvage value of the asset.

`life` – A `int` which specifies the number of periods over which the asset is being depreciated.

`per` – A `int` which specifies the period.

Returns

A `double` which specifies the sum-of-years digits depreciation of an asset.

Remarks

It is computed using the following:

$$(cost - salvage)(per)\ \frac{(life+1)\,(life)}{2}$$

Vdb

```
static public double Vdb(double cost, double salvage, int life, int
firstPeriod, int lastPeriod, double factor, bool noSL)
```

Description

Returns the depreciation of an asset for any given period using the variable-declining balance method.

Parameters

`cost` – A `double` which specifies the initial cost of the asset.

`salvage` – A `double` which specifies the salvage value of the asset.

`life` – A `int` which specifies the number of periods over which the asset is being depreciated.

`firstPeriod` – A `int` which specifies the first period for the calculation.

`lastPeriod` – A `int` which specifies the last period for the calculation.

`factor` – A `double` which specifies the rate at which the balance declines.

`noSL` – A `bool` flag. If `true`, do not switch to straight-line depreciation even when the depreciation is greater than the declining balance calculation.

Returns

A `double` which specifies the depreciation of the asset.

Remarks

It is computed using the following:

If $no\_sl = 0$,

$$\sum_{i=firstPeriod+1}^{lastPeriod} ddb_i$$

If $no\_sl \neq 0$,

$$A + \sum_{i=k}^{lastPeriod} \frac{cost - A - salvage}{lastPeriod - k + 1}$$

where ddb_i is computed from `Ddb` for the i-th period. k = the first period where straight line depreciation is greater than the depreciation using the double-declining balance method.

$$A = \sum_{i=firstPeriod+1}^{k-1} ddb_i$$

Xirr

```
static public double Xirr(double[] pmt, System.DateTime[] dates)
```

Description

Returns the internal rate of return for a schedule of cash flows.

Parameters

`pmt` – A `double` array which contains cash flow values which correspond to a schedule of payments in dates.

`dates` – A `DateTime` array which contains a schedule of payment dates.

Returns

A `double` which specifies the internal rate of return.

Remarks

It is not necessary that the cash flows be periodic. It can be found by solving the following:

$$0 = \sum_{i=1}^{count} \frac{value_i}{(1+rate)^{\frac{d_i - d_1}{365}}}$$

In the equation above, d_i represents the *i*th payment date. d_1 represents the 1st payment date. *value* represents the *i*th cash flow. *rate* is the internal rate of return.

Xirr

```
static public double Xirr(double[] pmt, System.DateTime[] dates, double guess)
```

Description

Returns the internal rate of return for a schedule of cash flows with a user supplied initial guess.

Parameters

`pmt` – A `double` array which contains cash flow values which correspond to a schedule of payments in dates.

`dates` – A `DateTime` array which contains a schedule of payment dates.

`guess` – A `double` value which represents an initial guess at the return value from this function.

Returns

A `double` which specifies the internal rate of return.

Remarks

It is not necessary that the cash flows be periodic. It can be found by solving the following:

$$0 = \sum_{i=1}^{count} \frac{value_i}{(1+rate)^{\frac{d_i - d_1}{365}}}$$

In the equation above, d_i represents the *i*th payment date. d_1 represents the 1st payment date. *value* represents the *i*th cash flow. *rate* is the internal rate of return.

Xnpv

```
static public double Xnpv(double rate, double[] cashFlow, System.DateTime[]
dates)
```

Description

Returns the present value for a schedule of cash flows.

Parameters

`rate` – A `double` which specifies the interest rate.

`cashFlow` – A `double` array containing the cash flows.

`dates` – A `DateTime` array which contains a schedule of payment dates.

Returns

A `double` which specifies the present value.

Remarks

It is not necessary that the cash flows be periodic. It is computed using the following:

$$\sum_{i=1}^{count} \frac{value_i}{(1+rate)^{(d_i-d_1)/365}}$$

In the equation above, d_i represents the ith payment date, d_1 represents the first payment date, and $value_i$ represents the ith cash flow.

Example: Cumulative Interest Example

The amount of interest paid in the first year of a 30 year fixed rate mortgage is computed. The amount financed is $200,000 at an interest rate of 7.25% for 30 years.

```
using System;
using Imsl.Finance;

public class cumipmtEx1
{
    public static void  Main(String[] args)
    {
        double rate = 0.0725 / 12;
        int periods = 12 * 30;
        double pv = 200000;
        int start = 1;
        int end = 12;
        double total = Finance.Cumipmt(rate, periods, pv, start, end,
                                       Finance.Period.AtEnd);
        Console.Out.WriteLine("First year interest = " +
                              total.ToString("C"));
    }
}
```

Output

```
First year interest = ($14,436.52)
```

Example: Cumulative Principal Example

The amount of principal paid in the first year of a 30 year fixed rate mortgage is computed. The amount financed is $200,000 at an interest rate of 7.25% for 30 years.

```
using System;
using Imsl.Finance;

public class cumprincEx1
```

```
{
    public static void  Main(String[] args)
    {
        double rate = 0.0725 / 12;
        int periods = 12 * 30;
        double pv = 200000;
        int start = 1;
        int end = 12;
        double total = Finance.Cumprinc(rate, periods, pv, start, end,
                                        Finance.Period.AtEnd);
        Console.Out.WriteLine("First year principal = " +
                              total.ToString("C"));
    }
}
```

Output

```
First year principal = ($1,935.71)
```

Example: Depreciation - Fixed Declining Balance Method

The depreciation of an asset with an initial cost of $2500 and a salvage value of $500 over a period of 3 years is calculated. Here month is 6 since the life of the asset did not begin until the seventh month of the first year.

```
using System;
using Imsl.Finance;

public class dbEx1
{
    public static void  Main(String[] args)
    {
        double cost = 2500;
        double salvage = 500;
        int life = 3;
        int month = 6;

        for (int period = 1; period <= life + 1; period++)
        {
            double db = Finance.Db(cost, salvage, life, period, month);
            Console.Out.WriteLine("For period " + period + "   " +
                                  db.ToString("C"));
        }
    }
}
```

Output

```
For period 1  $518.75
For period 2  $822.22
For period 3  $481.00
For period 4  $140.69
```

Example: Depreciation - Double-Declining Balance Method

The depreciation of an asset with an initial cost of $2500 and a salvage value of $500 over a period of 2 years is calculated. A factor of 2 is used (the double-declining balance method).

```
using System;
using Imsl.Finance;

public class ddbEx1
{
    public static void  Main(String[] args)
    {
        double cost = 2500;
        double salvage = 500;
        double factor = 2;
        int life = 24;

        for (int period = 1; period <= life; period++)
        {
            double ddb = Finance.Ddb(cost, salvage, life, period,
                                     factor);
            Console.Out.WriteLine("For period " + period +
                                  "    ddb = " + ddb.ToString("C"));
        }
    }
}
```

Output

```
For period 1    ddb = $208.33
For period 2    ddb = $190.97
For period 3    ddb = $175.06
For period 4    ddb = $160.47
For period 5    ddb = $147.10
For period 6    ddb = $134.84
For period 7    ddb = $123.60
For period 8    ddb = $113.30
For period 9    ddb = $103.86
For period 10    ddb = $95.21
For period 11    ddb = $87.27
For period 12    ddb = $80.00
For period 13    ddb = $73.33
For period 14    ddb = $67.22
For period 15    ddb = $61.62
For period 16    ddb = $56.48
For period 17    ddb = $51.78
For period 18    ddb = $47.46
For period 19    ddb = $22.09
For period 20    ddb = $0.00
For period 21    ddb = $0.00
For period 22    ddb = $0.00
For period 23    ddb = $0.00
For period 24    ddb = $0.00
```

Example: Price Conversion - Fractional Dollars

A fractional dollar price, in this case 1 3/8, is converted to a decimal price.

```
using System;
using Imsl.Finance;

public class dollardeEx1
{
    public static void  Main(String[] args)
    {
        double fractionalDollar = 1.3;
        int fraction = 8;

        double dollardec = Finance.Dollarde(fractionalDollar,
                                            fraction);
        Console.Out.WriteLine("The fractional dollar 1.3 = " +
                              dollardec.ToString("C"));
    }
}
```

Output

```
The fractional dollar 1.3 = $1.38
```

Example: Price Conversion - Decimal Dollars

A decimal dollar price, in this case $1.38, is converted to a fractional price.

```
using System;
using Imsl.Finance;

public class dollarfrEx1
{
    public static void  Main(String[] args)
    {
        double decimalDollar = 1.38;
        int fraction = 8;

        double dollarfrc = Finance.Dollarfr(decimalDollar, fraction);
        Console.Out.WriteLine("The decimal dollar $1.38 as a fractional"
                              + " dollar = " +
                              dollarfrc.ToString("0.00"));
    }
}
```

Output

```
The decimal dollar $1.38 as a fractional dollar = 1.30
```

Example: Effective Rate

In this example the effective interest rate is computed given that the nominal rate is 6.0% and that the interest will be compounded quarterly.

```
using System;
using Imsl.Finance;

public class effectEx1
{
    public static void  Main(String[] args)
    {
        double nominalRate = .06;
        int nper = 4;
        double effectiveRate;

        effectiveRate = Finance.Effect(nominalRate, nper);
        Console.Out.WriteLine("The effective rate of the nominal rate,"
                              + " 6.0%, " + "compounded quarterly is "
                              + effectiveRate.ToString("P"));
    }
}
```

Output

```
The effective rate of the nominal rate, 6.0%, compounded quarterly is 6.14 %
```

Example: Future Value of an Investment

A couple starts setting aside $30,000 a year when they are 45 years old. They expect to earn 5% interest on the money compounded yearly. The future value of the investment is computed for a 20 year period.

```
using System;
using Imsl.Finance;

public class fvEx1
{
    public static void  Main(String[] args)
    {
        double rate = .05;
        int nper = 20;
        double payment = - 30000.00;
        double pv = - 30000.00;

        double fv = Finance.Fv(rate, nper, payment, pv,
                               Finance.Period.AtBeginning);
        Console.Out.WriteLine("After 20 years, the value of the " +
                              "investments " + "will be " +
                              fv.ToString("C"));
    }
}
```

Output

```
After 20 years, the value of the investments will be $1,121,176.49
```

Example: Future Value - Adustable Rates

An investment of $10,000 is made. The investment will grow at the rate of 5.1% the first year, with the rate increasing by .1% each year thereafter for a total of 5 years. The future value of the investment is computed.

```
using System;
using Imsl.Finance;

public class fvscheduleEx1
{
    public static void  Main(String[] args)
    {
        double principal = 10000.0;
        double[] schedule = new double[]{.050, .051, .052, .053, .054};
        double fvschedule;

        fvschedule = Finance.Fvschedule(principal, schedule);
        Console.Out.WriteLine("After 5 years the $10,000 investment " +
                              "will have " + "grown to " +
                              fvschedule.ToString("C"));
    }
}
```

Output

```
After 5 years the $10,000 investment will have grown to $12,884.77
```

Example: Interest Payments

The interest due the second year on a $100,000 25 year loan is calculated. The loan is at 8%.

```
using System;
using Imsl.Finance;

public class ipmtEx1
{
    public static void  Main(String[] args)
    {
        double rate = .08;
        int per = 2;
        int nper = 25;
        double pv = 100000.00;
        double fv = 0.0;

        double ipmt = Finance.Ipmt(rate, per, nper, pv, fv,
                                   Finance.Period.AtEnd);
        Console.Out.WriteLine("The interest due the second year on the"
                              + " $100,000 loan is " +
                              ipmt.ToString("C"));
    }
}
```

Output

```
The interest due the second year on the $100,000 loan is ($7,890.57)
```

Example: Internal Rate of Return

A farmer buys 10 young cows and a bull for $4500. The first year he does not expect to sell any calves, he just expects to feed them. Thereafter, he expects to be able to sell calves to offset the cost of feed. He expects them to be productive for 9 years, after which time he will liquidate the herd. The internal rate of return is computed after 9 years.

```
using System;
using Imsl.Finance;

public class irrEx1
{
    public static void  Main(String[] args)
    {
        double[] pmt = new double[]
                        {- 4500.0, - 800.0,
                            800.0, 800.0,
                            600.0, 600.0,
                            800.0, 800.0,
                            700.0, 3000.0};

        double irr = Finance.Irr(pmt);
        Console.Out.WriteLine("After 9 years, the internal rate of " +
                            "return on the cows is " +
                            irr.ToString("P"));
    }
}
```

Output

```
After 9 years, the internal rate of return on the cows is 7.21 %
```

Example: Modified Internal Rate of Return

A farmer uses a $4500 loan to buy 10 young cows and a bull. The interest rate on the loan is 8%. He expects to reinvest the profits received in any one year in the money market and receive 5.5%. The first year he does not expect to sell any calves, he just expects to feed them. Thereafter, he expects to be able to sell calves to offset the cost of feed. He expects them to be productive for 9 years, after which time he will liquidate the herd. The modified internal rate of return is computed after 9 years.

```
using System;
using Imsl.Finance;

public class mirrEx1
{
    public static void  Main(String[] args)
    {
        double[] x = new double[]{- 4500.0, - 800.0,
```

```
                                    800.0, 800.0,
                                    600.0, 600.0,
                                    800.0, 800.0,
                                    700.0, 3000.0};
        double financeRate = .08;
        double reinvestRate = .055;
        double mirr = Finance.Mirr(x, financeRate, reinvestRate);

        Console.Out.WriteLine("After 9 years, the modified internal " +
                              "rate of return \non the cows is " +
                              mirr.ToString("P"));
    }
}
```

Output

```
After 9 years, the modified internal rate of return
on the cows is 6.66 %
```

Example: Nominal Rate

In this example the nominal interest rate is computed given that the effective rate is 6.14% and that the interest has been compounded quarterly.

```
using System;
using Imsl.Finance;

public class nominalEx1
{
    public static void  Main(String[] args)
    {
        double effectiveRate = .0614;
        int nper = 4;

        double nominalRate = Finance.Nominal(effectiveRate, nper);
        Console.Out.WriteLine("The nominal rate of the effective rate,"
                              + "6.14%, \ncompounded quarterly is " +
                              nominalRate.ToString("P"));
    }
}
```

Output

```
The nominal rate of the effective rate,6.14%,
compounded quarterly is 6.00 %
```

Example: Number of Periods for an Investment

Someone obtains a $20,000 loan at 7.25% to buy a car. They want to make $350 a month payments. Here, the number of payments necessary to pay off the loan is computed.

```
using System;
```

```
using Imsl.Finance;

public class nperEx1
{
    public static void  Main(String[] args)
    {
        double rate = 0.0725 / 12;
        double pmt = - 350.0;
        double pv = 20000;
        double fv = 0.0;
        double nperiods;
        nperiods = Finance.Nper(rate, pmt, pv, fv,
                                Finance.Period.AtBeginning);
        Console.Out.WriteLine("Number of payment periods = "
                              + nperiods);
    }
}
```

Output

```
Number of payment periods = 69.7805113662826
```

Example: Net Present Value of an Investment

A lady wins a $10 million lottery. The money is to be paid out at the end of each year in $500,000 payments for 20 years. The current treasury bill rate of 6% is used as the discount rate. Here, the net present value of her prize is computed.

```
using System;
using Imsl.Finance;

public class npvEx1
{
    public static void  Main(String[] args)
    {
        double rate = 0.06;
        double[] value_Renamed = new double[20];

        for (int i = 0; i < 20; i++)
            value_Renamed[i] = 500000.0;
        double npv = Finance.Npv(rate, value_Renamed);

        Console.Out.WriteLine("The net present value of the $10 " +
                              "million prize is " +    npv.ToString("C"));
    }
}
```

Output

```
The net present value of the $10 million prize is $5,734,960.61
```

Example: Periodic Payments

The payment due each year on a 25 year, $100,000 loan is calculated. The loan is at 8%.

```
using System;
using Imsl.Finance;

public class pmtEx1
{
    public static void  Main(String[] args)
    {
        double rate = .08;
        int nper = 25;
        double pv = 100000.00;
        double fv = 0.0;

        double pmt = Finance.PeriodicPayment(rate, nper, pv, fv,
                                             Finance.Period.AtEnd);
        Console.Out.WriteLine("The payment due each year on the " +
                              "$100,000 loan is " + pmt.ToString("C"));
    }
}
```

Output

```
The payment due each year on the $100,000 loan is ($9,367.88)
```

Example: Principal Payments

The payment on the principal the first year on a 25 year, $100,000 loan is calculated. The loan is at 8%.

```
using System;
using Imsl.Finance;

public class ppmtEx1
{
    public static void  Main(String[] args)
    {
        double rate = .08;
        int per = 1;
        int nper = 25;
        double pv = 100000.00;
        double fv = 0.0;

        double ppmt = Finance.Ppmt(rate, per, nper, pv, fv,
                                   Finance.Period.AtEnd);
        Console.Out.WriteLine("The payment on the principal the first "
                              + "year \nof the $100,000 loan is " +
                              ppmt.ToString("C"));
    }
}
```

Output

```
The payment on the principal the first year
of the $100,000 loan is ($1,367.88)
```

Example: Present Value of an Investment

A lady wins a $10 million lottery. The money is to be paid out at the end of each year in $500,000 payments for 20 years. The current treasury bill rate of 6% is used as the discount rate. Here, the present value of her prize is computed.

```
using System;
using Imsl.Finance;

public class pvEx1
{
    public static void  Main(String[] args)
    {
        double rate = 0.06;
        double pmt = 500000.0;
        double fv = 0.0;
        int nper = 20;

        double pv = Finance.Pv(rate, nper, pmt, fv,
                               Finance.Period.AtEnd);

        Console.Out.WriteLine("The present value of the $10 million " +
                              "prize is " + pv.ToString("C"));
    }
}
```

Output

```
The present value of the $10 million prize is ($5,734,960.61)
```

Example: Interest Rate

Someone obtains a $20,000 loan to buy a car. They make $350 a month payments for 70 months. Here, the interest rate of the loan is computed.

```
using System;
using Imsl.Finance;

public class rateEx1
{
    public static void  Main(String[] args)
    {
        int nper = 70;
        double pmt = - 350.0;
        double pv = 20000;
        double fv = 0.0;

        double rate = 12.0 * Finance.Rate(nper, pmt, pv, fv,
```

```
                                        Finance.Period.AtBeginning);
        Console.Out.WriteLine("The computed interest rate on the loan "
                              + "is " + rate.ToString("P"));
    }
}
```

Output

```
The computed interest rate on the loan is 7.35 %
```

Example: Depreciation - Straight Line Method

The straight line depreciation for one period of an asset with a life of 24 months, an initial cost of $2500 and a salvage value of $500 is computed.

```
using System;
using Imsl.Finance;

public class slnEx1
{
    public static void  Main(String[] args)
    {
        double cost = 2500;
        double salvage = 500;
        int life = 24;

        double sln = Finance.Sln(cost, salvage, life);
        Console.Out.WriteLine("The straight line depreciation of the " +
                              "asset for one period is " +
                              sln.ToString("C"));
    }
}
```

Output

```
The straight line depreciation of the asset for one period is $83.33
```

Example: Depreciation - Sum-of-years' Digits

The sum-of-years' digits depreciation for the 14th year of an asset with a life of 15 years, an initial cost of $25000 and a salvage value of $5000 is computed.

```
using System;
using Imsl.Finance;

public class sydEx1
{
    public static void  Main(String[] args)
    {
        double cost = 25000;
        double salvage = 5000;
```

```
        int life = 15;
        int per = 14;

        double syd = Finance.Syd(cost, salvage, life, per);
        Console.Out.WriteLine("The depreciation allowance for the 14th"
                              +" year is " + syd.ToString("C"));
    }
}
```

Output

```
The depreciation allowance for the 14th year is $333.33
```

Example: Depreciation - Variable Declining Balance

The depreciation between the 10th and 15th year of an asset with a life of 15 years, an initial cost of $25000 and a salvage value of $5000 is computed. The variable-declining balance method is used.

```
using System;
using Imsl.Finance;

public class vdbEx1
{
    public static void  Main(String[] args)
    {
        double cost = 25000;
        double salvage = 5000;
        int life = 15;
        int start = 10;
        int end = 15;
        double factor = 2.0;
        bool no_sl = false;

        double vdb = Finance.Vdb(cost, salvage, life, start, end,
                                 factor, no_sl);
        Console.Out.WriteLine("The depreciation allowance between the " +
                              "10th and 15th year is " +
                              vdb.ToString("C"));
    }
}
```

Output

```
The depreciation allowance between the 10th and 15th year is $976.69
```

Example: Internal Rate of Return - Variable Schedule

A farmer buys 10 young cows and a bull for $4500. The first year he does not expect to sell any calves, he just expects to feed them. Thereafter, he expects to be able to sell calves to offset the cost of feed. He expects them to be productive for 9 years, after which time he will liquidate the herd. The internal rate of return is computed after 9 years.

```
using System;
using Imsl.Finance;

public class xirrEx1
{
    public static void  Main(String[] args)
    {
        double[] pmt = new double[]{- 4500.0, - 800.0,
                                     800.0, 800.0,
                                     600.0, 600.0,
                                     800.0, 800.0,
                                     700.0, 3000.0};
        System.DateTime[] dates =
            new System.DateTime[]{DateTime.Parse("1/1/98"),
                                  DateTime.Parse("10/1/98"),
                                  DateTime.Parse("5/5/99"),
                                  DateTime.Parse("5/5/00"),
                                  DateTime.Parse("6/1/01"),
                                  DateTime.Parse("7/1/02"),
                                  DateTime.Parse("8/30/03"),
                                  DateTime.Parse("9/15/04"),
                                  DateTime.Parse("10/15/05"),
                                  DateTime.Parse("11/1/06")};
        double xirr = Finance.Xirr(pmt, dates);

        Console.Out.WriteLine("After approximately 9 years, the " +
                              "internal rate of return \n" +
                              "on the cows is " + xirr.ToString("P"));
    }
}
```

Output

```
After approximately 9 years, the internal rate of return
on the cows is 7.69 %
```

Example: Present Value of a Schedule of Cash Flows

In this example, the present value of 3 payments, $1,000, $2,000, and $1,000, with an interest rate of 5% made on January 3, 1997, January 3, 1999, and January 3, 2000 is computed.

```
using System;
using Imsl.Finance;

public class xnpvEx1
{
    public static void  Main(String[] args)
    {
        double rate = 0.05;
        double[] value_Renamed = new double[]{1000.0, 2000.0, 1000.0};
        System.DateTime[] dates =
            new System.DateTime[]{DateTime.Parse("1/3/1997"),
                                  DateTime.Parse("1/3/1999"),
                                  DateTime.Parse("1/3/2000")};
```

```
        double pv = Finance.Xnpv(rate, value_Renamed, dates);
        Console.Out.WriteLine("The present value of the schedule of " +
                              "cash flows is " + pv.ToString("C"));
    }
}
```

Output

```
The present value of the schedule of cash flows is $3,677.90
```

Finance.Period Enumeration

```
public enumeration Imsl.Finance.Finance.Period
```

Used to indicate that payment is made at the beginning or end of each period.

Fields

AtBeginning

```
public Imsl.Finance.Finance.Period AtBeginning
```

Description

Indicates payment is made at the beginning of each period.

AtEnd

```
public Imsl.Finance.Finance.Period AtEnd
```

Description

Indicates payment is made at the end of each period.

Bond Class

```
public class Imsl.Finance.Bond
```

Collection of bond functions.

Definitions

rate is an annualized rate of return based on the par value of the bills.

yield is an annualized rate based on the purchase price and reflects the actual yield to maturity.

coupons are interest payments on a bond.

redemption is the amount a bond pays at maturity.

frequency is the number of times a year that a bond makes interest payments.

basis is the method used to calculate dates. For example, sometimes computations are done assuming 360 days in a year.

issue is the day a bond is first sold.

settlement is the day a purchaser aquires a bond.

maturity is the day a bond's principal is repaid.

Discount Bonds

Discount bonds, also called *zero-coupon* bonds, do not pay interest during the life of the security, instead they sell at a discount to their value at maturity. The discount bond methods all have *settlement*, *maturity*, *basis* and *redemption* as arguments. In the following list these common arguments are ommitted.

- price = Pricedisc(rate) (p. 1266)
- price = Priceyield(yield) (p. 1268)
- price = Pricemat(issue, rate, yield) (p. 1267)
- rate = Disc(price) (p. 1263)
- yield = Yielddisc(price) (p. 1272)

A related method is Accrintm (p. 1258), which returns the interest that has accumulated on the discount bond.

Treasury Bills

US Treasury bills are a special case of discount bonds. The *basis* is fixed for treasury bills and the redemption value is assumed to be $100. So these functions have only *settlement* and *maturity* as common arguments.

- price = Tbillprice(rate) (p. 1270)
- yield = Tbillyield(Price) (p. 1270)
- yield = Tbilleq(rate) (p. 1269)

Interest Paying Bonds

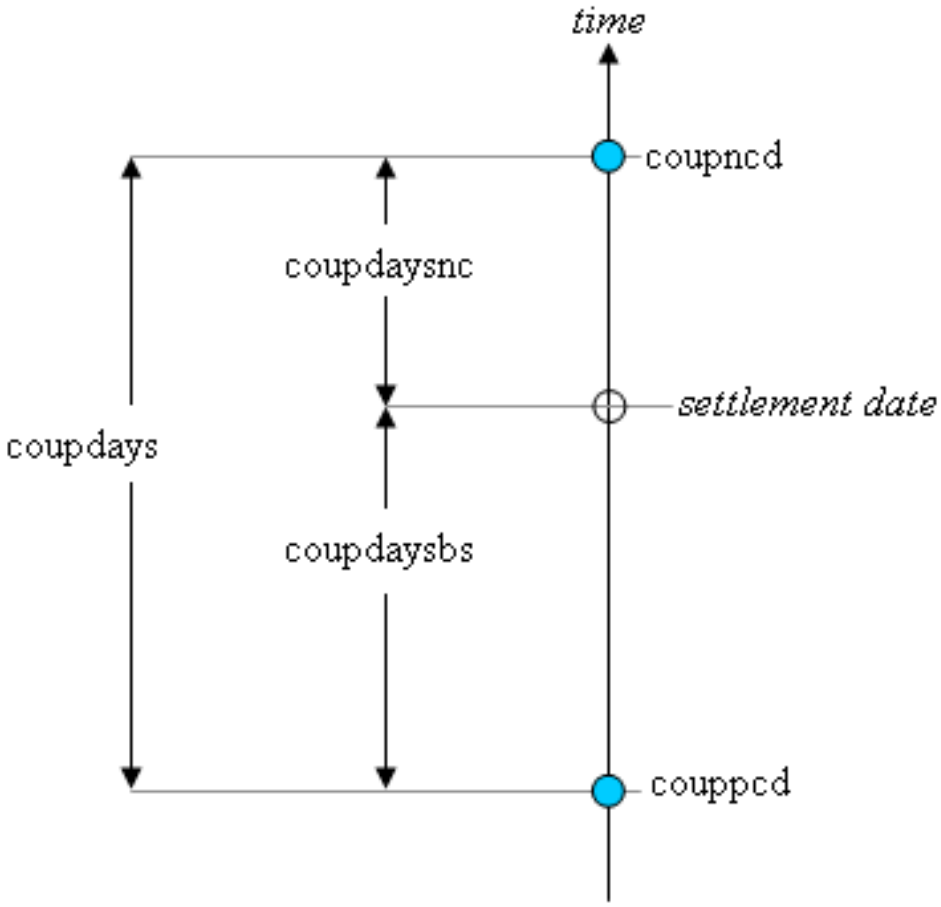
Most bonds pay interest periodically. The interest paying bond methods all have *settlement*, *maturity*, *basis* and *frequency* as arguments. Again supressing the common arguments,

- price = Price(rate, yield, redemption) (p. 1266)
- yield = Yield(rate, Price, redemption) (p. 1271)
- redemption = Received(Price, rate) (p. 1268)

A related method is Accrint (p. 1257), which returns the interest that has accumulated at settlement from the previous coupon date.

Coupon days

In this diagram, the settlement date is shown as a hollow circle and the adjacent coupon dates are shown as filled circles.



- Coupppcd (p. 1263) is the coupon date immediately prior to the settlement date.
- Coupncd (p. 1262) is the coupon date immediately after the settlement date.
- Coupdaybs (p. 1260) is the number of days from the immediately prior coupon date to the settlement date.
- Coupdaysnc (p. 1261) is the number of days from the settlement date to the next Coupon date.
- Coupdays (p. 1261) is the number of days between these two coupon dates.

A related method is Coupnum (p. 1262), which returns the number of coupons payable between settlement and maturity.

Another related method is Yearfrac (p. 1271), which returns the fraction of the year between two days.

Duration

Duration is used to measure the sensitivity of a bond to changes in interest rates. Convexity is a measure of the sensitivity of duration.

- Duration (p. 1264)
- DayCountBasis modified duration (p. 1265)
- Convexity (p. 1259)

Methods

Accrint

```
static public double Accrint(System.DateTime issue, System.DateTime
firstCoupon, System.DateTime settlement, double rate, double par,
Imsl.Finance.Bond.Frequency frequency, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the interest which has accrued on a security that pays interest periodically.

Parameters

`issue` – The `DateTime` issue date of the security.

`firstCoupon` – The `DateTime` date of the security's first interest date.

`settlement` – The `DateTime` settlement date of the security.

`rate` – A `double` which specifies the security's annual coupon rate.

`par` – A `double` which specifies the security's par value.

`frequency` – A `int` which specifies the number of coupon payments per year (1 for annual, 2 for semiannual, 4 for quarterly).

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the accrued interest.

Remarks

In the equation below, A_i represents the number of days which have accrued for the ith quasi-coupon period within the odd Frequency. (The quasi-coupon periods are periods obtained by extending the series of equal payment periods to before or after the actual payment periods.) NC represents the number of quasi-coupon periods within the odd period, rounded to the next highest integer. (The odd period is a period between payments that differs from the usual equally spaced periods at which payments are made.) NL_i represents the length of the normal ith quasi-coupon period within the odd Frequency. NL_i is expressed in days. `Accrint` solves the following:

$$par\left(\frac{rate}{frequency}\sum_{i=1}^{NC}\frac{A_i}{NL_i}\right)$$

Accrintm

```
static public double Accrintm(System.DateTime issue, System.DateTime maturity,
double rate, double par, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the interest which has accrued on a security that pays interest at maturity.

Parameters

`issue` – Ahe `DateTime` issue date of the security.

`maturity` – The `DateTime` date of the security's maturity.

`rate` – A `double` which specifies the security's annual coupon rate.

`par` – A `double` which specifies the security's par value.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the accrued interest.

Remarks

$$= par \times rate \times \frac{A}{D}$$

In the above equation, A represents the number of days starting at issue date to maturity date and D represents the annual basis.

Amordegrc

```
static public double Amordegrc(double cost, System.DateTime issue,
System.DateTime firstPeriod, double salvage, int period, double rate,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the depreciation for each accounting Frequency.

Parameters

`cost` – A `double` which specifies the cost of the asset.

`issue` – The `DateTime` issue date of the asset.

`firstPeriod` – The `DateTime` date of the end of the first period.

`salvage` – A `double` which specifies the asset's salvage value at the end of the life of the asset.

`period` – A `int` which specifies the period.

`rate` – A `double` which specifies the rate of depreciation.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the depreciation.

Remarks

This function is similar to `Amorlinc`. However, in this function a depreciation coefficient based on the asset life is applied during the evaluation of the function.

Amorlinc

```
static public double Amorlinc(double cost, System.DateTime issue,
System.DateTime firstPeriod, double salvage, int period, double rate,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the depreciation for each accounting Frequency.

Parameters

`cost` – A `double` which specifies the cost of the asset.

`issue` – The `DateTime` issue date of the asset.

`firstPeriod` – The `DateTime` date of the end of the first period.

`salvage` – A `double` which specifies the asset's salvage value at the end of the life of the asset.

`period` – A `int` which specifies the period.

`rate` – A `double` which specifies the rate of depreciation.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the depreciation.

Remarks

This function is similar to `Amordegrc`, except that `Amordegrc` has a depreciation coefficient that is applied during the evaluation that is based on the asset life.

Convexity

```
static public double Convexity(System.DateTime settlement, System.DateTime
maturity, double coupon, double yield, Imsl.Finance.Bond.Frequency frequency,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the convexity for a security.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`coupon` – A `double` which specifies the security's annual coupon rate.

`yield` – A `double` which specifies the security's annual yield.

`frequency` – A `int` which specifies the number of coupon payments per year (1 for annual, 2 for semiannual, 4 for quarterly).

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the convexity for a security.

Remarks

Convexity is the sensitivity of the duration of a security to changes in yield. It is computed using the following:

$$\frac{\frac{1}{(q\times frequency)^2}\left\{\sum_{t=1}^{n} t\,(t+1)\left(\frac{coupon}{frequency}\right)q^{-t}+n\,(n+1)\,q^{-n}\right\}}{\left(\sum_{t=1}^{n}\left(\frac{coupon}{frequency}\right)q^{-t}+q^{-n}\right)}$$

where n is calculated from `Coupnum`, and $q = 1 + \frac{yield}{frequency}$.

Coupdaybs

```
static public int Coupdaybs(System.DateTime settlement, System.DateTime
maturity, Imsl.Finance.Bond.Frequency frequency, Imsl.Finance.DayCountBasis
basis)
```

Description

Returns the number of days starting with the beginning of the coupon period and ending with the settlement date.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`frequency` – A `int` which specifies the number of coupon payments per year.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `int` which specifies the number of days from the beginning of the coupon period to the settlement date.

Remarks

For a good discussion on day count basis, see *SIA Standard Securities Calculation Methods* 1993, vol. 1, pages 17-35.

Coupdays

```
static public double Coupdays(System.DateTime settlement, System.DateTime
maturity, Imsl.Finance.Bond.Frequency frequency, Imsl.Finance.DayCountBasis
basis)
```

Description

Returns the number of days in the coupon period containing the settlement date.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`frequency` – A `int` which specifies the number of coupon payments per year.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `int` which specifies the number of days in the coupon period that contains the settlement date.

Remarks

For a good discussion on day count basis, see *SIA Standard Securities Calculation Methods* 1993, vol. 1, pages 17-35.

Coupdaysnc

```
static public int Coupdaysnc(System.DateTime settlement, System.DateTime
maturity, Imsl.Finance.Bond.Frequency frequency, Imsl.Finance.DayCountBasis
basis)
```

Description

Returns the number of days starting with the settlement date and ending with the next coupon date.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`frequency` – A `int` which specifies the number of coupon payments per year.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `int` which specifies the number of days from the settlement date to the next coupon date.

Remarks

For a good discussion on day count basis, see *SIA Standard Securities Calculation Methods* 1993, vol. 1, pages 17-35.

Coupncd

```
static public System.DateTime Coupncd(System.DateTime settlement,
System.DateTime maturity, Imsl.Finance.Bond.Frequency frequency,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the first coupon date which follows the settlement date.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`frequency` – A `int` which specifies the number of coupon payments per year.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `int` which specifies the next coupon date after the settlement date.

Remarks

For a good discussion on day count basis, see *SIA Standard Securities Calculation Methods* 1993, vol. 1, pages 17-35.

Coupnum

```
static public int Coupnum(System.DateTime settlement, System.DateTime maturity,
Imsl.Finance.Bond.Frequency frequency, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the number of coupons payable between the settlement date and the maturity date.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`frequency` – A `int` which specifies the number of coupon payments per year.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `int` which specifies the number of coupons payable between the settlement date and maturity date.

Remarks

For a good discussion on day count basis, see *SIA Standard Securities Calculation Methods* 1993, vol. 1, pages 17-35.

Couppcd

```
static public System.DateTime Couppcd(System.DateTime settlement,
System.DateTime maturity, Imsl.Finance.Bond.Frequency frequency,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the coupon date which immediately precedes the settlement date.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`frequency` – A `int` which specifies the number of coupon payments per year.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `int` which specifies the previous coupon date before the settlement date.

Remarks

For a good discussion on day count basis, see *SIA Standard Securities Calculation Methods* 1993, vol. 1, pages 17-35.

Disc

```
static public double Disc(System.DateTime settlement, System.DateTime maturity,
double price, double redemption, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the implied interest rate of a discount bond.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`price` – A `double` which specifies the security's price per $100 face value.

`redemption` – A `double` which the security's redemption value per $100 face value.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the discount rate for a security.

Remarks

The discount rate is the interest rate implied when a security is sold for less than its value at maturity in lieu of interest payments. It is computed using the following:

$$\frac{redemption - price}{price} \times \frac{B}{DSM}$$

In the equation above, *B* represents the number of days in a year based on the annual basis and *DSM* represents the number of days starting with the settlement date and ending with the maturity date.

Duration

```
static public double Duration(System.DateTime settlement, System.DateTime
maturity, double coupon, double yield, Imsl.Finance.Bond.Frequency frequency,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the Macauley's duration of a security where the security has periodic interest payments.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`coupon` – A `double` which specifies the security's annual coupon rate.

`yield` – A `double` which specifies the security's annual yield.

`frequency` – A `int` which specifies the number of coupon payments per year (1 for annual, 2 for semiannual, 4 for quarterly).

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the annual duration of a security with periodic interest payments.

Remarks

The Macauley's duration is the weighted-average time to the payments, where the weights are the present value of the payments. It is computed using the following:

$$\left(\frac{\frac{\left(N-1+\frac{DSC}{E}\right)\times 100}{\left(1+\frac{yield}{freq}\right)^{\left(N-1+\frac{DSC}{E}\right)}} + \sum_{k=1}^{N} \left(\left(\frac{100\times coupon}{freq\times\left(1+\frac{yield}{freq}\right)^{\left(k-1+\frac{DSC}{E}\right)}} \right) \times \left(k-1+\frac{DSC}{E}\right) \right)}{\frac{100}{\left(1+\frac{yield}{freq}\right)^{\left(N-1+\frac{DSC}{E}\right)}} + \sum_{k=1}^{N} \left(\frac{100\times coupon}{freq\times\left(1+\frac{yield}{freq}\right)^{\left(k-1+\frac{DSC}{E}\right)}} \right)} \right) \times \frac{1}{freq}$$

In the equation above, *DSC* represents the number of days starting with the settlement date and ending with the next coupon date. *E* represents the number of days within the coupon Frequency. *N* represents the number of coupons payable from the settlement date to the maturity date. *freq* represents the frequency of the coupon payments annually.

Intrate

```
static public double Intrate(System.DateTime settlement, System.DateTime
maturity, double investment, double redemption, Imsl.Finance.DayCountBasis
basis)
```

Description

Returns the interest rate of a fully invested security.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`investment` – A `double` which specifies the amount invested.

`redemption` – A `double` which specifies the amount to be received at maturity.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the interest rate for a fully invested security.

Remarks

It is computed using the following:

$$\frac{redemption - investment}{investment} \times \frac{B}{DSM}$$

In the equation above, *B* represents the number of days in a year based on the annual basis, and *DSM* represents the number of days in the period starting with the settlement date and ending with the maturity date.

Mduration

```
static public double Mduration(System.DateTime settlement, System.DateTime
maturity, double coupon, double yield, Imsl.Finance.Bond.Frequency frequency,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the modified Macauley duration for a security with an assumed par value of $100.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`coupon` – A `double` which specifies the security's annual coupon rate.

`yield` – A `double` which specifies the security's annual yield.

`frequency` – A `int` which specifies the number of coupon payments per year (1 for annual, 2 for semiannual, 4 for quarterly).

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the modified Macauley duration for a security with an assumed par value of $100.

Remarks

It is computed using the following:

$$\frac{duration}{1+\frac{yield}{frequency}}$$

where *duration* is calculated from `Mduration`.

Price

```
static public double Price(System.DateTime settlement, System.DateTime
maturity, double rate, double yield, double redemption,
Imsl.Finance.Bond.Frequency frequency, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the price, per $100 face value, of a security that pays periodic interest.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`rate` – A `double` which specifies the security's annual coupon rate.

`yield` – A `double` which specifies the security's annual yield.

`redemption` – A `double` which specifies the security's redemption value per $100 face value.

`frequency` – A `int` which specifies the number of coupon payments per year (1 for annual, 2 for semiannual, 4 for quarterly).

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the price per $100 face value of a security that pays periodic interest.

Remarks

It is computed using the following:

$$\frac{redemption}{\left(1+\frac{yield}{frequency}\right)^{\left(N-1+\frac{DSC}{E}\right)}}+\sum_{k=1}^{N}\frac{100\times\frac{rate}{frequency}}{\left(1+\frac{yield}{frequency}\right)^{\left(k-1+\frac{DSC}{E}\right)}}-\left(100\times\frac{rate}{frequency}\times\frac{A}{E}\right)$$

In the above equation, *DSC* represents the number of days in the period starting with the settlement date and ending with the next coupon date. *E* represents the number of days within the coupon Frequency. *N* represents the number of coupons payable in the timeframe from the settlement date to the redemption date. *A* represents the number of days in the timeframe starting with the beginning of coupon period and ending with the settlement date.

Pricedisc

```
static public double Pricedisc(System.DateTime settlement, System.DateTime
maturity, double rate, double redemption, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the price of a discount bond given the discount rate.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`rate` – A `double` which specifies the security's discount rate.

`redemption` – A `double` which specifies the security's redemption value per $100 face value.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the price per $100 face value of a discounted security.

Remarks

It is computed using the following:

$$redemption - rate \times redemption \times \frac{DSM}{B}$$

In the equation above, *DSM* represents the number of days starting at the settlement date and ending with the maturity date. *B* represents the number of days in a year based on the annual basis.

Pricemat

```
static public double Pricemat(System.DateTime settlement, System.DateTime
maturity, System.DateTime issue, double rate, double yield,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the price, per $100 face value, of a discount bond.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`issue` – The `DateTime` issue date of the security.

`rate` – A `double` which specifies the security's interest rate at issue date.

`yield` – A `double` which specifies the security's annual yield.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the price per $100 face value of a security that pays interest at maturity.

Remarks

It is computed using the following:

$$\frac{100 + \left(\frac{DIM}{B} \times rate \times 100\right)}{1 + \left(\frac{DSM}{B} \times yield\right)} - \frac{A}{B} \times rate \times 100$$

In the equation above, *B* represents the number of days in a year based on the annual basis. *DSM* represents the number of days in the period starting with the settlement date and ending with the maturity date. *DIM* represents the number of days in the period starting with the issue date and ending with the maturity date. *A* represents the number of days in the period starting with the issue date and ending with the settlement date.

Priceyield

```
static public double Priceyield(System.DateTime settlement, System.DateTime
maturity, double yield, double redemption, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the price of a discount bond given the yield.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`yield` – A `double` which specifies the security's yield.

`redemption` – A `double` which specifies the security's redemption value per $100 face value.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the price per $100 face value of a discounted security.

Remarks

It is computed using the following:

$$\frac{redemption}{1 + \left(\frac{DSM}{B}\right) yield}$$

In the equation above, *DSM* represents the number of days starting at the settlement date and ending with the maturity date. *B* represents the number of days in a year based on the annual basis.

Received

```
static public double Received(System.DateTime settlement, System.DateTime
maturity, double investment, double rate, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the amount one receives when a fully invested security reaches the maturity date.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`investment` – A `double` which specifies the amount invested in the security.

`rate` – A `double` which specifies the security's rate at issue date.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the amount received at maturity for a fully invested security.

Remarks

It is computed using the following:

$$\frac{investment}{1-\left(rate\times\frac{DIM}{B}\right)}$$

In the equation above, *B* represents the number of days in a year based on the annual basis, and *DIM* represents the number of days in the period starting with the issue date and ending with the maturity date.

Tbilleq

```
static public double Tbilleq(System.DateTime settlement, System.DateTime
maturity, double rate)
```

Description

Returns the bond-equivalent yield of a Treasury bill.

Parameters

`settlement` – The `DateTime` settlement date of the Treasury bill.

`maturity` – The `DateTime` maturity date of the Treasury bill. The maturity cannot be more than a year after the settlement.

`rate` – A `double` which specifies the Treasury bill's discount rate at issue date. The discount rate is an annualized rate of return based on the par value of the bills. The discount rate is calculated on a 360-day basis (twelve 30-day months).

Returns

A `double` which specifies the bond-equivalent yield for the Treasury bill. This is an annualized rate based on the purchase price of the bills and reflects the actual yield to maturity.

Remarks

It is computed using the following: If $DSM <= 182$

$$\frac{365\times rate}{360-rate\times DSM}$$

otherwise,

$$\frac{-\frac{DSM}{365}+\sqrt{\left(\frac{DSM}{365}\right)^2-\left(2\times\frac{DSM}{365}-1\right)\times\frac{rate\times DSM}{rate\times DSM-360}}}{\frac{DSM}{365}-0.5}$$

In the above equation, *DSM* represents the number of days starting at settlement date to maturity date.

Tbillprice

```
static public double Tbillprice(System.DateTime settlement, System.DateTime
maturity, double rate)
```

Description

Returns the price, per $100 face value, of a Treasury bill.

Parameters

`settlement` – The `DateTime` settlement date of the Treasury. bill.

`maturity` – The `DateTime` maturity date of the Treasury bill. The maturity cannot be more than a year after the settlement.

`rate` – A `double` which specifies the Treasury bill's discount rate at issue date. The discount rate is an annualized rate of return based on the par value of the bills. The discount rate is calculated on a 360-day basis (twelve 30-day months).

Returns

A `double` which specifies the price per $100 face value for the Treasury bill.

Remarks

It is computed using the following:

$$100\left(1-\frac{rate\times DSM}{360}\right)$$

In the equation above, *DSM* represents the number of days in the period starting with the settlement date and ending with the maturity date (any maturity date that is more than one calendar year after the settlement date is excluded).

Tbillyield

```
static public double Tbillyield(System.DateTime settlement, System.DateTime
maturity, double price)
```

Description

Returns the yield of a Treasury bill.

Parameters

`settlement` – The `DateTime` settlement date of the Treasury bill.

`maturity` – The `DateTime` maturity date of the Treasury bill. The maturity cannot be more than a year after the settlement.

`price` – A `double` which specifies the Treasury bill's price per $100 face value.

Returns

A `double` which specifies the yield for the Treasury bill. This is an annualized rate based on the purchase price of the bills and reflects the actual yield to maturity.

Remarks

It is computed using the following:

$$\frac{100 - price}{price} \times \frac{360}{DSM}$$

In the equation above, *DSM* represents the number of days in the period starting with the settlement date and ending with the maturity date (any maturity date that is more than one calendar year after the settlement date is excluded).

Yearfrac

```
static public double Yearfrac(System.DateTime startDate, System.DateTime
endDate, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the fraction of a year represented by the number of whole days between two dates.

Parameters

`startDate` – The `DateTime` start date of the security.

`endDate` – The `DateTime` end date of the security.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the annual yield of a security that pays interest at maturity.

Remarks

It is computed using the following:

$$A/D$$

where *A* equals the number of days from `start` to `end`, *D* equals annual basis.

Yield

```
static public double Yield(System.DateTime settlement, System.DateTime
maturity, double rate, double price, double redemption,
Imsl.Finance.Bond.Frequency frequency, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the yield of a security that pays periodic interest.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`rate` – A `double` which specifies the security's annual coupon rate.

`price` – A `double` which specifies the security's price per $100 face value.

`redemption` – A `double` which specifies the security's redemption value per $100 face value.

`frequency` – A `int` which specifies the number of coupon payments per year (1 for annual, 2 for semiannual, 4 for quarterly).

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the yield of a security that pays periodic interest.

Remarks

If there is one coupon period use the following:

$$\frac{\left(\frac{redemption}{100}+\frac{rate}{frequency}\right)-\left[\frac{price}{100}+\left(\frac{A}{E}\times\frac{rate}{frequency}\right)\right]}{\frac{price}{100}+\left(\frac{A}{E}\times\frac{rate}{frequency}\right)}\times\frac{frequency\times E}{DSR}$$

In the equation above, *DSR* represents the number of days in the period starting with the settlement date and ending with the redemption date. *E* represents the number of days within the coupon Frequency. *A* represents the number of days in the period starting with the beginning of coupon period and ending with the settlement date.

If there is more than one coupon period use the following:

$$price-\frac{redemption}{\left(\frac{1+yield}{frequency}\right)^{\frac{N-1+DSC}{E}}}-\left(\sum_{k=1}^{N}\frac{100\times\frac{rate}{frequency}}{\left(\frac{1+yield}{frequency}\right)^{\frac{k-1+DSC}{E}}}\right)+100\times\frac{rate}{frequency}\times\frac{A}{E}=0$$

In the equation above, *DSC* represents the number of days in the period from the settlement to the next coupon date. *E* represents the number of days within the coupon Frequency.*N* represents the number of coupons payable in the period starting with the settlement date and ending with the redemption date. *A* represents the number of days in the period starting with the beginning of the coupon period and ending with the settlement date.

Yielddisc

```
static public double Yielddisc(System.DateTime settlement, System.DateTime
maturity, double price, double redemption, Imsl.Finance.DayCountBasis basis)
```

Description

Returns the annual yield of a discount bond.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`price` – A `double` which specifies the security's price per $100 face value.

`redemption` – A `double` which specifies the security's redemption value per $100 face value.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the annual yield for a discounted security.

Remarks

It is computed using the following:

$$\frac{redemption - price}{price} \times \frac{B}{DSM}$$

In the equation above, *B* represents the number of days in a year based on the annual basis, and *DSM* represents the number of days starting with the settlement date and ending with the maturity date.

Yieldmat

```
static public double Yieldmat(System.DateTime settlement, System.DateTime
maturity, System.DateTime issue, double rate, double price,
Imsl.Finance.DayCountBasis basis)
```

Description

Returns the annual yield of a security that pays interest at maturity.

Parameters

`settlement` – The `DateTime` settlement date of the security.

`maturity` – The `DateTime` maturity date of the security.

`issue` – The `DateTime` issue date of the security.

`rate` – A `double` which specifies the security's interest rate at date of issue.

`price` – A `double` which specifies the security's price per $100 face value.

`basis` – A `DayCountBasis` object which contains the type of day count basis to use.

Returns

A `double` which specifies the annual yield of a security that pays interest at maturity.

Remarks

It is computed using the following:

$$\frac{\left[1 + \left(\frac{DIM}{B} \times rate\right)\right] - \left[\frac{price}{100} + \left(\frac{A}{B} \times rate\right)\right]}{\frac{price}{100} + \left(\frac{A}{B} \times rate\right)} \times \frac{B}{DSM}$$

In the equation above, *DIM* represents the number of days in the period starting with the issue date and ending with the maturity date. *DSM* represents the number of days in the period starting with the settlement date and ending with the maturity date. *A* represents the number of days in the period starting with the issue date and ending with the settlement date. *B* represents the number of days in a year based on the annual basis.

Example: Accrued Interest - Periodic Payments

In this example, the accrued interest is calculated for a bond which pays interest semiannually. The day count basis used is 30/360.

```
using System;
using Imsl.Finance;

public class accrintEx1
{
    public static void  Main(String[] args)
    {
        DateTime issue = DateTime.Parse("10/1/91");
        DateTime firstCoupon = DateTime.Parse("3/31/92");
        DateTime settlement = DateTime.Parse("11/3/91");
        double rate = .06;
        double par = 1000.0;
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double accrint = Bond.Accrint(issue, firstCoupon, settlement,
                                      rate, par, freq, dcb);
        Console.Out.WriteLine("The accrued interest is " + accrint);
    }
}
```

Output

```
The accrued interest is 5.33333333333333
```

Example: Accrued Interest - Payment at Maturity

In this example, the accrued interest is calculated for a bond which pays at maturity. The day count basis used is 30/360.

```
using System;
using Imsl.Finance;

public class accrintmEx1
{
    public static void  Main(String[] args)
    {
        DateTime issue = DateTime.Parse("10/1/91");
        DateTime settlement = DateTime.Parse("11/3/91");
        double rate = .06;
        double par = 1000.0;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double accrintm = Bond.Accrintm(issue, settlement, rate, par, dcb);
        Console.Out.WriteLine("The accrued interest is " + accrintm);
    }
}
```

Output

```
The accrued interest is 5.33333333333333
```

Example: Depreciation - French Accounting System

In this example, the depreciation for the second accounting period is calculated for an asset.

```
using System;
using Imsl.Finance;

public class amordegrcEx1
{
    public static void  Main(String[] args)
    {
        double cost = 2400.0;
        DateTime issue = DateTime.Parse("11/1/92");
        DateTime firstPeriod = DateTime.Parse("11/30/93");
        double salvage = 300.0;
        int period = 2;
        double rate = .15;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double amordegrc = Bond.Amordegrc(cost, issue, firstPeriod,
                                          salvage, period, rate, dcb);
        Console.Out.WriteLine("The depreciation for the second accounting "
                              + "period is " + amordegrc);
    }
}
```

Output

```
The depreciation for the second accounting period is 334
```

Example: Depreciation - French Accounting System

In this example, the depreciation for the second accounting period is calculated for an asset.

```
using System;
using Imsl.Finance;

public class amorlincEx1
{
    public static void  Main(String[] args)
    {
        double cost = 2400.0;
        DateTime issue = DateTime.Parse("11/1/92");
        DateTime firstPeriod = DateTime.Parse("11/30/93");
        double salvage = 300.0;
        int period = 2;
        double rate = .15;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double amorlinc = Bond.Amorlinc(cost, issue, firstPeriod,
                                        salvage, period, rate, dcb);
        Console.Out.WriteLine("The depreciation for the second accounting "
                              + "period is " + amorlinc);
    }
}
```

Output

```
The depreciation for the second accounting period is 360
```

Example: Convexity for a Security

The convexity of a 10 year bond which pays interest semiannually is returned in this example.

```
using System;
using Imsl.Finance;

public class convexityEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/90");
        DateTime maturity = DateTime.Parse("7/1/00");
        double coupon = .075;
        double yield = .09;
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        double convexity = Bond.Convexity(settlement, maturity, coupon,
                                          yield, freq, dcb);
        Console.Out.WriteLine("The convexity of the bond with semiannual "
                              + "interest payments is " + convexity);
    }
}
```

Output

```
The convexity of the bond with semiannual interest payments is 59.4049912915856
```

Example: Days - Beginning of Period to Settlement Date

In this example, the settlement date is 11/11/86. The number of days from the beginning of the coupon period to the settlement date is returned.

```
using System;
using Imsl.Finance;

public class coupdaybsEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("11/11/86");
        DateTime maturity = DateTime.Parse("3/1/99");
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        int coupdaybs = Bond.Coupdaybs(settlement, maturity, freq, dcb);
        Console.Out.WriteLine("The number of days from the beginning of the"
                              + "\ncoupon period to the settlement date is "
                              + coupdaybs);
    }
}
```

Output

```
The number of days from the beginning of the
```

```
coupon period to the settlement date is 71
```

Example: Days in the Settlement Date Period

In this example, the settlement date is 11/11/86. The number of days in the coupon period containing this date is returned.

```
using System;
using Imsl.Finance;

public class coupdaysEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("11/11/86");
        DateTime maturity = DateTime.Parse("3/1/99");
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        double coupdays = Bond.Coupdays(settlement, maturity, freq, dcb);
        Console.Out.WriteLine("The number of days in the coupon period that "
                              + "contains the settlement date is "
                              + coupdays);
    }
}
```

Output

```
The number of days in the coupon period that contains the settlement date is 182.5
```

Example: Days - Settlement Date to Next Coupon Date

In this example, the settlement date is 11/11/86. The number of days from this date to the next coupon date is returned.

```
using System;
using Imsl.Finance;

public class coupdaysncEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("11/11/86");
        DateTime maturity = DateTime.Parse("3/1/99");
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        int coupdaysnc = Bond.Coupdaysnc(settlement, maturity, freq,
                                         dcb);
        Console.Out.WriteLine("The number of days from the settlement date "
                              + "to the next coupon date is " + coupdaysnc);
    }
}
```

Output

```
The number of days from the settlement date to the next coupon date is 110
```

Example: Next Coupon Date After the Settlement Date

In this example, the settlement date is 11/11/86. The previous coupon date before this date is returned.

```
using System;
using Imsl.Finance;

public class coupncdEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("11/11/86");
        DateTime maturity = DateTime.Parse("3/1/99");
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        DateTime coupncd = Bond.Coupncd(settlement, maturity, freq,
                                        dcb);
        Console.Out.WriteLine("The next coupon date after the " +
                              "settlement date is " + coupncd);
    }
}
```

Output

```
The next coupon date after the settlement date is 3/1/1987 12:00:00 AM
```

Example: Number of Payable Coupons

In this example, the settlement date is 11/11/86. The number of payable coupons between this date and the maturity date is returned.

```
using System;
using Imsl.Finance;

public class coupnumEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("11/11/86");
        DateTime maturity = DateTime.Parse("3/1/99");
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        int coupnum = Bond.Coupnum(settlement, maturity, freq, dcb);
        Console.Out.WriteLine("The number of coupons payable between" +
                              " the \nsettlement date and the maturity"
                              + " date is " + coupnum);
    }
}
```

Output

```
The number of coupons payable between the
settlement date and the maturity date is 25
```

Example: Previous Coupon Date Before the Settlement Date

In this example, the settlement date is 11/11/86. The previous coupon date before this date is returned.

```
using System;
using Imsl.Finance;

public class couppcdEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("11/11/86");
        DateTime maturity = DateTime.Parse("3/1/99");
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        DateTime couppcd = Bond.Couppcd(settlement, maturity, freq,
                                        dcb);
        Console.Out.WriteLine("The previous coupon date before the " +
                              "settlement \ndate is " +
                              couppcd.ToLongDateString());
    }
}
```

Output

```
The previous coupon date before the settlement
date is Monday, September 01, 1986
```

Example: Discount Rate for a Security

In this example, the discount rate for a security is returned.

```
using System;
using Imsl.Finance;

public class discEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("2/15/92");
        DateTime maturity = DateTime.Parse("6/10/92");
        double price = 97.975;
        double redemption = 100.0;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        double disc = Bond.Disc(settlement, maturity, price,
                                redemption, dcb);
        Console.Out.WriteLine("The discount rate for the security is "
                              + disc);
```

```
    }
}
```

Output

```
The discount rate for the security is 0.0637176724137933
```

Example: Duration of a Security with Periodic Payments

The annual duration of a 10 year bond which pays interest semiannually is returned in this example.

```
using System;
using Imsl.Finance;

public class durationEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double coupon = .075;
        double yield = .09;
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        double duration = Bond.Duration(settlement, maturity, coupon,
                                        yield, freq, dcb);
        Console.Out.WriteLine("The annual duration of the bond with" +
                              "\nsemiannual interest payments is " +
                              duration);
    }
}
```

Output

```
The annual duration of the bond with
semiannual interest payments is 7.04195337797215
```

Example: Interest Rate of a Fully Invested Security

The discount rate of a 10 year bond is returned in this example.

```
using System;
using Imsl.Finance;

public class intrateEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double investment = 7000.0;
```

```
        double redemption = 10000.0;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        double intrate = Bond.Intrate(settlement, maturity, investment,
                                      redemption, dcb);
        Console.Out.WriteLine("The interest rate of the bond is " +
                              intrate);
    }
}
```

Output

```
The interest rate of the bond is 0.0428336723517446
```

Example: Modified Macauley Duration of a Security with Periodic Payments

The modified Macauley duration of a 10 year bond which pays interest semiannually is returned in this example.

```
using System;
using Imsl.Finance;

public class mdurationEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double coupon = .075;
        double yield = .09;
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        double mduration = Bond.Mduration(settlement, maturity, coupon,
                                          yield, freq, dcb);
        Console.Out.WriteLine("The modified Macauley duration " +
                              "of the bond");
        Console.Out.WriteLine("with semiannual interest payments is "
                              + mduration);
    }
}
```

Output

```
The modified Macauley duration of the bond
with semiannual interest payments is 6.73871136648053
```

Example: Price of a Security

The price per $100 face value of a 10 year bond which pays interest semiannually is returned in this example.

```
using System;
using Imsl.Finance;

public class priceEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double rate = .06;
        double yield = .07;
        double redemption = 105.0;
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double price = Bond.Price(settlement, maturity, rate, yield,
                                  redemption, freq, dcb);
        Console.Out.WriteLine("The price of the bond is " +
                              price.ToString("C"));
    }
}
```

Output

```
The price of the bond is $95.41
```

Example: Price of a Discounted Security

The price per $100 face value of a discounted 1 year bond is returned in this example.

```
using System;
using Imsl.Finance;

public class pricediscEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/86");
        double rate = .05;
        double redemption = 100.0;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double pricedisc = Bond.Pricedisc(settlement, maturity, rate,
                                          redemption, dcb);
        Console.Out.WriteLine("The price of the discounted bond is " +
                              pricedisc.ToString("C"));
    }
}
```

Output

```
The price of the discounted bond is $95.00
```

Example: Price of a Security that Pays at Maturity

The price per $100 face value of 1 year bond that pays interest at maturity is returned in this example.

```
using System;
using Imsl.Finance;

public class pricematEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("8/1/85");
        DateTime maturity = DateTime.Parse("7/1/86");
        DateTime issue = DateTime.Parse("7/1/85");
        double rate = .05;
        double yield = .05;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double pricemat = Bond.Pricemat(settlement, maturity, issue,
                                        rate, yield, dcb);
        Console.Out.WriteLine("The price of the bond is " + pricemat);
    }
}
```

Output

```
The price of the bond is 99.9817397078353
```

Price of a Discounted Security

The price of a discounted 1 year bond is returned in this example.

priceyieldEx1

```
using System;
using Imsl.Finance;

public class priceyieldEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double yield = 0.010055244588347783;
        double redemption = 105.0;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double priceyield = Bond.Priceyield(settlement, maturity,
                                            yield, redemption, dcb);
        Console.Out.WriteLine("The price of the discounted bond is " +
                              priceyield);
    }
}
```

Output

```
The price of the discounted bond is 95.40663
```

Example: Amount Received at Maturity for a Fully Invested Security

The amount to be received at maturity for a 10 year bond is returned in this example.

```
using System;
using Imsl.Finance;

public class receivedEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double investment = 7000.0;
        double discount = .06;
        DayCountBasis dcb = DayCountBasis.BasisActual365;
        double received = Bond.Received(settlement, maturity,
                                        investment, discount, dcb);
        Console.Out.WriteLine("The amount received at maturity for the"
                              + " bond is " + received.ToString("C"));
    }
}
```

Output

```
The amount received at maturity for the bond is $17,514.40
```

Example: Bond-Equivalent Yield

The bond-equivalent yield for a 1 year Treasury bill is returned in this example.

```
using System;
using Imsl.Finance;

public class tbilleqEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/86");
        double discount = .05;
        double tbilleq = Bond.Tbilleq(settlement, maturity, discount);
        Console.Out.WriteLine("The bond-equivalent yield for the " +
                              "T-bill is " + tbilleq.ToString("P"));
    }
}
```

Output

```
The bond-equivalent yield for the T-bill is 5.27 %
```

Example: Treasury Bill Price

The price per $100 face value for a 1 year Treasury bill is returned in this example.

```
using System;
using Imsl.Finance;

public class tbillpriceEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/86");
        double discount = .05;
        double tbillprice = Bond.Tbillprice(settlement, maturity,
                                            discount);
        Console.Out.WriteLine("The price per $100 face value for the " +
                              "T-bill is " + tbillprice.ToString("C"));
    }
}
```

Output

```
The price per $100 face value for the T-bill is $94.93
```

Example: Treasury Bill Yield

The yield for a 1 year Treasury bill is returned in this example.

```
using System;
using Imsl.Finance;

public class tbillyieldEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/86");
        double price = 94.93;
        double tbillyield = Bond.Tbillyield(settlement, maturity, price);
        Console.Out.WriteLine("The yield for the T-bill is " +
            tbillyield.ToString("P"));
    }
}
```

Output

```
The yield for the T-bill is 5.27 %
```

Example: Year Fraction

The year fraction of a 30/360 year starting 8/1/85 and ending 7/1/86 is returned in this example.

```
using System;
using Imsl.Finance;

public class yearfracEx1
{
    public static void  Main(String[] args)
    {
        DateTime start = DateTime.Parse("8/1/85");
        DateTime end = DateTime.Parse("7/1/86");
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double yearfrac = Bond.Yearfrac(start, end, dcb);
        Console.Out.WriteLine("The year fraction of the 30/360 period "
                              + "is " + yearfrac);
    }
}
```

Output

```
The year fraction of the 30/360 period is 0.916666666666667
```

Example: Yield on a Security

The yield on a 10 year bond which pays interest semiannually is returned in this example.

```
using System;
using Imsl.Finance;

public class yieldEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double rate = .06;
        double price = 95.40663;
        double redemption = 105.0;
        Bond.Frequency freq = Bond.Frequency.SemiAnnual;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double yield = Bond.Yield(settlement, maturity, rate, price,
                                  redemption, freq, dcb);
        Console.Out.WriteLine("The yield of the bond is " + yield);
    }
}
```

Output

```
The yield of the bond is 0.0699999968284289
```

Example: Yield on a Discounted Security

The yield on a discounted 10 year bond is returned in this example.

```
using System;
using Imsl.Finance;

public class yielddiscEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("7/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        double price = 95.40663;
        double redemption = 105.0;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double yielddisc = Bond.Yielddisc(settlement, maturity, price,
                                          redemption, dcb);
        Console.Out.WriteLine("The yield on the discounted bond is " +
                              yielddisc);
    }
}
```

Output

```
The yield on the discounted bond is 0.0100552445883478
```

Example: Yield on a Security Which Pays at Maturity

The yield on a bond which pays at maturity is returned in this example.

```
using System;
using Imsl.Finance;

public class yieldmatEx1
{
    public static void  Main(String[] args)
    {
        DateTime settlement = DateTime.Parse("8/1/85");
        DateTime maturity = DateTime.Parse("7/1/95");
        DateTime issue = DateTime.Parse("7/1/85");
        double rate = .06;
        double price = 95.40663;
        DayCountBasis dcb = DayCountBasis.BasisNASD;
        double yieldmat = Bond.Yieldmat(settlement, maturity, issue,
                                        rate, price, dcb);
        Console.Out.WriteLine("The yield on a bond which pays at " +
                              "maturity is " + yieldmat);
    }
}
```

Output

```
The yield on a bond which pays at maturity is 0.0673905127809195
```

Bond.Frequency Enumeration

```
public enumeration Imsl.Finance.Bond.Frequency
```

Frequency of the bond's coupon payments.

Fields

Annual

```
public Imsl.Finance.Bond.Frequency Annual
```

Description

Indicates interest is paid once a year.

BiMonthly

```
public Imsl.Finance.Bond.Frequency BiMonthly
```

Description

Indicates interest is paid six times a year.

Monthly

```
public Imsl.Finance.Bond.Frequency Monthly
```

Description

Indicates interest is paid twelve times a year.

Quarterly

```
public Imsl.Finance.Bond.Frequency Quarterly
```

Description

Indicates interest is paid four times a year.

SemiAnnual

```
public Imsl.Finance.Bond.Frequency SemiAnnual
```

Description

Indicates interest is paid twice a year.

DayCountBasis Class

```
public class Imsl.Finance.DayCountBasis
```

The Day Count Basis.

Rules for computing the number of days between two dates or number of days in a year. For many securities, computations are based on rules other than on the actual calendar.

Fields

Basis30e360

```
public Imsl.Finance.DayCountBasis Basis30e360
```

Description

Computations based on the assumption of 30 days per month and 360 days per year.

See Also: Imsl.Finance.DayCountBasis.BasisPart30E360 (p. 1290)

BasisActual360

```
public Imsl.Finance.DayCountBasis BasisActual360
```

Description

Computations are based on the number of days in a month based on the actual calendar value and the number of days, but assuming 360 days per year.

See Also: Imsl.Finance.DayCountBasis.BasisPartActual (p. 1290), Imsl.Finance.DayCountBasis.BasisPartNASD (p. 1290)

BasisActual365

```
public Imsl.Finance.DayCountBasis BasisActual365
```

Description

Computations are based on the number of days in a month based on the actual calendar value and the number of days, but assuming 365 days per year.

See Also: Imsl.Finance.DayCountBasis.BasisPartActual (p. 1290), Imsl.Finance.DayCountBasis.BasisPart365 (p. 1290)

BasisActualActual

```
public Imsl.Finance.DayCountBasis BasisActualActual
```

Description

Computations are based on the actual calendar.

See Also: Imsl.Finance.DayCountBasis.BasisPartActual (p. 1290)

BasisNASD

```
public Imsl.Finance.DayCountBasis BasisNASD
```

Description

Computations based on the assumption of 30 days per month and 360 days per year.

See Also: Imsl.Finance.DayCountBasis.BasisPartNASD (p. 1290)

BasisPart30E360

```
public Imsl.Finance.IBasisPart BasisPart30E360
```

Description

Computations based on the assumption of 30 days per month and 360 days per year. This computes the number of days between two dates differently than BasisPartNASD for months with other than 30 days.

BasisPart365

```
public Imsl.Finance.IBasisPart BasisPart365
```

Description

Computations based on the assumption of 365 days per year.

BasisPartActual

```
public Imsl.Finance.IBasisPart BasisPartActual
```

Description

Computations are based on the actual calendar.

BasisPartNASD

```
public Imsl.Finance.IBasisPart BasisPartNASD
```

Description

Computations based on the assumption of 30 days per month and 360 days per year.

Properties

MonthBasis

```
public Imsl.Finance.IBasisPart MonthBasis {get; }
```

Description

The (days in month) portion of the Day Count Basis.

Property Value

A `IBasisPart` object which represents the month Basis for this `DayCountBasis`.

YearBasis

```
public Imsl.Finance.IBasisPart YearBasis {get; }
```

Description

The (days in year) portion of the Day Count Basis.

Property Value

A `IBasisPart` object which represents the year Basis for this `DayCountBasis`.

Constructor

DayCountBasis

```
public DayCountBasis(Imsl.Finance.IBasisPart monthBasis,
Imsl.Finance.IBasisPart yearBasis)
```

Description

Creates a new DayCountBasis.

Parameters

`monthBasis` – A `IBasisPart` which specifies the month basis.

`yearBasis` – A `IBasisPart` which specifies the year basis.

IBasisPart Interface

```
public interface Imsl.Finance.IBasisPart
```

Component of DayCountBasis.

The day count basis consists of a month basis and a yearly basis. Each of these components implements this interface.

See Also

Imsl.Finance.DayCountBasis (p. 1289)

Methods

DaysBetween

```
abstract public int DaysBetween(System.DateTime date1, System.DateTime date2)
```

Description

Returns the number of days from `date1` to `date2`.

Parameters

`date1` – A `DateTime` object containing the initial date.

`date2` – A `DateTime` object containing the final date.

Returns

A `int` which specifies the number of days from `date1` to `date2`.

DaysInPeriod

```
abstract public double DaysInPeriod(System.DateTime finalDate,
Imsl.Finance.Bond.Frequency frequency)
```

Description

Returns the number of days in a coupon period.

Parameters

`finalDate` – A `DateTime` object containing the final date of the coupon period.

`frequency` – The `Frequency` specifying the number of coupon periods per year. This is typically 1, 2 or 4.

Returns

A `int` containing the number of days in the coupon period.

GetDaysInYear

```
abstract public int GetDaysInYear(System.DateTime date)
```

Description

Returns the number of days in the year.

Parameter

`date` – A `DateTime` object containing the date.

Returns

A `int` which specifies the number of days in the year.

Chapter 24: Chart2D

Types

AbstractChartNode Class

`public class Imsl.Chart2D.AbstractChartNode`

The base class of all of the nodes in 2D chart trees.

Fields

AUTOSCALE_DATA
`public int AUTOSCALE_DATA`

Description
An `int` that indicates autoscaling is to be done by scanning the data nodes.

See Also: Imsl.Chart2D.AbstractChartNode.AutoscaleInput (p. 1298)

AUTOSCALE_DENSITY
`public int AUTOSCALE_DENSITY`

Description
An `int` that indicates autoscaling is to adjust the “Density” attribute.

Remarks
This applies only to time axes.

See Also: Imsl.Chart2D.AbstractChartNode.AutoscaleOutput (p. 1299)

AUTOSCALE_NUMBER
`public int AUTOSCALE_NUMBER`

Description
An `int` that indicates autoscaling is to adjust the “Number” attribute.

See Also: Imsl.Chart2D.AbstractChartNode.AutoscaleOutput (p. 1299)

AUTOSCALE_OFF
`public int AUTOSCALE_OFF`

Description
An `int` that indicates autoscaling is turned off.

See Also: Imsl.Chart2D.AbstractChartNode.AutoscaleInput (p. 1298), Imsl.Chart2D.AbstractChartNode.AutoscaleOutput (p. 1299)

AUTOSCALE_WINDOW
`public int AUTOSCALE_WINDOW`

Description
An `int` that indicates autoscaling is to be done by using the “Window” attribute.

See Also: Imsl.Chart2D.AbstractChartNode.AutoscaleInput (p. 1298), Imsl.Chart2D.AbstractChartNode.AutoscaleOutput (p. 1299)

AXIS_X
`public int AXIS_X`

Description

An `int` that indicates the x-axis.

See Also: Type (p. 1356)

AXIS_Y

```
public int AXIS_Y
```

Description

An `int` that indicates the y-axis.

See Also: Type (p. 1356)

DAY

```
public int DAY
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is a day.

HOUR

```
public int HOUR
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is in hours.

LABEL_TYPE_NONE

```
public int LABEL_TYPE_NONE
```

Description

An `int` used to indicate the an element is not to be labeled.

See Also: Imsl.Chart2D.AbstractChartNode.LabelType (p. 1302)

LABEL_TYPE_TITLE

```
public int LABEL_TYPE_TITLE
```

Description

An `int` used to indicate that an element is to be labeled with the value of its title attribute.

See Also: Imsl.Chart2D.AbstractChartNode.LabelType (p. 1302)

LABEL_TYPE_X

```
public int LABEL_TYPE_X
```

Description

An `int` used to indicate that an element is to be labeled with the value of its x-coordinate.

See Also: Imsl.Chart2D.AbstractChartNode.LabelType (p. 1302)

LABEL_TYPE_Y

```
public int LABEL_TYPE_Y
```

Description

An `int` used to indicate that an element is to be labeled with the value of its y-coordinate.

See Also: Imsl.Chart2D.AbstractChartNode.LabelType (p. 1302)

MILLISECOND

```
public int MILLISECOND
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is in milliseconds.

MINUTE

```
public int MINUTE
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is in minutes.

MONTH

```
public int MONTH
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is a month.

SECOND

```
public int SECOND
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is in seconds.

TRANSFORM_CUSTOM

```
public int TRANSFORM_CUSTOM
```

Description

An `int` used to indicate that the axis using a custom transformation.

See Also: Imsl.Chart2D.AbstractChartNode.Transform (p. 1305)

TRANSFORM_LINEAR

```
public int TRANSFORM_LINEAR
```

Description

An `int` used to indicate that the axis uses linear scaling.

See Also: Imsl.Chart2D.AbstractChartNode.Transform (p. 1305)

TRANSFORM_LOG

```
public int TRANSFORM_LOG
```

Description

An `int` used to indicate that the axis uses logarithmic scaling.

See Also: Imsl.Chart2D.AbstractChartNode.Transform (p. 1305)

WEEK

```
public int WEEK
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is a week of the year.

YEAR

```
public int YEAR
```

Description

An `int` which specifies a minimum tick mark interval for an autoscaled time axis where the time resolution is a year.

Properties

AbstractParent

```
virtual public Imsl.Chart2D.AbstractChartNode AbstractParent {get; }
```

Description

Indicates the parent of this `AbstractChartNode`.

Property Value

An `AbstractChartNode` that is the parent of this node.

Remarks

If this is the root node in the chart tree the value is `null`.

Note that this is *not* an attribute setting.

Note that there is no `SetParent` method or property assignment.

AutoscaleInput

```
virtual public int AutoscaleInput {get; set; }
```

Description

Indicates what inputs are used for autoscaling.

Property Value

An `int` containing the “AutoscaleInput” attribute value.

Remarks

Legal values are:

Value	Behavior
AUTOSCALE_OFF	Do not do autoscaling.
AUTOSCALE_DATA	Use the data values. This is the default.
AUTOSCALE_WINDOW	Use the “Window” attribute value.

AutoscaleMinimumTimeInterval

```
virtual public int AutoscaleMinimumTimeInterval {get; set; }
```

Description

Specifies the minimum tick mark interval for autoscaled time axes.

Property Value

An `int` containing the “AutoscaleMinimumTimeInterval” attribute value.

Remarks

Legal values are:

- AbstractChartNode.MILLISECOND
- AbstractChartNode.SECOND
- AbstractChartNode.MINUTE
- AbstractChartNode.HOUR
- AbstractChartNode.DAY
- AbstractChartNode.WEEK
- AbstractChartNode.MONTH
- AbstractChartNode.YEAR

AutoscaleOutput

```
virtual public int AutoscaleOutput {get; set; }
```

Description

Specifies what attributes to change as a result of autoscaling.

Property Value

An `int` containing the “AutoscaleOutput” attribute value.

Remarks

Legal values are bitwise-or combinations of the following:

Value	Behavior
AUTOSCALE_OFF	Do not do autoscaling.
AUTOSCALE_WINDOW	Change the “Window” attribute value.
AUTOSCALE_NUMBER	Change the “Number” attribute value.
AUTOSCALE_DENSITY	Change the “Density” attribute value.

By default, `AutoscaleOutput=`|AUTOSCALE_NUMBER | AUTOSCALE_WINDOW | AUTOSCALE_DENSITY).

CultureInfo

```
virtual public System.Globalization.CultureInfo CultureInfo {get; set; }
```

Description

Adds support for Windows supported locales.

Property Value

A `CultureInfo` which contains the “CultureInfo” attribute.

Remarks

By default, `CultureInfo` = CurrentCulture .

CustomTransform

```
virtual public Imsl.Chart2D.Transform CustomTransform {get; set; }
```

Description

Allows for the specification of a custom transform.

Property Value

An `Object` implimenting the `Transform` interface which specifies the “CustomTransform” attribute value.

Remarks

This is used only if the “Transform” attribute is set to `TRANSFORM_CUSTOM`.

Density

```
virtual public int Density {get; set; }
```

Description

Specifies the number of minor tick marks in the interval between major tick marks.

Property Value

An `int` containing the “Density” attribute value.

Remarks

By default, `Density` = 4.

FillColor

```
virtual public System.Drawing.Color FillColor {get; set; }
```

Description

Specifies a color that will be used to fill an area.

Property Value

A `Color` that contains the “FillColor” attribute value.

Remarks

By default, `FillColor` = `Color.Black`.

Font

```
virtual public System.Drawing.Font Font {get; set; }
```

Description

Defines a particular format for text, including font name, size, and style attributes.

Property Value

A `Font` that contains information about the text font to be used.

FontName

```
virtual public string FontName {get; set; }
```

Description

Specifies the font to be used by name.

Property Value

A `String` containing the “FontName” attribute value.

Remarks

By default, `FontName` = Sanserif.

FontSize

```
virtual public float FontSize {get; set; }
```

Description

Specifies the font size.

Property Value

A `float` containing the “FontSize” attribute value.

Remarks

By default, `FontSize` = 8.

FontStyle

```
virtual public System.Drawing.FontStyle FontStyle {get; set; }
```

Description

Specifies the font style to be used.

Property Value

A `FontStyle` containing the “FontStyle” attribute value.

Remarks

By default, `FontStyle = FontStyle.Regular` .

ImageAttr

```
virtual public System.Drawing.Image ImageAttr {get; set; }
```

Description

An image that is to rendered when this `ChartNode` is displayed.

Property Value

An `Image` which contains the “Image” attribute value.

IsVisible

```
virtual public bool IsVisible {get; set; }
```

Description

Specifies if this node and its children will be drawn.

Property Value

A `bool` that contains the “IsVisible” attribute value.

Remarks

If `false`, this node and its children are not drawn. By default, `IsVisible = true`.

LabelType

```
virtual public int LabelType {get; set; }
```

Description

Specifies the type of label to display.

Property Value

An `int` which contains the “LabelType” attribute value.

Remarks

This indicates how a data point is to be labeled. The default is to not label data points, `LABEL_TYPE_NONE`.

See Also: Imsl.Chart2D.AbstractChartNode.LABEL_TYPE_NONE (p. 1296), Imsl.Chart2D.AbstractChartNode.LABEL_TYPE_TITLE (p. 1296), Imsl.Chart2D.AbstractChartNode.LABEL_TYPE_X (p. 1296), Imsl.Chart2D.AbstractChartNode.LABEL_TYPE_Y (p. 1297)

LineColor

```
virtual public System.Drawing.Color LineColor {get; set; }
```

Description

Specifies the line color for this node.

Property Value

A `Color` which contains the “LineColor” attribute value.

Remarks

By default, `LineColor = Color.Black`.

LineWidth

```
virtual public double LineWidth {get; set; }
```

Description

Specifies the line width for this node.

Property Value

A `double` which contains the “LineWidth” attribute value.

Remarks

By default, `LineWidth` = 1.0.

MarkerColor

```
virtual public System.Drawing.Color MarkerColor {get; set; }
```

Description

Specifies what color will be used when rendering marker.

Property Value

A `Color` containing the “MarkerColor” attribute value.

Remarks

By default, `MarkerColor = Color.Black`.

MarkerSize

```
virtual public double MarkerSize {get; set; }
```

Description

Specifies the size of markers.

Property Value

A `double` which contains the “MarkerSize” attribute value.

Remarks

By default, `MarkerSize` = 1.0.

Name

```
virtual public string Name {get; set; }
```

Description

Specifies the name of this node.

Property Value

A `String` that contains the “Name” attribute value.

Number

```
virtual public int Number {get; set; }
```

Description

Specifies the number of tick marks along an axis.

Property Value

An `int` which contains the “Number” attribute value.

Remarks

By default, `Number` = 0.

SkipWeekends

```
virtual public bool SkipWeekends {get; set; }
```

Description

Specifies whether to skip weekends.

Property Value

A `bool` which contains the “SkipWeekends” attribute value.

Remarks

By default, `SkipWeekends` = `false`.

See Also: Imsl.Chart2D.AbstractChartNode.AutoscaleMinimumTimeInterval (p. 1299)

SmoothingMode

```
virtual public System.Drawing.Drawing2D.SmoothingMode SmoothingMode {get; set;
}
```

Description

Specifies the SmoothingMode for this node.

Property Value

A `SmoothingMode` containing the “SmoothingMode” attribute value.

Remarks

By default, `SmoothingMode` = SmoothingMode.None (no antialiasing).

TextColor

```
virtual public System.Drawing.Color TextColor {get; set; }
```

Description

Specifies the text color.

Property Value

A `Color` containing the “TextColor” attribute value.

Remarks

The default value is `Color.Black`.

TextFormat

```
virtual public string TextFormat {get; set; }
```

Description

Specifies the “TextFormat” attribute value.

Property Value

A `string` that contains the “TextFormat” attribute value.

Remarks

The default is “0.00” that allows exactly two digits after the decimal.

TextFormatProvider

```
virtual public System.IFormatProvider TextFormatProvider {get; set; }
```

Description

Specifies the “TextFormatProvider” attribute value.

Property Value

An `IFormatProvider` that contains the “TextFormatProvider” attribute value.

Remarks

The default is `null`.

TickLength

```
virtual public double TickLength {get; set; }
```

Description

This scales the length of the tick mark lines.

Property Value

A `double` which contains the “TickLength” attribute value.

Remarks

A value of 2.0 makes the tick marks twice as long as normal. A negative value causes the tick marks to be drawn pointing into the plot area. By default, `TickLength` = 1.0.

Transform

```
virtual public int Transform {get; set; }
```

Description

Specifies whether the axis is linear, logarithmic or a custom transform.

Property Value

An `int` that contains the “Transform” attribute value.

Remarks

Legal values are Imsl.Chart2D.AbstractChartNode.TRANSFORM_LINEAR (p. 1297) (the default), Imsl.Chart2D.AbstractChartNode.TRANSFORM_LOG (p. 1298) and Imsl.Chart2D.AbstractChartNode.TRANSFORM_CUSTOM (p. 1297).

Constructor

AbstractChartNode

```
public AbstractChartNode(Imsl.Chart2D.AbstractChartNode parent)
```

Description

This interface contains members that will be common to chart objects in a variety of dimentions.

Parameter

`parent` – A chart node which is the parent node of this object.

Methods

GetAttribute

```
virtual public object GetAttribute(string name)
```

Description

Gets the value of an attribute.

Parameter

`name` – A `String` which specifies attribute that will have its value retrieved.

Returns

An `Object` which contains the specified attribute value.

GetBooleanAttribute

```
virtual public bool GetBooleanAttribute(string name, bool defaultValue)
```

Description

Convenience routine to get a Boolean-valued attribute.

Parameters

`name` – A `String` which contains the name of the attribute to be assessed.

`defaultValue` – A `bool` specifying the default value of the attribute.

Returns

A `bool` containing the attribute value.

Remarks

The value of an attribute is returned if it is defined and its value is of type `bool`. Otherwise the `defaultValue` is returned.

GetColorAttribute

```
virtual public System.Drawing.Color GetColorAttribute(string name)
```

Description

Convenience routine to get a Color-valued attribute.

Parameter

`name` – A `String` which contains the name of the attribute to be assessed.

Returns

A `Color` containing the attribute value.

Remarks

The value of an attribute is returned if it is defined and its value is of type `Color`. Otherwise the `Color.Black` is returned.

GetDoubleAttribute

```
virtual public double GetDoubleAttribute(string name, double defaultValue)
```

Description

Convenience routine to get a Double-valued attribute.

Parameters

`name` – A `String` which contains the name of the attribute to be assessed.

`defaultValue` – A `double` specifying the default value of the attribute.

Returns

A `double` containing the attribute value.

Remarks

The value of an attribute is returned if it is defined and its value is of type `double`. Otherwise the `defaultValue` is returned.

GetIntegerAttribute

```
virtual public int GetIntegerAttribute(string name, int defaultValue)
```

Description

Convenience routine to get an Integer-valued attribute.

Parameters

`name` – A `String` which contains the name of the attribute to be assessed.

`defaultValue` – An `int` specifying the default value of the attribute.

Returns

An `int` containing the attribute value.

Remarks

The value of an attribute is returned if it is defined and its value is of type `int`. Otherwise the `defaultValue` is returned.

GetStringAttribute

```
virtual public string GetStringAttribute(string name)
```

Description

Convenience routine to get a String-valued attribute.

Parameter

`name` – A `String` which contains the name of the attribute to be assessed.

Returns

The `String` value of the attribute.

Remarks

The attribute value is returned if it is defined and its value is of type `String`.

GetX

```
virtual public double[] GetX()
```

Description

Returns the “X” attribute value.

Returns

A `double[]` which contains the “X” attribute value.

GetY

```
virtual public double[] GetY()
```

Description

Returns the “Y” attribute value.

Returns

A `double[]` which contains the “Y” attribute value.

IsAncestorOf

```
virtual public bool IsAncestorOf(Imsl.Chart2D.AbstractChartNode node)
```

Description

Determines if this node is an ancestor of the argument node.

Parameter

`node` – An `AbstractChartNode` object that will have it’s relationship checked.

Returns

A `bool`, `true` if this node is an ancestor of the argument, node.

IsAttributeSet

```
virtual public bool IsAttributeSet(string name)
```

Description

Determines if an attribute is defined (may have been inherited).

Parameter

`name` – A `String` which contains the name of the attribute.

Returns

A `bool`, `true` if the attribute is defined for this node. The definition may have been inherited from its parent node.

IsAttributeSetAtThisNode

```
virtual public bool IsAttributeSetAtThisNode(string name)
```

Description

Determines if an attribute is defined in this node (not inherited).

Parameter

`name` – A `String` which contains the name of the attribute to be checked.

Returns

A `bool`, `true` if the attribute is defined in this node.

Remarks

The definition must have been set directly in this node, not just inherited from its parent node.

IsBitSet

```
static public bool IsBitSet(int flag, int mask)
```

Description

Returns `true` if the bit set in flag is set in mask.

Parameters

`flag` – An `int` which contains the bit to be tested against the mask.

`mask` – A `int` which is used as the mask.

Returns

A `bool`, `true` if the bit set in flag is set in mask.

Remove

```
public void Remove()
```

Description

Removes the node from its parents list of children.

SetAttribute

```
virtual public void SetAttribute(string name, object value)
```

Description

Sets an attribute.

Parameters

`name` – A `String` which contains the name of the attribute to be set.

`value` – An `Object` which contains the attribute value.

SetX

```
virtual public void SetX(object x)
```

Description

Sets the “X” attribute value.

Parameter

`x` – An `Object` that specifies the “X” attribute value.

SetY

```
virtual public void SetY(object y)
```

Description

Sets the “Y” attribute value.

Parameter

`y` – An `Object` that specifies the “Y” attribute value.

ToString

```
override public string ToString()
```

Description

Returns the name of this chart node.

Returns

A `String`, the name of this chart node.

ChartNode Class

```
public class Imsl.Chart2D.ChartNode :  AbstractChartNode
```

The base class of all of the nodes in the 2D chart tree.

Fields

AXIS_X_TOP
`public int AXIS_X_TOP`

Description

Flag to indicate x-axis placed on top of the chart.

AXIS_Y_RIGHT
`public int AXIS_Y_RIGHT`

Description

Flag to indicate y-axis placed to the right of the chart.

BAR_TYPE_HORIZONTAL
`public int BAR_TYPE_HORIZONTAL`

Description

Flag to indicate a horizontal bar chart.

See Also: Imsl.Chart2D.ChartNode.BarType (p. 1319)

BAR_TYPE_VERTICAL
`public int BAR_TYPE_VERTICAL`

Description

Flag to indicate a vertical bar chart.

See Also: Imsl.Chart2D.ChartNode.BarType (p. 1319)

DASH_PATTERN_DASH
`public double[] DASH_PATTERN_DASH`

Description

A `double[]` flag that specifies the rendering of a dashed line.

See Also: Imsl.Chart2D.ChartNode.SetLineDashPattern(System.Double[]) (p. 1330)

DASH_PATTERN_DASH_DOT
`public double[] DASH_PATTERN_DASH_DOT`

Description

A `double[]` flag that specifies the rendering of a dash-dot patterned line.

See Also: Imsl.Chart2D.ChartNode.SetLineDashPattern(System.Double[]) (p. 1330)

DASH_PATTERN_DOT
`public double[] DASH_PATTERN_DOT`

Description

A `double[]` flag that specifies the rendering of a dotted line.

See Also: Imsl.Chart2D.ChartNode.SetLineDashPattern(System.Double[]) (p. 1330)

DASH_PATTERN_SOLID

```
public double[] DASH_PATTERN_SOLID
```

Description

A `double[]` flag that specifies the rendering of a solid line.

See Also: Imsl.Chart2D.ChartNode.SetLineDashPattern(System.Double[]) (p. 1330)

DATA_TYPE_FILL

```
public int DATA_TYPE_FILL
```

Description

An `int` which when assigned to attribute “DataType” indicates that the area between the lines connecting data points and the horizontal reference line (y = attribute “Reference”) should be filled.

Remarks

This is an area chart.

DATA_TYPE_LINE

```
public int DATA_TYPE_LINE
```

Description

An `int` which when assigned to attribute “DataType” indicates that data points should be connected with line segments.

Remarks

This is the default setting.

DATA_TYPE_MARKER

```
public int DATA_TYPE_MARKER
```

Description

An `int` which when assigned to attribute “DataType” indicates that a marker should be drawn at each data point.

DATA_TYPE_PICTURE

```
public int DATA_TYPE_PICTURE
```

Description

An `int` which when assigned to attribute “DataType” indicates that an image (attribute “Image”) should be drawn at each data point.

Remarks

This can be used to draw fancy markers.

DENDROGRAM_TYPE_HORIZONTAL

```
public int DENDROGRAM_TYPE_HORIZONTAL
```

Description

Flag to indicate a horizontal dendrogram.

DENDROGRAM_TYPE_VERTICAL

```
public int DENDROGRAM_TYPE_VERTICAL
```

Description

Flag to indicate a vertical dendrogram.

FILL_TYPE_GRADIENT

```
public int FILL_TYPE_GRADIENT
```

Description

An `int` which indicates that a region will be drawn in a color gradient as specified by the attribute "Gradient".

Remarks

This constant may be used with the Imsl.Chart2D.ChartNode.FillType (p. 1322) property.

See Also: Imsl.Chart2D.ChartNode.SetGradient(System.Drawing.Color,System.Drawing.Color,System.Drawing.Color,System.Drawing.Color) (p. 1330), Imsl.Chart2D.ChartNode.GetGradient (p. 1327)

FILL_TYPE_NONE

```
public int FILL_TYPE_NONE
```

Description

An `int` which indicates that a region is not to be drawn.

Remarks

When Imsl.Chart2D.ChartNode.FillType (p. 1322) and Imsl.Chart2D.ChartNode.FillOutlineType (p. 1321) are set to this value the region will not be rendered

FILL_TYPE_PAINT

```
public int FILL_TYPE_PAINT
```

Description

An `int` which indicates that a region will be drawn using the texture specified by the "FillPaint" attribute.

See Also: Imsl.Chart2D.ChartNode.SetFillPaint(System.Drawing.Brush) (p. 1329), Imsl.Chart2D.ChartNode.GetFillPaint (p. 1327)

FILL_TYPE_SOLID

```
public int FILL_TYPE_SOLID
```

Description

An `int` which indicates that a region will be drawn using the solid color specified by Imsl.Chart2D.ChartNode.FillType (p. 1322) and Imsl.Chart2D.ChartNode.FillOutlineType (p. 1321).

LABEL_TYPE_PERCENT

```
public int LABEL_TYPE_PERCENT
```

Description

An `int` which indicates that a pie slice is to be labeled with a percentage value.

Remarks

This flag only applies to pie charts.

See Also: LabelType (p. 1302)

MARKER_TYPE_ASTERISK

```
public int MARKER_TYPE_ASTERISK
```

Description

An `int` that indicates an asterisk is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_CIRCLE_CIRCLE

```
public int MARKER_TYPE_CIRCLE_CIRCLE
```

Description

An `int` that indicates a circle in a circle is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_CIRCLE_PLUS

```
public int MARKER_TYPE_CIRCLE_PLUS
```

Description

An `int` that indicates an plus in a circle is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_CIRCLE_X

```
public int MARKER_TYPE_CIRCLE_X
```

Description

An `int` that indicates an x in a circle is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_DIAMOND_PLUS

```
public int MARKER_TYPE_DIAMOND_PLUS
```

Description

An `int` that indicates a plus in a diamond is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_FILLED_CIRCLE

```
public int MARKER_TYPE_FILLED_CIRCLE
```

Description

An `int` that indicates a filled circle is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_FILLED_DIAMOND

```
public int MARKER_TYPE_FILLED_DIAMOND
```

Description

An `int` that indicates a filled diamond is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_FILLED_SQUARE

```
public int MARKER_TYPE_FILLED_SQUARE
```

Description

An `int` that indicates a filled square is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_FILLED_TRIANGLE

```
public int MARKER_TYPE_FILLED_TRIANGLE
```

Description

An `int` that indicates a filled triangle is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_HOLLOW_CIRCLE

```
public int MARKER_TYPE_HOLLOW_CIRCLE
```

Description

An `int` that indicates a hollow circle is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_HOLLOW_DIAMOND

```
public int MARKER_TYPE_HOLLOW_DIAMOND
```

Description

An `int` that indicates a hollow diamond is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_HOLLOW_SQUARE

```
public int MARKER_TYPE_HOLLOW_SQUARE
```

Description

An `int` that indicates a hollow square is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_HOLLOW_TRIANGLE

```
public int MARKER_TYPE_HOLLOW_TRIANGLE
```

Description

An `int` that indicates a hollow triangle is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_OCTAGON_PLUS

```
public int MARKER_TYPE_OCTAGON_PLUS
```

Description

An `int` that indicates a plus in an octagon is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_OCTAGON_X

```
public int MARKER_TYPE_OCTAGON_X
```

Description

An `int` that indicates a x in an octagon is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_PLUS

```
public int MARKER_TYPE_PLUS
```

Description

An `int` that indicates a plus-shaped data marker is to be drawn.

Remarks

This is the default value of Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_SQUARE_PLUS

```
public int MARKER_TYPE_SQUARE_PLUS
```

Description

An `int` that indicates a x in a square is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_SQUARE_X

```
public int MARKER_TYPE_SQUARE_X
```

Description

An `int` that indicates a x in a diamond is to be drawn as the data marker.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

MARKER_TYPE_X

```
public int MARKER_TYPE_X
```

Description

An `int` that indicates a x-shaped data marker is to be drawn.

See Also: Imsl.Chart2D.ChartNode.MarkerType (p. 1323)

TEXT_X_CENTER

```
public int TEXT_X_CENTER
```

Description

An `int` which indicates that text should be centered.

See Also: Alignment (p. 1386)

TEXT_X_LEFT

```
public int TEXT_X_LEFT
```

Description

An `int` which indicates that text should be left justified.

See Also: Alignment (p. 1386)

TEXT_X_RIGHT

```
public int TEXT_X_RIGHT
```

Description

An `int` which indicates that text should be right justified.

See Also: Alignment (p. 1386)

TEXT_Y_BOTTOM

```
public int TEXT_Y_BOTTOM
```

Description

An `int` which indicates that text should be drawn on the baseline.

See Also: Alignment (p. 1386)

TEXT_Y_CENTER

```
public int TEXT_Y_CENTER
```

Description

An `int` which indicates that text should be vertically centered.

See Also: Alignment (p. 1386)

TEXT_Y_TOP

```
public int TEXT_Y_TOP
```

Description

An `int` which indicates that text should be drawn with the top of the letters touching the top of the drawing region.

See Also: Alignment (p. 1386)

WebCtrl

```
protected internal bool WebCtrl
```

Description

A `bool` indicating if this `ChartNode` is a `WebControl`.

Properties

ALT

```
virtual public string ALT {get; set; }
```

Description

Used to construct an “alt” attribute value in client side image maps.

Property Value

A `String` which contains the “ALT” attribute value.

Remarks

The “alt” attribute is used when client-side image maps are generated. A client-side image map has an entry for each node in which the chart attribute `HREF` is defined. Some browsers use the alt tag value as tooltip text.

Axis

```
virtual public Imsl.Chart2D.Axis Axis {get; }
```

Description

Typically provides a mapping for children from the user coordinate space to the device (screen) space.

Property Value

An `Axis` that contains the “Axis” attribute value.

Background

```
virtual public Imsl.Chart2D.Background Background {get; }
```

Description

The base graphic layer displayed behind other `ChartNode` objects in the tree.

Property Value

A `Background` which contains the “Background” attribute value.

BarGap

```
virtual public double BarGap {get; set; }
```

Description

Specifies the gap between bars in a group.

Property Value

A `double` which contains the “BarGap” attribute value.

Remarks

A gap of 1.0 means that space between bars is the same as the width of an individual bar in the group. By default, `BarGap` = 0.0.

BarType

```
virtual public int BarType {get; set; }
```

Description

Specifies the orientation of the `BarChart`.

Property Value

An `int` which contains the BarType“ attribute value.

Remarks

Legal values are Imsl.Chart2D.ChartNode.BAR_TYPE_VERTICAL (p. 1311) or Imsl.Chart2D.ChartNode.BAR_TYPE_HORIZONTAL (p. 1311).

BarWidth

```
virtual public double BarWidth {get; set; }
```

Description

The width of all of the groups of bars at each index.

Property Value

A `double` which contains the “BarWidth” attribute value.

Remarks

By default, `BarWidth` = 0.5.

Chart

```
virtual public Imsl.Chart2D.Chart Chart {get; }
```

Description

This is the root node of the chart tree.

Property Value

A `Chart` that contains the “Chart” attribute value.

ChartTitle

```
virtual public Imsl.Chart2D.ChartTitle ChartTitle {get; set; }
```

Description

Specifies a title for the chart.

Property Value

A `ChartTitle` which contains the “ChartTitle” attribute value.

Remarks

This is effective only in the `ChartNode`, where it replaces the existing `ChartTitle` node.

ClipData

```
virtual public bool ClipData {get; set; }
```

Description

Specifies whether the data elements are to be clipped to the current window.

Property Value

A `bool` which contains the “ClipData” attribute value.

Remarks

By default, `ClipData = true`

DataType

```
virtual public int DataType {get; set; }
```

Description

Specifies how the data is to be rendered.

Property Value

An `int` that contains the “DataType” attribute value.

Remarks

This should be some xor-ed combination of Imsl.Chart2D.ChartNode.DATA_TYPE_LINE (p. 1312), Imsl.Chart2D.ChartNode.DATA_TYPE_MARKER (p. 1312). By default, `DataType` = `DATA_TYPE_LINE` (p. 1312).

DoubleBuffering

```
virtual public bool DoubleBuffering {get; set; }
```

Description

Specifies whether double is active.

Property Value

A `bool` which contains the "DoubleBuffering" attribute value.

Remarks

Double buffering reduces flicker when the screen is updated. This attribute only has an effect if it is set at the root node of the chart tree.

Explode

```
virtual public double Explode {get; set; }
```

Description

Specifies how far from the center pie slices are drawn.

Property Value

A `double` containing the "Explode" attribute value.

Remarks

The scale is proportional to the pie chart's radius. By default, `Explode` = 0.0.

FillOutlineColor

```
virtual public System.Drawing.Color FillOutlineColor {get; set; }
```

Description

Specifies a color that will be used to outline this node.

Property Value

A `Color` which contains the "FillOutlineColor" attribute value.

Remarks

The default value is `Color.Black`.

FillOutlineType

```
virtual public int FillOutlineType {get; set; }
```

Description

Specifies a fill pattern type for the outline of this node.

Property Value

An `int` that contains the "FillOutlineType" attribute value.

Remarks

By default, FillOutlineType = FILL_TYPE_SOLID (p. 1313).

FillType

```
virtual public int FillType {get; set; }
```

Description

Specifies a fill pattern type for this node.

Property Value

An int which contains the "FillType" attribute value.

Remarks

By default, FillType = FILL_TYPE_SOLID (p. 1313).

See Also: Imsl.Chart2D.ChartNode.FILL_TYPE_NONE (p. 1313), Imsl.Chart2D.ChartNode.FILL_TYPE_GRADIENT (p. 1313), Imsl.Chart2D.ChartNode.FILL_TYPE_PAINT (p. 1313)

HREF

```
virtual public string HREF {get; set; }
```

Description

Used to specify an "activated" object in an image map.

Property Value

A String which contains the "HREF" attribute value.

Remarks

The "HREF" attribute is used when client-side image maps are generated. A client-side image map has an entry for each node in which the chart attribute HREF is defined. The values of HREF attributes are URLs. Such regions treated by the browser as hyperlinks.

ImageAttr

```
virtual public System.Drawing.Image ImageAttr {get; set; }
```

Description

An image that is to rendered when this ChartNode is displayed.

Property Value

An Image which contains the "Image" attribute value.

IsWebControl

```
public bool IsWebControl {get; }
```

Description

Indicates whether this is a web control.

Property Value

A `bool` which is `true` if the node is a `WebControl`.

Legend

```
virtual public Imsl.Chart2D.Legend Legend {get; }
```

Description

Legend information associated with this `ChartNode`.

Property Value

A `Legend` containing legend information associated with this `ChartNode`.

MarkerThickness

```
virtual public double MarkerThickness {get; set; }
```

Description

Specifies the line thickness to be used when rendering the markers.

Property Value

A `double` that contains the “MarkerThickness” attribute value.

Remarks

If “MarkerThickness” is 2.0 then markers are drawn twice as thick as normal. By default, `MarkerThickness` = 1.0.

MarkerType

```
virtual public int MarkerType {get; set; }
```

Description

Specifies the type of data marker to be drawn.

Property Value

The `int` which contains the “MarkerType” attribute value.

Remarks

By default, `MarkerType` = Imsl.Chart2D.ChartNode.MARKER_TYPE_PLUS (p. 1316).

See Also: Imsl.Chart2D.ChartNode.MARKER_TYPE_ASTERISK (p. 1314), Imsl.Chart2D.ChartNode.MARKER_TYPE_X (p. 1317), Imsl.Chart2D.ChartNode.MARKER_TYPE_HOLLOW_SQUARE (p. 1316), Imsl.Chart2D.ChartNode.MARKER_TYPE_HOLLOW_TRIANGLE (p. 1316), Imsl.Chart2D.ChartNode.MARKER_TYPE_HOLLOW_DIAMOND (p. 1315), Imsl.Chart2D.ChartNode.MARKER_TYPE_DIAMOND_PLUS (p. 1314), Imsl.Chart2D.ChartNode.MARKER_TYPE_SQUARE_X (p. 1317), Imsl.Chart2D.ChartNode.MARKER_TYPE_SQUARE_PLUS (p. 1316), Imsl.Chart2D.ChartNode.MARKER_TYPE_OCTAGON_X (p. 1316), Imsl.Chart2D.ChartNode.MARKER_TYPE_OCTAGON_PLUS (p. 1316), Imsl.Chart2D.ChartNode.MARKER_TYPE_HOLLOW_CIRCLE (p. 1315), Imsl.Chart2D.ChartNode.MARKER_TYPE_CIRCLE_X (p. 1314),

Imsl.Chart2D.ChartNode.MARKER_TYPE_CIRCLE_PLUS (p. 1314),
Imsl.Chart2D.ChartNode.MARKER_TYPE_CIRCLE_CIRCLE (p. 1314),
Imsl.Chart2D.ChartNode.MARKER_TYPE_FILLED_SQUARE (p. 1315),
Imsl.Chart2D.ChartNode.MARKER_TYPE_FILLED_TRIANGLE (p. 1315),
Imsl.Chart2D.ChartNode.MARKER_TYPE_FILLED_DIAMOND (p. 1315),
Imsl.Chart2D.ChartNode.MARKER_TYPE_FILLED_CIRCLE (p. 1315)

Parent

```
virtual public Imsl.Chart2D.ChartNode Parent {get; }
```

Description

Indicates the parent of this `ChartNode`.

Property Value

A `ChartNode` object which contains this node's parent.

Remarks

This is `null` in the case of the root node of the chart tree, since that node has no parent.

Note that this is *not* an attribute setting.

Note that there is no function to set the Parent.

Reference

```
virtual public double Reference {get; set; }
```

Description

Indicates the baseline in drawing area charts.

Property Value

A `double` which contains the "Referenc" attribute value.

Remarks

In the case of a pie chart, this specifies the angle (in degrees) of the first slice. By default, `Reference` = 0.0.

ScreenAxis

```
virtual public Imsl.Chart2D.AxisXY ScreenAxis {get; }
```

Description

Provides a default mapping from the user coordinates [0,1] by [0,1] to the screen.

Property Value

An `AxisXY` that contains the "ScreenAxis" attribute value.

Remarks

This is set by the root `ChartNode`, so there is no `set ScreenAxis` accessor.

See Also: Imsl.Chart2D.Chart (p. 1332)

ScreenSize

```
virtual public System.Drawing.Size ScreenSize {get; set; }
```

Description

Indicates the size of this `ChartNode`.

Property Value

A `Size` that contains the “ScreenSize” attribute value.

Remarks

If this attribute has not been defined the size of the “Control” attribute is returned. If neither attribute is defined `null` is returned.

Size

```
virtual public System.Drawing.Size Size {get; set; }
```

Description

Specifies the drawing size.

Property Value

A `Size` which contains the “Size” attribute value.

TextAngle

```
virtual public int TextAngle {get; set; }
```

Description

An angle, in degrees, at which text is to be drawn.

Property Value

An `int` that contains the angle at which text is to be drawn.

Remarks

Only multiples of 90 are allowed at this time. By default, `TextAngle` = 0.

ToolTip

```
virtual public string ToolTip {get; set; }
```

Description

Text that can be displayed in the case where tool tips are used.

Property Value

A `String` which contains the “ToolTip” attribute value.

Constructor

ChartNode

```
public ChartNode(Imsl.Chart2D.ChartNode parent)
```

Description

Constructs a `ChartNode` object.

Parameter

`parent` – The `ChartNode` which is the parent of this object.

Methods

FirePickListeners

```
virtual public void FirePickListeners(System.Windows.Forms.MouseEventArgs e)
```

Description

Invokes the pick delegates defined at this node and at all of its ancestors, if the event "hits" the node.

Parameter

`e` – A `MouseEventArgs` which determines which nodes have been selected.

GetChildren

```
virtual public Imsl.Chart2D.ChartNode[] GetChildren()
```

Description

Gets the list of child nodes.

Returns

A `ChartNode[]` which contains the children of this node.

Remarks

If there are no children, a 0-length array is returned.

GetComponent

```
virtual public System.Windows.Forms.Control GetComponent()
```

Description

Gets the "Component" attribute value.

Returns

A `Control` that contains the "Component" attribute value.

GetConcatenatedViewport

```
virtual public double[] GetConcatenatedViewport()
```

Description

Returns the value of the "Viewport" attribute concatenated with the "Viewport" attributes set in its ancestor nodes.

Returns

A `double[4]` containing xmin, xmax, ymin and ymax.

Remarks

By default, `GetConcatenatedViewport = {0.0, 1.0, 0.0, 1.0}`.

GetFillPaint

```
virtual public System.Drawing.Brush GetFillPaint()
```

Description

Returns the “FillPaint” attribute value.

Returns

The value of the “FillPaint” attribute, if defined. Otherwise, `null` is returned.

GetGradient

```
virtual public System.Drawing.Color[] GetGradient()
```

Description

Returns the value of the “Gradient” attribute.

Returns

A `Color[4]` array which contains the color value of the “Gradient” attribute.

Remarks

The array is of length four, containing {`colorLL`, `colorLR`, `colorUR`, `colorUL`}. By default, `GetGradient = null`.

GetLineDashPattern

```
virtual public double[] GetLineDashPattern()
```

Description

Returns the “LineDashPattern” attribute value.

Returns

A `double[]` that contains the line “LineDashPattern” attribute value.

Remarks

Returns `null` if the attribute has not been defined.

GetMarkerDashPattern

```
virtual public double[] GetMarkerDashPattern()
```

Description

Returns the “MarkerDashPattern” attribute value.

Returns

A `double[]` that contains the “MarkerDashPattern” attribute value.

Remarks

Returns `null` if the attribute has not been defined.

GetScreenViewport

```
virtual public int[] GetScreenViewport()
```

Description

Returns the value of the “Viewport” attribute scaled by the screen size.

Returns

An `int[4]` containing the “Viewport” attribute value.

Remarks

The value returned is scaled by the screen size containing the pixel coordinates for xmin, xmax, ymin and ymax.

GetTitle

```
virtual public Imsl.Chart2D.Text GetTitle()
```

Description

Returns the value of the “Title” attribute.

Returns

A `Text` which contains the “Title” attribute value.

GetViewport

```
virtual public double[] GetViewport()
```

Description

Returns the value of the “Viewport” attribute.

Returns

A `double[4]` containing xmin, xmax, ymin and ymax.

Remarks

By default, `GetViewport = {0.0, 1.0, 0.0, 1.0}`.

GetWebComponent

```
virtual public System.Web.UI.WebControls.WebControl GetWebComponent()
```

Description

Gets the “Component” attribute value.

Returns

A `WebControl` that contains the “Component” attribute value.

IsBitSet

```
static public bool IsBitSet(int flag, int mask)
```

Description

Determins if the bit set in `flag` is set in `mask`.

Parameters

`flag` – An `int` which contains the bit to be tested against `mask`.

`mask` – An `int` which is to be used as teh mask.

Returns

A `bool` whichis `true` if the bit set in `flag` is set in `mask`.

OnPick

```
void OnPick(Imsl.Chart2D.PickEventArgs eventParam)
```

Description

Invokes delegates registered with the Pick event.

Parameter

`eventParam` – A `PickEventArgs` that specifies the event data.

Paint

```
abstract public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

PrePaint

```
virtual public void PrePaint()
```

Description

The PrePaint method is called in all nodes in a chart just before the chart is painted. The default implementation does nothing. Override this method to do computations just before painting in a chart node.

SetFillPaint

```
virtual public void SetFillPaint(System.Drawing.Brush brush)
```

Description

Sets the value of the “FillPaint” attribute.

Parameter

`brush` – A `Brush` which specifies the “FillPaint” attribute value.

SetFillPaint

```
virtual public void SetFillPaint(System.Drawing.Image imageIcon)
```

Description

Sets the “FillPaint” attribute value.

Parameter

`imageIcon` – A `Image` that specifies the “FillPaint” attribute value.

SetFillPaint

```
virtual public void SetFillPaint(System.Uri uriImage)
```

Description

Sets the “FillPaint” attribute value.

Parameter

`uriImage` – A `Uri` which specifies the location of an image used to set the “FillPaint” attribute.

SetGradient

```
virtual public void SetGradient(System.Drawing.Color colorLL,
System.Drawing.Color colorLR, System.Drawing.Color colorUR,
System.Drawing.Color colorUL)
```

Description

Sets the value of the “Gradient” attribute.

Parameters

`colorLL` – A `Color` value which specifies the color of the lower left corner.

`colorLR` – A `Color` value which specifies the color of the lower right corner.

`colorUR` – A `Color` value which specifies the color of the upper right corner.

`colorUL` – A `Color` value which specifies the color of the upper left corner.

Remarks

This attribute defines a color gradient used to fill regions. Only two of the four colors given are actually used.

Parameter Values	Result
`colorLL==colorLR` and `colorUL==colorUR`	A vertical gradient is drawn.
`colorLL==colorUL` and `colorLR==colorUR`	A horizontal gradient is drawn.
`colorLR== null` and `colorUL== null`	A diagonal gradient is drawn.
`colorLL== null` and `colorUR== null`	A diagonal gradient is drawn.

SetGradient

```
virtual public void SetGradient(System.Drawing.Color[] colorGradient)
```

Description

Sets the value of the “Gradient” attribute.

Parameter

`colorGradient` – A `Color[4]` containing the colors at the lower left, lower right, upper right and upper left corners of the bounding box of the regions being filled.

SetLineDashPattern

```
virtual public void SetLineDashPattern(double[] lineDashPattern)
```

Description

Sets the “LineDashPattern” attribute value.

Parameter

`lineDashPattern` – A `double[]` which specifies the line dash patten to be rendered.

SetMarkerDashPattern

```
virtual public void SetMarkerDashPattern(double[] markerDashPattern)
```

Description

Sets the “MarkerDashPattern” attribute value.

Parameter

`markerDashPattern` – A `double[]` that specifies the “MarkerDashPattern” attribute value.

SetTitle

```
virtual public void SetTitle(string title)
```

Description

Sets the value of the “Title” attribute.

Parameter

`title` – A `String` which specifies the “Title” attribute value.

SetTitle

```
virtual public void SetTitle(Imsl.Chart2D.Text title)
```

Description

Sets the value of the “Title” attribute.

Parameter

`title` – A `Text` which specifies the “Title” attribute value.

SetViewport

```
virtual public void SetViewport(double[] viewport)
```

Description

Used to specify the viewport location.

Parameter

`viewport` – A `double[4]` which specifies the “Viewport” attribute value.

Remarks

The viewport is the subregion of the drawing surface where the plot is to be drawn. “Viewport” coordinates are [0,1] by [0,1] with (0,0) in the upper left corner. The “Viewport” attribute affects only Axis nodes, since they contain the mappings to device space. The elements of `viewport` corrispond to xmin, xmax, ymin and ymax.

SetViewport

```
virtual public void SetViewport(double xmin, double xmax, double ymin, double
ymax)
```

Description

Used to specify the viewport location.

Parameters

`xmin` – A `double` specifying the left side of the viewport.

`xmax` – A `double` specifying the right side of the viewport.

`ymin` – A `double` specifying the bottom of the viewport.

`ymax` – A `double` specifying the top of the viewport.

Remarks

The viewport is the subregion of the drawing surface where the plot is to be drawn. “Viewport” coordinates are [0,1] by [0,1] with (0,0) in the lower left corner. The “Viewport” attribute affects only Axis nodes, since they contain the mappings to device space.

Chart Class

```
public class Imsl.Chart2D.Chart :  ChartNode :  ICloneable
```

The root node of the chart tree.

This chart node creates the following child nodes: Imsl.Chart2D.Background (p. 1338), Imsl.Chart2D.ChartTitle (p. 1339) and Imsl.Chart2D.Legend (p. 1341).

Constructors

Chart

```
public Chart()
```

Description

This is the root of our tree, it has no parent.

Remarks

This creates the `Chart` with a `null` component.

Chart

```
public Chart(System.Windows.Forms.Control component)
```

Description

This is the root of our tree, it has no parent.

Parameter

`component` – A `Component` that contains the chart.

Remarks

This creates the `Chart` with the named `Component`.

Chart

```
public Chart(System.Web.UI.WebControls.WebControl component)
```

Description

This is the root of our tree, it has no parent.

Parameter

`component` – This creates the `Chart` with the named `WebControl`.

Chart

```
public Chart(System.Drawing.Image image)
```

Description

This is the root of our tree, it has no parent.

Parameter

`image` – An `Image` into which the `Chart` is to be drawn.

Remarks

This creates the `Chart` drawn into the `Image`.

Methods

AddLegendItem

```
virtual public void AddLegendItem(int type, Imsl.Chart2D.ChartNode node)
```

Description

Adds a `Legend` to a `ChartNode`.

Parameters

`type` – An `int` which specifies the `LegendItem` type.

`node` – A `ChartNode` to which a `Legend` is to be added.

Remarks

This method is intended to be called from within the `Paint` method of classes which explicitly paint their own child chart nodes. The child chart nodes to be included in the legend must be added to the legend during each call to `Paint` of the parent chart node.

Typical users of the chart library do not need to call this routine. This method is for use by those writing new charting classes.

The possible legend types are:

1. DATA_TYPE_NONE
2. DATA_TYPE_LINE
3. DATA_TYPE_MARKER
4. DATA_TYPE_FILL

Clone

```
virtual public object Clone()
```

Description

Returns a clone of the graphics tree.

Returns

An `Object` which is a clone of this graphics tree.

Copy

```
virtual public void Copy()
```

Description

Copy the chart to the clipboard.

Finalize

```
override void Finalize()
```

Description

Finalize disposes the image buffer.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

Paint

```
virtual public void Paint(System.Drawing.Graphics g)
```

Description

Paints this node and all of its children.

Parameter

g – A `Graphics` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

PaintChart

```
virtual public void PaintChart(System.Drawing.Graphics graphics)
```

Description

Draw the chart using the given `Graphics` object.

Parameter

`graphics` – The `Graphics` object.

PaintImage

```
virtual public System.Drawing.Image PaintImage()
```

Description

Returns an Image of the chart.

Returns

An `Image` containing a picture of the chart.

Pick

```
virtual public void Pick(System.Windows.Forms.MouseEventArgs mouseEvent)
```

Description

Invoke the pick delegates for the nodes hit by the event.

Parameter

`mouseEvent` – A `MouseEventArgs` whose position determines which nodes have been selected.

PrintGraphics

```
public void PrintGraphics(object sender,
System.Drawing.Printing.PrintPageEventArgs e)
```

Description

This method prints the chart on a single page.

Parameters

`sender` – A `Object` that specifies the sender of an event.

`e` – A `PrintPageEventArgs` containing data for the PrintPage event.

Remarks

The output is scaled to fill the page as much as possible while preserving the aspect ratio.

Repaint

```
virtual public void Repaint()
```

Description

Prepares the chart to be repainted by deleting any double buffering image.

SetComponent

```
virtual public void SetComponent(System.Windows.Forms.Control component)
```

Description

Sets the "Component" attribute value.

Parameter

`component` – A `Control` that contains a component of the `Chart`.

Update

```
virtual public void Update(System.Drawing.Graphics g)
```

Description

Update the chart when its component is damaged. The component can be damaged when the window containing it is iconified or is overlapped by another window.

Parameter

`g` – is a `Graphics` object used for drawing.

WritePNG

```
virtual public void WritePNG(System.IO.Stream os, int width, int height)
```

Description

Writes the chart as an PNG file.

Parameters

`os` – A `Stream` containing the output stream to which the PNG image is to be written.

`width` – An `int` which specifies the width of the output image in pixels.

`height` – An `int` which specifies the height of the output image in pixels.

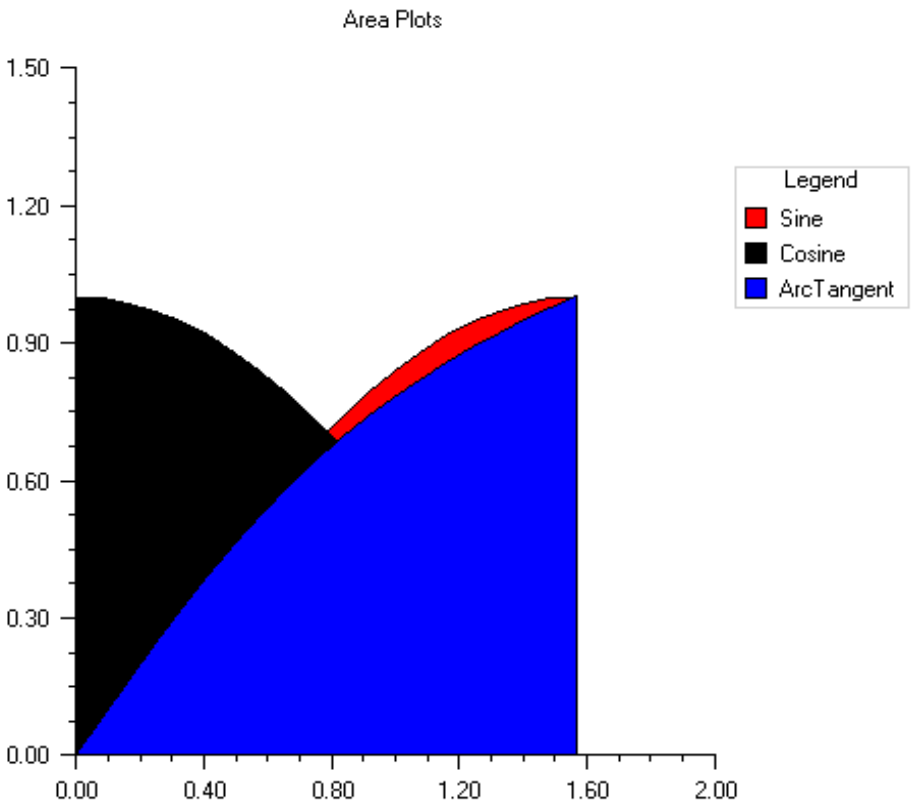
Example: Area Chart

An area chart is constructed in this example. Three data sets are used and a legend is added to the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class AreaEx1 : FrameChart
{
    public AreaEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int npoints = 20;
```

```
        double dx = .5 * System.Math.PI / (npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];
        double[] y3 = new double[npoints];

        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            y2[i] = System.Math.Cos(x[i]);
            y3[i] = System.Math.Atan(x[i]);
        }
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);
        Data d3 = new Data(axis, x, y3);

        //    Set Data Type to Fill Area
        axis.DataType = Imsl.Chart2D.Data.DATA_TYPE_FILL;

        //    Set Line Colors
        d1.LineColor = System.Drawing.Color.Red;
        d2.LineColor = System.Drawing.Color.Black;
        d3.LineColor = System.Drawing.Color.Blue;

        //    Set Fill Colors
        d1.FillColor = System.Drawing.Color.Red;
        d2.FillColor = System.Drawing.Color.Black;
        d3.FillColor = System.Drawing.Color.Blue;

        //    Set Data Labels
        d1.SetTitle("Sine");
        d2.SetTitle("Cosine");
        d3.SetTitle("ArcTangent");

        //    Add a Legend
        Legend legend = chart.Legend;
        legend.SetTitle(new Text("Legend"));
        legend.IsVisible = true;

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Area Plots");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new AreaEx1());
    }
}
```

Output



Background Class

```
public class Imsl.Chart2D.Background :  AxisXY
```

The background of a chart.

Grid is created by Imsl.Chart2D.Chart (p. 1332) as its child. It can be retrieved using the method Imsl.Chart2D.ChartNode.Background (p. 1319).

Fill attributes (specified with FillType (p. 1322)) in this node control the drawing of the background.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

ChartTitle Class

```
public class Imsl.Chart2D.ChartTitle :  AxisXY
```

The main title of a chart.

`ChartTitle` is created by Chart (p. 1332) as its child. It can be retrieved using the method ChartTitle (p. 1339).

The axis title is the "Title" attribute value at this node. Text attributes (specified with CultureInfo , NumberFormatInfo.CurrentInfo and DateTimeFormatInfo.CurrentInfo members) in this node control the drawing of the title.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

Grid Class

```
public class Imsl.Chart2D.Grid :  ChartNode
```

Draws the grid lines perpendicular to an axis.

`Grid` is created by Imsl.Chart2D.Axis1D (p. 1353) as its child. It can be retrieved using the Imsl.Chart2D.Axis1D.Grid (p. 1355) property.

Line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) in this node control the drawing of the grid lines.

Property

Type

```
virtual public int Type {get; }
```

Description

Specifies the type of `Axis1D`.

Property Value

An `int` which contains the type of axis this `Grid` is associated with.

Remarks

The Axis types are:

- Imsl.Chart2D.AbstractChartNode.AXIS_X (p. 1295)
- Imsl.Chart2D.AbstractChartNode.AXIS_Y (p. 1296)
- Imsl.Chart2D.ChartNode.AXIS_X_TOP (p. 1311)
- Imsl.Chart2D.ChartNode.AXIS_Y_RIGHT (p. 1311)

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

Legend Class

```
public class Imsl.Chart2D.Legend :  AxisXY
```

A Imsl.Chart2D.Chart (p. 1332) legend.

`Legend` is created by `Chart` as its child. It can be retrieved using the Imsl.Chart2D.ChartNode.Legend (p. 1323) property.

By default the legend is not drawn. To have it drawn, set `chart.IsVisisble = true;`

Imsl.Chart2D.Data (p. 1372) objects that have their "Title" attribute defined are automatically entered into the legend.

The "Viewport" attribute for this node is set to [0.83,0.98] by [0.1,0.6].

The drawing of the background of the legend box is controlled by the Fill attributes (specified with FillType (p. 1322)) in this node. Text attributes (specified with CultureInfo , NumberFormatInfo.CurrentInfo and DateTimeFormatInfo.CurrentInfo members) in this node control the drawing of the text strings in the box.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

Annotation Class

```
public class Imsl.Chart2D.Annotation :  Data
```

Draws an annotation.

Locations are defined in chart (user) coordinates. To convert between device and user coordinates see Axis.MapDeviceToUser (p. 1347) and Axis.MapUserToDevice (p. 1347).

Properties

String

```
public string String {get; set; }
```

Description

Sets the `string` for the `Text` object to render.

Property Value

The `String` for the `Text` object to render.

Text

```
public Imsl.Chart2D.Text Text {get; set; }
```

Description

The `Text` for this `Annotation` instance.

Property Value

A `Text` object

Constructors

Annotation

```
public Annotation(Imsl.Chart2D.ChartNode parent, Imsl.Chart2D.Text text, double
x, double y)
```

Description

Creates a `Text` object at the specific x,y location in chart coordinates.

Parameters

`parent` – The `ChartNode` parent of this data node, usually an `Axis` object.

`text` – The `Text` object to draw.

x – The x location in user coordinates.

y – The y location in user coordinates.

Annotation

```
public Annotation(Imsl.Chart2D.ChartNode parent, string s, double x, double y)
```

Description

Creates a string object at the specific x,y location in chart coordinates.

Parameters

parent – The ChartNode parent of this data node, usually an Axis object.

s – The string object to draw.

x – The x location in user coordinates.

y – The y location in user coordinates.

Annotation

```
public Annotation(Imsl.Chart2D.ChartNode parent, System.Drawing.Image img,
double x, double y)
```

Description

Renders an Image object centered at an x,y location in chart coordinates.

Parameters

parent – The ChartNode parent of this data node, usually an Axis object.

img – The Image object to draw.

x – The x location in user coordinates.

y – The y location in user coordinates.

Methods

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

draw – A Draw object which is to be painted.

Remarks

This is normally called only by the Paint method in this node's parent.

SetLocation

```
public void SetLocation(double x, double y)
```

Description

Update the location of this `Annotation` instance.

Parameters

`x` – A `double` specifying the new x location in chart coordinates.

`y` – A `double` specifying the new y location in chart coordinates.

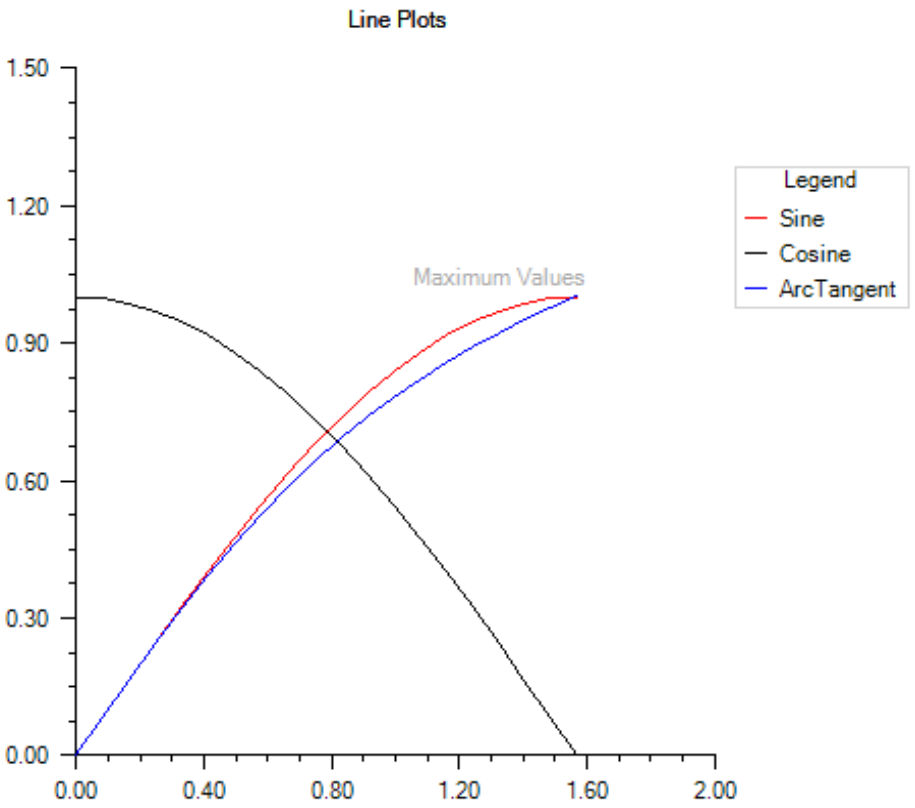
Example: Line Chart

A simple line chart is constructed in this example. Three data sets are used and a legend is added to the chart. An annotation is included indicating a feature on the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class LineEx1 : FrameChart
{

    public LineEx1()
    {
        Chart chart = this.Chart;

        AxisXY axis = new AxisXY(chart);

        int npoints = 20;
        double dx = .5 * System.Math.PI / (npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];
        double[] y3 = new double[npoints];

        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            y2[i] = System.Math.Cos(x[i]);
            y3[i] = System.Math.Atan(x[i]);
        }
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);
        Data d3 = new Data(axis, x, y3);

        //    Set Data Type to Line
        axis.DataType = Imsl.Chart2D.AxisXY.DATA_TYPE_LINE;

        //    Set Line Colors
        d1.LineColor = System.Drawing.Color.Red;
        d2.LineColor = System.Drawing.Color.Black;
        d3.LineColor = System.Drawing.Color.Blue;

        //    Set Data Labels
```

```
        d1.SetTitle("Sine");
        d2.SetTitle("Cosine");
        d3.SetTitle("ArcTangent");

        //    Add a Legend
        Legend legend = chart.Legend;
        legend.SetTitle(new Text("Legend"));
        legend.IsVisible = true;

        // Add an Annotation
        Text label = new Text("Maximum Values");
        label.Alignment = Axis.TEXT_X_RIGHT;
        Annotation ann = new Annotation(axis, label, 1.6, 1.02);
        ann.TextColor = System.Drawing.Color.DarkGray;

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Line Plots");

    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new LineEx1());
    }
}
```

Output



Axis Class

```
public class Imsl.Chart2D.Axis :  ChartNode
```

The `Axis` node provides the mapping for all of its children from the user coordinate space to the device (screen) space.

Constructor

Axis

```
Axis(Imsl.Chart2D.Chart chart)
```

Description

Contructs an `Axis` node.

Parameter

`chart` – A `Chart` object which is the parent of this node.

Remarks

The parent must be a `Chart` node. This node's "Axis" attribute has itself as a value, so that decendent nodes can easily obtain their controlling axis node.

Methods

MapDeviceToUser

```
abstract public void MapDeviceToUser(int devX, int devY, double[] userXY)
```

Description

Maps the device coordinates to user coordinates.

Parameters

`devX` – An `int` which specifies the device x-coordinate.

`devY` – An `int` which specifies the device y-coordinate.

`userXY` – An `int[2]` array on input, on output, the user coordinates.

MapUserToDevice

```
abstract public void MapUserToDevice(double userX, double userY, int[] devXY)
```

Description

Maps the user coordinates (`userX`, `userY`) to the device coordinates `devXY`.

Parameters

`userX` – A `double` which specifies the user x-coordinate.

`userY` – A `double` which specifies the user y-coordinate.

`devXY` – An `int[2]` array on input, on output, the device coordinates.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` object which specifies the chart tree to be rendered on the screen.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetUpMapping

```
abstract public void SetUpMapping()
```

Description

Initializes the mappings between user and coordinate space.

Remarks

This must be called whenever the screen size, the window or the viewport may have changed. Generally, it is safest to call this each time the chart is repainted.

AxisXY Class

```
public class Imsl.Chart2D.AxisXY : Axis
```

The axes for an x-y chart.

This node is used when the mapping to and from user and device space can be decomposed into an x and a y mapping. This is when the mapping map(userX,userY) = (deviceX,deviceY) can be written as map(userX,userY) = (mapX(userX), mapY(userY) = (deviceX,deviceY).

Properties

AxisX

```
virtual public Imsl.Chart2D.Axis1D AxisX {get; }
```

Description

The X axis associated with this node.

Property Value

An `Axis1D` which contains the "AxisX" attribute value.

Remarks

The X axis is a child of this node.

AxisY

```
virtual public Imsl.Chart2D.Axis1D AxisY {get; }
```

Description

The Y axis associated with this node.

Property Value

An `Axis1D` which contains the “AxisY” attribute value.

Remarks

The Y axis is a child of this node.

Constructor

AxisXY

```
public AxisXY(Imsl.Chart2D.Chart chart)
```

Description

Creates an `AxisXY`.

Parameter

`chart` – A `Chart` which is the parent of this node.

Remarks

This also creates two `Axis1D` nodes as children of this node. They hold the decomposed mapping. The “Viewport” attribute for this node is set to [0.2,0.8] by [0.2,0.8].

Methods

GetCross

```
virtual public double[] GetCross()
```

Description

Returns the “Cross” attribute value.

Returns

A `double[2]` containing the “Cross” attribute value.

Remarks

The value is the point where the X and Y axes intersect, (`xcross`,`ycross`). If “Cross” is not defined then `null` is returned.

MapDeviceToUser

```
override public void MapDeviceToUser(int devX, int devY, double[] userXY)
```

Description

Maps the device coordinates to user coordinates.

Parameters

`devX` – An `int` which specifies the device x-coordinate.

`devY` – An `int` which specifies the device y-coordinate.

`userXY` – An `int[2]` array on input, on output, the user coordinates.

MapUserToDevice

```
override public void MapUserToDevice(double userX, double userY, int[] devXY)
```

Description

Maps the user coordinates (`userX`, `userY`) to the device coordinates `devXY`.

Parameters

`userX` – A `double` which specifies the user x-coordinate.

`userY` – A `double` which specifies the user y-coordinate.

`devXY` – An `int[2]` array on input, on output, the device coordinates.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetCross

```
virtual public void SetCross(double xcross, double ycross)
```

Description

Sets the "Cross" attribute value.

Parameters

`xcross` – A `double` which specifies the x-coordinate where the axes cross.

`ycross` – A `double` which specifies the y-coordinate where the axes cross.

Remarks

This defines the point where the X and Y axes intersect. If "Cross" is not defined then the attribute "Window" is used to determine the crossing point.

SetCross

```
virtual public void SetCross(double[] cross)
```

Description

Sets the “Cross” attribute value.

Parameter

`cross` – A `double[2]` containing the x and y-coordinate where the axes cross.

Remarks

This defines the point where the X and Y axes intersect. If “Cross” is not defined then the attribute “Window” is used to determine the crossing point.

SetUpMapping

```
override public void SetUpMapping()
```

Description

Initializes the mappings between user and coordinate space.

Remarks

This must be called whenever the screen size, the window or the viewport may have changed. Generally, it is safest to call this each time the chart is repainted.

SetWindow

```
virtual public void SetWindow(double[] window)
```

Description

Sets the window for an `AxisXY`.

Parameter

`window` – A `double[2]` containing the “Window” attribute value.

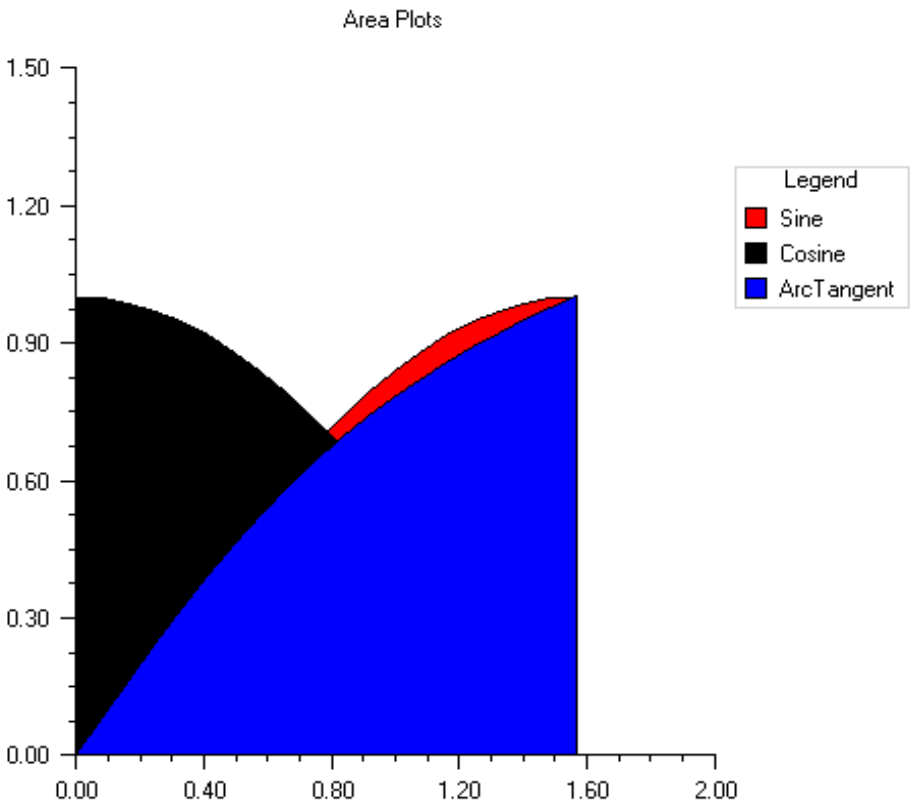
Example: Area Chart

An area chart is constructed in this example. Three data sets are used and a legend is added to the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class AreaEx1 : FrameChart
{
    public AreaEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int npoints = 20;
        double dx = .5 * System.Math.PI / (npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];
        double[] y3 = new double[npoints];
```

```
        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            y2[i] = System.Math.Cos(x[i]);
            y3[i] = System.Math.Atan(x[i]);
        }
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);
        Data d3 = new Data(axis, x, y3);

        //    Set Data Type to Fill Area
        axis.DataType = Imsl.Chart2D.Data.DATA_TYPE_FILL;

        //    Set Line Colors
        d1.LineColor = System.Drawing.Color.Red;
        d2.LineColor = System.Drawing.Color.Black;
        d3.LineColor = System.Drawing.Color.Blue;

        //    Set Fill Colors
        d1.FillColor = System.Drawing.Color.Red;
        d2.FillColor = System.Drawing.Color.Black;
        d3.FillColor = System.Drawing.Color.Blue;

        //    Set Data Labels
        d1.SetTitle("Sine");
        d2.SetTitle("Cosine");
        d3.SetTitle("ArcTangent");

        //    Add a Legend
        Legend legend = chart.Legend;
        legend.SetTitle(new Text("Legend"));
        legend.IsVisible = true;

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Area Plots");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new AreaEx1());
    }
}
```

Output



Axis1D Class

```
public class Imsl.Chart2D.Axis1D : ChartNode
```

An x-axis or a y-axis.

`Axis1D` is created by Imsl.Chart2D.AxisXY (p. 1348) as its child. It can be retrieved using the method Imsl.Chart2D.AxisXY.AxisX (p. 1348) or Imsl.Chart2D.AxisXY.AxisY (p. 1348).

It in turn creates the following child nodes: Imsl.Chart2D.Axis1D.AxisLine (p. 1354),

Imsl.Chart2D.Axis1D.AxisLabel (p. 1354), Imsl.Chart2D.Axis1D.AxisTitle (p. 1354), Imsl.Chart2D.Axis1D.AxisUnit (p. 1354), Imsl.Chart2D.Axis1D.MajorTick (p. 1355), Imsl.Chart2D.Axis1D.MinorTick (p. 1355) and Imsl.Chart2D.Axis1D.Grid (p. 1355).

The number of tick marks ("Number" attribute) is set to 5, but autoscaling can change this value.

Properties

AxisLabel

```
virtual public Imsl.Chart2D.AxisLabel AxisLabel {get; }
```

Description

The label node of this `Axis1D`.

Property Value

An `AxisLabel` which contains the "AxisLabel" attribute value.

Remarks

This is a child of the axis node.

AxisLine

```
virtual public Imsl.Chart2D.AxisLine AxisLine {get; }
```

Description

The line node of this `Axis1D`.

Property Value

An `AxisLine` that is the axis line node associated with this axis.

Remarks

This is a child of the axis node.

AxisTitle

```
virtual public Imsl.Chart2D.AxisTitle AxisTitle {get; }
```

Description

The title node of this `Axis1D`.

Property Value

An `AxisTitle` that is the title node associated with this axis.

Remarks

This is a child of the axis node.

AxisUnit

```
virtual public Imsl.Chart2D.AxisUnit AxisUnit {get; }
```

Description

The unit node of this `Axis1D`.

Property Value

An `AxisUnit` that is the unit node associated with this axis.

Remarks

This is a child of the axis node.

FirstTick

```
virtual public double FirstTick {get; set; }
```

Description

This indicates the location of the first tick.

Property Value

A `double` which contains the “FistTick” attribute value.

Remarks

By default, `FirstTick = GetWindow()[0]`.

Grid

```
virtual public Imsl.Chart2D.Grid Grid {get; }
```

Description

The grid node of this `Axis1D`.

Property Value

A `Grid` which contains the “Grid” attribute value.

Remarks

This is a child of the axis node.

MajorTick

```
virtual public Imsl.Chart2D.MajorTick MajorTick {get; }
```

Description

The major tick node of this `Axis1D`.

Property Value

A `MajorTick` which contains the “MajorTick” attribute value.

Remarks

This is a child of the axis node.

MinorTick

```
virtual public Imsl.Chart2D.MinorTick MinorTick {get; }
```

Description

The minor tick node of this `Axis1D`.

Property Value

A `MajorTick` which contains the “MinorTick” attribute value.

Remarks

This is a child of the axis node.

TickInterval

```
virtual public double TickInterval {get; set; }
```

Description

The tick interval node of this `Axis1D`.

Property Value

A `double` which contains the “TickInterval” attribute value.

Type

```
virtual public int Type {get; set; }
```

Description

Specifies the type of this `Axis1D`.

Property Value

An `int` which contains the type of this axis node.

Remarks

The node types are:

- Imsl.Chart2D.AbstractChartNode.AXIS_X (p. 1295)
- Imsl.Chart2D.AbstractChartNode.AXIS_Y (p. 1296)
- Imsl.Chart2D.ChartNode.AXIS_X_TOP (p. 1311)
- Imsl.Chart2D.ChartNode.AXIS_Y_RIGHT (p. 1311)

Methods

GetTicks

```
virtual public double[] GetTicks()
```

Description

Returns the “Ticks” attribute value.

Returns

A `double[]` containing the “Ticks” attribute value.

Remarks

If not set, then computed tick values are returned based on the type of axis (linear, log or custom), and the attributes “Number” and “TickInterval”.

GetWindow

```
virtual public double[] GetWindow()
```

Description

Returns the window for an `AxisiD`.

Returns

A `double[2]` containing the range of this axis.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

SetTicks

```
virtual public void SetTicks(double[] ticks)
```

Description

Sets the “Ticks” attribute value.

Parameter

`ticks` – A `double[]` which contains the location, in user coordinates, of the major tick marks.

SetWindow

```
virtual public void SetWindow(double min, double max)
```

Description

Sets the window for an `Axis1D`.

Parameters

`min` – A `double` which specifies the value of the left/bottom end of the axis.

`max` – A `double` which specifies the value of the right/top end of the axis.

SetWindow

```
virtual public void SetWindow(double[] window)
```

Description

Sets the window for an `Axis1D`.

Parameter

`window` – A `double[2]` containing the range of this axis.

AxisLabel Class

```
public class Imsl.Chart2D.AxisLabel :  ChartNode
```

The labels on an axis.

`AxisLabel` is created by Imsl.Chart2D.Axis1D (p. 1353) as its child. It can be retrieved using the method Imsl.Chart2D.Axis1D.AxisLabel (p. 1354).

Axis labels are placed at the tick mark locations. The number of tick marks is determined by the attribute "Number". Tick marks are evenly spaced. If the attribute "Labels" is defined then it is used to label the tick marks.

If "Labels" is not defined, the ticks are labeled numerically. The endpoint label values are obtained from the attribute "Window". The numbers are formatted using the attribute "TextFormat".

Text attributes (specified with CultureInfo , NumberFormatInfo.CurrentInfo and DateTimeFormatInfo.CurrentInfo members) in this node control the drawing of the axis labels.

Methods

GetLabels

```
virtual public Imsl.Chart2D.Text[] GetLabels()
```

Description

Returns the "Labels" attribute.

Returns

A `Text[]` containing the axis labels.

Remarks

By default, `GetLabels = null`.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` containing the object to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetLabels

```
virtual public void SetLabels(string[] value)
```

Description

Sets the axis label values for this node to be used instead of the default numbers.

Parameter

`value` – A `String[]` specifying the labels for the major tick marks.

Remarks

The attribute "Number" is also set to `value.Length`.

AxisLine Class

```
public class Imsl.Chart2D.AxisLine :  ChartNode
```

The axis line.

`AxisLine` is created by Imsl.Chart2D.Axis1D (p. 1353) as its child. It can be retrieved using the method Imsl.Chart2D.Axis1D.AxisLine (p. 1354).

Line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) in this node control the drawing of the axis line.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` with the object to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

AxisTitle Class

```
public class Imsl.Chart2D.AxisTitle :  ChartNode
```

The title on an axis.

`AxisTitle` is created by Imsl.Chart2D.Axis1D (p. 1353) as its child. It can be retrieved using the method Imsl.Chart2D.Axis1D.AxisTitle (p. 1354).

The axis title is the "Title" attribute value at this node. Text attributes (specified with CultureInfo , NumberFormatInfo.CurrentInfo and DateTimeFormatInfo.CurrentInfo members) in this node control the drawing of the axis title.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

AxisUnit Class

```
public class Imsl.Chart2D.AxisUnit :  ChartNode
```

The unit on an axis.

`AxisUnit` is created by Imsl.Chart2D.Axis1D (p. 1353) as its child. It can be retrieved using the method Imsl.Chart2D.Axis1D.AxisUnit (p. 1354).

The axis title is the “Title” attribute value at this node. Text attributes (specified with CultureInfo , NumberFormatInfo.CurrentInfo and DateTimeFormatInfo.CurrentInfo members) in this node control the drawing of the unit title.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

MajorTick Class

```
public class Imsl.Chart2D.MajorTick :  ChartNode
```

The major tick marks.

`MajorTick` is created by Imsl.Chart2D.Axis1D (p. 1353) as its child. It can be retrieved using the Imsl.Chart2D.Axis1D.MajorTick (p. 1355) property.

Line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) in this node control the drawing of the major tick marks.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

MinorTick Class

```
public class Imsl.Chart2D.MinorTick :  ChartNode
```

The minor tick marks.

`MinorTick` is created by Imsl.Chart2D.Axis1D (p. 1353) as its child. It can be retrieved using the Imsl.Chart2D.Axis1D.MinorTick (p. 1355) property.

Line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) in this node control the drawing of the minor tick marks.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

Transform Interface

```
public interface Imsl.Chart2D.Transform
```

Defines a custom transformation along an axis.

`Axis1D` has built in support for linear and logarithmic transformations. Additional transformations can be specified by setting the "CustomTransform" attribute in an `Axis1D` to an `Object` that implements this interface.

The interface consists of two methods that must be implemented. Each method is the inverse of the other.

Methods

MapUnitToUser

```
abstract public double MapUnitToUser(double unit)
```

Description

Maps points in the interval [0,1] to user coordinates.

Parameter

`unit` – A `double` which contains a location in unit coordinates to be converted to user coordinates.

MapUserToUnit

```
abstract public double MapUserToUnit(double user)
```

Description

Maps user coordinates to the interval [0,1].

Parameter

`user` – A `double` which contains a location in user coordinates to be converted to unit coordinates.

Remarks

The user coordinate interval is specified by the “Window” attribute for the axis with which the transform is associated.

SetupMapping

```
abstract public void SetupMapping(Imsl.Chart2D.Axis1D axis1d)
```

Description

Initializes the mappings between user and coordinate space.

Parameter

`axis1d` – An `Axis1D` that specifies the axis to which the transform is to be associated.

TransformDate Class

```
public class Imsl.Chart2D.TransformDate :  Imsl.Chart2D.Transform
```

Defines a transformation along an axis that skips weekend dates.

Constructor

TransformDate

```
public TransformDate()
```

Description

Initializes a new instance of the Imsl.Chart2D.TransformDate (p. 1363) class.

Methods

IsWeekday

```
virtual public bool IsWeekday(System.DateTime dateTime)
```

Description

Indicates whether the specified date is a weekday.

Parameter

`dateTime` – A `DateTime` indicating the day to be confirmed a day other than Saturday or Sunday.

Returns

A `bool` indicating whether this is neither Saturday nor Sunday.

Remarks

Returns `false` if the specified day is a Saturday or Sunday.

MapUnitToUser

```
virtual public double MapUnitToUser(double unit)
```

Description

Maps points in the interval [0,1] to user coordinates.

Parameter

`unit` – A `double` which contains a location in unit coordinates to be converted to user coordinates.

MapUserToUnit

```
virtual public double MapUserToUnit(double user)
```

Description

Maps user coordinates to the interval [0,1].

Parameter

`user` – A `double` which contains a location in user coordinates to be converted to unit coordinates.

Remarks

The user coordinate interval is specified by the “Window” attribute for the axis with which the transform is associated.

SetupMapping

```
virtual public void SetupMapping(Imsl.Chart2D.Axis1D axis1d)
```

Description

Initializes the mappings between user and coordinate space.

Parameter

`axis1d` – An `Axis1D` that specifies the axis to which the transform is to be associated.

AxisR Class

```
public class Imsl.Chart2D.AxisR : ChartNode
```

The R-axis in a polar plot.

`AxisR` is created by Imsl.Chart2D.Polar (p. 1495) as its child. It can be retrieved using the Imsl.Chart2D.Polar.AxisR (p. 1495).

It in turn creates the following child nodes: Imsl.Chart2D.AxisR.AxisRLine (p. 1365), Imsl.Chart2D.AxisR.AxisRLabel (p. 1365) and Imsl.Chart2D.AxisR.AxisRMajorTick (p. 1366).

The number of tick marks (“Number” attribute) is set to 4, but autoscaling can change this value.

See Also

Imsl.Chart2D.Polar (p. 1495)

Properties

AxisRLabel

```
virtual public Imsl.Chart2D.AxisRLabel AxisRLabel {get; }
```

Description

A `AxisRLabel` which specifies the label node associated with this axis.

AxisRLine

```
virtual public Imsl.Chart2D.AxisRLine AxisRLine {get; }
```

Description

Specifies the line node associated with this axis.

Property Value

A `AxisRLine` which contains the line node associated with this axis.

AxisRMajorTick

```
virtual public Imsl.Chart2D.AxisRMajorTick AxisRMajorTick {get; }
```

Description

Specifies the major tick associated with this axis.

Property Value

A `AxisRMajorTick` which contains the “AxisRMajorTick” attribute value.

Remarks

This is a child of the axis node.

TickInterval

```
virtual public double TickInterval {get; set; }
```

Description

The tick interval node of this `AxisR`.

Property Value

A `double` which contains the “TickInterval” attribute value.

Window

```
virtual public double Window {get; set; }
```

Description

The radius at which `AxisTheta` is drawn.

Property Value

A `double` which contains the “Window” attribute value.

Remarks

The window has a maximum value of R. The R-axis always starts at 0. By default, `Window` = 1.0.

Methods

GetTicks

```
virtual public double[] GetTicks()
```

Description

Returns the “Ticks” attribute value.

Returns

A `double[]` containing the “Ticks” attribute value.

Remarks

If not set, then computed tick values are returned based on the type of axis (linear, log or custom), the attributes “Number” and “TickInterval”.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

AxisRLabel Class

```
public class Imsl.Chart2D.AxisRLabel :  ChartNode
```

The labels on an axis.

`AxisRLabel` is created by Imsl.Chart2D.AxisR (p. 1365) as its child. It can be retrieved using the method Imsl.Chart2D.AxisR.AxisRLabel (p. 1365).

Axis labels are placed at the tick mark locations. The number of tick marks is determined by the attribute “Number”. Tick marks are evenly spaced. If the attribute “Labels” is defined then it is used to label the tick marks.

If “Labels” is not defined, the ticks are labeled numerically. The endpoint label values are obtained from the attribute “Window”. The numbers are formatted using the attribute “TextFormat”.

Text attributes (specified with CultureInfo , NumberFormatInfo.CurrentInfo and DateTimeFormatInfo.CurrentInfo members) in this node control the drawing of the axis labels.

See Also

Imsl.Chart2D.Polar (p. 1495), Imsl.Chart2D.AxisR (p. 1365)

Methods

GetLabels

```
virtual public Imsl.Chart2D.Text[] GetLabels()
```

Description

Returns the “Labels” attribute.

Returns

A `Text[]` containing the axis labels.

Remarks

By default, `GetLabels = null`.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

SetLabels

```
virtual public void SetLabels(string[] value)
```

Description

Sets the axis label values for this node to be used instead of the default numbers.

Parameter

`value` – A `String[]` specifying the labels for the major tick marks.

Remarks

The attribute “Number” is also set to `value.Length`.

AxisRLine Class

```
public class Imsl.Chart2D.AxisRLine :  ChartNode
```

The radius axis line in a polar plot.

`AxisRLine` is created by Imsl.Chart2D.AxisR (p. 1365) as its child. It can be retrieved using the method Imsl.Chart2D.AxisR.AxisRLine (p. 1365).

Line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) in this node control the drawing of the axis line.

See Also

Imsl.Chart2D.Polar (p. 1495), Imsl.Chart2D.AxisR (p. 1365)

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

AxisRMajorTick Class

```
public class Imsl.Chart2D.AxisRMajorTick :  ChartNode
```

The major tick marks for the radius axis in a polar plot.

`AxisRMajorTick` is created by Imsl.Chart2D.AxisR (p. 1365) as its child. It can be retrieved using the method Imsl.Chart2D.AxisR.AxisRMajorTick (p. 1366).

Line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) in this node control the drawing of the major tick marks.

See Also

Imsl.Chart2D.Polar (p. 1495), Imsl.Chart2D.AxisR (p. 1365)

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

AxisTheta Class

```
public class Imsl.Chart2D.AxisTheta :  ChartNode
```

The angular axis in a polar plot.

`AxisTheta` is created by Imsl.Chart2D.Polar (p. 1495) as its child. It can be retrieved using the Imsl.Chart2D.Polar.AxisTheta (p. 1495) property.

The angles are labeled using the "TextFormat" attribute, which is set to `''0.##\\u00b0''`, where `\\u00b0` is the Unicode character for degrees. This labels the angles in degrees. More generally, "TextFormat" can be set to a `NumberFormat` object to format the angles in degrees.

"TextFormat" can also be set to a `MessageFormat` object. In this case, field {0} is the value in degrees, field {1} is the value in radians and field {2} is the value in radians/π. So, for labels like `1.5\\u03c0`, where `\\u03c0` is the Unicode character for π, set "TextFormat" to `new MessageFormat(''{2,number,0.##\\u03c0}'')`.

The number of tick marks ("Number" attribute) is set to 9, but autoscaling can change this value.

See Also

Imsl.Chart2D.Polar (p. 1495)

Methods

GetTicks

```
virtual public double[] GetTicks()
```

Description

Returns the "Ticks" attribute value.

Returns

A `double[]` containing the “Ticks” attribute value.

Remarks

These are the positions at which the angles are labeled. The ticks are in radians, not degrees.

GetWindow

```
virtual public double[] GetWindow()
```

Description

Returns the window for an `AxisTheta`.

Returns

A `double` array of length two containing the angular range of the window.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

SetWindow

```
virtual public void SetWindow(double min, double max)
```

Description

Sets the window for an `AxisTheta`.

Parameters

`min` – A `double` which specifies the initial angular value, in radians value.

`max` – A `double` which specifies the final angular value, in radians.

Remarks

The default “Window” is [0,2pi].

SetWindow

```
virtual public void SetWindow(double[] window)
```

Description

Sets the window for an `AxisTheta`.

Parameter

`window` – A `double` array of length two containing the angular range.

Remarks

The default “Window” is [0,2pi].

GridPolar Class

```
public class Imsl.Chart2D.GridPolar :  ChartNode
```

Draws the grid lines for a polar plot.

`GridPolar` is created by Imsl.Chart2D.Polar (p. 1495) as its child. It can be retrieved using the Imsl.Chart2D.Polar.GridPolar (p. 1495) property.

Line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) in this node control the drawing of the grid lines.

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

Data Class

```
public class Imsl.Chart2D.Data :  ChartNode
```

A data node in the chart tree.

Drawing of a `Data` node is determined by the DataType (p. 1320) property. Multiple bits can be set in “DataType”.

If the DATA_TYPE_LINE (p. 1312) bit is set, the line attributes are active.

If the DATA_TYPE_MARKER (p. 1312) bit is set, the marker attributes are active.

If the DATA_TYPE_FILL (p. 1312) bit is set, the fill attributes are active.

If LabelType (p. 1302) is set to something other than the default (`LABEL_TYPE_NONE`), then the data points are labeled. The contents of the labels are determined by the value of the `LabelType` property.

The drawing of the labels is controlled with CultureInfo , NumberFormatInfo.CurrentInfo and DateTimeFormatInfo.CurrentInfo members) in this node control the drawing of the title.

Constructors

Data

```
public Data(Imsl.Chart2D.ChartNode parent)
```

Description

Creates a data node.

Parameter

`parent` – A `ChartNode` which specifies the parent of this data node.

Data

```
public Data(Imsl.Chart2D.ChartNode parent, double[] y)
```

Description

Creates a `Data` node with y values.

Parameters

`parent` – A `ChartNode` which specifies the parent of this data node.

`y` – A `double` array containing the dependant values for this node.

Remarks

The x values are set to the double array containing {`0,1,...,y.Length-1`}.

Data

```
public Data(Imsl.Chart2D.ChartNode parent, Imsl.Chart2D.ChartFunction cf,
double a, double b)
```

Description

Creates a `Data` node with y values.

Parameters

`parent` – A `ChartNode` which specifies the parent of this data node.

`cf` – A `ChartFunction` that defines the function to be plotted.

`a` – A `double` that contains the left endpoint.

`b` – A `double` that contains the right endpoint.

Remarks

The x values are set to the double array containing {0,1,...,y.Length-1}.

Data

```
public Data(Imsl.Chart2D.ChartNode parent, double[] x, double[] y)
```

Description

Creates a Data node with x and y values.

Parameters

parent – A ChartNode which specifies the parent of this data node.

x – A double array containing the independant values for this node.

y – A double array containing the dependant values for this node.

Methods

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

draw – A Draw which is to be painted.

Remarks

This is normally called only by the Paint method in this node's parent.

SetDataRange

```
virtual public void SetDataRange(double[] range)
```

Description

Update the data range, range = {xmin,xmax,ymin,ymax}

Parameter

range – a double array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument range is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

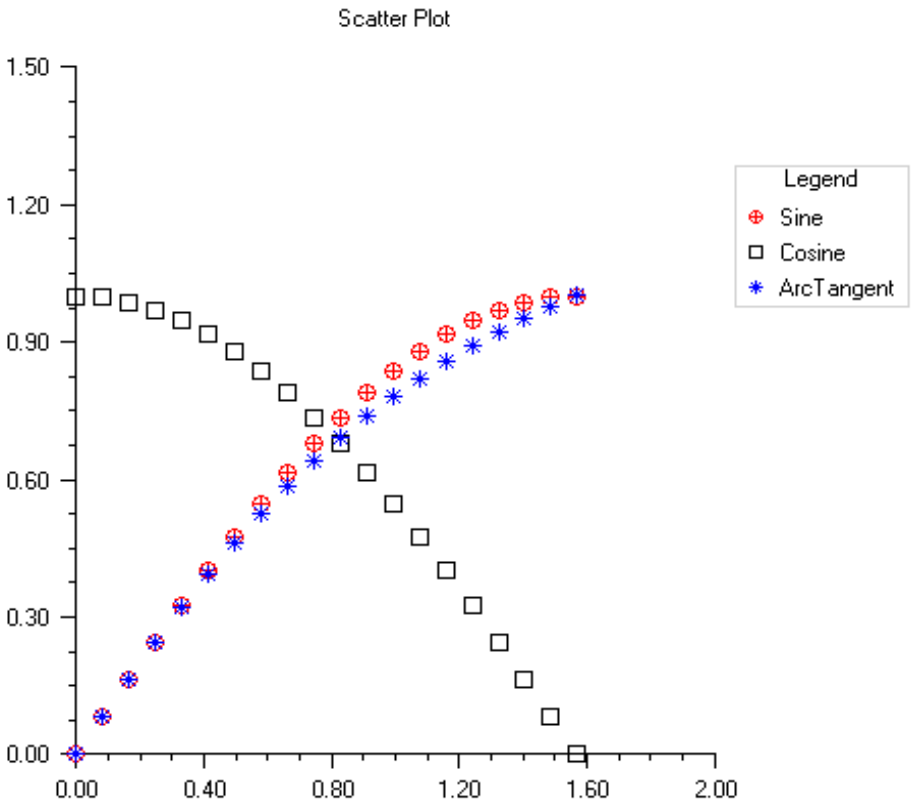
Example: Scatter Chart

A scatter plot is constructed in this example. Three data sets are used and a legend is added to the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class ScatterEx1 : FrameChart
{
    public ScatterEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int npoints = 20;
        double dx = .5 * System.Math.PI / (npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];
        double[] y3 = new double[npoints];

        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            y2[i] = System.Math.Cos(x[i]);
            y3[i] = System.Math.Atan(x[i]);
        }
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);
        Data d3 = new Data(axis, x, y3);

        //    Set Data Type to Marker
        d1.DataType = Imsl.Chart2D.Data.DATA_TYPE_MARKER;
        d2.DataType = Imsl.Chart2D.Data.DATA_TYPE_MARKER;
        d3.DataType = Imsl.Chart2D.Data.DATA_TYPE_MARKER;

        //    Set Marker Types
        d1.MarkerType = Data.MARKER_TYPE_CIRCLE_PLUS;
        d2.MarkerType = Data.MARKER_TYPE_HOLLOW_SQUARE;
        d3.MarkerType = Data.MARKER_TYPE_ASTERISK;

        //    Set Marker Colors
        d1.MarkerColor = System.Drawing.Color.Red;
        d2.MarkerColor = System.Drawing.Color.Black;
        d3.MarkerColor = System.Drawing.Color.Blue;

        //    Set Data Labels
        d1.SetTitle("Sine");
        d2.SetTitle("Cosine");
        d3.SetTitle("ArcTangent");

        //    Add a Legend
        Legend legend = chart.Legend;
```

```
        legend.SetTitle(new Text("Legend"));
        legend.IsVisible = true;

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Scatter Plot");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new ScatterEx1());
    }
}
```

Output



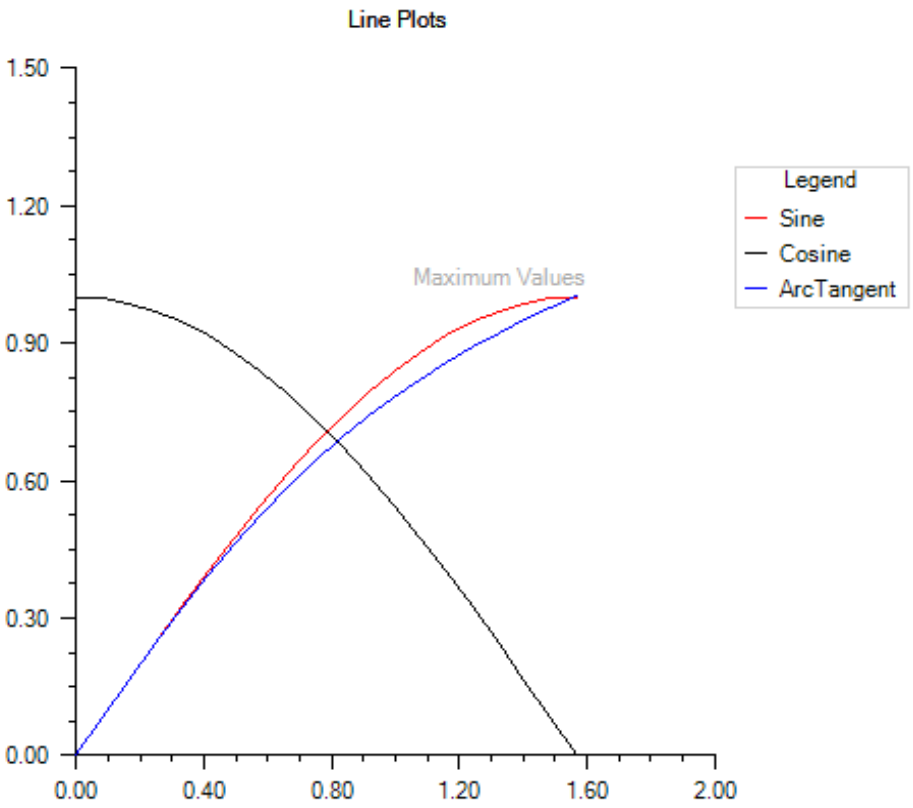
Example: Line Chart

A simple line chart is constructed in this example. Three data sets are used and a legend is added to the chart. An annotation is included indicating a feature on the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class LineEx1 : FrameChart
{

    public LineEx1()
    {
        Chart chart = this.Chart;

        AxisXY axis = new AxisXY(chart);

        int npoints = 20;
        double dx = .5 * System.Math.PI / (npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];
        double[] y3 = new double[npoints];

        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            y2[i] = System.Math.Cos(x[i]);
            y3[i] = System.Math.Atan(x[i]);
        }
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);
        Data d3 = new Data(axis, x, y3);

        //    Set Data Type to Line
        axis.DataType = Imsl.Chart2D.AxisXY.DATA_TYPE_LINE;

        //    Set Line Colors
        d1.LineColor = System.Drawing.Color.Red;
        d2.LineColor = System.Drawing.Color.Black;
        d3.LineColor = System.Drawing.Color.Blue;

        //    Set Data Labels
        d1.SetTitle("Sine");
        d2.SetTitle("Cosine");
        d3.SetTitle("ArcTangent");

        //    Add a Legend
        Legend legend = chart.Legend;
        legend.SetTitle(new Text("Legend"));
        legend.IsVisible = true;
```

```
        // Add an Annotation
        Text label = new Text("Maximum Values");
        label.Alignment = Axis.TEXT_X_RIGHT;
        Annotation ann = new Annotation(axis, label, 1.6, 1.02);
        ann.TextColor = System.Drawing.Color.DarkGray;

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Line Plots");

    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new LineEx1());
    }
}
```

Output



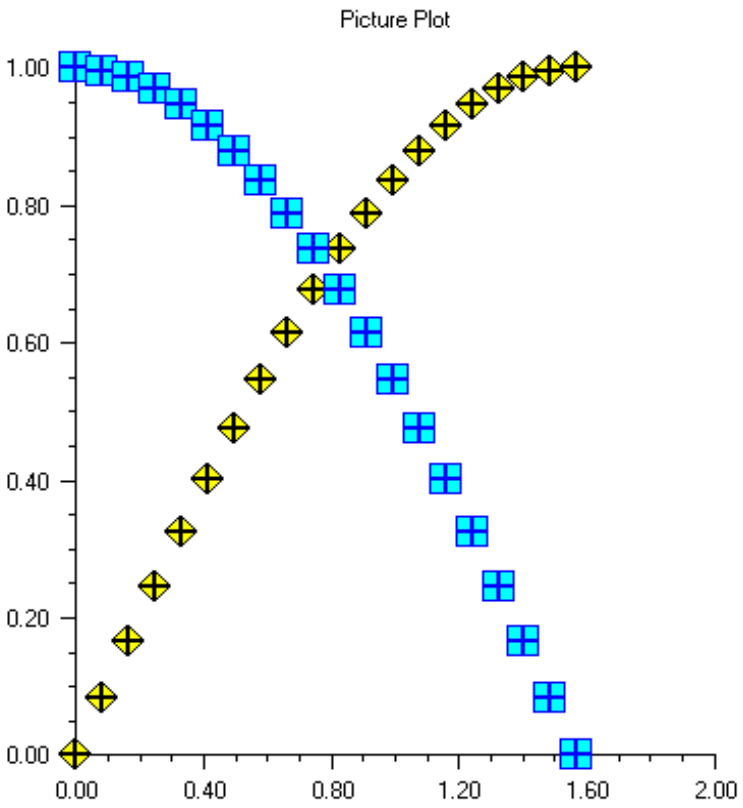
Example: Picture Chart

A picture plot is constructed in this example.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;
using System.Drawing;

public class PictureEx1 : FrameChart
{

    public PictureEx1()
    {

        string appPath = Application.ExecutablePath;
```

```
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int npoints = 20;
        double dx = .5 * System.Math.PI / (npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];

        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            y2[i] = System.Math.Cos(x[i]);
        }
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);

        //    Load Images
        d1.DataType = Data.DATA_TYPE_PICTURE;
        d1.ImageAttr = new Bitmap(@"IMSL.NET\Example\Chart2D\marker.gif", true);
        d2.DataType = Data.DATA_TYPE_PICTURE;
        d2.ImageAttr = new Bitmap(@"IMSL.NET\Example\Chart2D\marker2.gif", true);

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Picture Plot");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new PictureEx1());
    }
}
```

Output



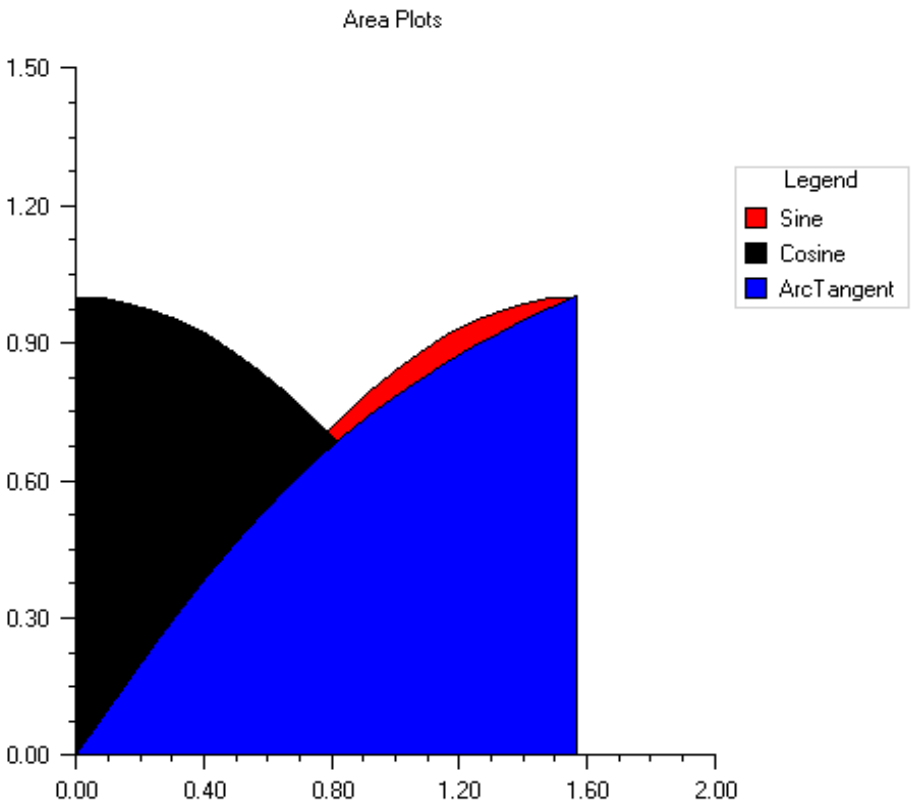
Example: Area Chart

An area chart is constructed in this example. Three data sets are used and a legend is added to the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class AreaEx1 : FrameChart
{
    public AreaEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int npoints = 20;
```

```
        double dx = .5 * System.Math.PI / (npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];
        double[] y3 = new double[npoints];

        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            y2[i] = System.Math.Cos(x[i]);
            y3[i] = System.Math.Atan(x[i]);
        }
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);
        Data d3 = new Data(axis, x, y3);

        //    Set Data Type to Fill Area
        axis.DataType = Imsl.Chart2D.Data.DATA_TYPE_FILL;

        //    Set Line Colors
        d1.LineColor = System.Drawing.Color.Red;
        d2.LineColor = System.Drawing.Color.Black;
        d3.LineColor = System.Drawing.Color.Blue;

        //    Set Fill Colors
        d1.FillColor = System.Drawing.Color.Red;
        d2.FillColor = System.Drawing.Color.Black;
        d3.FillColor = System.Drawing.Color.Blue;

        //    Set Data Labels
        d1.SetTitle("Sine");
        d2.SetTitle("Cosine");
        d3.SetTitle("ArcTangent");

        //    Add a Legend
        Legend legend = chart.Legend;
        legend.SetTitle(new Text("Legend"));
        legend.IsVisible = true;

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Area Plots");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new AreaEx1());
    }
}
```

Output



ChartFunction Interface

`public interface Imsl.Chart2D.ChartFunction`

An interface that allows a function to be plotted.

See Also

Imsl.Chart2D.Data (p. 1372)

Method

F

```
abstract public double F(double x)
```

Description

Function to be charted.

Parameter

x – A `double[]` which specifies the independent data.

Returns

A `double[]` containing the dependant data.

ChartSpline Class

```
public class Imsl.Chart2D.ChartSpline :  Imsl.Chart2D.ChartFunction
```

Wraps a spline into a `ChartFunction` to be plotted.

Constructors

ChartSpline

```
public ChartSpline(Imsl.Math.Spline spline)
```

Description

Creates a `ChartSpline`.

Parameter

`spline` – A `Spline` used to construct this `ChartSpline`.

ChartSpline

```
public ChartSpline(Imsl.Math.Spline spline, int ideriv)
```

Description

Creates a `ChartSpline`.

Parameters

`spline` – A `Spline` which is to have its derivative plotted.

`ideriv` – An `int` that specifies what derivative is to be plotted.

Remarks

If zero, the function value is plotted.

If one, the first derivative is plotted, etc.

Method

F

```
virtual public double F(double x)
```

Description

Function to be charted.

Parameter

`x` – A `double` specifying the point at which the function is to be evaluated.

Returns

A `double` containing the function evaluation.

Text Class

```
public class Imsl.Chart2D.Text
```

The value of the attribute “Title”.

A title is a multi-line `string` with alignment information.

Line breaks are indicated by the newline character (’“n’) within the `string`.

Titles are drawn relative to a reference point. Alignment determines the position of the reference point on the horizontally-aligned box that bounds the text.

Properties

Alignment

```
virtual public int Alignment {get; set; }
```

Description

The alignment for this `Text` object.

Property Value

An `int` indicating the alignment for this `Text` object.

Remarks

The alignment determines the position of the reference point on the horizontally aligned box containing the drawn text. It is the bitwise combination of the following:

- TEXT_X_LEFT (p. 1317)
- TEXT_X_CENTER (p. 1317)
- TEXT_X_RIGHT (p. 1317)
- TEXT_Y_BOTTOM (p. 1317)
- TEXT_Y_CENTER (p. 1318)
- TEXT_Y_TOP (p. 1318)

DefaultAlignment

```
virtual public int DefaultAlignment {set; }
```

Description

The default alignment for this `Text` object.

Property Value

An `int` indicating the alignment for this `Text` object.

Remarks

The alignment determines the position of the reference point on the horizontally aligned box containing the drawn text. It is the bitwise combination of the following:

- TEXT_X_LEFT (p. 1317)
- TEXT_X_CENTER (p. 1317)
- TEXT_X_RIGHT (p. 1317)
- TEXT_Y_BOTTOM (p. 1317)
- TEXT_Y_CENTER (p. 1318)
- TEXT_Y_TOP (p. 1318)

DefaultOffset

```
virtual public double DefaultOffset {set; }
```

Description

The default value of the offset.

Property Value

A `double` which contains the default offset for this `Text` object.

Remarks

Offset is in units of the default marker size. `Text` drawn is offset in the direction of the alignment.

Offset

```
virtual public double Offset {get; set; }
```

Description

The offset for this `Text` object.

Property Value

A `double` which contains the offset for this `Text` object.

Remarks

Offset is in units of the default marker size. `Text` drawn is offset in the direction of the alignment.

String

```
virtual public string String {get; set; }
```

Description

A `string` representation of this `Text` object.

Property Value

A `string` containing the value of this `Text` object.

Constructors

Text

```
public Text(string text)
```

Description

Constructs a `Text` object from a `string`.

Parameter

`text` – A `string` that is to be converted to a `Text` object.

Text

```
public Text(string text, int alignment)
```

Description

Constructs a `Text` object from a `string` with specified alignment.

Parameters

`text` – The `String` that is to be converted to a `Text` object.

`alignment` – An `int` which specifies the alignment.

Remarks

The alignment determines the position of the reference point on the horizontally aligned box containing the drawn text. It is the bitwise combination of the following:

- TEXT_X_LEFT (p. 1317)
- TEXT_X_CENTER (p. 1317)
- TEXT_X_RIGHT (p. 1317)
- TEXT_Y_BOTTOM (p. 1317)
- TEXT_Y_CENTER (p. 1318)
- TEXT_Y_TOP (p. 1318)

Text

```
public Text(string format, System.IFormatProvider formatProvider,
System.IFormattable obj)
```

Description

Constructs a `Text` object given a format `string`, an `IFormatProvider` and the value to be formatted.

Parameters

`format` – A `string` containing the format.

`formatProvider` – An `IFormatProvider` like NumberFormat or DateTimeFormat .

`obj` – A `IFormattable` that is to be converted into a `Text` object.

ToolTip Class

```
public class Imsl.Chart2D.ToolTip :  ChartNode
```

A tool tip for a chart element.

This class requires that the chart's component be a subclass of ComponentModel . The `ComponentModel` class can be subclassed to provide different behaviors for displaying tool tips.

To use, create an instance of `ToolTip` to activate the `ToolTips` in a node and in the node's descendants. The `ToolTip` string is the value of a node's "ToolTip" attribute or, if it is `null`, the node's "Title" attribute.

Constructor

ToolTip

```
public ToolTip(Imsl.Chart2D.ChartNode parent)
```

Description

Creates a `ToolTip` node that enables tool tips on charts.

Parameter

`parent` – The `ChartNode` parent of this node.

Remarks

Do not use the root `ChartNode` for this argument, because it will normally select only the `Background` node.

Methods

MouseMoved

```
virtual public void MouseMoved(object sender,
System.Windows.Forms.MouseEventArgs e)
```

Description

The `MouseMoved` delegate added to the `Chart` when a `ToolTip` is created.

Parameters

`sender` – A `Object` that specifies the sender of an event.

`e` – A `MouseEventArgs` that provides data for the MouseUp, MouseDown, and MouseMove events.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

FillPaint Class

```
public class Imsl.Chart2D.FillPaint
```

A collection of methods to create `Brush` objects for fill areas.

All of the `Brush` objects returned by the methods in this class are serializable.

Methods

Checkerboard

```
static public System.Drawing.Brush Checkerboard(int n, System.Drawing.Color
colorA, System.Drawing.Color colorB)
```

Description

Returns a checkerboard pattern.

Parameters

`n` – An `int` that specifies the pattern size in pixels.

`colorA` – A `Color` which specifies the first color in the checkerboard pattern.

`colorB` – A `Color` which specifies the second color in the checkerboard pattern.

Returns

A `Brush` containing the checkerboard pattern.

Crosshatch

```
static public System.Drawing.Brush Crosshatch(int n, int p,
System.Drawing.Color colorBackground, System.Drawing.Color colorLine)
```

Description

Returns a crosshatch pattern.

Parameters

`n` – An `int` that specifies the pattern size in pixels.

`p` – An `int` which specifies the number of pixels between the crosshatched lines.

`colorBackground` – A `Color` which specifies the background color.

`colorLine` – A `Color` which specifies the color of the line.

Returns

A Brush containing the pattern.

DefaultReadObject

```
static public void
DefaultReadObject(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context, object instance)
```

Description

Reads the serialized fields written by the DefaultWriteObject method.

Parameters

info – A SerializationInfo parameter from the special deserialization constructor.

context – A StreamingContext parameter from the special deserialization constructor.

instance – An Object to deserialize.

DefaultWriteObject

```
static public void
DefaultWriteObject(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context, object instance)
```

Description

Writes the serializable fields to the SerializationInfo object, which stores all the data needed to serialize the specified Object.

Parameters

info – A SerializationInfo parameter from the GetObjectData method.

context – A StreamingContext parameter from the GetObjectData method.

instance – An Object to serialize.

Diagonal

```
static public System.Drawing.Brush Diagonal(int n, System.Drawing.Color colorA,
System.Drawing.Color colorB)
```

Description

Returns a diagonal pattern.

Parameters

n – An int that specifies the pattern size in pixels.

colorA – A Color which specifies the first color in the diagonal pattern.

colorB – A Color which specifies the second color in the diagonal pattern.

Returns

A Brush containing the diagonal pattern.

Diamond

```
static public System.Drawing.Brush Diamond(int n, int p, System.Drawing.Color
colorBackground, System.Drawing.Color colorLine)
```

Description

Returns a diamond pattern (a checkerboard rotated 45 degrees).

Parameters

`n` – An `int` that specifies the pattern size in pixels.

`p` – An `int` which specifies the line thickness.

`colorBackground` – A `Color` which specifies the background color.

`colorLine` – A `Color` which specifies the color of the line.

Returns

A `Brush` containing the diamond pattern.

DiamondHatch

```
static public System.Drawing.Brush DiamondHatch(int n, int p,
System.Drawing.Color colorBackground, System.Drawing.Color colorLine)
```

Description

Returns a crosshatch pattern on a 45 degree angle.

Parameters

`n` – An `int` that specifies the pattern size in pixels.

`p` – An `int` which specifies the number of pixels between the crosshatched lines.

`colorBackground` – A `Color` which specifies the background color.

`colorLine` – A `Color` which specifies the color of the line.

Returns

A `Brush` containing the pattern.

Dot

```
static public System.Drawing.Brush Dot(int n, int r, System.Drawing.Color
colorBackground, System.Drawing.Color colorCircle)
```

Description

Returns a pattern that is an array of circles.

Parameters

`n` – An `int` that specifies the pattern size in pixels.

`r` – An `int` which specifies the radius of circles in the pattern in pixels.

`colorBackground` – A `Color` which specifies the background color.

`colorCircle` – A `Color` which specifies the color of circles in the pattern.

Returns

A `Brush` containing the pattern.

HorizontalStripe

```
static public System.Drawing.Brush HorizontalStripe(int n, int p,
System.Drawing.Color colorBackground, System.Drawing.Color colorLine)
```

Description

Returns a horizontally striped pattern.

Parameters

`n` – An `int` that specifies the pattern size in pixels.

`p` – An `int` which specifies the number of pixels between horizontally lines.

`colorBackground` – A `Color` which specifies the background color.

`colorLine` – A `Color` which specifies the color of the line.

Returns

A `Brush` containing the pattern.

Image

```
static public System.Drawing.Brush Image(System.Drawing.Image imageIcon)
```

Description

Returns a tiling of an image.

Parameter

`imageIcon` – An `Image` that specifies the image to be tiled.

Returns

A `Brush` containing the tiling of the image.

VerticalStripe

```
static public System.Drawing.Brush VerticalStripe(int n, int p,
System.Drawing.Color colorBackground, System.Drawing.Color colorLine)
```

Description

Returns a vertically striped pattern.

Parameters

`n` – An `int` that specifies the pattern size in pixels.

`p` – An `int` which specifies the number of pixels between vertical lines.

`colorBackground` – A `Color` which specifies the background color.

`colorLine` – A `Color` which specifies the color of the line.

Returns

A `Brush` containing the pattern.

Draw Class

```
public class Imsl.Chart2D.Draw
```

Renders the chart tree to the screen.

Properties

ClipBounds

```
virtual public System.Drawing.Rectangle ClipBounds {get; set; }
```

Description

Contains the rectangle to be used for clipping.

Property Value

A `Rectangle` object which contains the clipping bounds.

DeviceMarkerSize

```
virtual public float DeviceMarkerSize {get; }
```

Description

The marker size in device coordinates.

Property Value

A `float` that contains the marker size.

Node

```
virtual public Imsl.Chart2D.ChartNode Node {set; }
```

Description

Specifies a `ChartNode` as the current node.

Property Value

A `ChartNode` containing the curent node.

Remarks

This is used to get drawing attributes from the tree.

ScaleFont

```
virtual public double ScaleFont {get; set; }
```

Description

The factor by which fonts are to be scaled.

Property Value

A `double` containing the font scaling factor.

Constructor

Draw

```
public Draw(System.Drawing.Graphics graphics, System.Drawing.Size bounds)
```

Description

Contructs a `Draw` object.

Parameters

`graphics` – A `Graphics` object encapsulating a GDI+ drawing surface.

`bounds` – A `Size` specifying the width and height of a rectangle.

Methods

CreateGradientBrush

```
static public System.Drawing.Drawing2D.LinearGradientBrush
CreateGradientBrush(float x1, float y1, System.Drawing.Color color1, float x2,
float y2, System.Drawing.Color color2)
```

Description

Creates an acyclic `GradientBrush`.

Parameters

`x1` – A `float` containing the x-coordinate of the upper-left corner of drawing area.

`y1` – A `float` containing the y-coordinate of the upper-left corner of drawing area.

`color1` – A `Color` structure that represents the starting color for the gradient.

`x2` – A `float` containing the x-coordinate of the lower-right corner of drawing area.

`y2` – A `float` containing the x-coordinate of the lower-right corner of drawing area.

`color2` – A `Color` structure that represents the ending color for the gradient.

Returns

A new instance of `LinearGradientBrush` with the colors and coordinates specified.

Remarks

This gradient is acyclic.

DrawArc

```
virtual public void DrawArc(int x, int y, int width, int height, int
startAngle, int arcAngle)
```

Description

Draws the outline of a circular or elliptical arc covering the specified rectangle.

Parameters

`x` – An `int` which contains the x-coordinate of the upper-left corner of the rectangle that defines the ellipse.

`y` – An `int` which contains the y-coordinate of the upper-left corner of the rectangle that defines the ellipse.

`width` – An `int` which contains the width of the rectangle that defines the ellipse.

`height` – An `int` which contains the height of the rectangle that defines the ellipse.

`startAngle` – An `int` which specifies the angle in degrees measured clockwise from the x-axis to the starting point of the arc.

`arcAngle` – An `int` which specifies an angle in degrees measured counter-clockwise from the `startAngle` parameter to the ending point of the arc.

Remarks

The center of the arc is center of this rectangle.

`startAngle = 0` is equivalent to the 3-o'clock position.

`DrawArc` draws the arc from `startAngle` to `startAngle+arcAngle`. A positive `arcAngle` indicates a counter-clockwise rotation. A negative `arcAngle` implies a clockwise rotation.

DrawClippedImage

```
virtual public void DrawClippedImage(System.Drawing.Image image, int x, int y)
```

Description

Draw an image with clipping.

Parameters

`image` – The `Image` object to draw.

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the drawn image.

`y` – An `int` which specifies the y-coordinate of the upper-left corner of the drawn image.

DrawErrorBar

```
virtual public void DrawErrorBar(int x0, int y0, int x1, int y1, int flag)
```

Description

Draws an error bar.

Parameters

`x0` – An `int` which specifies the x-coordinate of the beginning reference point.

`y0` – An `int` which specifies the y-coordinate of the beginning reference point.

`x1` – An `int` which specifies the x-coordinate of the ending reference point.

`y1` – An `int` which specifies the y-coordinate of the ending reference point.

`flag` – An `int` that indicates which caps to draw.

Remarks

Legal values: 0=none, 1=bottom, 2=top, 3=both

DrawImage

```
virtual public void DrawImage(System.Drawing.Image image, int x, int y)
```

Description

Draws the specified image at the location specified by a coordinate pair.

Parameters

`image` – The `Image` object to draw.

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the drawn image.

`y` – An `int` which specifies the x-coordinate of the upper-left corner of the drawn image.

DrawLine

```
virtual public void DrawLine(int x0, int y0, int x1, int y1)
```

Description

Draws a line from between two points.

Parameters

`x0` – An `int` which specifies the x-coordinate of the line origin, (x0,y0).

`y0` – An `int` which specifies the y-coordinate of the line origin, (x0,y0).

`x1` – An `int` which specifies the x-coordinate of the line destination, (x1,y1).

`y1` – An `int` which specifies the y-coordinate of the line destination, (x1,y1).

DrawMarker

```
virtual public void DrawMarker(int x, int y)
```

Description

Draws a marker.

Parameters

`x` – An `int` which specifies the x-coordinate of the marker destination, (x,y).

`y` – An `int` which specifies the y-coordinate of the marker destination, (x,y).

DrawText

```
virtual public System.Drawing.Size DrawText(Imsl.Chart2D.Text text, int x, int
y)
```

Description

Draws a `Text` object.

Parameters

`text` – A `Text` object to be drawn.

`x` – An `int` which specifies the abscissa of the (x,y) point at which to start drawing the text.

`y` – An `int` which specifies the ordinate of the (x,y) point at which to start drawing the text.

Returns

A `Size` containing the bounds of the `Text` to be drawn.

EndErrorBar

```
virtual public void EndErrorBar()
```

Description

Finish drawing an error bar.

EndFill

```
virtual public void EndFill()
```

Description

Finish drawing a filled region.

EndImage

```
virtual public void EndImage()
```

Description

Finish drawing an image.

EndLine

```
virtual public void EndLine()
```

Description

Finish drawing lines.

EndMarker

```
virtual public void EndMarker()
```

Description

Finish drawing markers.

EndText

```
virtual public void EndText()
```

Description

Finish drawing text.

FillArc

```
virtual public void FillArc(int x, int y, int width, int height, int
startAngle, int arcAngle)
```

Description

Fills a circular or elliptical arc covering the specified rectangle.

Parameters

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the rectangular region that defines the ellipse from which the arc is drawn.

`y` – An `int` which specifies the y-coordinate of the upper-left corner of the rectangular region that defines the ellipse from which the arc is drawn.

`width` – An `int` which specifies the width of the rectangular region that defines the ellipse from which the arc is drawn.

`height` – An `int` which specifies the height of the rectangular region that defines the ellipse from which the arc is drawn.

`startAngle` – An `int` which specifies the starting angle of the arc, measured in degrees clockwise from the x-axis.

`arcAngle` – An `int` which specifies an angle in degrees measured counter-clockwise from the `startAngle` parameter to the ending point of the arc.

Remarks

The center of the arc is center of this rectangle.

`startAngle = 0` is equivalent to the 3-o'clock position.

`DrawArc` draws the arc from `startAngle` to `startAngle+arcAngle`. A positive `arcAngle` indicates a counter-clockwise rotation. A negative `arcAngle` implies a clockwise rotation.

FillPolygon

```
virtual public void FillPolygon(System.Drawing.Drawing2D.GraphicsPath polygon)
```

Description

Fill a polygon defined by a `Polygon` object.

Parameter

`polygon` – A `Polygon` object which specifies the polygon to be filled.

FillPolygon

```
virtual public void FillPolygon(int[] xpoints, int[] ypoints, int npoints)
```

Description

Fills a polygon.

Parameters

`xpoints` – An `int` array which contains the abscissae of the points which define the polygon.

`ypoints` – An `int` array which contains the ordinates of the points which define the polygon.

`npoints` – An `int` which specifies the number of pointsto add to the graphics path.

FillRectangle

```
virtual public void FillRectangle(int x, int y, int width, int height)
```

Description

Fill a rectangle.

Parameters

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the rectangle.

`y` – An `int` which specifies the y-coordinate of the upper-left corner of the rectangle.

`width` – An `int` which specifies the width of the rectangle.

`height` – An `int` which specifies the height of the rectangle.

GetStringWidth

```
static public int GetStringWidth(string target, System.Drawing.Font font)
```

Description

Gets the width of a string.

Parameters

`target` – A `string` to measure.

`font` – A `Font` object that defines the text format of the string.

Returns

An `int` that represents the size, in pixels, of the `string` specified by `target` as drawn with `font`.

Start

```
virtual public void Start(Imsl.Chart2D.Chart chart)
```

Description

Called just before a chart is drawn.

Parameter

`chart` – The `Chart` object to draw.

StartErrorBar

```
virtual public void StartErrorBar()
```

Description

Start drawing an `ErrorBar`.

StartFill

```
virtual public void StartFill()
```

Description

Start drawing a filled region.

StartImage

```
virtual public void StartImage()
```

Description

Start drawing an image.

StartLine

```
virtual public void StartLine()
```

Description

Start drawing a line.

StartMarker

```
virtual public void StartMarker()
```

Description

Start drawing a marker.

StartText

```
virtual public void StartText()
```

Description

Start drawing text.

Stop

```
virtual public void Stop()
```

Description

Called when a chart is finished being drawn.

Translate

```
virtual public void Translate(int x, int y)
```

Description

Prepends the specified translation to the transformation matrix of this Graphics object.

Parameters

x – An `int` which specifies *dx*, the x component of the translation.

y – An `int` which specifies *dy*, the y component of the translation.

FrameChart Class

```
public class Imsl.Chart2D.FrameChart :  Form :
System.ComponentModel.IComponent, System.IDisposable,
System.Windows.Forms.IDropTarget, System.ComponentModel.ISynchronizeInvoke,
System.Windows.Forms.IWin32Window, System.Windows.Forms.IBindableComponent,
System.Windows.Forms.IContainerControl
```

FrameChart is a Form that contains a chart.

Properties

Chart

```
virtual public Imsl.Chart2D.Chart Chart {get; set; }
```

Description

Specifies the chart to be handled.

Property Value

A `Chart` that specifies the chart to be displayed by this container.

Panel

```
virtual public Imsl.Chart2D.PanelChart Panel {get; }
```

Description

Specifies a Panel that contains the `Chart` to be drawn.

Property Value

A `PanelChart` into which the chart is to be drawn.

Constructors

FrameChart

```
public FrameChart()
```

Description

Creates new `FrameChart` to display a chart.

FrameChart

```
public FrameChart(Imsl.Chart2D.Chart chart)
```

Description

Creates new `FrameChart` to display a given chart.

Parameter

`chart` – A `Chart` containing the chart to be displayed.

Method

Dispose

```
override void Dispose(bool disposing)
```

Description

Clean up any resources being used.

Parameter

`disposing` – A `bool`indicating whether to release both managed and unmanaged resources.

Remarks

`true` to release both managed and unmanaged resources; `false` to release only unmanaged resources.

PanelChart Class

```
public class Imsl.Chart2D.PanelChart :  Panel :
System.ComponentModel.IComponent, System.IDisposable,
System.Windows.Forms.IDropTarget, System.ComponentModel.ISynchronizeInvoke,
System.Windows.Forms.IWin32Window, System.Windows.Forms.IBindableComponent
```

A Windows.Forms.Panel that contains a chart.

This class causes the contained chart to be redrawn as necessary.

Properties

Chart

```
virtual public Imsl.Chart2D.Chart Chart {get; set; }
```

Description

Specifies the Chart to be rendered for in this panel.

Property Value

A `Chart` to be handled by this container.

DefaultSize

```
override System.Drawing.Size DefaultSize {get; }
```

Description

The default size for a PanelChart is 200x200.

Constructors

PanelChart

```
public PanelChart()
```

Description

Creates a new `PanelChart`.

Remarks

This creates a new `Chart` object.

PanelChart

```
public PanelChart(Imsl.Chart2D.Chart chart)
```

Description

Creates new `PanelChart` using a given `Chart` object.

Parameter

`chart` – A `Chart` to be displayed in this panel.

Methods

OnPaint

```
override void OnPaint(System.Windows.Forms.PaintEventArgs painteventargs)
```

Description

Calls the UI delegate's `Paint` method, if the UI delegate is non-null.

Parameter

`painteventargs` – The `PaintEventArgs` with the `Graphics` property for painting the chart.

Remarks

We pass the delegate a copy of the `Graphics` object to protect the rest of the `Paint` code from irrevocable changes (for example, `Graphics.translate`).

If you override this in a subclass you should not make permanent changes to the passed in `Graphics`. For example, you should not alter the clip `Rectangle` or modify the transform. If you need to do these operations you may find it easier to create a new `Graphics` from the passed in `Graphics` and manipulate it.

Further, if you do not invoker super's implementation you must honor the opaque property, that is if this component is opaque, you must completely fill in the background in a non-opaque color. If you do not honor the opaque property you will likely see visual artifacts.

OnResize

```
override void OnResize(System.EventArgs eventargs)
```

Description

When the `PanelChart` is resized, `Refresh()` is called.

Parameter

`eventargs` – The `EventArgs`.

Print

```
virtual public void Print()
```

Description

Print the `Chart` centered on a page.

DrawPick Class

```
public class Imsl.Chart2D.DrawPick :  Draw
```

The DrawPick class.

Properties

Node

```
override public Imsl.Chart2D.ChartNode Node {set; }
```

Description

Specifies the current node of the chart tree.

Property Value

A `ChartNode` containing the current node of this tree.

Remarks

This is used to get drawing attributes from the tree.

Tolerance

```
virtual public int Tolerance {get; set; }
```

Description

The minimum distance that an event can be from a point or a line and still be considered a hit.

Property Value

An `int` containing the minimum distance that an event can be from a point or a line and still be considered a hit.

Constructor

DrawPick

```
public DrawPick(System.Windows.Forms.MouseEventArgs mouseEventArgs,
System.Drawing.Graphics graphics, System.Drawing.Size bounds)
```

Description

Contructs a `DrawPick` object.

Parameters

`mouseEventArgs` – A `MouseEvent` that provides data for the MouseUp, MouseDown, and MouseMove events.

`graphics` – A `Graphics` object encapsulating a GDI+ drawing surface.

`bounds` – A `Size` specifying the width and height of a rectangle.

Methods

DrawArc

```
override public void DrawArc(int x, int y, int width, int height, int
startAngle, int arcAngle)
```

Description

Draws the outline of a circular or elliptical arc covering the specified rectangle.

Parameters

`x` – An `int` which contains the x-coordinate of the upper-left corner of the rectangle that defines the ellipse.

`y` – An `int` which contains the y-coordinate of the upper-left corner of the rectangle that defines the ellipse.

`width` – An `int` which contains the width of the rectangle that defines the ellipse.

`height` – An `int` which contains the height of the rectangle that defines the ellipse.

`startAngle` – An `int` which specifies the angle in degrees measured clockwise from the x-axis to the starting point of the arc.

`arcAngle` – An `int` which specifies an angle in degrees measured counter-clockwise from the `startAngle` parameter to the ending point of the arc.

Remarks

The center of the arc is center of this rectangle.

`startAngle = 0` is equivalent to the 3-o'clock position.

`DrawArc` draws the arc from `startAngle` to `startAngle+arcAngle`. A positive `arcAngle` indicates a counter-clockwise rotation. A negative `arcAngle` implies a clockwise rotation.

DrawErrorBar

```
override public void DrawErrorBar(int x0, int y0, int x1, int y1, int flag)
```

Description

Draws an error bar.

Parameters

`x0` – An `int` which specifies the x-coordinate of the beginning reference point.

`y0` – An `int` which specifies the y-coordinate of the beginning reference point.

`x1` – An `int` which specifies the x-coordinate of the ending reference point.

`y1` – An `int` which specifies the y-coordinate of the ending reference point.

`flag` – An `int` that indicates which caps to draw.

DrawImage

```
override public void DrawImage(System.Drawing.Image image, int x, int y)
```

Description

Draws the specified image at the location specified by a coordinate pair.

Parameters

`image` – The `Image` object to draw.

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the drawn image.

`y` – An `int` which specifies the x-coordinate of the upper-left corner of the drawn image.

DrawLine

```
override public void DrawLine(int x0, int y0, int x1, int y1)
```

Description

Draws a line from between two points.

Parameters

`x0` – An `int` which specifies the x-coordinate of the line origin, (x0,y0).

`y0` – An `int` which specifies the y-coordinate of the line origin, (x0,y0).

`x1` – An `int` which specifies the x-coordinate of the line destination, (x1,y1).

`y1` – An `int` which specifies the y-coordinate of the line destination, (x1,y1).

DrawMarker

```
override public void DrawMarker(int x, int y)
```

Description

Draws a marker.

Parameters

`x` – An `int` which specifies the x-coordinate of the marker destination, (x,y).

`y` – An `int` which specifies the y-coordinate of the marker destination, (x,y).

DrawText

```
override public System.Drawing.Size DrawText(Imsl.Chart2D.Text text, int x, int
y)
```

Description

Draws a `Text` object.

Parameters

`text` – A `Text` object to be drawn.

`x` – An `int` which specifies the abscissa of the (x,y) point at which to start drawing the text.

`y` – An `int` which specifies the ordinate of the (x,y) point at which to start drawing the text.

Returns

A `Size` containing the bounds of the `Text` to be drawn.

EndErrorBar

```
override public void EndErrorBar()
```

Description

Finsih drawing an error bar.

EndFill

```
override public void EndFill()
```

Description

Finish drawing a filled region.

EndImage

```
override public void EndImage()
```

Description

Finsih drawing an image.

EndLine

```
override public void EndLine()
```

Description

Finish drawing lines.

EndMarker

```
override public void EndMarker()
```

Description

Finish drawing markers.

EndText

```
override public void EndText()
```

Description

Finish drawing text.

FillArc

```
override public void FillArc(int x, int y, int width, int height, int
startAngle, int arcAngle)
```

Description

Fills a circular or elliptical arc covering the specified rectangle.

Parameters

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the rectangular region that defines the ellipse from which the arc is drawn.

`y` – An `int` which specifies the y-coordinate of the upper-left corner of the rectangular region that defines the ellipse from which the arc is drawn.

`width` – An `int` which specifies the width of the rectangular region that defines the ellipse from which the arc is drawn.

`height` – An `int` which specifies the height of the rectangular region that defines the ellipse from which the arc is drawn.

`startAngle` – An `int` which specifies the starting angle of the arc, measured in degrees clockwise from the x-axis.

`arcAngle` – An `int` which specifies an angle in degrees measured counter-clockwise from the `startAngle` parameter to the ending point of the arc.

Remarks

The center of the arc is center of this rectangle.

`startAngle = 0` is equivalent to the 3-o'clock position.

`DrawArc` draws the arc from `startAngle` to `startAngle`+`arcAngle`. A positive `arcAngle` indicates a counter-clockwise rotation. A negative `arcAngle` implies a clockwise rotation.

FillPolygon

```
override public void FillPolygon(System.Drawing.Drawing2D.GraphicsPath polygon)
```

Description

Fill a polygon defined by a `Polygon` object.

Parameter

`polygon` – A `Polygon` object which specifies the polygon to be filled.

FillPolygon

```
override public void FillPolygon(int[] xpoints, int[] ypoints, int npoints)
```

Description

Fills a polygon.

Parameters

`xpoints` – An `int` array which contains the abscissae of the points which define the polygon.

`ypoints` – An `int` array which contains the ordinates of the points which define the polygon.

`npoints` – An `int` which specifies the number of pointsto add to the graphics path.

FillRectangle

```
override public void FillRectangle(int x, int y, int width, int height)
```

Description

Fill a rectangle.

Parameters

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the rectangle.

`y` – An `int` which specifies the y-coordinate of the upper-left corner of the rectangle.

`width` – An `int` which specifies the width of the rectangle.

`height` – An `int` which specifies the height of the rectangle.

Fire

```
virtual public void Fire()
```

Description

Invoke the delegates for all of the picked nodes.

StartErrorBar

```
override public void StartErrorBar()
```

Description

Start drawing an error bar.

StartFill

```
override public void StartFill()
```

Description

Start drawing a filled region.

StartImage

```
override public void StartImage()
```

Description

Start drawing an image.

StartLine

```
override public void StartLine()
```

Description

Start drawing lines.

StartMarker

```
override public void StartMarker()
```

Description

Start drawing markers.

StartText

```
override public void StartText()
```

Description

Start drawing text.

Translate

```
override public void Translate(int x, int y)
```

Description

Prepends the specified translation to the transformation matrix of this Graphics object.

Parameters

x – An `int` which specifies *dx*, the x component of the translation.

y – An `int` which specifies *dy*, the y component of the translation.

PickEventArgs Class

```
public class Imsl.Chart2D.PickEventArgs :  MouseEventArgs
```

An event that indicates that a chart element has been selected.

Provides data for the `PickPerformed` event.

See Also

(p. 1413)

Property

Node

```
virtual public Imsl.Chart2D.ChartNode Node {get; set; }
```

Description

The `ChartNode` associated with the pick event.

Property Value

A `ChartNode` which specifies the node associated with the pick event.

Constructor

PickEventArgs

```
public PickEventArgs(System.Windows.Forms.MouseEventArgs mouseEvent)
```

Description

Initializes a new instance of the `PickEventArgs` class.

Parameter

`mouseEvent` – A `MouseEventArgs` that provides data for the MouseUp, MouseDown, and MouseMove events.

Method

PointToLine

```
static public double PointToLine(int Px, int Py, int[] devA, int[] devB)
```

Description

Compute the distance from the point (`Px`, `Py`) to the line segment AB.

Parameters

`Px` – An `int` which specifies the x coordinate of the point (`Px`,`Py`).

`Py` – An `int` which specifies the y coordinate of the point (`Px`,`Py`).

`devA` – An `int[]` which specifies the point that defines the head of the line segment.

`devB` – An `int[]` which specifies the point that defines the tail of the line segment.

Returns

A `double` which contains the distance from the point (`Px`,`Py`) to the line segment AB.

Remarks

If the closest point from P to the line AB is not between A and B then the distance to the closer of A and B is returned.

PickEventHandler Delegate

```
public delegate Imsl.Chart2D.PickEventHandler
```

The delegate for receiving pick events. Represents the method that will handle the `PickPerformed` event of a chart node.

See Also

Imsl.Chart2D.PickEventArgs (p. 1411)

WebChart Class

```
public class Imsl.Chart2D.WebChart :  Panel :
System.ComponentModel.IComponent, System.IDisposable,
System.Web.UI.IParserAccessor, System.Web.UI.IUrlResolutionService,
System.Web.UI.IDataBindingsAccessor, System.Web.UI.IControlBuilderAccessor,
System.Web.UI.IControlDesignerAccessor, System.Web.UI.IExpressionsAccessor,
System.Web.UI.IAttributeAccessor
```

A `WebChart` provides a component to use in ASP.NET applications that holds a `Chart` object.

Property

Chart

```
public Imsl.Chart2D.Chart Chart {get; set; }
```

Description

The `Chart` object associated with this `WebChart`.

Property Value

A `Chart` where the drawing is to rendered.

Constructor

WebChart

```
public WebChart()
```

Description

Default constructor.

Methods

OnInit

```
override void OnInit(System.EventArgs e)
```

Description

Initializes the object.

Parameter

`e` – The `EventArgs` object that contains the event data.

Render

```
override void Render(System.Web.UI.HtmlTextWriter output)
```

Description

Renders the WebChart to the specified HTML writer.

Parameter

`output` – The `HtmlTextWriter` that receives the control content.

DrawMap Class

```
public class Imsl.Chart2D.DrawMap :  Draw
```

Creates an HTML client-side imagemap from a chart tree.

Entries in the imagemap correspond to nodes that define the HREF attribute.

Properties

Map

```
virtual public string Map {get; }
```

Description

Returns the body of the HTML imagemap.

Property Value

A `string` that contains the body of the HTML client-side imagemap. The actual <map> and </map> tags are not included, so that the client code can more easily add attributes to the <map> tag.

Node

```
override public Imsl.Chart2D.ChartNode Node {set; }
```

Description

Specifies the current node of the chart tree.

Property Value

A `ChartNode` containing the current node of this tree.

Remarks

This is used to get drawing attributes from the tree.

Tolerance

```
virtual public int Tolerance {get; set; }
```

Description

The minimum distance that an event can be from a point or a line and still be considered a hit.

Property Value

An `int` containing the minimum distance that an event can be from a point or a line and still be considered a hit.

Constructor

DrawMap

```
public DrawMap(System.Drawing.Graphics graphics, System.Drawing.Size bounds)
```

Description

Contructs a `DrawMap` object.

Parameters

`graphics` – A `Graphics` context in which to draw.

`bounds` – A `Size` object containing the width and height of the chart to be drawn.

Methods

DrawArc

```
override public void DrawArc(int x, int y, int width, int height, int
startAngle, int arcAngle)
```

Description

Draws the outline of a circular or elliptical arc covering the specified rectangle.

Parameters

`x` – An `int` which contains the x-coordinate of the upper-left corner of the rectangle that defines the ellipse.

`y` – An `int` which contains the y-coordinate of the upper-left corner of the rectangle that defines the ellipse.

`width` – An `int` which contains the width of the rectangle that defines the ellipse.

`height` – An `int` which contains the height of the rectangle that defines the ellipse.

`startAngle` – An `int` which specifies the angle in degrees measured clockwise from the x-axis to the starting point of the arc.

`arcAngle` – An `int` which specifies an angle in degrees measured counter-clockwise from the `startAngle` parameter to the ending point of the arc.

Remarks

The center of the arc is center of this rectangle.

`startAngle = 0` is equivalent to the 3-o'clock position.

`DrawArc` draws the arc from `startAngle` to `startAngle+arcAngle`. A positive `arcAngle` indicates a counter-clockwise rotation. A negative `arcAngle` implies a clockwise rotation.

DrawErrorBar

```
override public void DrawErrorBar(int x0, int y0, int x1, int y1, int flag)
```

Description

Draws an error bar.

Parameters

`x0` – An `int` which specifies the x-coordinate of the beginning reference point.

`y0` – An `int` which specifies the y-coordinate of the beginning reference point.

`x1` – An `int` which specifies the x-coordinate of the ending reference point.

`y1` – An `int` which specifies the y-coordinate of the ending reference point.

`flag` – An `int` that indicates which caps to draw.

Remarks

Legal values: 0=none, 1=bottom, 2=top, 3=both

DrawImage

```
override public void DrawImage(System.Drawing.Image image, int x, int y)
```

Description

Draws the specified image at the location specified by a coordinate pair.

Parameters

`image` – The `Image` object to draw.

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the drawn image.

`y` – An `int` which specifies the x-coordinate of the upper-left corner of the drawn image.

DrawLine

```
override public void DrawLine(int x0, int y0, int x1, int y1)
```

Description

Draws a line from between two points.

Parameters

`x0` – An `int` which specifies the x-coordinate of the line origin, (x0,y0).

`y0` – An `int` which specifies the y-coordinate of the line origin, (x0,y0).

`x1` – An `int` which specifies the x-coordinate of the line destination, (x1,y1).

`y1` – An `int` which specifies the y-coordinate of the line destination, (x1,y1).

DrawMarker

```
override public void DrawMarker(int x, int y)
```

Description

Draws a marker.

Parameters

`x` – An `int` which specifies the x-coordinate of the marker destination, (x,y).

`y` – An `int` which specifies the y-coordinate of the marker destination, (x,y).

EndErrorBar

```
override public void EndErrorBar()
```

Description

Finish drawing an error bar.

EndFill

```
override public void EndFill()
```

Description

Finish drawing a filled region.

EndImage

```
override public void EndImage()
```

Description

Finish drawing an image.

EndLine

```
override public void EndLine()
```

Description

Finish drawing lines.

EndMarker

```
override public void EndMarker()
```

Description

Finish drawing markers.

EndText

```
override public void EndText()
```

Description

Finsih drawing text.

FillArc

```
override public void FillArc(int x, int y, int width, int height, int
startAngle, int arcAngle)
```

Description

Fills a circular or elliptical arc covering the specified rectangle.

Parameters

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the rectangular region that defines the ellipse from which the arc is drawn.

`y` – An `int` which specifies the y-coordinate of the upper-left corner of the rectangular region that defines the ellipse from which the arc is drawn.

`width` – An `int` which specifies the width of the rectangular region that defines the ellipse from which the arc is drawn.

`height` – An `int` which specifies the height of the rectangular region that defines the ellipse from which the arc is drawn.

`startAngle` – An `int` which specifies the starting angle of the arc, measured in degrees clockwise from the x-axis.

`arcAngle` – An `int` which specifies an angle in degrees measured counter-clockwise from the `startAngle` parameter to the ending point of the arc.

Remarks

The center of the arc is center of this rectangle.

`startAngle = 0` is equivalent to the 3-o'clock position.

`DrawArc` draws the arc from `startAngle` to `startAngle+arcAngle`. A positive `arcAngle` indicates a counter-clockwise rotation. A negative `arcAngle` implies a clockwise rotation.

FillPolygon

```
override public void FillPolygon(System.Drawing.Drawing2D.GraphicsPath polygon)
```

Description

Fill a polygon defined by a `Polygon` object.

Parameter

`polygon` – A `Polygon` object which specifies the polygon to be filled.

FillPolygon

```
override public void FillPolygon(int[] xpoints, int[] ypoints, int npoints)
```

Description

Fills a polygon.

Parameters

`xpoints` – An `int` array which contains the abscissae of the points which define the polygon.

`ypoints` – An `int` array which contains the ordinates of the points which define the polygon.

`npoints` – An `int` which specifies the number of pointsto add to the graphics path.

FillRectangle

```
override public void FillRectangle(int x, int y, int width, int height)
```

Description

Fill a rectangle.

Parameters

`x` – An `int` which specifies the x-coordinate of the upper-left corner of the rectangle.

`y` – An `int` which specifies the y-coordinate of the upper-left corner of the rectangle.

`width` – An `int` which specifies the width of the rectangle.

`height` – An `int` which specifies the height of the rectangle.

StartErrorBar

```
override public void StartErrorBar()
```

Description

Start drawing an error bar.

StartFill

```
override public void StartFill()
```

Description

Start drawing a filled region.

StartImage

```
override public void StartImage()
```

Description

Start drawing an image.

StartLine

```
override public void StartLine()
```

Description

Start drawing lines.

StartMarker

```
override public void StartMarker()
```

Description

Start drawing markers.

StartText

```
override public void StartText()
```

Description

Start drawing text.

Translate

```
override public void Translate(int x, int y)
```

Description

Prepends the specified translation to the transformation matrix of this Graphics object.

Parameters

x – An `int` which specifies *dx*, the x component of the translation.

y – An `int` which specifies *dy*, the y component of the translation.

BoxPlot Class

```
public class Imsl.Chart2D.BoxPlot :  Data
```

Draws a multiple-group Box plot.

For each group of observations, the box limits represent the lower quartile (25th percentile) and upper quartile (75th percentile). The median is displayed as a line across the box. Whiskers are drawn from the upper quartile to the upper adjacent value, and from the lower quartile to the lower adjacent value.

Optional notches may be displayed to show a 95 percent confidence interval about the median, at $\pm 1.58\ IRQ\ /\sqrt{n}$, where *IRQ* is the interquartile range and *n* is the number of observations. Outside and far outside values may be displayed as symbols. Outside values are outside the inner fence. Far out values are outside the outer fence.

The `BoxPlot` has several child nodes. Any of these nodes can be disabled by setting their "IsVisible" attribute to `false`.

- The "Bodies" attribute has the main body of the box plot elements. Its fill attributes determine the drawing of (notched) rectangle. Its line attributes determine the drawing of the median line. The width of the box is controlled by the "MarkerSize" attribute.
- The "Whiskers" attribute draws the lines to the upper and lower quartile. Its drawing is affected by the marker attributes.
- The "FarMarkers" attribute hold the far markers. Its drawing is affected by the marker attributes.
- The "OutsideMarkers" attribute hold the outside markers. Its drawing is affected by the marker attributes.

Fields

BOXPLOT_TYPE_HORIZONTAL

```
public int BOXPLOT_TYPE_HORIZONTAL
```

Description

Value for attribute "BoxPlotType" indicating that this is a horizontal box plot.

Remarks

Used in connection with `BoxPlot` nodes.

BOXPLOT_TYPE_VERTICAL

```
public int BOXPLOT_TYPE_VERTICAL
```

Description

Value for attribute "BoxPlotType" indicating that this is a horizontal box plot.

Remarks

Used in connection with `BoxPlot` nodes.

Properties

Bodies

```
virtual public Imsl.Chart2D.ChartNode Bodies {get; }
```

Description

The main body of the `BoxPlot` elements.

Property Value

A `ChartNode` which contains the “Bodies” attribute value.

BoxPlotType

```
virtual public int BoxPlotType {get; set; }
```

Description

Specifies the orientation of the `BoxPlot`.

Property Value

An `int` which contains the “BoxPlotType” attribute value.

Remarks

Legal values are Imsl.Chart2D.BoxPlot.BOXPLOT_TYPE_VERTICAL (p. 1421) or Imsl.Chart2D.BoxPlot.BOXPLOT_TYPE_HORIZONTAL (p. 1421).

FarMarkers

```
virtual public Imsl.Chart2D.ChartNode FarMarkers {get; }
```

Description

The far markers of the `BoxPlot` elements.

Property Value

A `ChartNode` which contains the “FarMarkers” attribute value.

Notch

```
virtual public bool Notch {get; set; }
```

Description

Specifies whether the optional notches, indicating the extent of data falling within the 95 percet confidence range, are displayed.

Property Value

A `bool` which contains the “Notch” attribute value.

Remarks

`true` indicates that notches are to be displayed. default: `false`

OutsideMarkers

```
virtual public Imsl.Chart2D.ChartNode OutsideMarkers {get; }
```

Description

The outside markers of the `BoxPlot` elements.

Property Value

A `ChartNode` which contains the “OutsideMarkers” attribute value.

ProportionalWidth

```
virtual public bool ProportionalWidth {get; set; }
```

Description

Specifies whether the box widths are to be proportional.

Property Value

A `bool` which contains the “ProportionalWidth” attribute value.

Remarks

`true` indicates the box widths are to be proportional to the square root of the number of observations. If `false` all of the boxes have the same width. Default: `false`

Whiskers

```
virtual public Imsl.Chart2D.ChartNode Whiskers {get; }
```

Description

The wiskers of the `BoxPlot` elements drawn to the upper and lower quartile.

Property Value

A `ChartNode` which contains the “Whiskers” attribute value.

Constructors

BoxPlot

```
public BoxPlot(Imsl.Chart2D.AxisXY axis, double[] x, double[][] obs)
```

Description

Constructs a box plot chart node with specified x values.

Parameters

`axis` – An `AxisXY` which is the parent of this node.

`x` – A `double[]` which contains the x values.

`obs` – A `double[]` which contains the observations for each `x`.

Remarks

The number of rows in `obs` must equal the length of `x`. The length of each row in `obs` must be at least 4.

BoxPlot

```
public BoxPlot(Imsl.Chart2D.AxisXY axis, double[] x,
Imsl.Chart2D.BoxPlot.Statistics[] statistics)
```

Description

Constructs a box plot chart node with specified x values.

Parameters

`axis` – An `AxisXY` which is the parent of this node.

`x` – a `double[]` which contains the x values.

`statistics` – A `BoxPlot.Statistics[]` containing the statistics for each element in `x`.

Remarks

The number of `BoxPlot.Statistics[]` must equal `x.Length`.

BoxPlot

```
public BoxPlot(Imsl.Chart2D.AxisXY axis, double[][] obs)
```

Description

Constructs a box plot chart.

Parameters

`axis` – An `AxisXY` which is the parent of this node.

`obs` – A `double[]` containing the observations.

Remarks

The length of each row in `obs` must be at least 4.

Methods

GetStatistics

```
virtual public Imsl.Chart2D.BoxPlot.Statistics[] GetStatistics()
```

Description

Returns statistics for each set of observations.

Returns

A `BoxPlot.Statistics[]` containing the statistics for each set of observations.

GetStatistics

```
virtual public Imsl.Chart2D.BoxPlot.Statistics GetStatistics(int iSet)
```

Description

Returns statistics for a set of observations.

Parameter

`iSet` – An `int` which specifies the index of a set whose statistics are to be returned.

Returns

A `BoxPlot.Statistics` containing the statistics for the `iSet` set of observations.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetLabels

```
virtual public void SetLabels(string[] labels)
```

Description

Sets up an axis with labels.

Parameter

`labels` – A `String[]` containing the axis labels.

Remarks

Sets up an axis with labels. This turns off the tick marks and sets the "BoxPlotType" attribute. It also turns off autoscaling for the axis and sets its "Window" and "Number" and "Ticks" attribute as appropriate for a labeled Box plot. The existing value of the "BoxPlotType" attribute is used to determine the axis to be modified.

SetLabels

```
virtual public void SetLabels(string[] labels, int type)
```

Description

Sets up an axis with labels.

Parameters

`labels` – A `String[]` containing the axis labels.

`type` – An `int` which specifies the “BoxPlotType” attribute value.

Remarks

This turns off the tick marks and sets the “BoxPlotType” attribute. It also turns off autoscaling for the axis and sets its “Window”, “Number” and “Ticks” attributes as appropriate for a labeled Box plot.

The number of labels must equal the number of items.

Legal values for `type` are Imsl.Chart2D.BoxPlot.BOXPLOT_TYPE_VERTICAL (p. 1421) or Imsl.Chart2D.BoxPlot.BOXPLOT_TYPE_HORIZONTAL (p. 1421). This determines the axis to be modified.

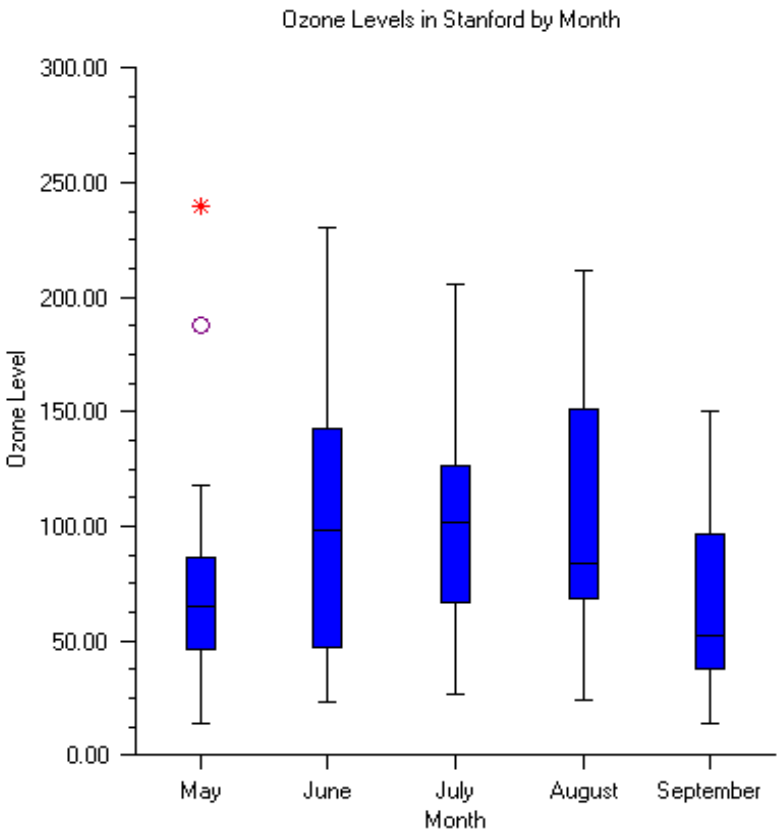
Example: Box Plot Chart

A simple box plot chart is constructed in this example. Display of far and outside values is turned on.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class BoxPlotEx1 : FrameChart
{
    public BoxPlotEx1()
    {
        Chart chart = this.Chart;

        double[][] obs = {new double[]{66.0, 52.0, 49.0, 64.0, 68.0, 26.0, 86.0,
                                       52.0, 43.0, 75.0, 87.0, 188.0, 118.0,
                                       103.0, 82.0, 71.0, 103.0, 240.0, 31.0,
                                       40.0, 47.0, 51.0, 31.0, 47.0, 14.0,
                                       71.0},

                                  new double[]{61.0, 47.0, 196.0, 131.0, 173.0, 37.0, 47.0,
                                          215.0, 230.0, 69.0, 98.0, 125.0, 94.0,
                                          72.0, 72.0, 125.0, 143.0, 192.0, 122.0,
                                          32.0, 114.0, 32.0, 23.0, 71.0, 38.0,
                                          136.0, 169.0},

                                  new double[]{152.0, 201.0, 134.0, 206.0, 92.0, 101.0,
                                          119.0, 124.0, 133.0, 83.0, 60.0, 124.0,
                                          142.0, 124.0, 64.0, 75.0, 103.0, 46.0,
                                          68.0, 87.0, 27.0, 73.0, 59.0, 119.0, 64.0,
                                          111.0},

                                  new double[]{80.0, 68.0, 24.0, 24.0, 82.0, 100.0, 55.0,
                                          91.0, 87.0, 64.0, 170.0, 86.0, 202.0,
                                          71.0, 85.0, 122.0, 155.0, 80.0, 71.0,
                                          28.0, 212.0, 80.0, 24.0, 80.0, 169.0,
```

```
                                    174.0, 141.0, 202.0},

                                new double[]{113.0, 38.0, 38.0, 28.0, 52.0, 14.0, 38.0,
                                    94.0, 89.0, 99.0, 150.0, 146.0, 113.0,
                                    38.0, 66.0, 38.0, 80.0, 80.0, 99.0, 71.0,
                                    42.0, 52.0, 33.0, 38.0, 24.0, 61.0,
                                    108.0, 38.0, 28.0}};

        double[] x = new double[]{1.0, 2.0, 3.0, 4.0, 5.0};
        System.String[] xLabels = new System.String[]{"May", "June", "July",
                                                      "August", "September"};

        //   Create an instance of a BoxPlot Chart
        AxisXY axis = new AxisXY(chart);
        BoxPlot boxPlot = new BoxPlot(axis, obs);
        boxPlot.SetLabels(xLabels);

        // Customize the fill color and the outside and far markers
        boxPlot.Bodies.FillColor = System.Drawing.Color.FromName("blue");
        boxPlot.OutsideMarkers.MarkerType =
         Imsl.Chart2D.BoxPlot.MARKER_TYPE_HOLLOW_CIRCLE;
        boxPlot.OutsideMarkers.MarkerColor =
         System.Drawing.Color.FromName("purple");
        boxPlot.FarMarkers.MarkerType =
         Imsl.Chart2D.BoxPlot.MARKER_TYPE_ASTERISK;
        boxPlot.FarMarkers.MarkerColor =
         System.Drawing.Color.FromName("red");

        // Set titles
        chart.ChartTitle.SetTitle("Ozone Levels in Stanford by Month");
        axis.AxisX.AxisTitle.SetTitle("Month");
        axis.AxisY.AxisTitle.SetTitle("Ozone Level");

    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new BoxPlotEx1());
    }
}
```

Output



BoxPlot.Statistics Class

`public class Imsl.Chart2D.BoxPlot.Statistics`

Computes the statistics for one set of observations in a `Boxplot`.

Properties

LowerAdjacentValue

```
virtual public double LowerAdjacentValue {get; }
```

Description

A `double` which contains the lower adjacent value.

LowerQuartile

```
virtual public double LowerQuartile {get; }
```

Description

A `double` which contains the lower quartile value (25th percentile).

MaximumValue

```
virtual public double MaximumValue {get; }
```

Description

A `double` which contains the maximum value of this set.

Median

```
virtual public double Median {get; }
```

Description

A `double` which contains the median value for a set of observations.

MedianLowerConfidenceInterval

```
virtual public double MedianLowerConfidenceInterval {get; }
```

Description

A `double` which contains the lower confidence interval for the median value of this set of observations.

MedianUpperConfidenceInterval

```
virtual public double MedianUpperConfidenceInterval {get; }
```

Description

A `double` which contains the upper confidence interval for the median value of this set of observations.

MinimumValue

```
virtual public double MinimumValue {get; }
```

Description

A `double` which contains the minimum value of this set.

NumberObservations

```
virtual public int NumberObservations {get; }
```

Description

An `int` which contains the number of observations in this set.

UpperAdjacentValue

```
virtual public double UpperAdjacentValue {get; }
```

Description

A `double` which contains the upper adjacent value.

UpperQuartile

```
virtual public double UpperQuartile {get; }
```

Description

A `double` which contains the upper quartile value (75th percentile).

Constructor

Statistics

```
public Statistics(double[] obs)
```

Description

Creates a new instance of `BoxPlot.Statistics`.

Parameter

`obs` – A `double[]` containing the set of observations.

Remarks

There must be at least 4 observations to compute the statistics.

Exception

`System.ArgumentException` is thrown if there are fewer than 4 observations.

Methods

GetFarMarkers

```
virtual public double[] GetFarMarkers()
```

Description

Returns the far markers.

Returns

A `double[]` which contains the far markers for this set.

GetOutsideMarkers

```
virtual public double[] GetOutsideMarkers()
```

Description

Returns the outside markers.

Returns

A `double[]` which contains the outside markers for this set.

Contour Class

```
public class Imsl.Chart2D.Contour :  Data
```

A `Contour` chart shows level curves of a two-dimensional function.

The function can be defined either as values on a rectangular grid or by scattered data points.

A set of ContourLevel (p. 1445) objects are created as children of this node. The number of `ContourLevels` is one more than the number of level curves. If the level curve values are $c_0, \ldots, c_{n-1}$ then the k-th `ContourLevel` child corresponds to $c_{k-1} < z \leq c_k$.

To change the look of the contour chart, change the line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) and fill attributes (specified with FillType (p. 1322) and FillColor (p. 1301))in the `ContourLevel` nodes.

A `Legend` object is also created as a child of this node. It should be used instead of the usual chart legend. By default, this legend is not shown. To show it, set `IsVisible = true`.

See Also

Imsl.Chart2D.ContourLevel (p. 1445)

Property

ContourLegend

```
virtual public Imsl.Chart2D.Contour.Legend ContourLegend {get; }
```

Description

Contains the legend information associated with this `Contour`.

Property Value

A `Legend` object for this `Contour`.

Remarks

By default, the legend is not drawn because `IsVisible` is set to `false`. To show the legend set `IsVisible = true`, i.e., `contour.ContourLegend.IsVisible = true;`

Constructors

Contour

```
public Contour(Imsl.Chart2D.AxisXY axis, double[] xGrid, double[] yGrid,
double[,] zData, double[] cLevel)
```

Description

Creates a `Contour` chart from rectangularly gridded data.

Parameters

`axis` – An `AxisXY` containing the parent node of this `Contour`.

`xGrid` – A `double[]` which contains the x-coordinate values of the grid.

`yGrid` – A `double[]` which contains the y-coordinate values of the grid.

`zData` – A `double[,]` which contains the function values to be contoured.

`cLevel` – A `double[]` which contains the values of the contour levels.

Remarks

The value of the function at (`xGrid[i]`,`yGrid[j]`) is given by `zData[i][j]`. The size of `zData` must be `xGrid.Length` by `yGrid.Length`.

Contour

```
public Contour(Imsl.Chart2D.AxisXY axis, double[] xGrid, double[] yGrid,
double[,] zData)
```

Description

Creates a `Contour` chart from rectangularly gridded data with computed contour levels.

Parameters

`axis` – An `AxisXY` containing the parent node of this `Contour`.

`xGrid` – A `double[]` which contains the x-coordinate values of the grid.

`yGrid` – A `double[]` which contains the y-coordinate values of the grid.

`zData` – A `double[,]` which contains the function values to be contoured.

Remarks

The contour levels are chosen to span the data and to be “nice” values. The value of the function at (`xGrid[i]`, `yGrid[j]`) is given by `zData[i][j]`. The size of `zData` must be `xGrid.Length` by `yGrid.Length`.

Contour

```
public Contour(Imsl.Chart2D.AxisXY axis, double[] x, double[] y, double[] z)
```

Description

Creates a `Contour` chart from scattered data with computed contour levels.

Parameters

`axis` – An `AxisXY` containing the parent node of this `Contour`.

`x` – A `double[]` which contains the x-values of the data points.

`y` – A `double[]` which contains the y-values of the data points.

`z` – A `double[]` which contains the x-values of the data points.

Remarks

The contour chart is created by using a radial basis approximation to estimate the functions value on a rectangular grid. The contour chart is then computed as for gridded data.

See Also: Imsl.Math.RadialBasis (p. 195)

Contour

```
public Contour(Imsl.Chart2D.AxisXY axis, double[] x, double[] y, double[] z,
double[] cLevel, int nCenters)
```

Description

Creates a `Contour` chart from scattered data with computed contour levels.

Parameters

`axis` – An `AxisXY` containing the parent node of this `Contour`.

`x` – A `double[]` which contains the x-values of the data points.

`y` – A `double[]` which contains the y-values of the data points.

`z` – A `double[]` which contains the x-values of the data points.

`cLevel` – A `double[]` which contains the values of the contour levels.

`nCenters` – An `int` specifying the number of centers to use for the radial basis approximation.

Remarks

The contour chart is created by using a radial basis approximation to estimate the functions value on a rectangular grid. The contour chart is then computed as for gridded data.

A larger number of centers will provide a closer, but noiser approximation.

See Also: Imsl.Math.RadialBasis (p. 195)

Methods

GetContourLevel

```
virtual public Imsl.Chart2D.ContourLevel[] GetContourLevel()
```

Description

Returns all of the contour levels.

Returns

A `ContourLevel[]` containing the “ContourLevel” attribute value.

GetContourLevel

```
virtual public Imsl.Chart2D.ContourLevel GetContourLevel(int k)
```

Description

Returns a specified `ContourLevel`.

Parameter

`k` – An `int` which indicates what `ContourLevel` to return.

Returns

A `ContourLevel` that corrisponds to the `k`-th level (`cLevel[k]`).

Remarks

The `k`-th contour level contains the level curve equal to `cLevel[k]` in the constructor. It also contains the fill areas for the values in the interval (`cLevel[k-1]`, `cLevel[k]`).

The first contour level (`k`=0) contains the fill area for values less than `cLevel[0]` and the level curves lines where the function value equals `cLevel[0]`.

The last contour level (`k=cLevel.Length`) contains the fill area for values greater than `cLevel[cLevel.length-1]`, but no level curve lines.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

Example: Contour Chart from Gridded Data

In the restricted three-body problem, two large objects (masses M_1 and M_2) a distance a apart, undergoing mutual gravitational attraction, circle a common center-of-mass. A third small object (mass m) is assumed to move in the same plane as M_1 and M_2 and is assumed to be two small to affect the large bodies. For simplicity, we use a coordinate system that has the center of mass at the origin. M_1 and M_2 are on the x-axis at x_1 and x_2, respectively.

In the center-of-mass coordinate system, the effective potential energy of the system is given by

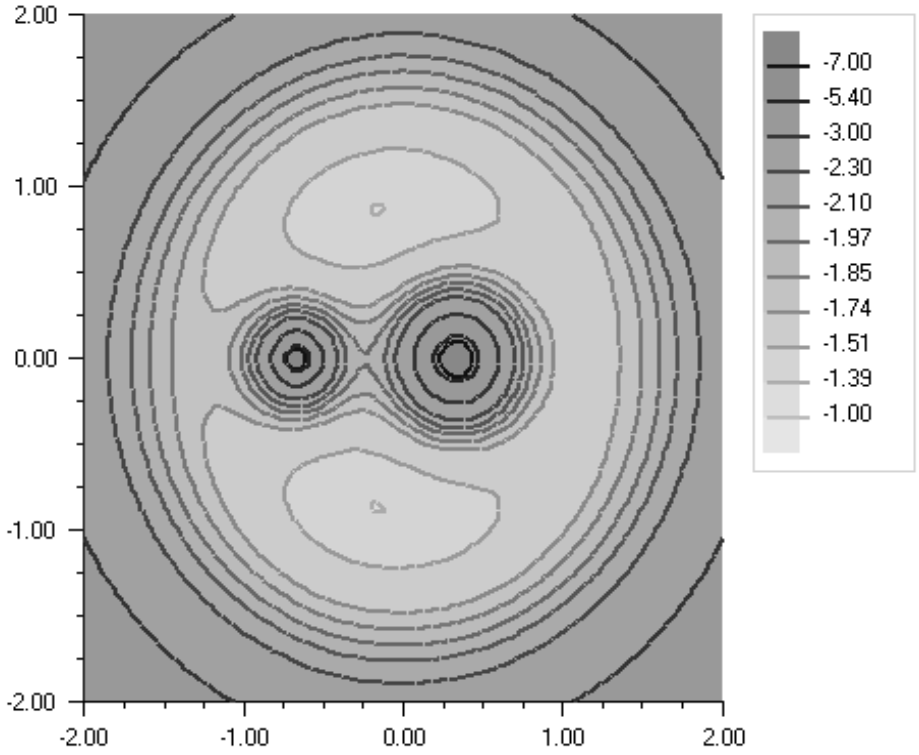
$$V = \frac{m(M_1+M_2)G}{a}\left[\frac{x_2}{\sqrt{(x-x_1)^2+y^2}} - \frac{x_1}{\sqrt{(x-x_2)^2+y^2}} - \frac{1}{2}\left(x^2+y^2\right)\right]$$

The universal gravitational constant is G. The following program plots the part of *V(x,y)* inside of the square bracket. The factor $\frac{m(M_1+M_2)G}{a}$ is ignored because it just scales the plot.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class ContourEx1 : FrameChart
{
    public ContourEx1()
    {
        Chart chart = this.Chart;

        int nx = 80;
        int ny = 80;

        // Allocate space
        double[] xGrid = new double[nx];
        double[] yGrid = new double[ny];
        double[,] zData = new double[nx,ny];

        // Setup the grids points
        for (int i = 0; i < nx; i++)
        {
            xGrid[i] = - 2 + 4.0 * i / (double) (nx - 1);
        }
        for (int j = 0; j < ny; j++)
```

```
        {
            yGrid[j] = - 2 + 4.0 * j / (double) (ny - 1);
        }

        // Evaluate the function at the grid points
        for (int i = 0; i < nx; i++)
        {
            for (int j = 0; j < ny; j++)
            {
                double x = xGrid[i];
                double y = yGrid[j];
                double rm = 0.5;
                double x1 = rm / (1.0 + rm);
                double x2 = x1 - 1.0;
                double d1 = System.Math.Sqrt((x - x1) * (x - x1) + y * y);
                double d2 = System.Math.Sqrt((x - x2) * (x - x2) + y * y);
                zData[i,j] = x2 / d1 - x1 / d2 - 0.5 * (x * x + y * y);
            }
        }

        // Create the contour chart, with user-specified levels and a legend
        AxisXY axis = new AxisXY(chart);
        double[] cLevel = new double[]{-7, -5.4, -3, -2.3, -2.1, -1.97, -1.85,
                                       -1.74, -1.51, -1.39, -1};
        Contour c = new Contour(axis, xGrid, yGrid, zData, cLevel);
        c.ContourLegend.IsVisible = true;

    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new ContourEx1());
    }
}
```

Output



Example: Contour Chart from Scattered Data

In this example, a contour chart is created from 150, randomly choosen, scattered data points. The function is $\sqrt{x^2+y^2}$, so the level curve should be circles.

The input data is shown on top of the contours as small green circles. The chart data nodes are drawn in the order in which they are added, so the input data marker node has to be added to the axis after the contour, so that the markers are not hidden.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class ContourEx2 : FrameChart
{
    public ContourEx2()
```

```
{
    Chart chart = this.Chart;

    int n = 150;

    // Allocate space
    double[] x = new double[n];
    double[] y = new double[n];
    double[] z = new double[n];

    System.Random random = new System.Random((System.Int32) 123457);

    double[] randomValue=new double[150];
    randomValue[0]=0.41312962995625035;
    randomValue[1]=0.8225528716547005;
    randomValue[2]=0.44364905186692527;
    randomValue[3]=0.9887088342522812;
    randomValue[4]=0.9647868112234352;
    randomValue[5]=0.5668831243079411;
    randomValue[6]=0.27386697614898103;
    randomValue[7]=0.8805853693809824;
    randomValue[8]=0.7180829622748057;
    randomValue[9]=0.6153607537410654;
    randomValue[10]=0.3158193853638753;
    randomValue[11]=0.10778543304578747;
    randomValue[12]=0.09275375134615693;
    randomValue[13]=0.9817642781628322;
    randomValue[14]=0.467363186309925;
    randomValue[15]=0.9066980293517674;
    randomValue[16]=0.31440695305815347;
    randomValue[17]=0.9991560762956562;
    randomValue[18]=0.785150345014761;
    randomValue[19]=0.7930129038729785;
    randomValue[20]=0.5695413465811706;
    randomValue[21]=0.7625752595574732;
    randomValue[22]=0.0482465474704169;
    randomValue[23]=0.09904819350827354;
    randomValue[24]=0.7013979421419555;
    randomValue[25]=0.8127581377189425;
    randomValue[26]=0.2160980302718407;
    randomValue[27]=0.2618716012466812;
    randomValue[28]=0.966175212476057;
    randomValue[29]=0.8929180151759015;
    randomValue[30]=0.9253777827882632;
    randomValue[31]=0.3192464623158826;
    randomValue[32]=0.6191390558809441;
    randomValue[33]=0.860615090126798;
    randomValue[34]=0.4202423262221493;
    randomValue[35]=0.3204335652731257;
    randomValue[36]=0.3501592792324697;
    randomValue[37]=0.08674811183862785;
    randomValue[38]=0.5605305915601296;
    randomValue[39]=0.6088802062708134;
    randomValue[40]=0.8382035138841133;
    randomValue[41]=0.9236987545556213;
    randomValue[42]=0.8024356174828979;
```

```
randomValue[43]=0.18382779454152387;
randomValue[44]=0.9443198089192774;
randomValue[45]=0.07466011736504485;
randomValue[46]=0.2961809553169247;
randomValue[47]=0.597869137157411;
randomValue[48]=0.3126393883707773;
randomValue[49]=0.9461805842458413;
randomValue[50]=0.4952325691501952;
randomValue[51]=0.0974865497453884;
randomValue[52]=0.39893060081096055;
randomValue[53]=0.31595422264648054;
randomValue[54]=0.9215776190059227;
randomValue[55]=0.963602405500786;
randomValue[56]=0.1962353914644036;
randomValue[57]=0.897888992070645;
randomValue[58]=0.9816014888911522;
randomValue[59]=0.2591728892012697;
randomValue[60]=0.177119526412298;
randomValue[61]=0.6364841570839579;
randomValue[62]=0.9770940229311096;
randomValue[63]=0.44085669522358406;
randomValue[64]=0.22206796609570068;
randomValue[65]=0.8125478558454153;
randomValue[66]=0.7059166517811799;
randomValue[67]=0.5417895331224579;
randomValue[68]=0.5535562377071471;
randomValue[69]=0.2922863750389211;
randomValue[70]=0.2968612011640126;
randomValue[71]=0.882495829596943;
randomValue[72]=0.9453297028667043;
randomValue[73]=0.5017962685731009;
randomValue[74]=0.17323198276725293;
randomValue[75]=0.516968989592425;
randomValue[76]=0.7264211901923515;
randomValue[77]=0.9589904164393783;
randomValue[78]=0.2896822052185578;
randomValue[79]=0.8709512849886136;
randomValue[80]=0.3494389711171513;
randomValue[81]=0.444989615581906;
randomValue[82]=0.03683604460307233;
randomValue[83]=0.2794447857758138;
randomValue[84]=0.5426558540369049;
randomValue[85]=0.14701055330017276;
randomValue[86]=0.45822765810918564;
randomValue[87]=0.3804843649168811;
randomValue[88]=0.31543075674256227;
randomValue[89]=0.35478179229078655;
randomValue[90]=0.6740882045962612;
randomValue[91]=0.5722042439512296;
randomValue[92]=0.336494210223919;
randomValue[93]=0.5425187147067986;
randomValue[94]=0.6565124760451249;
randomValue[95]=0.9902292520993252;
randomValue[96]=0.4546287589180955;
randomValue[97]=0.9184888233730713;
randomValue[98]=0.7505359876181693;
```

```
randomValue[99]=0.7124220647583559;
randomValue[100]=0.3812755838294607;
randomValue[101]=0.7741986381086996;
randomValue[102]=0.5856540334323093;
randomValue[103]=0.1480175568946106;
randomValue[104]=0.8045988425857213;
randomValue[105]=0.21523348843743784;
randomValue[106]=0.2723138761466122;
randomValue[107]=0.8181756787842892;
randomValue[108]=0.45453852386561255;
randomValue[109]=0.10578123947146922;
randomValue[110]=0.027911361401003143;
randomValue[111]=0.9849840119600158;
randomValue[112]=0.8883835561320729;
randomValue[113]=0.30887148321746527;
randomValue[114]=0.6268231326584466;
randomValue[115]=0.8359413755618763;
randomValue[116]=0.01639605006272593;
randomValue[117]=0.5543612693431772;
randomValue[118]=0.3190057747399081;
randomValue[119]=0.18095345468573598;
randomValue[120]=0.6370180793354232;
randomValue[121]=0.5166986319820245;
randomValue[122]=0.11169309885740164;
randomValue[123]=0.8688720220933366;
randomValue[124]=0.5011922442391221;
randomValue[125]=0.9344952771865647;
randomValue[126]=0.5587227111699117;
randomValue[127]=0.3806089260426023;
randomValue[128]=0.6753272961079825;
randomValue[129]=0.8539394715414731;
randomValue[130]=0.4520234874494251;
randomValue[131]=0.3058558270067878;
randomValue[132]=0.2224399403890832;
randomValue[133]=0.3280806679102708;
randomValue[134]=0.05979465629761105;
randomValue[135]=0.660441325427476;
randomValue[136]=0.4710041931991943;
randomValue[137]=0.15401687157352573;
randomValue[138]=0.8059082103579294;
randomValue[139]=0.25135648562180013;
randomValue[140]=0.3910396401490016;
randomValue[141]=0.48001615607289505;
randomValue[142]=0.5350655938328643;
randomValue[143]=0.5464799882069644;
randomValue[144]=0.8469694582001581;
randomValue[145]=0.3646033096669923;
randomValue[146]=0.7582401994865531;
randomValue[147]=0.7560344451536601;
randomValue[148]=0.7467799442143332;
randomValue[149]=0.619643401693058;

double[] randomValueY=new double[150];
randomValueY[0]=0.15995876895053263;
randomValueY[1]=0.48794367683379836;
randomValueY[2]=0.20896329070872555;
```

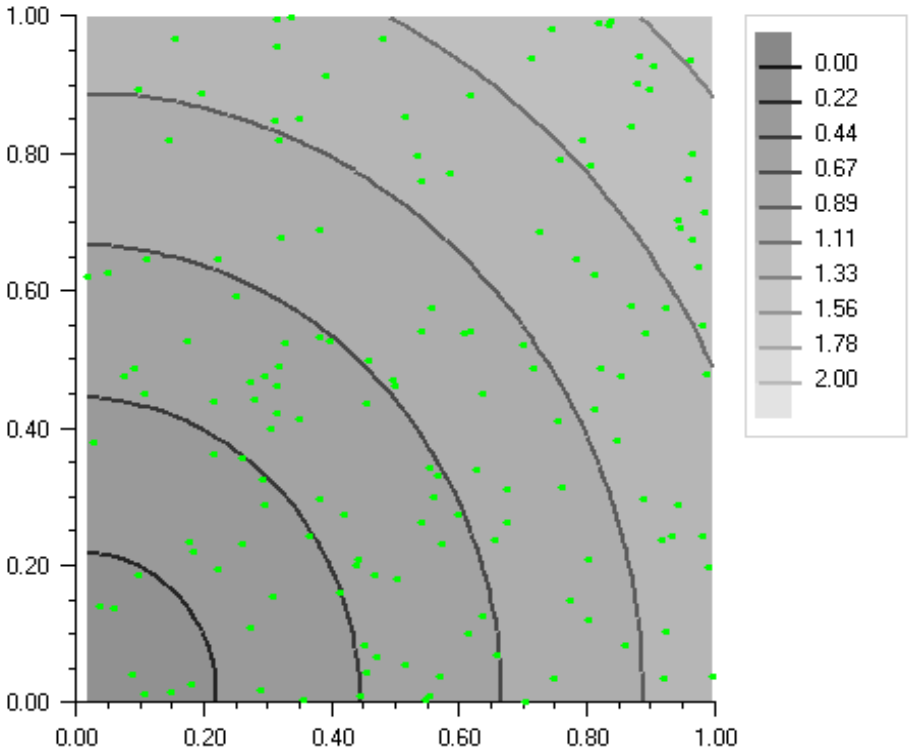
```
randomValueY[3]=0.4781765623804778;
randomValueY[4]=0.6732389937186418;
randomValueY[5]=0.33081942994459734;
randomValueY[6]=0.10880787186704965;
randomValueY[7]=0.901138442534768;
randomValueY[8]=0.48723656383264413;
randomValueY[9]=0.10153552805288812;
randomValueY[10]=0.9558058275075961;
randomValueY[11]=0.011829287599608884;
randomValueY[12]=0.4859902873228249;
randomValueY[13]=0.5505301300240635;
randomValueY[14]=0.18652444274911184;
randomValueY[15]=0.9272326533193322;
randomValueY[16]=0.4215880116306273;
randomValueY[17]=0.0386317648903991;
randomValueY[18]=0.6451521871931544;
randomValueY[19]=0.819301055474355;
randomValueY[20]=0.039285689951912395;
randomValueY[21]=0.31325564481720314;
randomValueY[22]=0.6272275622766595;
randomValueY[23]=0.8934533907186641;
randomValueY[24]=0.5212913217641422;
randomValueY[25]=0.6237725863035143;
randomValueY[26]=0.3611731793838059;
randomValueY[27]=0.23163547542978535;
randomValueY[28]=0.7999943624102621;
randomValueY[29]=0.5393314259940907;
randomValueY[30]=0.10341603798162413;
randomValueY[31]=0.48822476962455685;
randomValueY[32]=0.5414223626279245;
randomValueY[33]=0.08241640235000847;
randomValueY[34]=0.27287579633296155;
randomValueY[35]=0.6770605504344167;
randomValueY[36]=0.8497059767892107;
randomValueY[37]=0.04142051621448373;
randomValueY[38]=0.30060172837976995;
randomValueY[39]=0.5378809821731352;
randomValueY[40]=0.9933333184285308;
randomValueY[41]=0.5755163489718148;
randomValueY[42]=0.12033991348116369;
randomValueY[43]=0.22044795260992822;
randomValueY[44]=0.7039752563092764;
randomValueY[45]=0.47510550779825345;
randomValueY[46]=0.47581191139276346;
randomValueY[47]=0.2746412789430772;
randomValueY[48]=0.8486627562667742;
randomValueY[49]=0.6911278265254134;
randomValueY[50]=0.47048601468635676;
randomValueY[51]=0.18480344365963364;
randomValueY[52]=0.5260974820985063;
randomValueY[53]=0.9965118715946334;
randomValueY[54]=0.03562254706322543;
randomValueY[55]=0.9366159496862719;
randomValueY[56]=0.8878769321024975;
randomValueY[57]=0.8930475165444577;
randomValueY[58]=0.24237426250726957;
```

```
randomValueY[59]=0.354788700886031;
randomValueY[60]=0.2354154511947073;
randomValueY[61]=0.1269624995880959;
randomValueY[62]=0.6337231423679252;
randomValueY[63]=0.19984371337284335;
randomValueY[64]=0.19334220894181153;
randomValueY[65]=0.42648351165619114;
randomValueY[66]=0.0020349209904862997;
randomValueY[67]=0.26227419862014245;
randomValueY[68]=0.010157565396595736;
randomValueY[69]=0.32466354319724255;
randomValueY[70]=0.2880125699286028;
randomValueY[71]=0.942360375989513;
randomValueY[72]=0.28692884801712293;
randomValueY[73]=0.18075667041036092;
randomValueY[74]=0.526829825487406;
randomValueY[75]=0.05392345053644676;
randomValueY[76]=0.6848072074260566;
randomValueY[77]=0.7634213162987096;
randomValueY[78]=0.017226310006998813;
randomValueY[79]=0.8402985996291047;
randomValueY[80]=0.41214609100356114;
randomValueY[81]=0.00903342798862894;
randomValueY[82]=0.13934521987605275;
randomValueY[83]=0.44080857560050446;
randomValueY[84]=0.5420034416544178;
randomValueY[85]=0.8183907621649894;
randomValueY[86]=0.49709491461841304;
randomValueY[87]=0.2960190585426765;
randomValueY[88]=0.4608082576003252;
randomValueY[89]=0.005089578506740633;
randomValueY[90]=0.3108158643301907;
randomValueY[91]=0.23005689707662969;
randomValueY[92]=0.9989728680293828;
randomValueY[93]=0.7588548659179764;
randomValueY[94]=0.23603371611553747;
randomValueY[95]=0.1982727511862804;
randomValueY[96]=0.04423243217165507;
randomValueY[97]=0.23710549829602878;
randomValueY[98]=0.03408034658051773;
randomValueY[99]=0.9385290439821878;
randomValueY[100]=0.6884926962578499;
randomValueY[101]=0.14803546698365633;
randomValueY[102]=0.7703636833850115;
randomValueY[103]=0.01439471413150828;
randomValueY[104]=0.2089671359503994;
randomValueY[105]=0.4384925493939328;
randomValueY[106]=0.466067663723164;
randomValueY[107]=0.9885280557996187;
randomValueY[108]=0.4343852116079696;
randomValueY[109]=0.4499354044927121;
randomValueY[110]=0.3790637460316687;
randomValueY[111]=0.7145286684532488;
randomValueY[112]=0.2970523498826292;
randomValueY[113]=0.15575074519991794;
randomValueY[114]=0.33981500752026883;
```

```
    randomValueY[115]=0.9855399747339232;
    randomValueY[116]=0.621543401362443;
    randomValueY[117]=0.3432116007462742;
    randomValueY[118]=0.8180541618673799;
    randomValueY[119]=0.027883366004455068;
    randomValueY[120]=0.45081070184878236;
    randomValueY[121]=0.8533577155496994;
    randomValueY[122]=0.6460168649513455;
    randomValueY[123]=0.5780055157336823;
    randomValueY[124]=0.46048777917596295;
    randomValueY[125]=0.24207983525545718;
    randomValueY[126]=0.574011233178295;
    randomValueY[127]=0.5310197638599929;
    randomValueY[128]=0.2621701535374652;
    randomValueY[129]=0.4756887402397726;
    randomValueY[130]=0.08410532225672551;
    randomValueY[131]=0.3991230601447665;
    randomValueY[132]=0.6464545787001537;
    randomValueY[133]=0.524250367439074;
    randomValueY[134]=0.13771323020945658;
    randomValueY[135]=0.06816969003124507;
    randomValueY[136]=0.06651758347488423;
    randomValueY[137]=0.965968335289986;
    randomValueY[138]=0.7828616693306287;
    randomValueY[139]=0.5906828761391884;
    randomValueY[140]=0.9130151004091689;
    randomValueY[141]=0.9658950710812012;
    randomValueY[142]=0.7969176634278117;
    randomValueY[143]=0.003585724779986199;
    randomValueY[144]=0.38108388460809595;
    randomValueY[145]=0.24225280334829336;
    randomValueY[146]=0.7905591927051523;
    randomValueY[147]=0.4089325882708409;
    randomValueY[148]=0.9802263978904657;
    randomValueY[149]=0.8836456558655017;

    for (int k = 0; k < n; k++)
    {
        x[k] = randomValue[k];
        y[k] = randomValueY[k];
        z[k] = System.Math.Sqrt(x[k] * x[k] + y[k] * y[k]);
    }

    // Setup the contour plot and its legend
    AxisXY axis = new AxisXY(chart);
    Contour contour = new Contour(axis, x, y, z);
    contour.ContourLegend.IsVisible = true;

    // Show the input data points as small green circles
    Data dataPoints = new Data(axis, x, y);
    dataPoints.DataType = Data.DATA_TYPE_MARKER;
    dataPoints.MarkerType = Data.MARKER_TYPE_FILLED_CIRCLE;
    dataPoints.MarkerColor = System.Drawing.Color.FromArgb(0, 255, 0);
    dataPoints.MarkerSize = 0.5;
}
```

```
    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new ContourEx2());
    }
}
```

Output



Contour.Legend Class

```
public class Imsl.Chart2D.Contour.Legend :  AxisXY
```

A legend for a contour chart.

This legend should be used for contour charts, instead of usual chart legend. The “Viewport” attribute for this node is set to [0.83,0.98] by [0.1,0.6].

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

ContourLevel Class

```
public class Imsl.Chart2D.ContourLevel :  ChartNode
```

`ContourLevel` draws a level curve line and the fill area between the level curve and the next smaller level curve.

`ContourLevel` objects are created by `Contour` as child nodes.

Each `ContourLevel` defines a filled areas and level curves. The drawing of the filled areas can be changed using the line attributes (specified with LineColor (p. 1302), LineWidth (p. 1303) and SetMarkerDashPattern (p. 1331)) and fill attributes (specified with FillType (p. 1322) and FillColor (p. 1301))in the `ContourLevel` nodes.

See Also

Imsl.Chart2D.Contour (p. 1431)

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

ErrorBar Class

```
public class Imsl.Chart2D.ErrorBar :  Data
```

Renders data points with error bars.

Fields

DATA_TYPE_ERROR_X

```
public int DATA_TYPE_ERROR_X
```

Description

Value for attribute "DataType" indicating that this is a horizontal error bar.

Remarks

Used in connection with `ErrorBar` nodes.

DATA_TYPE_ERROR_Y

```
public int DATA_TYPE_ERROR_Y
```

Description

Value for attribute "DataType" indicating that this is a vertical error bar.

Remarks

Used in connection with `ErrorBar` nodes.

Constructor

ErrorBar

```
public ErrorBar(Imsl.Chart2D.AxisXY axis, double[] x, double[] y, double[] low,
double[] high)
```

Description

Creates a set of error bars centered at (x[k],y[k]) and with extents low[k],high[k].

Parameters

axis – An Axis containing the parent of this node.

x – A double[] which contains the x-coordinates of the points at which the error bars will be centered.

y – A double[] which contains the y-coordinates of the points at which the error bars will be centered.

low – A double[] which contains the values which define the minimum extent of the error bars.

high – A double[] which contains the values which define the maximum extent of the error bars.

Remarks

If DataType (p. 1320) has the bit Imsl.Chart2D.ErrorBar.DATA_TYPE_ERROR_X (p. 1446) set then this is a horizontal error bar. If the bit Imsl.Chart2D.ErrorBar.DATA_TYPE_ERROR_Y (p. 1446) is set then this is a vertical error bar. If neither bit is set then no error bar is drawn.

A Data node with the same x and y values can be used to put markers at the center of each error bar.

Each of the array arguements have an associated attribute. That is, “X”, “Y”, “Low” and “High”.

Methods

GetHigh

```
virtual public double[] GetHigh()
```

Description

Returns the maximum extent of the error bars.

Returns

A double[] which contains the values for the maximum extent of the error bars.

GetLow

```
virtual public double[] GetLow()
```

Description

Returns the minimum extent of the error bars.

Returns

A `double[]` which contains the values for the minimum extent of the error bars.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetHigh

```
virtual public void SetHigh(double[] high)
```

Description

Sets the maximum extent of the error bars.

Parameter

`high` – A `double[]` which contains the values for the maximum extent of the error bars.

SetLow

```
virtual public void SetLow(double[] low)
```

Description

Sets the minimum extent of the error bars.

Parameter

`low` – A `double[]` which contains the values for the minimum extent of the error bars.

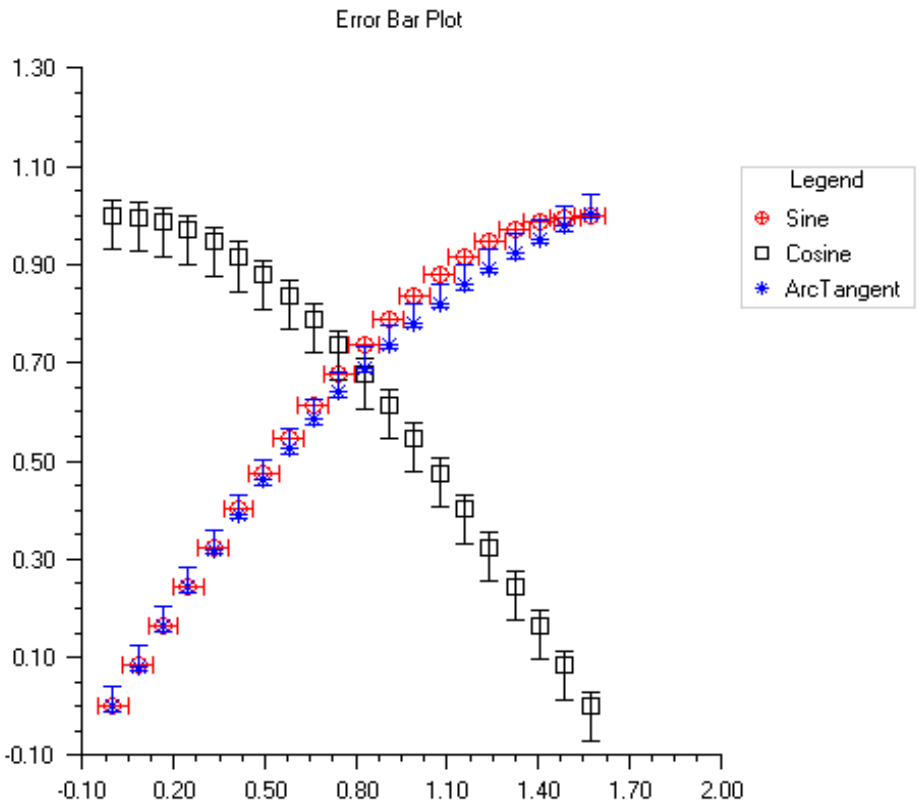
Example: ErrorBar Chart

An ErrorBar chart is constructed in this example. Three data sets are used and a legend is added to the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class ErrorBarEx1 : FrameChart
{
    public ErrorBarEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int    npoints = 20;
        double dx = .5 * Math.PI/(npoints - 1);
        double[] x = new double[npoints];
        double[] y1 = new double[npoints];
        double[] y2 = new double[npoints];
        double[] y3 = new double[npoints];
        double[] low1 = new double[npoints];
        double[] low2 = new double[npoints];
        double[] low3 = new double[npoints];
        double[] hi1 = new double[npoints];
        double[] hi2 = new double[npoints];
        double[] hi3 = new double[npoints];

        //    Generate some data
        for (int i = 0; i < npoints; i++)
        {
            x[i] = i * dx;
            y1[i] = System.Math.Sin(x[i]);
            low1[i] = x[i] - .05;
            hi1[i] = x[i] + .05;
            y2[i] = System.Math.Cos(x[i]);
            low2[i] = y2[i] - .07;
            hi2[i] = y2[i] + .03;
            y3[i] = System.Math.Atan(x[i]);
            low3[i] = y3[i] - .01;
            hi3[i] = y3[i] + .04;
        }

        // Data
        Data d1 = new Data(axis, x, y1);
        Data d2 = new Data(axis, x, y2);
        Data d3 = new Data(axis, x, y3);

        //    Set Data Type to Marker
        d1.DataType = Data.DATA_TYPE_MARKER;
        d2.DataType = Data.DATA_TYPE_MARKER;
        d3.DataType = Data.DATA_TYPE_MARKER;

        //    Set Marker Types
```

```
        d1.MarkerType = Data.MARKER_TYPE_CIRCLE_PLUS;
        d2.MarkerType = Data.MARKER_TYPE_HOLLOW_SQUARE;
        d3.MarkerType = Data.MARKER_TYPE_ASTERISK;

        //    Set Marker Colors
        d1.MarkerColor = System.Drawing.Color.Red;
        d2.MarkerColor = System.Drawing.Color.Black;
        d3.MarkerColor = System.Drawing.Color.Blue;

        //    Create an instances of ErrorBars
        ErrorBar ebar1 = new ErrorBar(axis, x, y1, low1, hi1);
        ErrorBar ebar2 = new ErrorBar(axis, x, y2, low2, hi2);
        ErrorBar ebar3 = new ErrorBar(axis, x, y3, low3, hi3);

        //    Set Data Type to Error_X
        ebar1.DataType = ErrorBar.DATA_TYPE_ERROR_X;
        ebar2.DataType = ErrorBar.DATA_TYPE_ERROR_Y;
        ebar3.DataType = ErrorBar.DATA_TYPE_ERROR_Y;

        //    Set Marker Colors
        ebar1.MarkerColor = System.Drawing.Color.Red;
        ebar2.MarkerColor = System.Drawing.Color.Black;
        ebar3.MarkerColor = System.Drawing.Color.Blue;

        //    Set Data Labels
        d1.SetTitle("Sine");
        d2.SetTitle("Cosine");
        d3.SetTitle("ArcTangent");

        //    Add a Legend
        Legend legend = chart.Legend;
        legend.SetTitle(new Text("Legend"));
        legend.IsVisible = true;

        //    Set the Chart Title
        chart.ChartTitle.SetTitle("Error Bar Plot");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new ErrorBarEx1());
    }
}
```

Output



HighLowClose Class

```
public class Imsl.Chart2D.HighLowClose :  Data
```

High-low-close plot of stock data.

Field

DAY

```
public double DAY
```

Description

Ticks per day.

Constructors

HighLowClose

```
public HighLowClose(Imsl.Chart2D.AxisXY axis, System.DateTime start, double[]
high, double[] low, double[] close)
```

Description

Constructs a high-low-close chart node beginning with specified start date.

Parameters

`axis` – An `Axis` specifying the parent of this node.

`start` – A `DateTime` which specifies the first date.

`high` – A `double[]` which contains the stock's high prices.

`low` – A `double[]` which contains the stock's low prices.

`close` – A `double[]` which contains the stock's closing prices.

Remarks

The `high`, `low` and `close` are used to specify the respective attributes. That is, "high", "low" and "close".

HighLowClose

```
public HighLowClose(Imsl.Chart2D.AxisXY axis, System.DateTime start, double[]
high, double[] low, double[] close, double[] open)
```

Description

Constructs a high-low-close-open chart node beginning with specified start date.

Parameters

`axis` – An `Axis` specifying the parent of this node.

`start` – A `DateTime` which specifies the first date.

`high` – A `double[]` which contains the stock's high prices.

`low` – A `double[]` which contains the stock's low prices.

`close` – A `double[]` which contains the stock's closing prices.

`open` – A `double[]` which contains the stock's opening prices.

Remarks

The `high`, `low`, `close` and `open` are used to specify the respective attributes. That is, “high”, “low”, “close” and “open”.

HighLowClose

```
public HighLowClose(Imsl.Chart2D.AxisXY axis, double[] x, double[] high,
double[] low, double[] close)
```

Description

Constructs a high-low-close chart node beginning with specified start date.

Parameters

`axis` – An `Axis` specifying the parent of this node.

`x` – A `double[]` which contains the axis points.

`high` – A `double[]` which contains the stock’s high prices.

`low` – A `double[]` which contains the stock’s low prices.

`close` – A `double[]` which contains the stock’s closing prices.

Remarks

The `X`, `high`, `low` and `close` are used to specify the respective attributes. That is, “X”, “high”, “low” and “close”.

HighLowClose

```
public HighLowClose(Imsl.Chart2D.AxisXY axis, double[] x, double[] high,
double[] low, double[] close, double[] open)
```

Description

Constructs a high-low-close-open chart node beginning with specified start date.

Parameters

`axis` – An `Axis` specifying the parent of this node.

`x` – A `double` array which contains the axis points.

`high` – A `double[]` which contains the stock’s high prices.

`low` – A `double[]` which contains the stock’s low prices.

`close` – A `double[]` which contains the stock’s closing prices.

`open` – A `double[]` which contains the stock’s opening prices.

Remarks

The `X`, `high`, `low` and `close` and `open` are used to specify the respective attributes. That is, “X”, “high”, “low”, “close” and “open”.

Methods

GetClose

```
virtual public double[] GetClose()
```

Description

Returns the stock prices at close.

Returns

A `double[]` containing the closing stock prices.

GetHigh

```
virtual public double[] GetHigh()
```

Description

Returns the high stock prices.

Returns

A `double[]` containing the high stock prices.

GetLow

```
virtual public double[] GetLow()
```

Description

Returns the low stock prices.

Returns

A `double[]` containing the low stock prices.

GetOpen

```
virtual public double[] GetOpen()
```

Description

Returns the opening stock prices.

Returns

A `double[]` containing the opening stock prices.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetClose

```
virtual public void SetClose(double[] close)
```

Description

Sets the closing stock prices.

Parameter

`close` – A `double[]` specifying the closing stock prices.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetDateAxis

```
virtual public void SetDateAxis(string labelFormat)
```

Description

Sets up the x-axis for high-low-close plot.

Parameter

`labelFormat` – A `string` used to format the date axis labels.

Remarks

This turns off autoscaling on the x-axis and sets the "Window" attribute depending on the number of dates being plotted. The Number attribute determines the number of intervals along the x-axis.

The `labelFormat` sets TextFormat (p. 1305) and TextFormatProvider (p. 1305) in the Imsl.Chart2D.AxisLabel (p. 1358) node.

SetHigh

```
virtual public void SetHigh(double[] high)
```

Description

Sets the high stock prices.

Parameter

`high` – A `double[]` specifying the high stock prices.

SetLow

```
virtual public void SetLow(double[] low)
```

Description

Sets the low stock prices.

Parameter

`low` – A `double[]` specifying the low stock prices.

SetOpen

```
virtual public void SetOpen(double[] open)
```

Description

Sets the opening stock prices.

Parameter

`open` – A `double[]` specifying the opening stock prices.

Example: High-Low-Close Chart

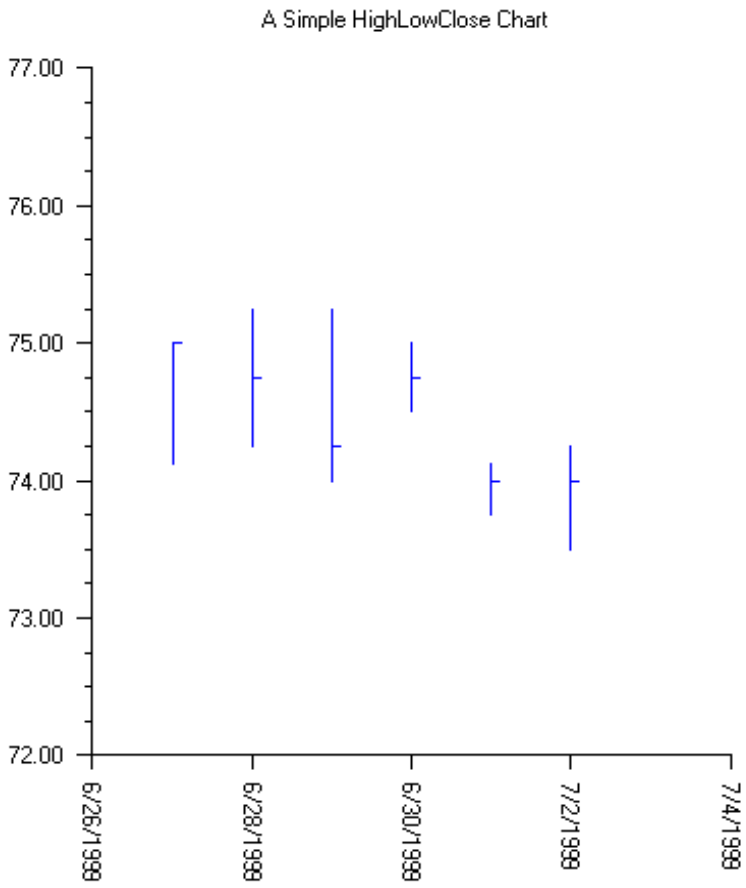
A simple high-low-close chart is constructed in this example.

Autoscaling does not properly handle time data, so autoscaling is turned off for the *x* (time) axis and the axis limits are set explicitly.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class HiLoEx1 : FrameChart
{
    public HiLoEx1()
    {
        Chart chart = this.Chart;

        AxisXY axis = new AxisXY(chart);

        // Date is June 27, 1999
        System.Globalization.GregorianCalendar temp_calendar;
        temp_calendar = new System.Globalization.GregorianCalendar();

        System.DateTime date = new DateTime(1999, 6, 27, temp_calendar);

        double[] high = new double[]{75.0, 75.25, 75.25, 75.0, 74.125, 74.25};
        double[] low = new double[]{74.125, 74.25, 74.0, 74.5, 73.75, 73.50};
        double[] close = new double[]{75.0, 74.75, 74.25, 74.75, 74.0, 74.0};

        //    Create an instance of a HighLowClose Chart
```

```
        HighLowClose hilo = new HighLowClose(axis, date, high, low, close);
        hilo.MarkerColor = System.Drawing.Color.Blue;

        //    Set the HighLowClose Chart Title
        chart.ChartTitle.SetTitle("A Simple HighLowClose Chart");

        // Configure the x-axis
        hilo.SetDateAxis("d");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new HiLoEx1());
    }
}
```

Output



Candlestick Class

```
public class Imsl.Chart2D.Candlestick :  HighLowClose
```

Candlestick plot of stock data.

Two nodes are created as children of this node. One for the up days and one for the down days.

Properties

Down

```
virtual public Imsl.Chart2D.CandlestickItem Down {get; }
```

Description

The down days of this `Candlestick`.

Property Value

A `CandlestickItem` which contains the “Down” attribute value.

Up

```
virtual public Imsl.Chart2D.CandlestickItem Up {get; }
```

Description

The up days of this `Candlestick`.

Property Value

A `CandlestickItem` which contains the “Up” attribute value.

Constructors

Candlestick

```
public Candlestick(Imsl.Chart2D.AxisXY axis, System.DateTime start, double[]
high, double[] low, double[] close, double[] open)
```

Description

Constructs a candlestick chart node beginning with specified start date.

Parameters

`axis` – An `AxisXY` which is the parent of this node.

`start` – A `DateTime` that specifies the first date.

`high` – A `double[]` which contains the stock’s high prices.

`low` – A `double[]` which contains the stock’s low prices.

`close` – A `double[]` which contains the stock’s closing prices.

`open` – A `double[]` which contains the stock’s opening prices.

Remarks

Each of the arguments are use to set the related attribute (e.g. “High”, “Low”, “Close” and “Open”).

Candlestick

```
public Candlestick(Imsl.Chart2D.AxisXY axis, double[] x, double[] high,
double[] low, double[] close, double[] open)
```

Description

Constructs a candlestick chart node beginning with specified axis points.

Parameters

`axis` – An `AxisXY` which is the parent of this node.

`x` – A `double[]` which contains the axis points.

`high` – A `double[]` which contains the stock's high prices.

`low` – A `double[]` which contains the stock's low prices.

`close` – A `double[]` which contains the stock's closing prices.

`open` – A `double[]` which contains the stock's opening prices.

Remarks

Each of the arguments are use to set the related attribute (e.g. "X", "High", "Low", "Close" and "Open").

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

CandlestickItem Class

```
public class Imsl.Chart2D.CandlestickItem :  Data
```

A candlestick for the up days or the down days.

`CandlestickItems` are created by Candlestick; one for up days and one for down days.

See Also

Imsl.Chart2D.Candlestick (p. 1458)

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SplineData Class

```
public class Imsl.Chart2D.SplineData :  Data
```

A data set created from a `Spline`.

See Also

Imsl.Math.Spline (p. 143)

Constructor

SplineData

```
public SplineData(Imsl.Chart2D.ChartNode parent, Imsl.Math.Spline spline)
```

Description

Creates a data node from `Spline` values.

Parameters

`parent` – A `ChartNode` which specifies the parent of this data node.

`spline` – A `Spline` which specifies the data to be plotted.

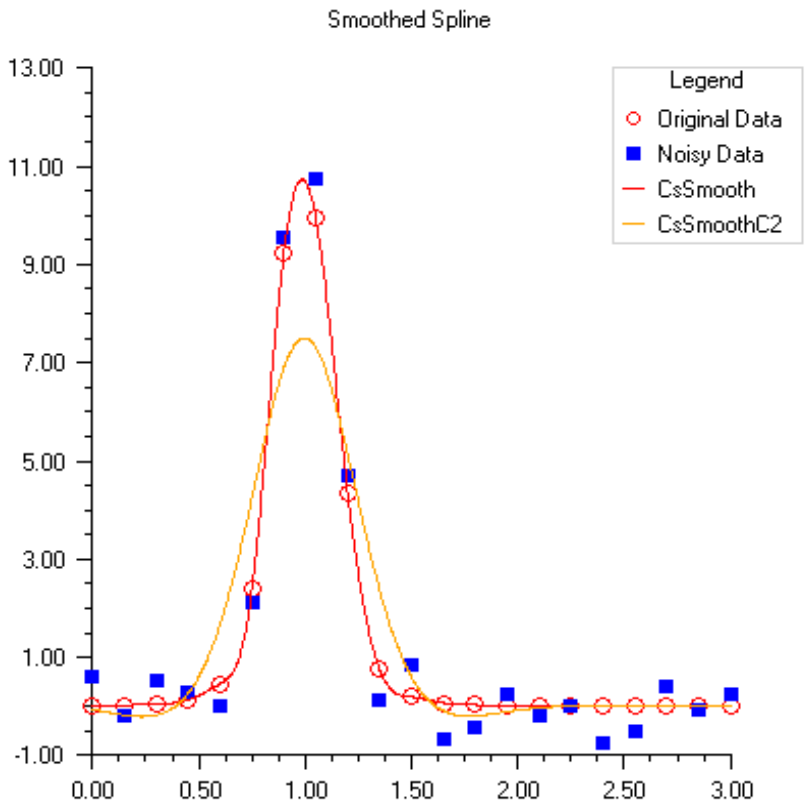
Example: SplineData Chart

This example makes use of the SplineData class as well as the two spline smoothing classes in `Imsl.Math`.

```
using System;
using System.Collections;
using System.ComponentModel;
using System.Drawing;
using System.Data;
using System.Windows.Forms;
using Imsl.Math;
using Imsl.Chart2D;
using Random = Imsl.Stat.Random;

public class SplineDataEx1 : FrameChart
{

    private const int nData = 21;
    private const int nSpline = 100;

    public SplineDataEx1()
    {

        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        chart.ChartTitle.SetTitle(new Text("Smoothed Spline"));

        Legend legend = chart.Legend;
        legend.SetTitle(new Text("Legend"));
        legend.SetViewport(0.7, 0.9, 0.1, 0.3);
        legend.IsVisible = true;

        // Original data
        double[] xData = grid(nData);
        double[] yData = new double[nData];
        for (int k = 0; k < nData; k++)
        {
            yData[k] = f(xData[k]);
        }

        Data data = new Data(axis, xData, yData);
        data.DataType = Imsl.Chart2D.Data.DATA_TYPE_MARKER;
        data.MarkerType = Data.MARKER_TYPE_HOLLOW_CIRCLE;
        data.MarkerColor = System.Drawing.Color.Red;
        data.SetTitle("Original Data");

        // Noisy data
        Random random = new Random(123457);
        double[] yNoisy = new double[nData];
        for (int k = 0; k < nData; k++)
        {
            yNoisy[k] = yData[k] + (2.0 * random.NextDouble() - 1.0);
        }
```

```
        data = new Data(axis, xData, yNoisy);
        data.DataType = Imsl.Chart2D.Data.DATA_TYPE_MARKER;
        data.MarkerType = Data.MARKER_TYPE_FILLED_SQUARE;
        data.MarkerSize = 0.75;
        data.MarkerColor = System.Drawing.Color.Blue;
        data.SetTitle("Noisy Data");

        chartSpline(axis, new CsSmooth(xData, yData), System.Drawing.Color.Red,
         "CsSmooth");
        chartSpline(axis, new CsSmoothC2(xData, yData, nData),
         System.Drawing.Color.Orange, "CsSmoothC2");
    }

    static private void chartSpline(AxisXY axis, Imsl.Math.Spline spline,
      System.Drawing.Color color, System.String title)
    {
        Data data = new SplineData(axis, spline);
        data.DataType = Imsl.Chart2D.Data.DATA_TYPE_LINE;
        data.LineColor = color;
        data.SetTitle(title);
    }

    static private double[] grid(int nData)
    {
        double[] xData = new double[nData];
        for (int k = 0; k < nData; k++)
        {
            xData[k] = 3.0 * k / (double) (nData - 1);
        }
        return xData;
    }

    static private double f(double x)
    {
        return 1.0 / (0.1 + System.Math.Pow(3.0 * (x - 1.0), 4));
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new SplineDataEx1());
    }
}
```

Output



Bar Class

```
public class Imsl.Chart2D.Bar :  Data
```

A bar chart.

The class `Bar` has children of class Imsl.Chart2D.BarItem (p. 1473). The attribute “BarItem” in class `Bar` is set to the `BarItem` array of children.

See Also

Imsl.Chart2D.BarSet (p. 1477), Imsl.Chart2D.BarItem (p. 1473)

Constructors

Bar

```
public Bar(Imsl.Chart2D.AxisXY axis)
```

Description

Constructs a bar chart.

Parameter

`axis` – A `AxisXY` which is the parent of this node.

Bar

```
public Bar(Imsl.Chart2D.AxisXY axis, double[] y)
```

Description

Constructs a simple bar chart using supplied y data.

Parameters

`axis` – A `AxisXY` which is the parent of this node.

`y` – A `double[]` which contains the y data for the simple bar chart

Bar

```
public Bar(Imsl.Chart2D.AxisXY axis, double[] x, double[] y)
```

Description

Constructs a simple bar chart using supplied x and y data.

Parameters

`axis` – A `AxisXY` which is the parent of this node.

`x` – A `double[]` which contains the x data for the simple bar chart.

`y` – A `double[]` which contains the y data for the simple bar chart.

Bar

```
public Bar(Imsl.Chart2D.AxisXY axis, double[][] y)
```

Description

Constructs a grouped bar chart using supplied x and y data.

Parameters

`axis` – A `AxisXY` which is the parent of this node.

`y` – A `double[]` which contains the `y` data for the grouped bar chart. The first index refers to the group and the second refers to the x position.

Bar

```
public Bar(Imsl.Chart2D.AxisXY axis, double[] x, double[][] y)
```

Description

Constructs a grouped bar chart using supplied `x` and `y` data.

Parameters

`axis` – A `AxisXY` which is the parent of this node.

`x` – A `double[]` which contains the `x` data for the grouped bar chart.

`y` – A `double[]` which contains the `y` data for the grouped bar chart. The first index refers to the group and the second refers to the x position.

Bar

```
public Bar(Imsl.Chart2D.AxisXY axis, double[][][] y)
```

Description

Constructs a stacked, grouped bar chart using supplied `y` data.

Parameters

`axis` – A `AxisXY` which is the parent of this node.

`y` – A `double[]` which contains the `y` data for the stacked, grouped bar chart. The first index refers to the stack, the second refers to the group and the third refers to the x position.

Bar

```
public Bar(Imsl.Chart2D.AxisXY axis, double[] x, double[][][] y)
```

Description

Constructs a stacked, grouped bar chart using supplied `x` and `y` data.

Parameters

`axis` – A `AxisXY` which is teh parent of this node.

`x` – A `double[]` which contains the `x` data for the stacked, grouped bar chart.

`y` – A `double[]` which contains the `y` data for the stacked, grouped bar chart. The first index refers to the “stack”, the second refers to the group and the third refers to the x position.

Methods

GetBarData

```
virtual public double[][][] GetBarData()
```

Description

Returns the “BarData” attribute value.

Returns

A `double[][][]` that contains the “BarData” attribute value.

Remarks

The value is an array of object that make up a bar chart. The first index refers to the “stack”, the second refers to the group and the third refers to the x position.

GetBarSet

```
virtual public Imsl.Chart2D.BarSet[][] GetBarSet()
```

Description

Returns the `BarSet` object.

Returns

A `BarSet[][]` containing the “BarSet” attribute value.

GetBarSet

```
virtual public Imsl.Chart2D.BarSet GetBarSet(int group)
```

Description

Returns the `BarSet` object.

Parameter

`group` – An `int` which specifies the group index.

Returns

A `BarSet[][]` containing the “BarSet” attribute value.

Remarks

The group index is assumed to be zero. This method is most useful for charts with only a single group.

GetBarSet

```
virtual public Imsl.Chart2D.BarSet GetBarSet(int stack, int group)
```

Description

Returns the `BarSet` object.

Parameters

`stack` – An `int` which specifies the stack index.

`group` – An `int` which specifies the group index.

Returns

A `BarSet[][]` containing the “BarSet” attribute value.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetBarData

```
virtual public void SetBarData(double[][][] bardata)
```

Description

Sets the "BarData" attribute value.

Parameter

`bardata` – A `double[][][]` that specifies the "BarData" attribute value.

Remarks

The value is an array of object that make up a bar chart. The first index refers to the "stack", the second refers to the group and the third refers to the x position.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetLabels

```
virtual public void SetLabels(string[] labels)
```

Description

Sets up an axis with bar labels.

Parameter

`labels` – A `String[]` array with which to label the axis.

Remarks

This turns off the tick marks and sets the “BarType” attribute. It also turns off autoscaling for the axis and sets its “Window” and “Number” and “Ticks” attribute as appropriate for a labeled bar chart. The existing value of the “BarType” attribute is used to determine the axis to be modified.

The number of labels must equal the number of items.

SetLabels

```
virtual public void SetLabels(string[] labels, int type)
```

Description

Sets up an axis with bar labels.

Parameters

`labels` – A `String[]` which specifes axis labels.

`type` – An `int` which specifies the “BarType”.

Remarks

This turns off the tick marks and sets the “BarType” attribute. It also turns off autoscaling for the axis and sets its “Window”, “Number” and “Ticks” attributes as appropriate for a labeled bar chart.

The number of labels must equal the number of items.

The bar type determines the axis to be modified. Legal values are:

- Imsl.Chart2D.ChartNode.BAR_TYPE_VERTICAL (p. 1311)
- Imsl.Chart2D.ChartNode.BAR_TYPE_HORIZONTAL (p. 1311)

Example: Stacked Bar Chart

A stacked bar chart is constructed in this example. Bar labels and colors are set and axis labels are set.

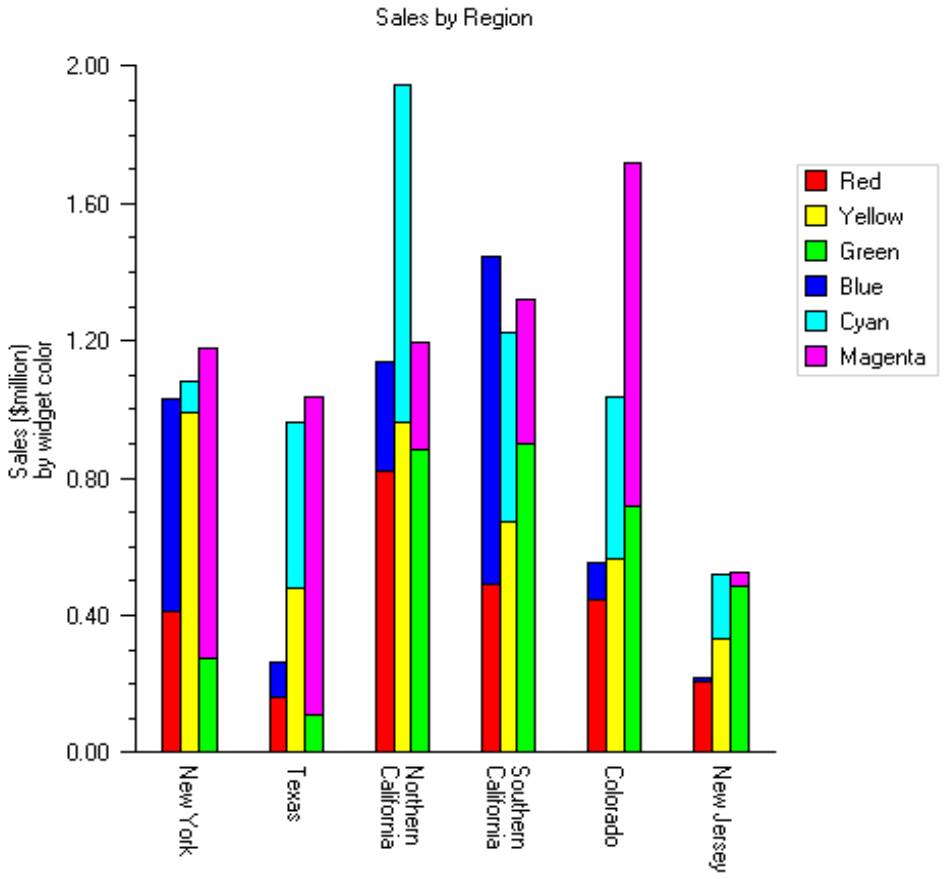
```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class BarEx1 : FrameChart
{

    public BarEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int nStacks = 2;
        int nGroups = 3;
        int nItems = 6;

        //    Generate some random data
        Imsl.Stat.Random r = new Imsl.Stat.Random(123457);

        double[] dbl = new double[50];
```

```
dbl[0]=0.41312962995625035;
dbl[1]=0.15995876895053263;
dbl[2]=0.8225528716547005;
dbl[3]=0.48794367683379836;
dbl[4]=0.44364905186692527;
dbl[5]=0.20896329070872555;
dbl[6]=0.9887088342522812;
dbl[7]=0.4781765623804778;
dbl[8]=0.9647868112234352;
dbl[9]=0.6732389937186418;
dbl[10]=0.5668831243079411;
dbl[11]=0.33081942994459734;
dbl[12]=0.27386697614898103;
dbl[13]=0.10880787186704965;
dbl[14]=0.8805853693809824;
dbl[15]=0.901138442534768;
dbl[16]=0.7180829622748057;
dbl[17]=0.48723656383264413;
dbl[18]=0.6153607537410654;
dbl[19]=0.10153552805288812;
dbl[20]=0.3158193853638753;
dbl[21]=0.9558058275075961;
dbl[22]=0.10778543304578747;
dbl[23]=0.011829287599608884;
dbl[24]=0.09275375134615693;
dbl[25]=0.4859902873228249;
dbl[26]=0.9817642781628322;
dbl[27]=0.5505301300240635;
dbl[28]=0.467363186309925;
dbl[29]=0.18652444274911184;
dbl[30]=0.9066980293517674;
dbl[31]=0.9272326533193322;
dbl[32]=0.31440695305815347;
dbl[33]=0.4215880116306273;
dbl[34]=0.9991560762956562;
dbl[35]=0.0386317648903991;
dbl[36]=0.785150345014761;
dbl[37]=0.6451521871931544;
dbl[38]=0.7930129038729785;
dbl[39]=0.819301055474355;
dbl[40]=0.5695413465811706;
dbl[41]=0.039285689951912395;
dbl[42]=0.7625752595574732;
dbl[43]=0.31325564481720314;
dbl[44]=0.0482465474704169;
dbl[45]=0.6272275622766595;
dbl[46]=0.09904819350827354;
dbl[47]=0.8934533907186641;
dbl[48]=0.7013979421419555;
dbl[49]=0.5212913217641422;

int z=0;

double[] x = new double[nItems];
double[][][] y = new double[nStacks][][];
for (int i = 0; i < nStacks; i++)
```

```
{
    y[i] = new double[nGroups][];
    for (int i2 = 0; i2 < nGroups; i2++)
    {
        y[i][i2] = new double[nItems];
    }
}
double dx = 0.5 * System.Math.PI / (x.Length - 1);
for (int istack = 0; istack < y.Length; istack++)
{
    for (int jgroup = 0; jgroup < y[istack].Length; jgroup++)
    {
        for (int kitem = 0; kitem < y[istack][jgroup].Length; kitem++)
        {
            y[istack][jgroup][kitem] = dbl[z];//r.NextDouble();
            z++;
        }
    }
}

//    Create an instance of a Bar Chart
Bar bar = new Bar(axis, y);

//    Set the Bar Chart Title
chart.ChartTitle.SetTitle("Sales by Region");

//    Set the fill outline type;
bar.FillOutlineType = Bar.FILL_TYPE_SOLID;

System.Drawing.Color GREEN = System.Drawing.Color.FromArgb(0, 255, 0);
//    Set the Bar Item fill colors
bar.GetBarSet(0, 0).FillColor = System.Drawing.Color.Red;
bar.GetBarSet(0, 1).FillColor = System.Drawing.Color.Yellow;
bar.GetBarSet(0, 2).FillColor = GREEN;
bar.GetBarSet(1, 0).FillColor = System.Drawing.Color.Blue;
bar.GetBarSet(1, 1).FillColor = System.Drawing.Color.Cyan;
bar.GetBarSet(1, 2).FillColor = System.Drawing.Color.Magenta;

chart.Legend.IsVisible = true;
bar.GetBarSet(0, 0).SetTitle("Red");
bar.GetBarSet(0, 1).SetTitle("Yellow");
bar.GetBarSet(0, 2).SetTitle("Green");
bar.GetBarSet(1, 0).SetTitle("Blue");
bar.GetBarSet(1, 1).SetTitle("Cyan");
bar.GetBarSet(1, 2).SetTitle("Magenta");

// Setup the vertical axis for a labeled bar chart.
System.String[] labels = new System.String[]{"New York", "Texas",
 "Northern\nCalifornia", "Southern\nCalifornia", "Colorado",
 "New Jersey"};
bar.SetLabels(labels, Imsl.Chart2D.Bar.BAR_TYPE_VERTICAL);

//    Set the text angle
axis.AxisX.AxisLabel.TextAngle = 270;

//    Set the Y axis title
```

```
        axis.AxisY.AxisTitle.SetTitle("Sales ($million)\nby " + "widget color");

    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new BarEx1());
    }
}
```

Output



BarItem Class

```
public class Imsl.Chart2D.BarItem :  Data
```

A single bar in a bar chart.

See Also

Imsl.Chart2D.Bar (p. 1464), Imsl.Chart2D.BarSet (p. 1477)

Methods

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

Example: Stacked Bar Chart

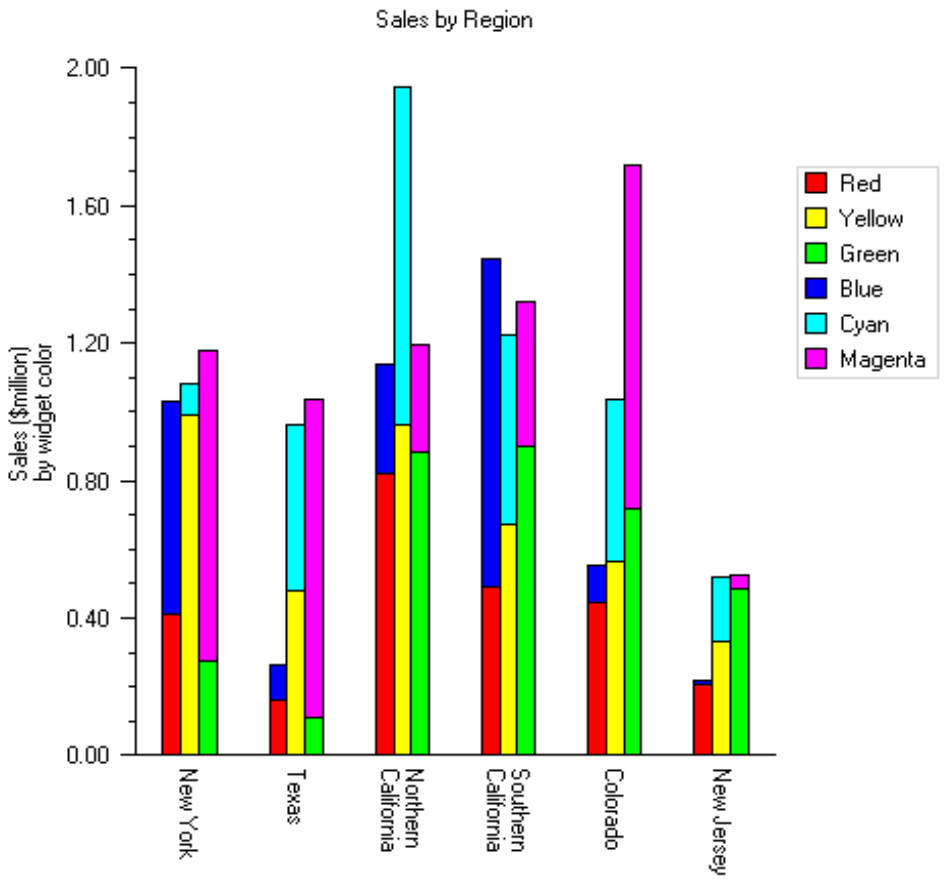
A stacked bar chart is constructed in this example. Bar labels and colors are set and axis labels are set.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class BarEx1 : FrameChart
{

    public BarEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        int nStacks = 2;
        int nGroups = 3;
        int nItems = 6;

        //    Generate some random data
        Imsl.Stat.Random r = new Imsl.Stat.Random(123457);

        double[] dbl = new double[50];
        dbl[0]=0.41312962995625035;
        dbl[1]=0.15995876895053263;
        dbl[2]=0.8225528716547005;
        dbl[3]=0.48794367683379836;
        dbl[4]=0.44364905186692527;
        dbl[5]=0.20896329070872555;
        dbl[6]=0.9887088342522812;
        dbl[7]=0.4781765623804778;
        dbl[8]=0.9647868112234352;
        dbl[9]=0.6732389937186418;
        dbl[10]=0.5668831243079411;
        dbl[11]=0.33081942994459734;
        dbl[12]=0.27386697614898103;
        dbl[13]=0.10880787186704965;
        dbl[14]=0.8805853693809824;
        dbl[15]=0.901138442534768;
        dbl[16]=0.7180829622748057;
        dbl[17]=0.48723656383264413;
        dbl[18]=0.6153607537410654;
        dbl[19]=0.10153552805288812;
        dbl[20]=0.3158193853638753;
        dbl[21]=0.9558058275075961;
        dbl[22]=0.10778543304578747;
        dbl[23]=0.011829287599608884;
        dbl[24]=0.09275375134615693;
        dbl[25]=0.4859902873228249;
        dbl[26]=0.9817642781628322;
        dbl[27]=0.5505301300240635;
        dbl[28]=0.467363186309925;
        dbl[29]=0.18652444274911184;
        dbl[30]=0.9066980293517674;
        dbl[31]=0.9272326533193322;
```

```
dbl[32]=0.31440695305815347;
dbl[33]=0.4215880116306273;
dbl[34]=0.9991560762956562;
dbl[35]=0.0386317648903991;
dbl[36]=0.785150345014761;
dbl[37]=0.6451521871931544;
dbl[38]=0.7930129038729785;
dbl[39]=0.819301055474355;
dbl[40]=0.5695413465811706;
dbl[41]=0.039285689951912395;
dbl[42]=0.7625752595574732;
dbl[43]=0.31325564481720314;
dbl[44]=0.0482465474704169;
dbl[45]=0.6272275622766595;
dbl[46]=0.09904819350827354;
dbl[47]=0.8934533907186641;
dbl[48]=0.7013979421419555;
dbl[49]=0.5212913217641422;

int z=0;

double[] x = new double[nItems];
double[][][] y = new double[nStacks][][];
for (int i = 0; i < nStacks; i++)
{
    y[i] = new double[nGroups][];
    for (int i2 = 0; i2 < nGroups; i2++)
    {
        y[i][i2] = new double[nItems];
    }
}
double dx = 0.5 * System.Math.PI / (x.Length - 1);
for (int istack = 0; istack < y.Length; istack++)
{
    for (int jgroup = 0; jgroup < y[istack].Length; jgroup++)
    {
        for (int kitem = 0; kitem < y[istack][jgroup].Length; kitem++)
        {
            y[istack][jgroup][kitem] = dbl[z];//r.NextDouble();
            z++;
        }
    }
}

//    Create an instance of a Bar Chart
Bar bar = new Bar(axis, y);

//    Set the Bar Chart Title
chart.ChartTitle.SetTitle("Sales by Region");

//    Set the fill outline type;
bar.FillOutlineType = Bar.FILL_TYPE_SOLID;

System.Drawing.Color GREEN = System.Drawing.Color.FromArgb(0, 255, 0);
//    Set the Bar Item fill colors
bar.GetBarSet(0, 0).FillColor = System.Drawing.Color.Red;
```

```
        bar.GetBarSet(0, 1).FillColor = System.Drawing.Color.Yellow;
        bar.GetBarSet(0, 2).FillColor = GREEN;
        bar.GetBarSet(1, 0).FillColor = System.Drawing.Color.Blue;
        bar.GetBarSet(1, 1).FillColor = System.Drawing.Color.Cyan;
        bar.GetBarSet(1, 2).FillColor = System.Drawing.Color.Magenta;

        chart.Legend.IsVisible = true;
        bar.GetBarSet(0, 0).SetTitle("Red");
        bar.GetBarSet(0, 1).SetTitle("Yellow");
        bar.GetBarSet(0, 2).SetTitle("Green");
        bar.GetBarSet(1, 0).SetTitle("Blue");
        bar.GetBarSet(1, 1).SetTitle("Cyan");
        bar.GetBarSet(1, 2).SetTitle("Magenta");

        // Setup the vertical axis for a labeled bar chart.
        System.String[] labels = new System.String[]{"New York", "Texas",
         "Northern\nCalifornia", "Southern\nCalifornia", "Colorado",
         "New Jersey"};
        bar.SetLabels(labels, Imsl.Chart2D.Bar.BAR_TYPE_VERTICAL);

        //    Set the text angle
        axis.AxisX.AxisLabel.TextAngle = 270;

        //    Set the Y axis title
        axis.AxisY.AxisTitle.SetTitle("Sales ($million)\nby " + "widget color");

    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new BarEx1());
    }
}
```

Output



BarSet Class

```
public class Imsl.Chart2D.BarSet :  ChartNode
```

A set of bars in a bar chart.

A `BarSet` is created by Imsl.Chart2D.Bar (p. 1464) and contains a collection of Imsl.Chart2D.BarItem (p. 1473). `Bar` creates a `BarSet` for each stack-group combination. Each `BarSet` contains the `BarItems` for that combination. Normally all of the `BarItems` in a `BarSet` have the same color, title, etc.

Methods

GetBarItem

```
virtual public Imsl.Chart2D.BarItem[] GetBarItem()
```

Description

Returns an array of `BarItems`.

Returns

A `BarItem[]` that contains the `BarItem` attribute value.

Remarks

This is the collection of all `BarItems` contained in this bar group.

GetBarItem

```
virtual public Imsl.Chart2D.BarItem GetBarItem(int index)
```

Description

Returns the `BarItem` given the index.

Parameter

`index` – An `int` which specifies the index.

Returns

A `BarItem` associated with the specified index.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetDataRange

```
virtual public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

Pie Class

```
public class Imsl.Chart2D.Pie :  Axis
```

A pie chart.

The angle of the first slice is determined by the attribute “Reference”.

`Pie` is derived from `Axis`, because it defines its own mapping to device space.

Constructors

Pie

```
public Pie(Imsl.Chart2D.Chart chart)
```

Description

Constructs a `Pie` chart object.

Parameter

`chart` – A `Chart` which specifies the parent of this node.

Remarks

The “Viewport” attribute for this node is set to [0.2,0.8] by [0.2,0.8].

Pie

```
public Pie(Imsl.Chart2D.Chart chart, double[] y)
```

Description

Constructs a `Pie` chart object with a specified number of slices.

Parameters

`chart` – A `Chart` which specifies the parent of this node.

`y` – A `double[]` which contains the values for the pie chart.

Remarks

An array of `y.length` Imsl.Chart2D.PieSlice (p. 1483) nodes are created as children of this node and this array is used to define the attribute “PieSlice” in this node.

The “Viewport” attribute for this node is set to [0.2,0.8] by [0.2,0.8].

Methods

GetPieSlice

```
virtual public Imsl.Chart2D.PieSlice[] GetPieSlice()
```

Description

Returns the `PieSlice` objects.

Returns

A `PieSlice[]` containing the pie slices to be associated with this node.

GetPieSlice

```
virtual public Imsl.Chart2D.PieSlice GetPieSlice(int index)
```

Description

Returns a specified `PieSlice`.

Parameter

`index` – An `int` specifying the pie slice to return.

Returns

A `PieSlice` which contains the specified slice.

Remarks

The “PieSlice” attribute is a 0 based index array.

MapDeviceToUser

```
override public void MapDeviceToUser(int devX, int devY, double[] userXY)
```

Description

Maps the device coordinates `devXY` to user coordinates (`userX`,`userY`).

Parameters

`devX` – An `int` which specifies the device x-coordinate.

`devY` – An `int` which specifies the device y-coordinate.

`userXY` – An `int[2]` in which the the user coordinates are returned.

MapUserToDevice

```
override public void MapUserToDevice(double userX, double userY, int[] devXY)
```

Description

Maps the user coordinates (`userX`,`userY`) to the device coordinates `devXY`.

Parameters

`userX` – A `double` which specifies the user x-coordinate.

`userY` – A `double` which specifies the user y-coordinate.

`devXY` – An `int[2]` in which the device coordinates are returned.

SetData

```
virtual public Imsl.Chart2D.PieSlice[] SetData(double[] y)
```

Description

Changes the data in a Pie chart object.

Parameter

`y` – A `double[]` which contains the values for the pie chart.

Returns

A `PieSlice[]` array containing the updated PieSlice.

Remarks

If the number of slices is unchanged then the existing pie slice array, defined by the attribute “PieSlice” in this node, is reused. If the number is different, a new array is allocated, using the existing `PieSlice` elements to initialize the new array.

SetUpMapping

```
override public void SetUpMapping()
```

Description

Initializes the mappings between user and coordinate space.

Remarks

This must be called whenever the screen size, the window or the viewport may have changed. Generally, it is safest to call this each time the chart is repainted.

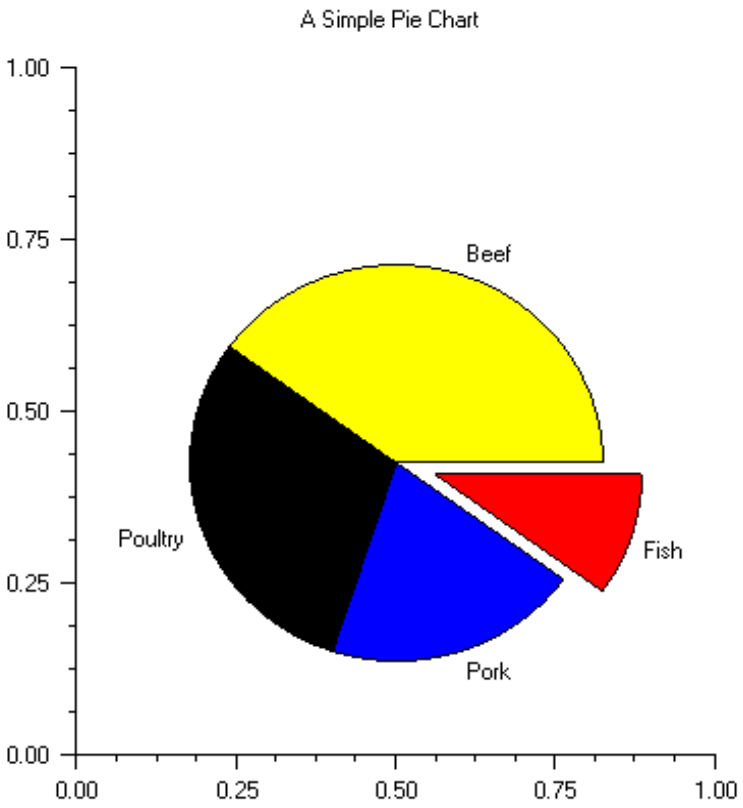
Example: Pie Chart

A simple Pie chart is constructed in this example. Pie slice labels and colors are set and one pie slice is exploded from the center. This class extends FrameChart, which manages the window.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class PieEx1 : FrameChart
{

    public PieEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);
```

```
        //    Create an instance of a Pie Chart
        double[] y = new double[]{10.0, 20.0, 30.0, 40.0};
        Pie pie = new Pie(chart, y);

        //    Set the Pie Chart Title
        chart.ChartTitle.SetTitle("A Simple Pie Chart");

        //    Set the colors of the Pie Slices
        PieSlice[] slice = pie.GetPieSlice();
        slice[0].FillColor = System.Drawing.Color.Red;
        slice[1].FillColor = System.Drawing.Color.Blue;
        slice[2].FillColor = System.Drawing.Color.Black;
        slice[3].FillColor = System.Drawing.Color.Yellow;

        //    Set the Pie Slice Labels
        pie.LabelType = Imsl.Chart2D.Pie.LABEL_TYPE_TITLE;
        slice[0].SetTitle("Fish");
        slice[1].SetTitle("Pork");
        slice[2].SetTitle("Poultry");
        slice[3].SetTitle("Beef");

        //    Explode a Pie Slice
        slice[0].Explode = 0.2;
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new PieEx1());
    }
}
```

Output



PieSlice Class

```
public class Imsl.Chart2D.PieSlice :  Data
```

One wedge of a pie chart.

Imsl.Chart2D.Pie (p. 1479) creates `PieSlice` objects as its children, one per pie wedge. A specific slice can be retrieved using the method Imsl.Chart2D.Pie.GetPieSlice(System.Int32) (p. 1480). All of the slices can be retrieved using the method Imsl.Chart2D.Pie.GetPieSlice (p. 1480).

The drawing of the slice is controlled by the fill attributes (specified with FillType (p. 1322)) in this node.

Methods

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetAngles

```
virtual protected internal void SetAngles(double angleA, double angleB)
```

Description

Sets the angles, in degrees, that determine the extent of this slice.

Parameters

`angleA` – A `double` that specifies the angle, in degrees, at which the slice begins.

`angleB` – A `double` that specifies the angle, in degrees, at which the slice ends.

Dendrogram Class

```
public class Imsl.Chart2D.Dendrogram :  Data
```

A `Dendrogram` chart for cluster analysis.

Properties

Coordinates

```
virtual public double[][] Coordinates {get; set; }
```

Description

The cluster coordinates in the `Dendrogram` object.

Property Value

An `doube[] []` containing the “Coordinates” attribute value.

LeftSons

```
virtual public int[] LeftSons {get; set; }
```

Description

The left sons of each merged cluster.

Property Value

An `int[]` containing the “LeftSons” attribute value.

Levels

```
virtual public double[] Levels {get; set; }
```

Description

Specifies the levels at which the clusters are joined.

Property Value

A `double[]` containing the “Levels” attribute value.

Order

```
virtual public int[] Order {get; set; }
```

Description

The cluster order in the `Dendrogram` object.

Property Value

An `int[]` containing the “Order” attribute value.

RightSons

```
virtual public int[] RightSons {get; set; }
```

Description

The right sons of each merged cluster.

Property Value

An `int[]` containing the “RightSons” attribute value.

Constructors

Dendrogram

```
public Dendrogram(Imsl.Chart2D.AxisXY axis, Imsl.Stat.ClusterHierarchical
clusterHierarchical)
```

Description

Constructs a vertical Dendrogram chart using a supplied ClusterHierarchical object.

Parameters

axis – An AxisXY specifying the parent of this node.

clusterHierarchical – A ClusterHierarchical used as a source object for the Dendrogram.

Dendrogram

```
public Dendrogram(Imsl.Chart2D.AxisXY axis, double[] clusterLevel, int[]
leftSons, int[] rightSons)
```

Description

Constructs a vertical Dendrogram chart using supplied data.

Parameters

axis – An AxisXY specifying the parent of this node.

clusterLevel – A double[] which contains the levels at which the clusters are joined.

leftSons – An int[] which contains the left sons of each merged cluster.

rightSons – An int[] which contains the right sons of each merged cluster.

Dendrogram

```
public Dendrogram(Imsl.Chart2D.AxisXY axis, Imsl.Stat.ClusterHierarchical
clusterHierarchical, int type)
```

Description

Constructs a Dendrogram chart using a supplied ClusterHierarchical object.

Parameters

axis – An AxisXY specifying the parent of this node.

clusterHierarchical – A ClusterHierarchical object used as a source for the Dendrogram.

type – An int which specifies the Dendrogram type.

Remarks

The types possible types of Dendrograms are DENDROGRAM_TYPE_VERTICAL (p. 1313) and DENDROGRAM_TYPE_HORIZONTAL (p. 1313).

Dendrogram

```
public Dendrogram(Imsl.Chart2D.AxisXY axis, double[] clusterLevel, int[]
leftSons, int[] rightSons, int type)
```

Description

Constructs a Dendrogram chart using supplied data.

Parameters

`axis` – An `AxisXY` specifying the parent of this node.

`clusterLevel` – A `double[]` which contains the levels at which the clusters are joined.

`leftSons` – An `int[]` which contains the left sons of each merged cluster.

`rightSons` – An `int[]` which contains the right sons of each merged cluster.

`type` – An `int` which specifies the `Dendrogram` type.

Remarks

The types possible types of `Dendrograms` are DENDROGRAM_TYPE_VERTICAL (p. 1313) and DENDROGRAM_TYPE_HORIZONTAL (p. 1313).

Methods

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetLabels

```
virtual public void SetLabels(string[] labels)
```

Description

Sets up the axis labels for `Dendrogram` plot.

Parameter

`labels` – A `String[]` containing the axis labels.

Remarks

The number of labels must equal the number of items.

This method turns off autoscaling on the axis and sets the Window attribute depending on the number of points being plotted.

Note that user-defined labels will be re-ordered to match the order of the clusters displayed in the plot.

SetLineColors

```
virtual public void SetLineColors(System.Drawing.Color[] colors)
```

Description

Define colors for individual clusters.

Parameter

`colors` – A `Color[]` which contains each color to use for the subclusters.

Remarks

The color of the top most level should be set using `Dendrogram.LineColor`. This property will color N clusters, where N is the number of elements in `colors`.

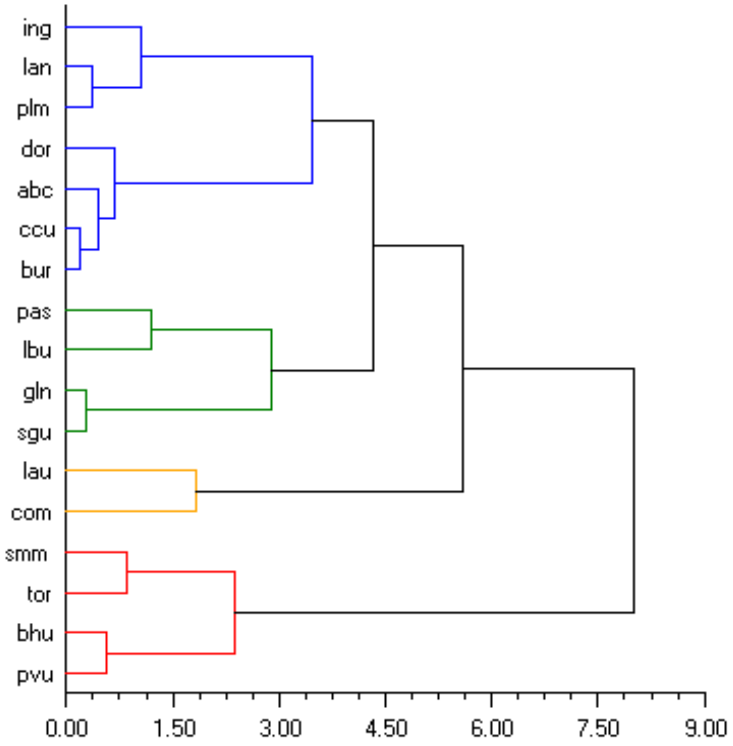
Example: Dendrogram

A Dendrogram.

```
using Imsl.Chart2D;
using Imsl.Stat;
using System;
using System.Windows.Forms;
using System.Drawing;

public class DendrogramEx1 : FrameChart
{
    public DendrogramEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);
        double[,] data = {{.38, 626.5, 601.3, 605.3},
                          {.18, 654.0, 647.1, 641.8},
                          {.07, 677.2, 676.5, 670.5},
                          {.09, 639.9, 640.3, 636.0},
                          {.19, 614.7, 617.3, 606.2},
                          {.12, 670.2, 666.0, 659.3},
                          {.20, 651.1, 645.2, 643.4},
                          {.41, 645.4, 645.8, 644.8},
                          {.07, 683.5, 682.9, 674.3},
                          {.39, 648.6, 647.8, 643.1},
                          {.21, 650.4, 650.8, 643.9},
                          {.24, 637.0, 636.9, 626.5},
```

```
                        {.09, 641.1, 628.8, 629.4},
                        {.12, 638.0, 627.7, 628.6},
                        {.11, 661.4, 659.0, 651.8},
                        {.22, 646.4, 646.2, 647.0},
                        {.33, 634.1, 632.0, 627.8}};

        System.String[] lab = new System.String[]{"lau", "ccu", "bhu", "ing",
         "com", "smm", "bur", "gln", "pvu", "sgu", "abc", "pas", "lan", "plm",
         "tor", "dor", "lbu"};

        Dissimilarities dist = new Dissimilarities(data, 0, 1, 1);
        double[,] distanceMatrix = dist.DistanceMatrix;
        ClusterHierarchical clink = new ClusterHierarchical(
           dist.DistanceMatrix, 4, 0);

        int nClusters = 4;
        int[] iclus = clink.GetClusterMembership(nClusters);
        int[] nclus = clink.GetObsPerCluster(nClusters);

        // use either method below to create the chart
        Dendrogram dc = new Dendrogram(axis, clink,
           Data.DENDROGRAM_TYPE_HORIZONTAL);

        dc.SetLabels(lab);
        dc.SetLineColors(new Color[] {Color.Blue, Color.Green, Color.Red,
                                      Color.Orange});
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new DendrogramEx1());
    }
}
```

Output



Example: Dendrogram

A Dendrogram.

```
using Imsl.Chart2D;
using Imsl.Stat;
using System;
using System.Windows.Forms;
using System.Drawing;

public class DendrogramEx2 : FrameChart
{
    public DendrogramEx2()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);
```

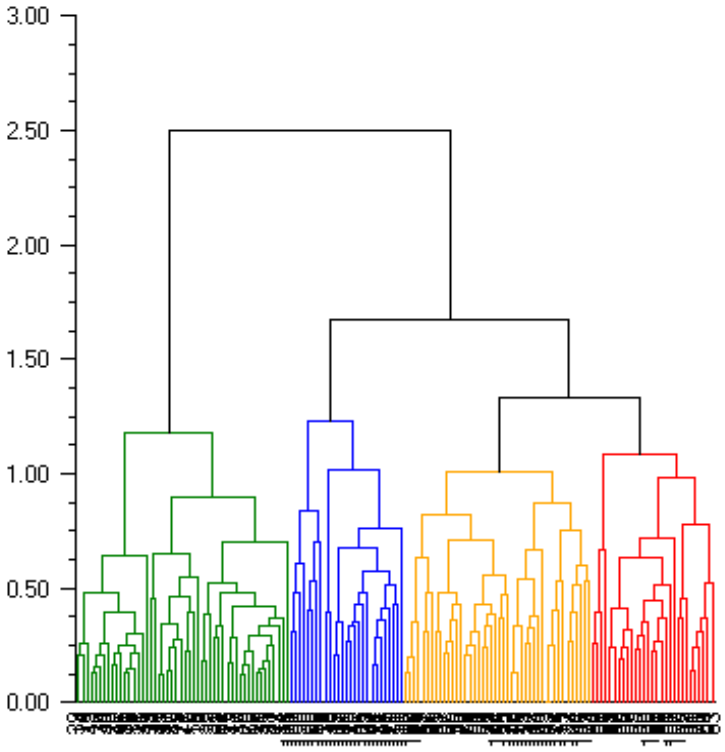
```
double[,] data = {{5.1, 3.5, 1.4, .2},
                  {4.9, 3.0, 1.4, .2},
                  {4.7, 3.2, 1.3, .2},
                  {4.6, 3.1, 1.5, .2},
                  {5.0, 3.6, 1.4, .2},
                  {5.4, 3.9, 1.7, .4},
                  {4.6, 3.4, 1.4, .3},
                  {5.0, 3.4, 1.5, .2},
                  {4.4, 2.9, 1.4, .2},
                  {4.9, 3.1, 1.5, .1},
                  {5.4, 3.7, 1.5, .2},
                  {4.8, 3.4, 1.6, .2},
                  {4.8, 3.0, 1.4, .1},
                  {4.3, 3.0, 1.1, .1},
                  {5.8, 4.0, 1.2, .2},
                  {5.7, 4.4, 1.5, .4},
                  {5.4, 3.9, 1.3, .4},
                  {5.1, 3.5, 1.4, .3},
                  {5.7, 3.8, 1.7, .3},
                  {5.1, 3.8, 1.5, .3},
                  {5.4, 3.4, 1.7, .2},
                  {5.1, 3.7, 1.5, .4},
                  {4.6, 3.6, 1.0, .2},
                  {5.1, 3.3, 1.7, .5},
                  {4.8, 3.4, 1.9, .2},
                  {5.0, 3.0, 1.6, .2},
                  {5.0, 3.4, 1.6, .4},
                  {5.2, 3.5, 1.5, .2},
                  {5.2, 3.4, 1.4, .2},
                  {4.7, 3.2, 1.6, .2},
                  {4.8, 3.1, 1.6, .2},
                  {5.4, 3.4, 1.5, .4},
                  {5.2, 4.1, 1.5, .1},
                  {5.5, 4.2, 1.4, .2},
                  {4.9, 3.1, 1.5, .2},
                  {5.0, 3.2, 1.2, .2},
                  {5.5, 3.5, 1.3, .2},
                  {4.9, 3.6, 1.4, .1},
                  {4.4, 3.0, 1.3, .2},
                  {5.1, 3.4, 1.5, .2},
                  {5.0, 3.5, 1.3, .3},
                  {4.5, 2.3, 1.3, .3},
                  {4.4, 3.2, 1.3, .2},
                  {5.0, 3.5, 1.6, .6},
                  {5.1, 3.8, 1.9, .4},
                  {4.8, 3.0, 1.4, .3},
                  {5.1, 3.8, 1.6, .2},
                  {4.6, 3.2, 1.4, .2},
                  {5.3, 3.7, 1.5, .2},
                  {5.0, 3.3, 1.4, .2},
                  {7.0, 3.2, 4.7, 1.4},
                  {6.4, 3.2, 4.5, 1.5},
                  {6.9, 3.1, 4.9, 1.5},
                  {5.5, 2.3, 4.0, 1.3},
                  {6.5, 2.8, 4.6, 1.5},
```

```
{5.7, 2.8, 4.5, 1.3},
{6.3, 3.3, 4.7, 1.6},
{4.9, 2.4, 3.3, 1.0},
{6.6, 2.9, 4.6, 1.3},
{5.2, 2.7, 3.9, 1.4},
{5.0, 2.0, 3.5, 1.0},
{5.9, 3.0, 4.2, 1.5},
{6.0, 2.2, 4.0, 1.0},
{6.1, 2.9, 4.7, 1.4},
{5.6, 2.9, 3.6, 1.3},
{6.7, 3.1, 4.4, 1.4},
{5.6, 3.0, 4.5, 1.5},
{5.8, 2.7, 4.1, 1.0},
{6.2, 2.2, 4.5, 1.5},
{5.6, 2.5, 3.9, 1.1},
{5.9, 3.2, 4.8, 1.8},
{6.1, 2.8, 4.0, 1.3},
{6.3, 2.5, 4.9, 1.5},
{6.1, 2.8, 4.7, 1.2},
{6.4, 2.9, 4.3, 1.3},
{6.6, 3.0, 4.4, 1.4},
{6.8, 2.8, 4.8, 1.4},
{6.7, 3.0, 5.0, 1.7},
{6.0, 2.9, 4.5, 1.5},
{5.7, 2.6, 3.5, 1.0},
{5.5, 2.4, 3.8, 1.1},
{5.5, 2.4, 3.7, 1.0},
{5.8, 2.7, 3.9, 1.2},
{6.0, 2.7, 5.1, 1.6},
{5.4, 3.0, 4.5, 1.5},
{6.0, 3.4, 4.5, 1.6},
{6.7, 3.1, 4.7, 1.5},
{6.3, 2.3, 4.4, 1.3},
{5.6, 3.0, 4.1, 1.3},
{5.5, 2.5, 4.0, 1.3},
{5.5, 2.6, 4.4, 1.2},
{6.1, 3.0, 4.6, 1.4},
{5.8, 2.6, 4.0, 1.2},
{5.0, 2.3, 3.3, 1.0},
{5.6, 2.7, 4.2, 1.3},
{5.7, 3.0, 4.2, 1.2},
{5.7, 2.9, 4.2, 1.3},
{6.2, 2.9, 4.3, 1.3},
{5.1, 2.5, 3.0, 1.1},
{5.7, 2.8, 4.1, 1.3},
{6.3, 3.3, 6.0, 2.5},
{5.8, 2.7, 5.1, 1.9},
{7.1, 3.0, 5.9, 2.1},
{6.3, 2.9, 5.6, 1.8},
{6.5, 3.0, 5.8, 2.2},
{7.6, 3.0, 6.6, 2.1},
{4.9, 2.5, 4.5, 1.7},
{7.3, 2.9, 6.3, 1.8},
{6.7, 2.5, 5.8, 1.8},
{7.2, 3.6, 6.1, 2.5},
{6.5, 3.2, 5.1, 2.0},
```

```
                        {6.4, 2.7, 5.3, 1.9},
                        {6.8, 3.0, 5.5, 2.1},
                        {5.7, 2.5, 5.0, 2.0},
                        {5.8, 2.8, 5.1, 2.4},
                        {6.4, 3.2, 5.3, 2.3},
                        {6.5, 3.0, 5.5, 1.8},
                        {7.7, 3.8, 6.7, 2.2},
                        {7.7, 2.6, 6.9, 2.3},
                        {6.0, 2.2, 5.0, 1.5},
                        {6.9, 3.2, 5.7, 2.3},
                        {5.6, 2.8, 4.9, 2.0},
                        {7.7, 2.8, 6.7, 2.0},
                        {6.3, 2.7, 4.9, 1.8},
                        {6.7, 3.3, 5.7, 2.1},
                        {7.2, 3.2, 6.0, 1.8},
                        {6.2, 2.8, 4.8, 1.8},
                        {6.1, 3.0, 4.9, 1.8},
                        {6.4, 2.8, 5.6, 2.1},
                        {7.2, 3.0, 5.8, 1.6},
                        {7.4, 2.8, 6.1, 1.9},
                        {7.9, 3.8, 6.4, 2.0},
                        {6.4, 2.8, 5.6, 2.2},
                        {6.3, 2.8, 5.1, 1.5},
                        {6.1, 2.6, 5.6, 1.4},
                        {7.7, 3.0, 6.1, 2.3},
                        {6.3, 3.4, 5.6, 2.4},
                        {6.4, 3.1, 5.5, 1.8},
                        {6.0, 3.0, 4.8, 1.8},
                        {6.9, 3.1, 5.4, 2.1},
                        {6.7, 3.1, 5.6, 2.4},
                        {6.9, 3.1, 5.1, 2.3},
                        {5.8, 2.7, 5.1, 1.9},
                        {6.8, 3.2, 5.9, 2.3},
                        {6.7, 3.3, 5.7, 2.5},
                        {6.7, 3.0, 5.2, 2.3},
                        {6.3, 2.5, 5.0, 1.9},
                        {6.5, 3.0, 5.2, 2.0},
                        {6.2, 3.4, 5.4, 2.3},
                        {5.9, 3.0, 5.1, 1.8}};

    Dissimilarities dist = new Dissimilarities(data, 0, 1, 1);
    double[,] distanceMatrix = dist.DistanceMatrix;
    ClusterHierarchical clink = new ClusterHierarchical(dist.DistanceMatrix,
     2, 0);

    int nClusters = 4;
    int[] iclus = clink.GetClusterMembership(nClusters);
    int[] nclus = clink.GetObsPerCluster(nClusters);

    // use either method below to create the chart
//    Dendrogram dc = new Dendrogram(axis, clink);
    Dendrogram dc = new Dendrogram(axis, clink.ClusterLevel,
       clink.ClusterLeftSons, clink.ClusterRightSons);

    // set colors
    dc.SetLineColors(new Color[] {Color.Blue, Color.Green, Color.Red,
```

```
                                        Color.Orange});
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new DendrogramEx2());
    }
}
```

Output



Polar Class

```
public class Imsl.Chart2D.Polar :  Axis
```

This `Axis` node is used for polar charts.

In a polar plot, the (x,y) coordinates in Imsl.Chart2D.Data (p. 1372) nodes are interpreted as (r,theta) values. The “Viewport” attribute for this node is set to [0.1,0.9] by [0.1,0.9].

Properties

AxisR

```
virtual public Imsl.Chart2D.AxisR AxisR {get; }
```

Description

Return the radius axis node.

AxisTheta

```
virtual public Imsl.Chart2D.AxisTheta AxisTheta {get; }
```

Description

Returns the angular axis node.

GridPolar

```
virtual public Imsl.Chart2D.GridPolar GridPolar {get; }
```

Description

A grid for the polar plot.

Constructor

Polar

```
public Polar(Imsl.Chart2D.Chart chart)
```

Description

Creates a `Polar` object.

Parameter

`chart` – A `Chart` which specifies the parent of this node.

Methods

MapDeviceToUser

```
override public void MapDeviceToUser(int devX, int devY, double[] userRT)
```

Description

Map the device coordinates to polar coordinates.

Parameters

`devX` – An `int` which specifies the device x-coordinate.

`devY` – An `int` which specifes the device y-coordinate.

`userRT` – A `double[2]` in which the user coordinates, (radius,theta), are returned.

MapUserToDevice

```
override public void MapUserToDevice(double userRadius, double userTheta, int[]
devXY)
```

Description

Map the polar coordinates (`userRadius`,`userAngle`) to the device coordinates `devXY`.

Parameters

`userRadius` – A `double` which specifies the user radius coordinate.

`userTheta` – A `double` which specifies the user angle coordinate.

`devXY` – An `int[2]` in which the device coordinates are returned.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetUpMapping

```
override public void SetUpMapping()
```

Description

Initializes the mappings between user and coordinate space.

Remarks

This must be called whenever the screen size, the window or the viewport may have changed.

Heatmap Class

```
public class Imsl.Chart2D.Heatmap :  Data
```

`Heatmap` creates a chart from a two-dimensional array of double precision values or Color values.

Optionally, each cell in the heatmap can be labeled.

If the input is a two-dimensional array of `double` values then a `Colormap` object is used to map the real values to colors.

See Also

Imsl.Chart2D.Heatmap.Colormap (p. 1497)

Properties

Colormap

```
virtual public Imsl.Chart2D.Colormap Colormap {get; set; }
```

Description

Specifies the value of the “Colormap” attribute.

Property Value

A `Colormap` which contains the value of the “Colormap” attribute.

Remarks

This is the `Colormap` associated with this `Heatmap`. By default, `Colormap = null`.

HeatmapLegend

```
virtual public Imsl.Chart2D.Heatmap.Legend HeatmapLegend {get; }
```

Description

Specifies the heatmap legend.

Property Value

A `Legend` object associated with the `Heatmap`.

Remarks

By default, the legend is not drawn because the IsVisible (p. 1302) property is set to `false`. To show the legend set `heatmap.HeatmapLegend.IsVisisble = true;`

Constructors

Heatmap

```
public Heatmap(Imsl.Chart2D.AxisXY axis, double xmin, double xmax, double ymin,
double ymax, System.Drawing.Color[,] color)
```

Description

Creates a `Heatmap` from an array of `Color` values.

Parameters

`axis` – An `AxisXY` which contains the parent of this node.

`xmin` – A `double` which specifies the minimum *x*-value of the color data.

`xmax` – A `double` which specifies the maximum *x*-value of the color data.

`ymin` – A `double` which specifies the minimum *y*-value of the color data.

`ymax` – A `double` which specifies the maximum *y*-value of the color data.

`color` – A `Color[,]` which specifies the color values.

Remarks

The value of `color[0,0]` is the color of the cell whose lower left corner is (`xmin`, `ymin`).

Heatmap

```
public Heatmap(Imsl.Chart2D.AxisXY axis, double xmin, double xmax, double ymin,
double ymax, double zmin, double zmax, double[,] data, Imsl.Chart2D.Colormap
colormap)
```

Description

Creates a `Heatmap` from a `double[,]` and a `Colormap`.

Parameters

`axis` – An `AxisXY` object which specifes the parent of this node.

`xmin` – A `double` which specifies the minimum *x*-value of the color data.

`xmax` – A `double` which specifies the maximum *x*-value of the color data.

`ymin` – A `double` which specifies the minimum *y*-value of the color data.

`ymax` – A `double` which specifies the maximum *y*-value of the color data.

`zmin` – A `double` which specifies the data value that corresponds to the initial (*t*=0) value in the `Colormap`.

`zmax` – A `double` which specifies the data value that corresponds to the final (*t*=1) value in the `Colormap`.

`data` – A `double[,]` containing the data values.

`colormap` – Maps the values in `data` to colors.

Remarks

The *x*-interval (`xmin`, `xmax`) is uniformly divided and mapped into the first index of `data`. The *y*-interval (`ymin`, `ymax`) is uniformly divided and mapped into the second index of `data`. So, the value of `data[0,0]` is used to determine the color of the cell whose lower left corner is (`xmin`, `ymin`).

If a cell has a data value equal to *t* then its color is the value of the colormap at *s*, where

$$s = \frac{t - \text{zmin}}{\text{zmax} - \text{zmin}}$$

.

Methods

GetHeatmapLabels

```
virtual public Imsl.Chart2D.Text[,] GetHeatmapLabels()
```

Description

Returns the value of the "HeatmapLabels" attribute.

Returns

A `Text[,]` that contains the values of the "HeatmapLabels" attribute.

Remarks

By default, `GetHeatmapLabels = null`.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetHeatmapLabels

```
virtual public void SetHeatmapLabels(string[,] labels)
```

Description

Sets the value of the “HeatmapLabels” attribute.

Parameter

`labels` – A `string[,]` used to create a `Text[,]` that specifies the `Heatmap` labels.

Remarks

Each `Text` object is created from the corresponding label value with `TEXT_X_CENTER|TEXT_Y_CENTER` alignment.

See Also: Imsl.Chart2D.Text (p. 1385), TEXT_X_CENTER (p. 1317), TEXT_Y_CENTER (p. 1318)

SetHeatmapLabels

```
virtual public void SetHeatmapLabels(Imsl.Chart2D.Text[,] labels)
```

Description

Sets the value of the “HeatmapLabels” attribute.

Parameter

`labels` – A `Text[,]` that specifies the `Heatmap` labels.

Remarks

The default alignment for `Text` is `TEXT_X_CENTER|TEXT_Y_CENTER`.

See Also: Imsl.Chart2D.Text (p. 1385), TEXT_X_CENTER (p. 1317), TEXT_Y_CENTER (p. 1318)

Example: Heatmap from Color array

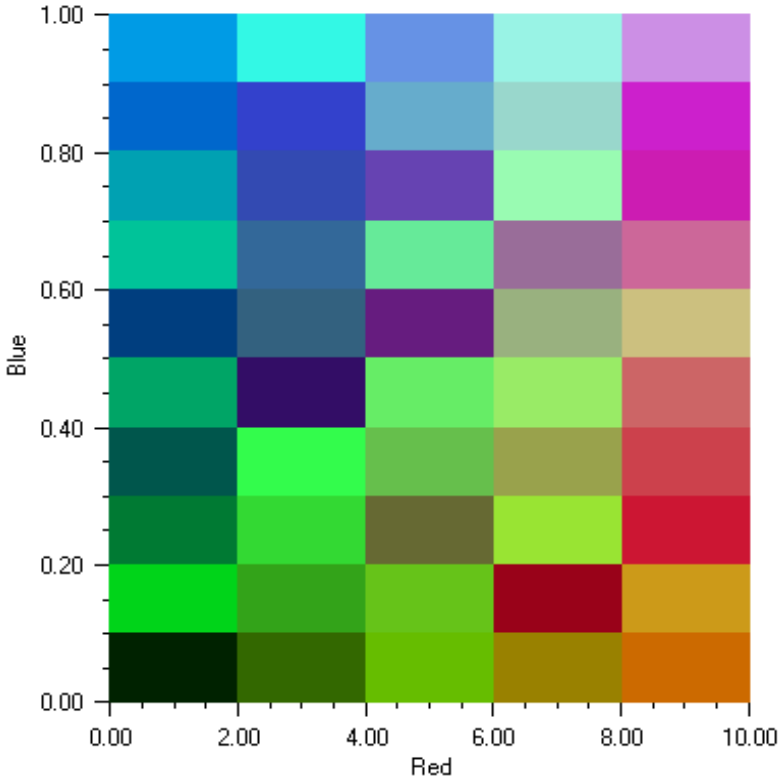
A 5 by 10 array of `Color` objects is created by linearly interpolating red along the x-axis, blue along the y-axis and mixing in a random amount of green. The data range is set to [0,10] by [0,1].

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class HeatmapEx1 : FrameChart
{
    public HeatmapEx1()
    {
        Chart chart = this.Chart;

        AxisXY axis = new AxisXY(chart);
```

```
double xmin = 0.0;
double xmax = 10.0;
double ymin = 0.0;
double ymax = 1.0;

int nxRed = 5;
int nyBlue = 10;

System.Random random = new System.Random((System.Int32) 123457L);
System.Drawing.Color[,] color = new System.Drawing.Color[nxRed,nyBlue];

int z=0;
int []d=new int[50];
d[0]=34;
d[1]=212;
d[2]=122;
d[3]=86;
d[4]=165;
d[5]=62;
d[6]=195;
d[7]=161;
d[8]=103;
d[9]=155;
d[10]=104;
d[11]=163;
d[12]=217;
d[13]=252;
d[14]=13;
d[15]=97;
d[16]=104;
d[17]=74;
d[18]=65;
d[19]=248;
d[20]=189;
d[21]=195;
d[22]=105;
d[23]=191;
d[24]=237;
d[25]=28;
d[26]=234;
d[27]=67;
d[28]=172;
d[29]=146;
d[30]=129;
d[31]=2;
d[32]=228;
d[33]=162;
d[34]=235;
d[35]=177;
d[36]=109;
d[37]=251;
d[38]=215;
d[39]=243;
d[40]=106;
d[41]=154;
```

```
        d[42]=22;
        d[43]=65;
        d[44]=101;
        d[45]=192;
        d[46]=103;
        d[47]=28;
        d[48]=32;
        d[49]=143;

        for (int i = 0; i < nxRed; i++)
        {
            for (int j = 0; j < nyBlue; j++)
            {
                int r = (int) (255.0 * i / nxRed);
                //
                int g =d[z];
                z++;

                int b = (int) (255.0 * j / nyBlue);
                color[i,j] = System.Drawing.Color.FromArgb(r, g, b);
            }
        }
        Heatmap heatmap = new Heatmap(axis, xmin, xmax, ymin, ymax, color);
        axis.AxisX.AxisTitle.SetTitle("Red");
        axis.AxisY.AxisTitle.SetTitle("Blue");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new HeatmapEx1());
    }
}
```

Output



Example: Heatmap from Color array

A 5 by 10 data array is created by linearly interpolating from the lower left corner to the upper right corner and adding in a uniform random variable. A red temperature color map is used. This maps the minimum data value to light green and the maximum data value to dark green.

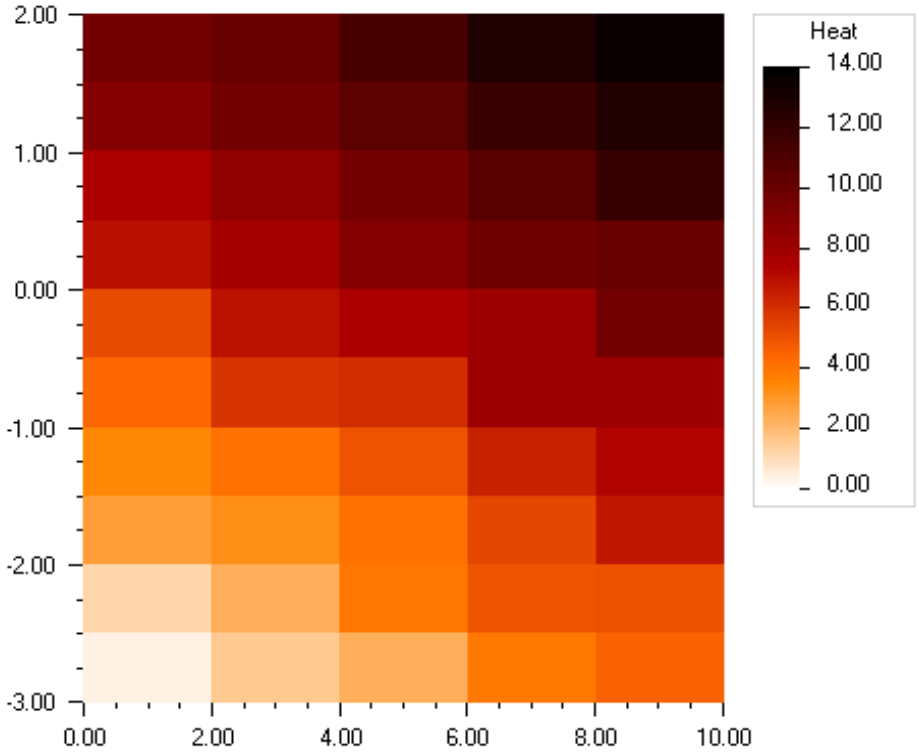
The legend is enabled by setting its paint attribute to true.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class HeatmapEx2 : FrameChart
{
    public HeatmapEx2()
    {
```

```
Chart chart = this.Chart;

AxisXY axis = new AxisXY(chart);

int nx = 5;
int ny = 10;
double xmin = 0.0;
double xmax = 10.0;
double ymin = - 3.0;
double ymax = 2.0;
double fmax = nx + ny - 1;

double[,] data = new double[nx,ny];

System.Random random = new System.Random((System.Int32) 123457L);

double[] dbl = new double[50];
dbl[0]=0.41312962995625035;
dbl[1]=0.15995876895053263;
dbl[2]=0.8225528716547005;
dbl[3]=0.48794367683379836;
dbl[4]=0.44364905186692527;
dbl[5]=0.20896329070872555;
dbl[6]=0.9887088342522812;
dbl[7]=0.4781765623804778;
dbl[8]=0.9647868112234352;
dbl[9]=0.6732389937186418;
dbl[10]=0.5668831243079411;
dbl[11]=0.33081942994459734;
dbl[12]=0.27386697614898103;
dbl[13]=0.10880787186704965;
dbl[14]=0.8805853693809824;
dbl[15]=0.901138442534768;
dbl[16]=0.7180829622748057;
dbl[17]=0.48723656383264413;
dbl[18]=0.6153607537410654;
dbl[19]=0.10153552805288812;
dbl[20]=0.3158193853638753;
dbl[21]=0.9558058275075961;
dbl[22]=0.10778543304578747;
dbl[23]=0.011829287599608884;
dbl[24]=0.09275375134615693;
dbl[25]=0.4859902873228249;
dbl[26]=0.9817642781628322;
dbl[27]=0.5505301300240635;
dbl[28]=0.467363186309925;
dbl[29]=0.18652444274911184;
dbl[30]=0.9066980293517674;
dbl[31]=0.9272326533193322;
dbl[32]=0.31440695305815347;
dbl[33]=0.4215880116306273;
dbl[34]=0.9991560762956562;
dbl[35]=0.0386317648903991;
dbl[36]=0.785150345014761;
dbl[37]=0.6451521871931544;
dbl[38]=0.7930129038729785;
```

```
        dbl[39]=0.819301055474355;
        dbl[40]=0.5695413465811706;
        dbl[41]=0.039285689951912395;
        dbl[42]=0.7625752595574732;
        dbl[43]=0.31325564481720314;
        dbl[44]=0.0482465474704169;
        dbl[45]=0.6272275622766595;
        dbl[46]=0.09904819350827354;
        dbl[47]=0.8934533907186641;
        dbl[48]=0.7013979421419555;
        dbl[49]=0.5212913217641422;

        int z=0;
        for (int i = 0; i < nx; i++)
        {
            for (int j = 0; j < ny; j++)
            {
                data[i,j] = i + j + dbl[z];
                z++;
            }
        }
        Heatmap heatmap = new Heatmap(axis, xmin, xmax, ymin, ymax, 0.0, fmax,
         data, Imsl.Chart2D.Colormap_Fields.RED_TEMPERATURE);
        heatmap.HeatmapLegend.IsVisible = true;
        heatmap.HeatmapLegend.SetTitle("Heat");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new HeatmapEx2());
    }
}
```

Output



Example: Heatmap with Labels

A 5 by 10 array of random data is created and a similarly sized array of strings is also created. These labels contain spreadsheet-like indices and the random data value expressed as a percentage.

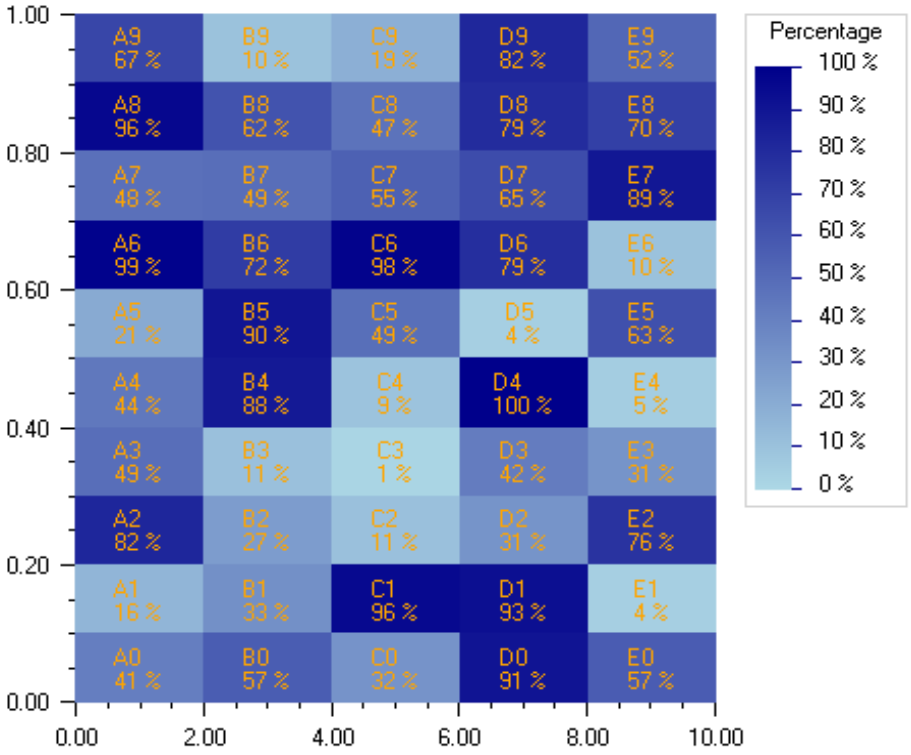
The legend is enabled by setting its paint attribute to true. The tick marks in the legend are formatted using the percentage `NumberFormat` object. A title is also set in the legend.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class HeatmapEx3 : FrameChart
{
    public HeatmapEx3()
    {
```

```
Chart chart = this.Chart;

AxisXY axis = new AxisXY(chart);

double xmin = 0.0;
double xmax = 10.0;
double ymin = 0.0;
double ymax = 1.0;

int nx = 5;
int ny = 10;
double[,] data = new double[nx,ny];

System.String[,] labels = new System.String[nx,ny];
System.Random random = new System.Random((System.Int32) 123457L);

double[] dbl = new double[50];
dbl[0]=0.41312962995625035;
dbl[1]=0.15995876895053263;
dbl[2]=0.8225528716547005;
dbl[3]=0.48794367683379836;
dbl[4]=0.44364905186692527;
dbl[5]=0.20896329070872555;
dbl[6]=0.9887088342522812;
dbl[7]=0.4781765623804778;
dbl[8]=0.9647868112234352;
dbl[9]=0.6732389937186418;
dbl[10]=0.5668831243079411;
dbl[11]=0.33081942994459734;
dbl[12]=0.27386697614898103;
dbl[13]=0.10880787186704965;
dbl[14]=0.8805853693809824;
dbl[15]=0.901138442534768;
dbl[16]=0.7180829622748057;
dbl[17]=0.48723656383264413;
dbl[18]=0.6153607537410654;
dbl[19]=0.10153552805288812;
dbl[20]=0.3158193853638753;
dbl[21]=0.9558058275075961;
dbl[22]=0.10778543304578747;
dbl[23]=0.011829287599608884;
dbl[24]=0.09275375134615693;
dbl[25]=0.4859902873228249;
dbl[26]=0.9817642781628322;
dbl[27]=0.5505301300240635;
dbl[28]=0.467363186309925;
dbl[29]=0.18652444274911184;
dbl[30]=0.9066980293517674;
dbl[31]=0.9272326533193322;
dbl[32]=0.31440695305815347;
dbl[33]=0.4215880116306273;
dbl[34]=0.9991560762956562;
dbl[35]=0.0386317648903991;
dbl[36]=0.785150345014761;
dbl[37]=0.6451521871931544;
dbl[38]=0.7930129038729785;
```

```
        dbl[39]=0.819301055474355;
        dbl[40]=0.5695413465811706;
        dbl[41]=0.039285689951912395;
        dbl[42]=0.7625752595574732;
        dbl[43]=0.31325564481720314;
        dbl[44]=0.0482465474704169;
        dbl[45]=0.6272275622766595;
        dbl[46]=0.09904819350827354;
        dbl[47]=0.8934533907186641;
        dbl[48]=0.7013979421419555;
        dbl[49]=0.5212913217641422;

        int z=0;
        for (int i = 0; i < nx; i++)
        {
            for (int j = 0; j < ny; j++)
            {
                data[i,j] = dbl[z];//random.NextDouble();
                z++;
                labels[i,j] = "ABCDE"[i] + System.Convert.ToString(j) + "\n" +
               data[i,j].ToString("P0");
            }
        }
        Heatmap heatmap = new Heatmap(axis, xmin, xmax, ymin, ymax, 0.0, 1.0,
         data, Imsl.Chart2D.Colormap_Fields.BLUE);
        heatmap.SetHeatmapLabels(labels);
        heatmap.TextColor =  System.Drawing.Color.FromName("orange");
        heatmap.HeatmapLegend.IsVisible = true;
        heatmap.HeatmapLegend.TextFormat = "P0";
        heatmap.HeatmapLegend.SetTitle("Percentage");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new HeatmapEx3());
    }
}
```

Output



Heatmap.Legend Class

```
public class Imsl.Chart2D.Heatmap.Legend :  AxisXY
```

A legend for use with a `Heatmap`.

This `Legend` should be used with `Heatmaps`, rather than the usual `Chart` legend. The “Viewport” attribute for this node is set to [0.83,0.98] by [0.1,0.6].

Method

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node's parent.

Treemap Class

```
public class Imsl.Chart2D.Treemap :  Data
```

`Treemap` creates a chart from two arrays of double precision values or one data array and one array of Color values. The size of each element is scaled using the first input array. The second array of values or colors is used to shade the corresponding area.

The algorithm is adapted from Bruls, Mark and Huizing, Kees and Wijk, Jarke J. van (2000) *Squarified Treemaps*. In Proceedings of the Joint Eurographics and IEEE TCVG Symposium on Visualization.

See Also

Imsl.Chart2D.Treemap.Colormap (p. 1510)

Properties

Colormap

```
virtual public Imsl.Chart2D.Colormap Colormap {get; set; }
```

Description

Specifies the value of the "Colormap" attribute.

Property Value

A `Colormap` which contains the value of the "Colormap" attribute.

Default: `Colormap = Colormap_Fields.BLUE`.

Remarks

This is the `Colormap` associated with this `Treemap`.

Orientation

```
virtual public Imsl.Chart2D.Treemap.OrientationMethod Orientation {get; set; }
```

Description

Specifies the value of the “Orientation” attribute.

Property Value

One of `Treemap.OrientationMethod.Automatic`, `Treemap.OrientationMethod.RowFirst`, or `TreemapOrientationMethod.ColumnFirst`. The default behavior is `TreemapOrientationMethod.Automatic` and filling the graph is based on the aspect ratio of the parent `Axis` object such that if the height is less than the width, columns are drawn first; otherwise rows are drawn first.

Default: `Orientation = TreemapOrientationMethod.Automatic`.

Remarks

This is the `Orientation` associated with this `Treemap`.

TreemapLegend

```
virtual public Imsl.Chart2D.Treemap.Legend TreemapLegend {get; }
```

Description

Specifies the treemap legend.

Property Value

A `Legend` object associated with the `Treemap`.

Remarks

Default: The legend is not drawn because the IsVisible (p. 1302) property is set to `false`. To show the legend set `treemap.TreemapLegend.IsVisisble = true;`

Constructors

Treemap

```
public Treemap(Imsl.Chart2D.AxisXY axis, double[] data, System.Drawing.Color[]
colors)
```

Description

Constructs a treemap using supplied data and color array.

Parameters

`axis` – The `AxisXY` parent of this node.

`data` – A `double` array containing the area values for the treemap in decreasing order.

`colors` – A `Color` array, one for each area.

Treemap

```
public Treemap(Imsl.Chart2D.AxisXY axis, double[] data, double[] shades,
Imsl.Chart2D.Colormap colormap)
```

Description

Constructs a treemap using the supplied data and a colormap.

Parameters

`axis` – The `AxisXY` parent of this node.

`data` – A `double` array containing the area values for the treemap in decreasing order.

`shades` – An array of `double` values to use for shading each area.

`colormap` – A `Colormap` to use for the shading.

Methods

GetTreemapLabels

```
virtual public Imsl.Chart2D.Text[] GetTreemapLabels()
```

Description

Returns the value of the “TreemapLabels” attribute.

Returns

A `Text[]` that contains the values of the “TreemapLabels” attribute. If the Labels have not been set, this method retuns `null`.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children.

Parameter

`draw` – A `Draw` object which is to be painted.

Remarks

This is normally called only by the `Paint` method in this node’s parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – a `double` array which contains the updated range, {xmin,xmax,ymin,ymax} or {xmin,xmax,ymin,ymax,zmin,zmax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetTreemapLabels

```
virtual public void SetTreemapLabels(string[] labels)
```

Description

Sets the value of the "TreemapLabels" attribute.

Parameter

`labels` – A `String[]` used to create a `Text[]` that specifies the `Treemap` labels.

Remarks

Each `Text` object is created from the corresponding label value with `TEXT_X_CENTER|TEXT_Y_CENTER` alignment.

Default: No labels are drawn.

See Also: Imsl.Chart2D.Text (p. 1385), TEXT_X_CENTER (p. 1317), TEXT_Y_CENTER (p. 1318)

SetTreemapLabels

```
virtual public void SetTreemapLabels(Imsl.Chart2D.Text[] labels)
```

Description

Sets the value of the "TreemapLabels" attribute.

Parameter

`labels` – A `Text[]` that specifies the `Treemap` labels.

Remarks

The default alignment for `Text` is `TEXT_X_CENTER|TEXT_Y_CENTER`.

Befault: No labels are drawn.

See Also: Imsl.Chart2D.Text (p. 1385), TEXT_X_CENTER (p. 1317), TEXT_Y_CENTER (p. 1318)

SetZRange

```
public void SetZRange(double[] range)
```

Description

Set the Z data (shading) range, `range = {zmin,zmax}`.

Parameter

`range` – A `double` array which contains the data range, {`zmin`,`zmax`}.

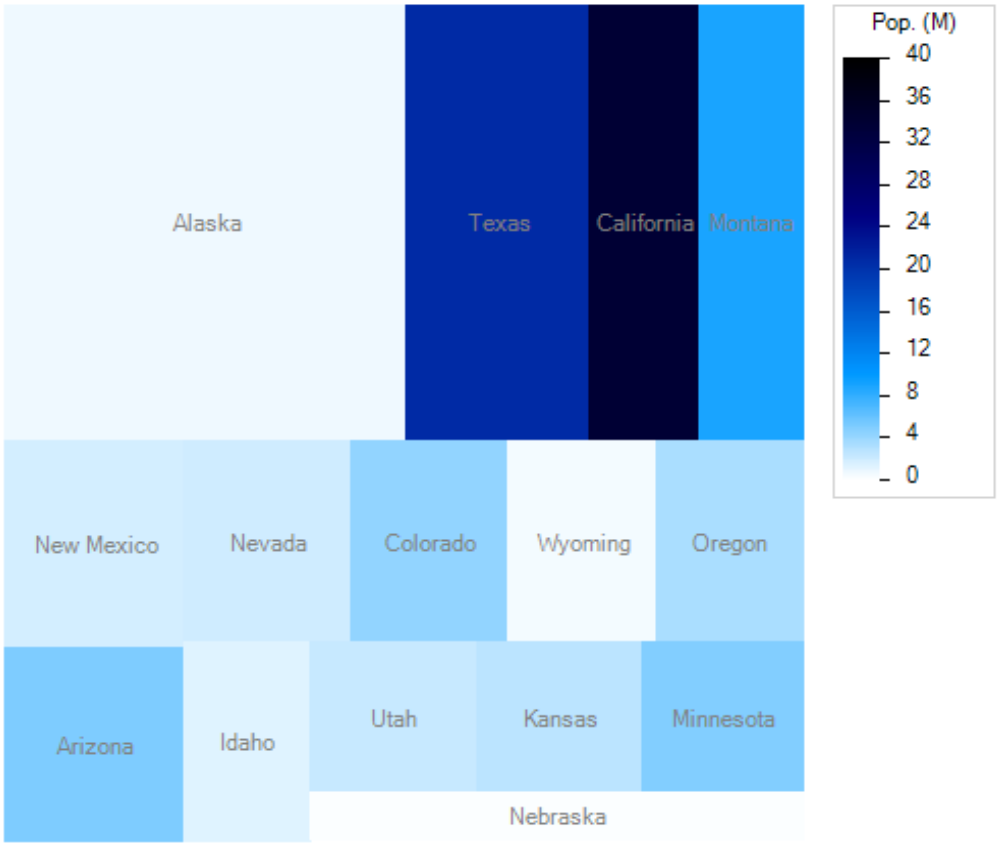
Default: Values computed from the input `shades` array or else is {0,1}.

Example: Treemap with sorted data

A treemap is constructed from area and population data of the 15 largest US states. Each rectangle is proportional to the state's area (in square miles) and is shaded by its population (in millions).

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class TreemapEx1 : FrameChart
{
    public TreemapEx1()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);

        double[] areas = {570374, 261914, 155973, 145556, 121364,
                          113642, 109806, 103729, 97105, 96002,
                          82751, 82168, 81823, 79617, 76878};

        double[] population = {0.626932, 20.851820, 33.871648, 9.02195,
                               1.819046, 5.130632, 1.998257, 4.301261,
                               0.493782, 3.421399, 1.293953, 2.233169,
                               2.688418, 4.919479, 0.1711263};

        string[] names = {"Alaska", "Texas", "California", "Montana",
                          "New Mexico", "Arizona", "Nevada", "Colorado",
                          "Wyoming", "Oregon", "Idaho", "Utah", "Kansas",
                          "Minnesota", "Nebraska"};

        Treemap treemap = new Treemap(axis, areas, population,
            Colormap_Fields.BLUE_WHITE);
        treemap.SetZRange(new double[]{0, 40});
        treemap.SetTreemapLabels(names);
        treemap.TextColor = System.Drawing.Color.Gray;
        treemap.TreemapLegend.IsVisible = true;
        treemap.TreemapLegend.SetTitle("Pop. (M)");
        treemap.TreemapLegend.TextFormat = "0";
        axis.SetViewport(0.05, 0.8, 0.1, 0.95);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new TreemapEx1());
    }
}
```

Output



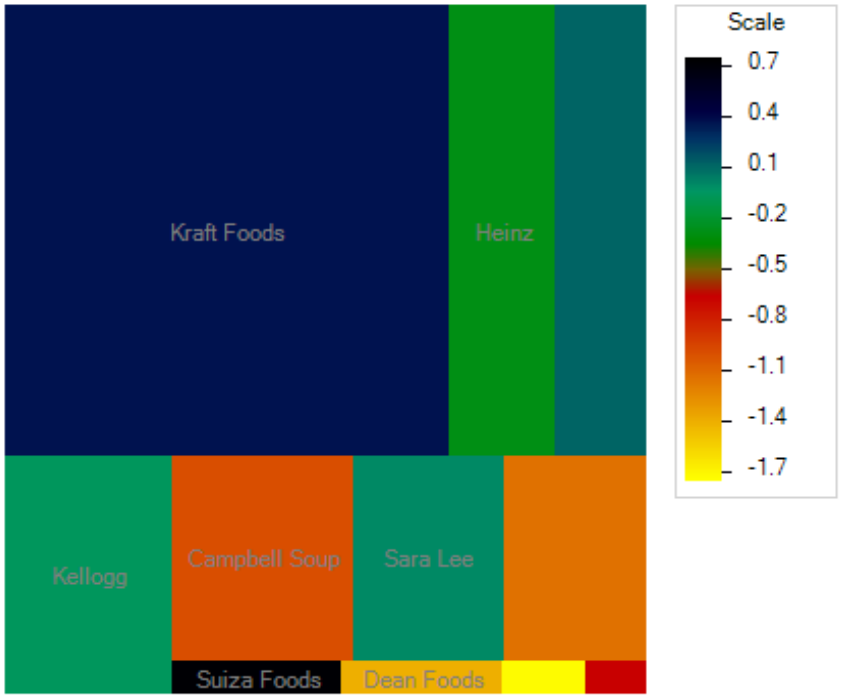
Example: Treemap with unsorted data

A treemap is constructed from business analytics data. The area is proportional to the company's sales volume and the color scale maps to a performance indicator. The raw data are unsorted and must be sorted by the area values before creating the chart.

```
using Imsl.Chart2D;
using System;
using System.Windows.Forms;

public class TreemapEx2 : FrameChart
{
    public TreemapEx2()
    {
        Chart chart = this.Chart;
        AxisXY axis = new AxisXY(chart);
```

```
        double[] areas = {834, 11359, 1621, 12282, 14646, 8686, 12114,
                          61402, 582, 9448, 1678};
        double[] perf = {-1.75, -.99, -1.4, 0.12, -0.28, -1.14, -0.06,
                          0.37, -0.66, 0, 0.75};
        string[] labels = {"Amer. It. Pasta", "Campbell Soup",
                           "Dean Foods", "General Mills", "Heinz",
                           "Hershey Foods", "Kellogg", "Kraft Foods",
                           "Ralston Purina", "Sara Lee", "Suiza Foods"};

        // sort data
        double[] s_perf = new double[perf.Length];
        String[] s_lab = new String[perf.Length];
        int[] idx = new int[perf.Length];
        Imsl.Stat.Sort.Descending(areas, idx);
        for (int i = 0; i < perf.Length; i++)
        {
            s_perf[i] = perf[idx[i]];
            s_lab[i] = labels[idx[i]];
        }

        Treemap treemap = new Treemap(axis, areas, s_perf,
            Colormap_Fields.BLUE_GREEN_RED_YELLOW);
        treemap.SetTreemapLabels(s_lab);
        treemap.TextColor = System.Drawing.Color.Gray;
        treemap.TreemapLegend.IsVisible = true;
        treemap.TreemapLegend.SetTitle("Scale");
        treemap.TreemapLegend.TextFormat = "0.0";
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new TreemapEx2());
    }
}
```

Output



Colormap Interface

`public interface Imsl.Chart2D.Colormap`

An interface used to implement colormaps.

Method

GetColor

```
abstract public System.Drawing.Color GetColor(double t)
```

Description

Maps the parameterization interval [0,1] into Colors.

Parameter

t – A double in the interval [0,1] to be mapped.

Returns

A Color corresponding to t.

Colormap_Fields Structure

```
public structure Imsl.Chart2D.Colormap_Fields
```

Colormaps are mappings from the unit interval to Colors.

They are a one-dimensional parameterized path through the color cube.

See Also

Imsl.Chart2D.Heatmap (p. 1497)

Fields

BLUE

```
public Imsl.Chart2D.Colormap BLUE
```

Description

A linear blue colormap.

BLUE_GREEN_RED_YELLOW

```
public Imsl.Chart2D.Colormap BLUE_GREEN_RED_YELLOW
```

Description

A blue, green, red and yellow colormap.

BLUE_RED

```
public Imsl.Chart2D.Colormap BLUE_RED
```

Description

A linear blue and red colormap.

BLUE_WHITE

```
public Imsl.Chart2D.Colormap BLUE_WHITE
```

Description

A linear blue and white colormap.

BW_LINEAR

```
public Imsl.Chart2D.Colormap BW_LINEAR
```

Description

A linear black and white (grayscale) colormap.

GREEN

```
public Imsl.Chart2D.Colormap GREEN
```

Description

A linear green colormap.

GREEN_PINK

```
public Imsl.Chart2D.Colormap GREEN_PINK
```

Description

A linear green and pink colormap.

GREEN_RED_BLUE_WHITE

```
public Imsl.Chart2D.Colormap GREEN_RED_BLUE_WHITE
```

Description

A green, red, blue and white colormap.

GREEN_WHITE_EXPONENTIAL

```
public Imsl.Chart2D.Colormap GREEN_WHITE_EXPONENTIAL
```

Description

An exponential green and white colormap.

GREEN_WHITE_LINEAR

```
public Imsl.Chart2D.Colormap GREEN_WHITE_LINEAR
```

Description

A linear green and white colormap.

PRISM

```
public Imsl.Chart2D.Colormap PRISM
```

Description

A prism colormap.

RED

```
public Imsl.Chart2D.Colormap RED
```

Description

A linear red colormap.

RED_PURPLE

```
public Imsl.Chart2D.Colormap RED_PURPLE
```

Description

A red and purple colormap.

RED_TEMPERATURE

```
public Imsl.Chart2D.Colormap RED_TEMPERATURE
```

Description

A linear red temperature colormap.

SPECTRAL

```
public Imsl.Chart2D.Colormap SPECTRAL
```

Description

A spectral colormap.

STANDARD_GAMMA

```
public Imsl.Chart2D.Colormap STANDARD_GAMMA
```

Description

A standard gamma colormap.

WB_LINEAR

```
public Imsl.Chart2D.Colormap WB_LINEAR
```

Description

A linear black and white (grayscale) colormap, the reverse of `BW_LINEAR`.

Chapter 25: Quality Control and Improvement Charts

Types

ShewhartControlChart Class

`public class Imsl.Chart2D.QC.ShewhartControlChart : Data`

`ShewhartControlChart` is the base class for the Shewhart control charts.

The control limits are generally calculated as

$$\text{center} + k\sigma$$

where the meaning of *center* and σ depend on the specific control chart being drawn. The variable k is the value of the attribute "ControlLimit" associated with the line being drawn. Typically there are three lines with $k = -3, 0, 3$.

If all of the samples sizes are equal, the lines are horizontal. If the sample sizes are unequal, the lines have a "stairstep" pattern. Horizontal lines have a title drawn just above the line and right-adjusted on the chart. The contents of the title is determined by the line's "Title" attribute. If the title contains the placeholder "{0}", it is replaced by the lines value. This replacement is done using `String.Format`.

Fields

d2_data

```
public double[] d2_data
```

Description

This field contains $d_{2,n}$ the mean of the ranges of n samples from a Gaussian distribution. The entries for n= 0 and 1 are `NaN` because these values are not defined. Valid entries in the table are for n=2 through 50.

d3_data

```
public double[] d3_data
```

Description

This field contains $d_{3,n}$ the standard deviation of the ranges of n samples from a Gaussian distribution. The entries for n= 0 and 1 are `NaN` because these values are not defined. Valid entries in the table are for n=2 through 50.

Properties

Center

```
virtual public double Center {get; set; }
```

Description

The value of the attribute "Center". This is used to position the center line.

CenterLine

```
virtual public Imsl.Chart2D.QC.ControlLimit CenterLine {get; }
```

Description

The center line.

Property Value

Its default value is zero.

ControlData

```
virtual public Imsl.Chart2D.Data ControlData {get; }
```

Description

The `Data` object for the control data.

LowerControlLimit

```
virtual public Imsl.Chart2D.QC.ControlLimit LowerControlLimit {get; }
```

Description

The lower control limit.

MeanSampleSize

```
virtual public double MeanSampleSize {get; }
```

Description

Value of the attribute “MeanSampleSize”, the average sample size.

Property Value

Its default value is zero.

UpperControlLimit

```
virtual public Imsl.Chart2D.QC.ControlLimit UpperControlLimit {get; }
```

Description

The upper control limit.

Constructor

ShewhartControlChart

```
public ShewhartControlChart(Imsl.Chart2D.AxisXY axis)
```

Description

Constructs a Shewhart control chart.

Parameter

`axis` – The `AxisXY` parent of this node.

Methods

AddCenterLine

```
virtual public Imsl.Chart2D.QC.ControlLimit AddCenterLine()
```

Description

Adds the center line to the control chart and returns the newly added line.

Returns

The `ControlLimit` object which draws the center line.

AddControlLimit

```
virtual public Imsl.Chart2D.QC.ControlLimit AddControlLimit()
```

Description

Adds a control limit to the chart. These can be used as tolerance limit lines or warning control limit lines.

Returns

The newly created `ControlLimit` object.

AddLowerControlLimit

```
virtual public Imsl.Chart2D.QC.ControlLimit AddLowerControlLimit()
```

Description

Creates lower `ControlLimit`, adds it to the control chart, and returns the newly created object.

Returns

Tthe `ControlLimit` object which draws the lower control limit line.

AddUpperControlLimit

```
virtual public Imsl.Chart2D.QC.ControlLimit AddUpperControlLimit()
```

Description

Creates upper `ControlLimit`, adds it to the control chart, and returns the newly created object.

Returns

The `ControlLimit` object which draws the upper control limit line.

AddWecoLimits

```
virtual public Imsl.Chart2D.QC.ControlLimit[] AddWecoLimits()
```

Description

Adds lines for the Western Electric Company Rules.

Returns

An array containing the four added lines. They are in order with the `ControlLimit` corresponding to *k* = -2 first and the `ControlLimit` corresponding to *k* = +2 last.

Remarks

These additional lines are at

$$center + k\sigma \quad \text{for } k = -2, -1, 1, 2$$

The meaning of *center* and σ depend on the chart to which these lines are added. The "LineColor" attribute for these lines is set to yellow.

GetSampleSize

```
virtual public int[] GetSampleSize()
```

Description

Returns the value of the attribute "SampleSize".

Returns

The value of the attribute "SampleSize". Its default value is an array of length one containing a one, (`new int[]{1}`).

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints this node and all of its children. This is normally called only by `Paint` (p. 1372) in this node's parent.

Parameter

`draw` – A `Draw` object which is used for drawing the chart.

RemoveControlLimit

```
virtual public void RemoveControlLimit(Imsl.Chart2D.QC.ControlLimit line)
```

Description

Removes a control limit from the chart.

Parameter

`line` – The `ControlLimit` object to be removed from this chart.

SetData

```
virtual public Imsl.Chart2D.Data SetData(int[] y)
```

Description

Sets the integer data in the control chart.

Parameter

`y` – An array containing the data values.

Returns

A `Data` object containing the data points.

Remarks

By default, the data points are drawn as filled circle markers connected by a line.

SetData

```
virtual public Imsl.Chart2D.Data SetData(double[] y)
```

Description

Sets the data in the control chart. By default, the data points are drawn as filled circle markers connected by a line.

Parameter

`y` – An array containing the data values.

Returns

A `Data` object containing the data points.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – A `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetSampleSize

```
virtual public void SetSampleSize(int sampleSize)
```

Description

Sets the value of the attribute “SampleSize” when there is only a single sample size.

Parameter

`sampleSize` – The value of the attribute “SampleSize”.

Remarks

Its default value is an array of length one containing a one, (`new int[]{1}`).

SetSampleSize

```
virtual public void SetSampleSize(int[] sampleSize)
```

Description

Sets the value of the attribute “SampleSize”. Its default value is an array of length one containing a one, (`new int[]{1}`).

Parameter

`sampleSize` – The value of the attribute “SampleSize”.

SetX

```
virtual public void SetX(double[] x)
```

Description

Sets the x-coordinates of the data. The “X” chart attribute of each `ControlLimit` added to this chart is also set. If not set, the values 0, 1, 2, ... are used.

Parameter

`x` – An array containing the x-coordinates. The length of `x` must equal the length of the data array used to construct this object.

ControlLimit Class

```
public class Imsl.Chart2D.QC.ControlLimit :  Data
```

`ControlLimit` is a control limit line on a process control chart.

This class draws either a horizontal line or a stair step line depending on the value of the attribute “Value”. Its value is an array. If the array has a just one entry then a horizontal line is drawn at *y* equal to this value. This line extends across the limit given by the x-axis window attribute. If the array has more than one entry then a stair step line is drawn using the array values as the *y*-coordinates of the stair step line.

Properties

ControlLimitValue

```
virtual public double ControlLimitValue {get; set; }
```

Description

The attribute “ControlLimitValue”. This is the *y*-coordinate at which the line is drawn. Its default value is zero.

MaximumValue

```
virtual public double MaximumValue {get; set; }
```

Description

The maximum value of this control limit line.

Property Value

Its default value is positive infinity.

MinimumValue

```
virtual public double MinimumValue {get; set; }
```

Description

The minimum value of this control limit line.

Property Value

Its default value is negative infinity.

Methods

GetValue

```
virtual public double[] GetValue()
```

Description

Returns the value of this control limit line.

Returns

The *y*-coordinate at which this control limit line is drawn.

Paint

```
override public void Paint(Imsl.Chart2D.Draw draw)
```

Description

Paints the horizontal control limit line as wide as the window.

Parameter

`draw` – A `Draw` object which is used for painting.

Remarks

This is normally called only by the `Paint` method in this node's parent.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – A `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

SetValue

```
virtual public void SetValue(double y)
```

Description

Sets the value of this control limit line. The actual value used is subject to minimum and maximum values set as attributes to this object.

Parameter

`y` – The *y*-coordinate at which this control limit line is drawn.

SetValue

```
virtual public void SetValue(double[] y)
```

Description

Sets the value of this control limit line to an array of values.

Parameter

`y` – An array containing the *y*-coordinates of a stair step line.

XbarR Class

```
public class Imsl.Chart2D.QC.XbarR : ShewhartControlChart
```

`XbarR` is an *X-bar* chart for monitoring a process using sample ranges.

The control limits are at

$$\bar{\bar{x}} + k\frac{\bar{R}}{d_{2,n}\sqrt{n}}$$

where $\bar{\bar{x}}$ is the grand mean of all of the observations, $\bar{R}$ is the mean of the observed ranges, n is the sample size, and k is the value of the "ControlLimit" attribute for the limit. Additionally, $d_{2,n} = E[R]/\sigma$, where R is the range of data from a Gaussian distribution. Therefore $\bar{R}/d_{2,n}$ is an estimate of the within sample standard deviation.

By default, the chart contains an upper control limit with k=3, a lower control limit with k=-3, and a central line equal to $\bar{\bar{x}}$. Additional control limits can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

Property

Rbar

```
virtual public double Rbar {get; set; }
```

Description

The value of the “Rbar” attribute, the mean of the ranges for a series of samples.

Constructors

XbarR

```
public XbarR(Imsl.Chart2D.AxisXY axis, double[][] x)
```

Description

Creates an X-bar chart from sample data using sample ranges.

Parameters

`axis` – The `AxisXY` parent of this node..

`x` – An array of arrays containing sample data. The data of the *i*-th sample is in `x[i]`. Each row must have between 2 and 50.

XbarR

```
public XbarR(Imsl.Chart2D.AxisXY axis, double[,] x)
```

Description

Creates an X-bar chart from sample data using sample ranges.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – A two-dimensional array containing sample data. The data of the *i*-th sample is in `x[i,*]`. The number of columns must be between 2 and 50 entries.

XbarR

```
public XbarR(Imsl.Chart2D.AxisXY axis, int sampleSize, double[] xbar, double[]
range)
```

Description

Creates an X-bar control chart given the means and ranges for a series of equally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – The number of observations in each sample. It must be between 2 and 50.

`xbar` – An array containing the mean values for a series of samples.

`range` – An array containing the ranges of the samples. The arrays `xbar` and `range` must have the same lengths.

XbarR

```
public XbarR(Imsl.Chart2D.AxisXY axis, int[] sampleSize, double[] xbar,
double[] range)
```

Description

Creates an X-bar control chart given the means and ranges for a series of unequally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – An array containing the number of observations in each sample. Each sample must contain between 2 and 50 observations.

`xbar` – An array containing the mean values for a series of samples.

`range` – An array containing the ranges of the samples.

Methods

CapabilityIndexCp

```
virtual public double CapabilityIndexCp(double lowerSpecificationLimit, double
upperSpecificationLimit)
```

Description

Returns the capability index c_p.

Parameters

`lowerSpecificationLimit` – The lower specification limit.

`upperSpecificationLimit` – The upper specification limit.

Returns

The capability index c_p.

Remarks

The capability index is

$$c_p = \frac{\text{USL} - \text{LSL}}{6\sigma}$$

where LSL and USL are the lower and upper specification limits and

$$\sigma = \frac{\bar{R}}{d_{2,n}\sqrt{n}}$$

.

CapabilityIndexCpk

```
virtual public double CapabilityIndexCpk(double lowerSpecificationLimit, double
upperSpecificationLimit)
```

Description

Returns the capability index c_{pk}.

Parameters

`lowerSpecificationLimit` – The lower specification limit.

`upperSpecificationLimit` – The upper specification limit.

Returns

The capability index c_{pk}.

Remarks

The capability index is

$$c_{pk} = \min\left[\frac{\text{USL} - \bar{x}}{3\sigma}, \frac{\bar{x} - \text{LSL}}{3\sigma}\right]$$

where LSL and USL are the lower and upper specification limits, $\bar{x}$ is the center line, and

$$\sigma = \frac{\bar{R}}{d_{2,n}\sqrt{n}}$$

.

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, double[][] x)
```

Description

Creates a combined `XbarR` chart and `RChart` from data.

Parameters

`chart` – The `Chart` object which is the parent of the two charts being created.

`x` – An array of arrays containing sample data. The data of the *i*-th sample is in `x[i]`. Each sample must contain at least 2 and no more than 50 observations.

Returns

An array of length two containing the `XbarR` chart and the `RChart`.

Remarks

The viewport of the `XbarR` chart is [0.2, 0.9] by [0.1, 0.4]. The viewport of the `RChart` chart is [0.2, 0.9] by [0.5, 0.8].

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, double[,] x)
```

Description

Creates a combined `XbarR` chart and `RChart` from data.

Parameters

`chart` – The `Chart` object which is the parent of the two charts being created.

`x` – A two-dimensional array containing sample data. The data of the *i*-th sample is in `x[i,*]`. Each sample must contain at least 2 and no more than 50 observations.

Returns

An array of length two containing the `XbarR` chart and the `RChart`.

Remarks

The viewport of the `XbarR` chart is [0.2, 0.9] by [0.1, 0.4]. The viewport of the `RChart` chart is [0.2, 0.9] by [0.5, 0.8].

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, int sampleSize, double[] xbar, double[]
range)
```

Description

Creates a combined `XbarR` chart and `RChart` given the means and ranges for a series of equally sized samples.

Parameters

`chart` – The `Chart` object which is the parent of the two charts being created.

`sampleSize` – The number of observations in each sample. It must be between 2 and 50.

`xbar` – An array containing the mean values for a series of samples.

`range` – An array containing the ranges of the samples.

Returns

An array of length two containing the `XbarR` chart and the `RChart`.

Remarks

The viewport of the `XbarR` chart is [0.2, 0.9] by [0.1, 0.4]. The viewport of the `RChart` chart is [0.2, 0.9] by [0.5, 0.8].

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, int[] sampleSize, double[] xbar,
double[] range)
```

Description

Creates a combined `XbarR` chart and `RChart` given the means and ranges for a series of unequally sized samples.

Parameters

`chart` – The `Chart` object which is the parent of the two charts being created.

`sampleSize` – An array containing the number of observations in each sample. Each sample must have between 2 and 50 observations.

`xbar` – An array containing the mean values for a series of samples.

`range` – An array containing the ranges of the samples.

Returns

An array of length two containing the `XbarR` chart and the `RChart`.

Remarks

The viewport of the `XbarR` chart is [0.2, 0.9] by [0.1, 0.4]. The viewport of the `RChart` chart is [0.2, 0.9] by [0.5, 0.8].

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

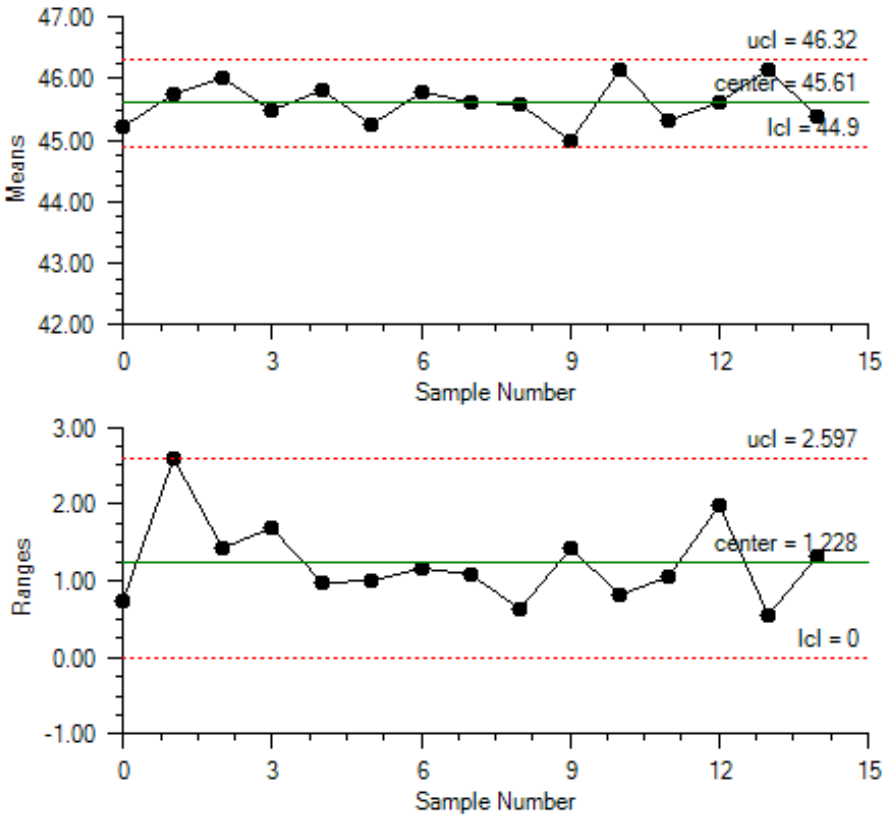
Example: XbarR Chart

During a manufacturing process 15 samples, each containing 5 items, were measured. An XbarR chart was contructed from the 15 sample ranges.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class XbarREx1 : FrameChart
{
    static private readonly double[][] data = {
        new double[]{44.73, 45.47, 45.39, 45.33, 45.24},
        new double[]{45.57, 46.87, 45.40, 46.68, 44.29},
        new double[]{46.39, 45.31, 46.74, 46.06, 45.51},
        new double[]{45.54, 46.27, 44.57, 45.36, 45.72},
        new double[]{45.58, 45.59, 46.02, 45.45, 46.42},
        new double[]{45.91, 45.38, 44.98, 44.91, 45.17},
        new double[]{45.98, 45.29, 45.50, 45.77, 46.44},
        new double[]{46.30, 45.65, 45.21, 45.43, 45.57},
        new double[]{45.77, 45.38, 45.65, 45.25, 45.89},
        new double[]{44.10, 45.44, 45.27, 45.53, 44.65},
        new double[]{45.95, 46.22, 46.71, 45.92, 45.90},
        new double[]{44.87, 44.98, 45.91, 45.18, 45.64},
        new double[]{44.70, 45.89, 46.67, 45.84, 45.07},
        new double[]{45.90, 45.80, 46.30, 46.34, 46.34},
        new double[]{44.90, 46.23, 45.31, 45.29, 45.16}
    };

    public XbarREx1()
```

```
    {
        XbarR.CreateCharts(this.Chart, data);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new XbarREx1());
    }
}
```

Output



RChart Class

```
public class Imsl.Chart2D.QC.RChart :  ShewhartControlChart
```

`RChart` is an R chart using sample ranges to monitor the variability of a process. Each sample must contain at least two observations. The range of a sample is the maximum observed value minus the minimum observed value.

The control limits are at

$$\bar{R} + k\frac{d_{3,n}}{d_{2,n}}\bar{R}$$

$\bar{R}$ is the mean of the observed ranges, n is the sample size, and k is the value of the "ControlLimit" attribute for the line. Additionally, $d_{2,n}$ is the mean of the distribution of the ranges of n samples from the normal distribution with mean zero and standard deviation one. The standard deviation of this distribution is $d_{3,n}$. Therefore

$$\frac{d_{3,n}}{d_{2,n}}\bar{R}$$

is an estimator of the standard deviation of the ranges.

By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line with k=0. Additional control limit lines can be added. The method `AddWeco` adds control limit lines with k = -2, -1, 1, 2.

Constructors

RChart

```
public RChart(Imsl.Chart2D.AxisXY axis, double[][] x)
```

Description

Creates an R chart given sample data.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – An array of arrays containing sample data. The data of the i-th sample is in `x[i]`. Each sample must contain at least 2 and no more than 50 observations.

Exception

`System.ArgumentException` is thrown if the number of samples is less than 2 or greater than 50.

RChart

```
public RChart(Imsl.Chart2D.AxisXY axis, double[,] x)
```

Description

Creates an R chart given sample data.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – A two-dimensional array containing sample data. The data of the *i*-th sample is in `x[i,*]`. Each sample must contain at least 2 and no more than 50 observations.

Exception

`System.ArgumentException` is thrown if the number of samples is less than 2 or greater than 50.

RChart

```
public RChart(Imsl.Chart2D.AxisXY axis, int sampleSize, double[] range)
```

Description

Creates an R chart given the ranges for a series of equally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – The number of observations in each sample. It must be at least 2 and no more than 50.

`range` – An array containing the data ranges for a series of samples.

RChart

```
public RChart(Imsl.Chart2D.AxisXY axis, int[] sampleSize, double[] range)
```

Description

Creates an R chart given the means for a series of equally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – An array containing the number of observations in each sample. It must be at least 2 and no more than 50.

`range` – An array containing the data ranges for a series of samples.

Method

PrePaint

```
override public void PrePaint()
```

Description

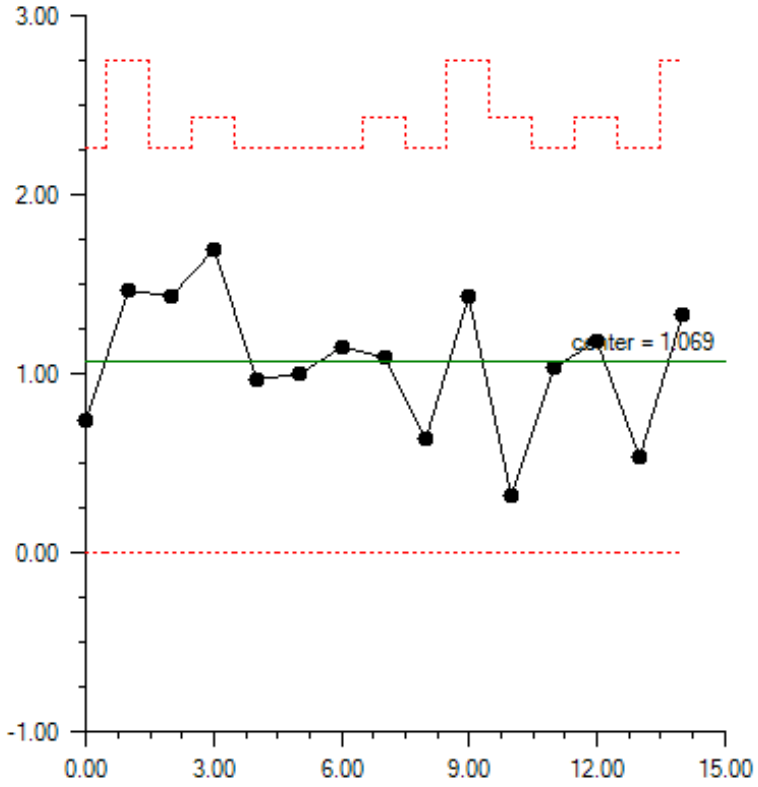
Setup chart with current settings.

Example: R Chart

During a manufacturing process 15 samples, each containing 3 to 5 items, were measured. An R chart was contructed from the 15 sample ranges. Since the sample sizes are unequal the upper control limit is a stair step line.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class RChartEx1 : FrameChart
{
    static private double[][] data = {
        new double[]{44.73, 45.47, 45.39, 45.33, 45.24},
        new double[]{45.57, 46.87, 45.40},
        new double[]{46.39, 45.31, 46.74, 46.06, 45.51},
        new double[]{45.54, 46.27, 44.57, 45.36},
        new double[]{45.58, 45.59, 46.02, 45.45, 46.42},
        new double[]{45.91, 45.38, 44.98, 44.91, 45.17},
        new double[]{45.98, 45.29, 45.50, 45.77, 46.44},
        new double[]{46.30, 45.65, 45.21, 45.43},
        new double[]{45.77, 45.38, 45.65, 45.25, 45.89},
        new double[]{44.10, 45.53, 44.65},
        new double[]{45.95, 46.22, 45.92, 45.90},
        new double[]{44.87, 44.98, 45.91, 45.18, 45.64},
        new double[]{44.70, 45.89, 45.84, 45.07},
        new double[]{45.90, 45.80, 46.30, 46.34, 46.34},
        new double[]{44.90, 46.23, 45.16}
    };

    public RChartEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        new RChart(axis, data);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new RChartEx1());
    }
}
```

Output



XbarS Class

`public class Imsl.Chart2D.QC.XbarS : ShewhartControlChart`

`XbarS` is an X-bar chart for monitoring a process using sample standard deviations.

The control limits are at

$$\bar{\bar{x}} + k\frac{\bar{s}}{c_{4,n}\sqrt{n}}$$

where $\bar{\bar{x}}$ is the grand mean of all of the observations, n is the sample size, and k is the value of the "ControlLimit" attribute for the limit. Additionally, $c_{4,n}$ is a factor such that $\bar{s}/c_{4,n}$ is an unbiased estimator of the within sample standard deviation.

By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line equal to $\bar{\bar{x}}$. Additional control limit lines can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

Property

Wbar

```
virtual public double Wbar {get; set; }
```

Description

The value of the "Wbar" attribute.

Remarks

This is the within sample variation for a series of samples. Its default value is set by the constructor to an estimate of the within sample variation

Constructors

XbarS

```
public XbarS(Imsl.Chart2D.AxisXY axis, double[][] x)
```

Description

Creates an `XBarS` chart from sample data using within sample standard deviations.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – An array of arrays containing sample data. The data of the i-th sample is in `x[i]`. Each row must have at least one entry.

XbarS

```
public XbarS(Imsl.Chart2D.AxisXY axis, double[,] x)
```

Description

Creates an `XBarS` chart from sample data using within sample standard deviations.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – A two-dimensional array containing sample data. The data of the i-th sample is in `x[i,*]`.

XbarS

```
public XbarS(Imsl.Chart2D.AxisXY axis, int sampleSize, double[] xbar, double[]
w)
```

Description

Creates an `XBarS` chart given the means and standard deviations for a series of equally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – The number of observations in each sample. It must be at least one.

`xbar` – An array containing the mean values for a series of samples.

`w` – An array containing the within sample variation for a series of samples.

Exception

`System.ArgumentException` is thrown if the two input arrays do not have the same length.

XbarS

```
public XbarS(Imsl.Chart2D.AxisXY axis, int[] sampleSize, double[] xbar,
double[] w)
```

Description

Creates an `XBarS` chart given the means and standard deviations for a series of unequally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – An array containing the number of observations in each sample. Each sample must have at least one observation.

`xbar` – An array containing the mean values for a series of samples.

`w` – An array containing the within sample variation for a series of samples.

Exception

`System.ArgumentException` is thrown if the three input arrays do not all have the same length.

Methods

CapabilityIndexCp

```
virtual public double CapabilityIndexCp(double lowerSpecificationLimit, double
upperSpecificationLimit)
```

Description

Returns the capability index c_p.

Parameters

`lowerSpecificationLimit` – The lower specification limit.

`upperSpecificationLimit` – The upper specification limit.

Returns

The capability index c_p.

Remarks

The capability index is

$$c_p = \frac{\text{USL} - \text{LSL}}{6\sigma}$$

where LSL and USL are the lower and upper specification limits and

$$\sigma = \frac{\bar{s}}{c_{4,n}\sqrt{n}}$$

.

CapabilityIndexCpk

```
virtual public double CapabilityIndexCpk(double lowerSpecificationLimit, double
upperSpecificationLimit)
```

Description

Returns the capability index c_{pk}.

Parameters

`lowerSpecificationLimit` – The lower specification limit.

`upperSpecificationLimit` – The upper specification limit.

Returns

The capability index c_{pk}.

Remarks

The capability index is

$$c_{pk} = \min\left[\frac{\text{USL} - \bar{x}}{3\sigma}, \frac{\bar{x} - \text{LSL}}{3\sigma}\right]$$

where LSL and USL are the lower and upper specification limits, $\bar{x}$ is the center line, and

$$\sigma = \frac{\bar{s}}{c_{4,n}\sqrt{n}}$$

.

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, double[][] x)
```

Description

Creates a combined `XBarS` chart and `SChart` from data.

Parameters

`chart` – The `Chart` object which is the parent of the two charts being created.

`x` – An array of arrays containing sample data. The data of the *i*-th sample is in `x[i]`. Each row must have at least one entry.

Returns

An array of length two containing the `XBarS` chart and the `SChart`.

Remarks

The viewport of the `XBarS` chart is [0.2, 0.9] by [0.1, 0.4]. The viewport of the `SChart` chart is [0.2, 0.9] by [0.5, 0.8].

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, double[,] x)
```

Description

Creates a combined `XBarS` chart and `SChart` from data.

Parameters

`chart` – The `Chart` object which is the parent of the two charts being created.

`x` – A two-dimensional array containing sample data. The data of the *i*-th sample is in `x[i,*]`.

Returns

An array of length two containing the `XBarS` chart and the `SChart`.

Remarks

The viewport of the `XBarS` chart is [0.2, 0.9] by [0.1, 0.4]. The viewport of the `SChart` chart is [0.2, 0.9] by [0.5, 0.8].

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, int sampleSize, double[] xbar, double[]
w)
```

Description

Creates a combined `XBarS` chart and `SChart` given the means and in sample standard deviations for a series of equally sized samples.

Parameters

`chart` – The `Chart` object which is the parent of the two charts being created.

`sampleSize` – The number of observations in each sample. It must be at least one.

`xbar` – An array containing the mean values for a series of samples.

`w` – An array containing the in sample standard deviations of the samples.

Returns

An array of length two containing the `XBarS` chart and the `SChart`.

Remarks

The viewport of the XBarS chart is [0.2, 0.9] by [0.1, 0.4]. The viewport of the SChart chart is [0.2, 0.9] by [0.5, 0.8].

CreateCharts

```
static public Imsl.Chart2D.QC.ShewhartControlChart[]
CreateCharts(Imsl.Chart2D.Chart chart, int[] sampleSize, double[] xbar,
double[] w)
```

Description

Creates a combined X-bar chart and S-chart given the means and in sample standard deviations for a series of unequally sized samples.

Parameters

chart – The Chart object which is the parent of the two charts being created.

sampleSize – An array containing the number of observations in each sample. Each sample must have at least one observation.

xbar – An array containing the mean values for a series of samples.

w – An array containing the in sample standard deviations of the samples.

Returns

An array of length two containing the XBarS chart and the SChart.

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

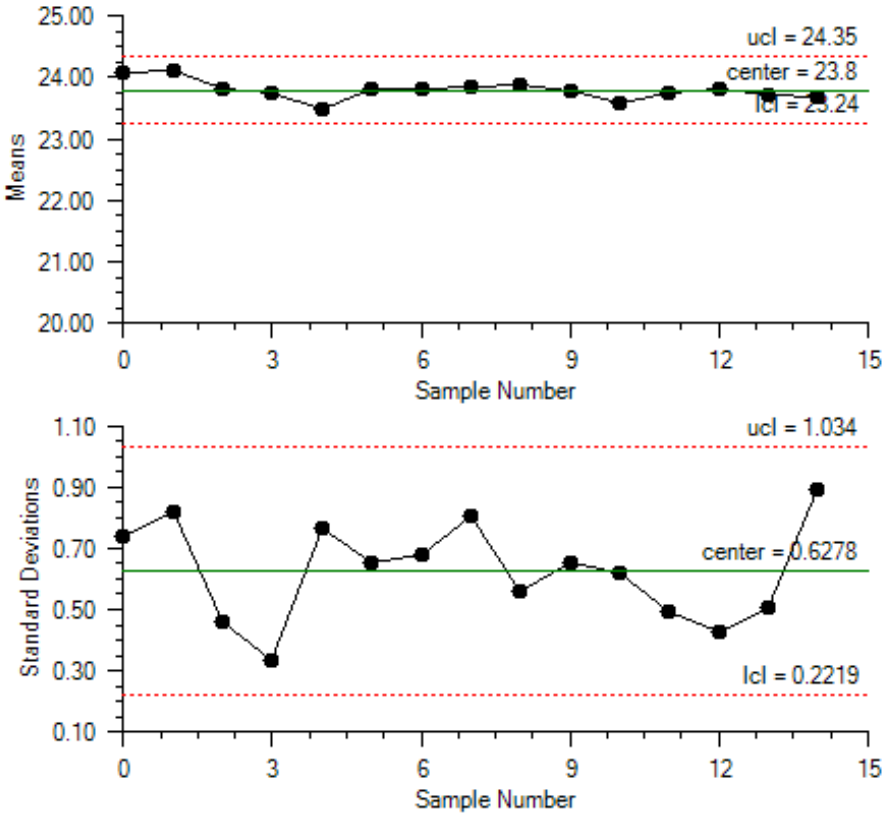
Example: XbarS Chart

During a manufacturing process 15 samples, each containing 12 items, were measured. An XbarS chart was contructed from the 15 sample standard deviations.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class XbarSEx1 : FrameChart
{
    static private readonly double[][] data = {
        new double[]{23.97, 24.08, 23.16, 23.49, 24.73, 25.26, 22.97, 23.12, 24.66,
                     24.20, 24.62, 24.56},
        new double[]{24.20, 24.50, 23.45, 22.22, 25.10, 24.41, 24.05, 23.75, 23.89,
                     24.83, 25.21, 23.70},
        new double[]{23.73, 22.70, 23.54, 24.37, 24.08, 23.74, 24.08, 23.95, 24.20,
                     23.43, 24.26, 23.61},
        new double[]{23.46, 23.14, 23.96, 23.37, 23.73, 24.29, 24.13, 23.62, 24.08,
```

```
                        23.73, 23.91, 23.65},
        new double[]{23.95, 24.13, 22.95, 24.72, 24.40, 22.82, 22.66, 22.71, 24.21,
                        23.39, 23.41, 22.56},
        new double[]{24.13, 24.28, 23.84, 24.55, 23.53, 23.77, 24.38, 22.58, 24.47,
                        23.63, 22.64, 24.12},
        new double[]{23.69, 24.19, 24.76, 23.29, 24.84, 24.12, 23.83, 22.60, 24.35,
                        22.96, 23.81, 23.46},
        new double[]{24.35, 23.11, 25.24, 24.10, 24.93, 22.93, 23.47, 23.55, 23.91,
                        24.08, 22.45, 24.13},
        new double[]{24.98, 24.58, 23.52, 24.42, 23.90, 23.55, 23.67, 24.25, 23.85,
                        23.08, 23.44, 23.43},
        new double[]{23.90, 24.04, 24.29, 23.62, 23.29, 23.16, 24.34, 24.37, 24.19,
                        24.33, 22.17, 23.66},
        new double[]{23.51, 24.98, 24.34, 23.87, 23.29, 23.96, 23.06, 23.47, 23.53,
                        22.87, 23.38, 22.86},
        new double[]{23.13, 23.17, 23.40, 23.68, 23.41, 23.67, 23.37, 24.40, 24.64,
                        24.16, 24.17, 23.88},
        new double[]{23.52, 24.23, 24.25, 24.31, 23.89, 24.02, 24.04, 23.83, 22.82,
                        23.93, 23.55, 23.40},
        new double[]{23.66, 23.59, 23.79, 24.07, 23.76, 23.34, 23.65, 23.73, 25.12,
                        23.65, 23.23, 23.13},
        new double[]{23.04, 23.75, 22.84, 23.46, 21.72, 23.81, 24.51, 24.01, 24.73,
                        23.88, 23.34, 24.98}
    };

    public XbarSEx1()
    {
        XbarS.CreateCharts(this.Chart, data);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new XbarSEx1());
    }
}
```

Output



SChart Class

```
public class Imsl.Chart2D.QC.SChart :  ShewhartControlChart
```

`SChart` is an S chart using sample standard deviations to monitor the variability of a process. This is normally used with sample sizes greater than 10.

The control limits are at

$$\bar{s} + k\frac{\bar{s}}{c_{4,n}}\sqrt{1-c_{4,n}^2}$$

where $\bar{s}$ is the mean of the within sample standard deviations, n is the sample size, and k is the value of the "ControlLimit" attribute for the line. Additionally,

$$c_{4,n} = \sqrt{\frac{2}{n-1}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}$$

is a factor such that $\bar{s}/c_{4,n}$ is an unbiased estimator of the within sample standard deviation. By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line equal to $\bar{s}$. Additional control limit lines can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

Constructors

SChart

```
public SChart(Imsl.Chart2D.AxisXY axis, double[][] x)
```

Description

Creates an S chart given sample data.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – An array of arrays containing sample data. The data of the i-th sample is in `x[i]`. Each row must have at least two entries.

SChart

```
public SChart(Imsl.Chart2D.AxisXY axis, double[,] x)
```

Description

Creates an S chart given sample data.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – A two-dimensional array containing sample data. The data of the i-th sample is in `x[i,*]`. There must be at least two columns.

SChart

```
public SChart(Imsl.Chart2D.AxisXY axis, int sampleSize, double[] s)
```

Description

Creates an S chart given the within sample standard deviations for a series of equally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – The number of observations in each sample. It must be at least 2.

`s` – An array containing the within sample standard deviations for a series of samples.

SChart

```
public SChart(Imsl.Chart2D.AxisXY axis, int[] sampleSize, double[] s)
```

Description

Creates an S chart given the within sample standard deviations for a series of unequally sized samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – An array containing the number of observations in each sample. All samples must have at least two observations.

`s` – An array containing the within sample standard deviations for a series of samples.

Exception

`System.ArgumentException` is thrown if the two input arrays do not have the same length.

Method

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

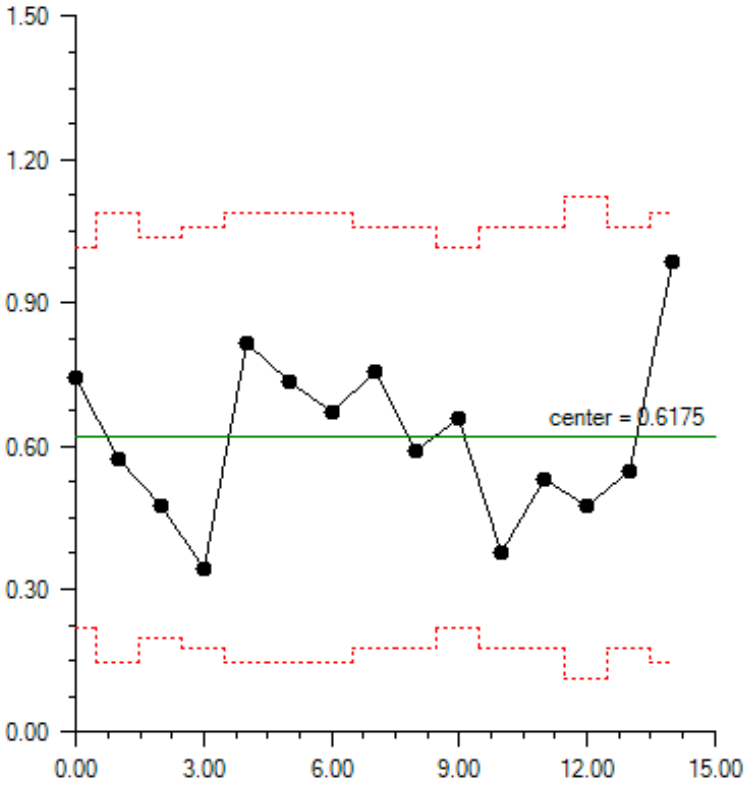
Example: S Chart

During a manufacturing process 15 samples, each containing up to 12 items, were measured. An S chart was contructed from the 15 sample standard deviations. Since the sample sizes are unequal the upper and lower control limit lines are stair step lines.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class SChartEx1 : FrameChart
{
    static private double[][] data = {
        new double[]{23.97, 24.08, 23.16, 23.49, 24.73, 25.26, 22.97, 23.12, 24.66,
                     24.20, 24.62, 24.56},
        new double[]{24.20, 24.50, 23.45, 24.05, 23.75, 23.89, 24.83, 25.21, 23.70},
        new double[]{23.73, 22.70, 23.54, 24.37, 23.74, 24.08, 23.95, 24.20, 23.43,
```

```
                    24.26, 23.61},
        new double[]{23.46, 23.14, 23.73, 24.29, 24.13, 23.62, 24.08, 23.73, 23.91,
                    23.65},
        new double[]{23.95, 22.95, 24.72, 24.40, 22.82, 22.66, 22.71, 23.41, 22.56},
        new double[]{24.13, 24.28, 23.84, 24.55, 22.58, 24.47, 23.63, 22.64, 24.12},
        new double[]{23.69, 24.19, 24.76, 23.83, 22.60, 24.35, 22.96, 23.81, 23.46},
        new double[]{24.35, 23.11, 25.24, 24.10, 23.47, 23.55, 23.91, 24.08, 22.45,
                    24.13},
        new double[]{24.98, 24.58, 23.52, 23.55, 23.67, 24.25, 23.85, 23.08, 23.44,
                    23.43},
        new double[]{23.90, 24.04, 24.29, 23.62, 23.29, 23.16, 24.34, 24.37, 24.19,
                    24.33, 22.17, 23.66},
        new double[]{23.51, 23.87, 23.29, 23.96, 23.06, 23.47, 23.53, 22.87, 23.38,
                    22.86},
        new double[]{23.13, 23.17, 23.40, 23.68, 23.37, 24.40, 24.64, 24.16, 24.17,
                    23.88},
        new double[]{23.52, 24.23, 24.25, 23.83, 22.82, 23.93, 23.55, 23.40},
        new double[]{23.66, 23.59, 23.79, 24.07, 23.65, 23.73, 25.12, 23.65, 23.23,
                    23.13},
        new double[]{23.04, 23.75, 22.84, 23.46, 21.72, 24.73, 23.88, 23.34, 24.98}
     };

    public SChartEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        new SChart(axis, data);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new SChartEx1());
    }
}
```

Output



XmR Class

`public class Imsl.Chart2D.QC.XmR : ShewhartControlChart`

`XmR` is an *XmR* chart for monitoring a process using moving ranges.

The moving range control chart uses the moving range of two successive observations to measure the process variability. This control chart is used for individual measurements (sample size = 1).

The *moving range* is defined to be $\mathrm{MR}_i = |x_i - x_{i-1}|$.

The control limits are at

$$\bar{x} + k\frac{\overline{MR}}{d_{2,2}}$$

where $\bar{x}$ is the mean of all of the individual observations, $\overline{MR}$ is the mean of the moving averages, and k is the value of the "ControlLimit" attribute for the line. Additionally, $d_{2,n} = E(R)/\sigma$ where R is the range of a Gaussian distribution. Therefore $\overline{MR}/d_{2,2}$ is an estimate of the standard deviation.

By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line equal to $\bar{x}$. Additionally control limits can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

Property

MRBar

```
virtual public double MRBar {get; set; }
```

Description

The expected mean of of all of the moving ranges of two observations.

Property Value

Its default value is the computed from the data passed to the constructor.

Constructor

XmR

```
public XmR(Imsl.Chart2D.AxisXY axis, double[] x)
```

Description

Creates an `XmR` chart given sample data.

Parameters

`axis` – The `AxisXY` parent of this node.

`x` – An array containing sample data.

Method

PrePaint

```
override public void PrePaint()
```

Description

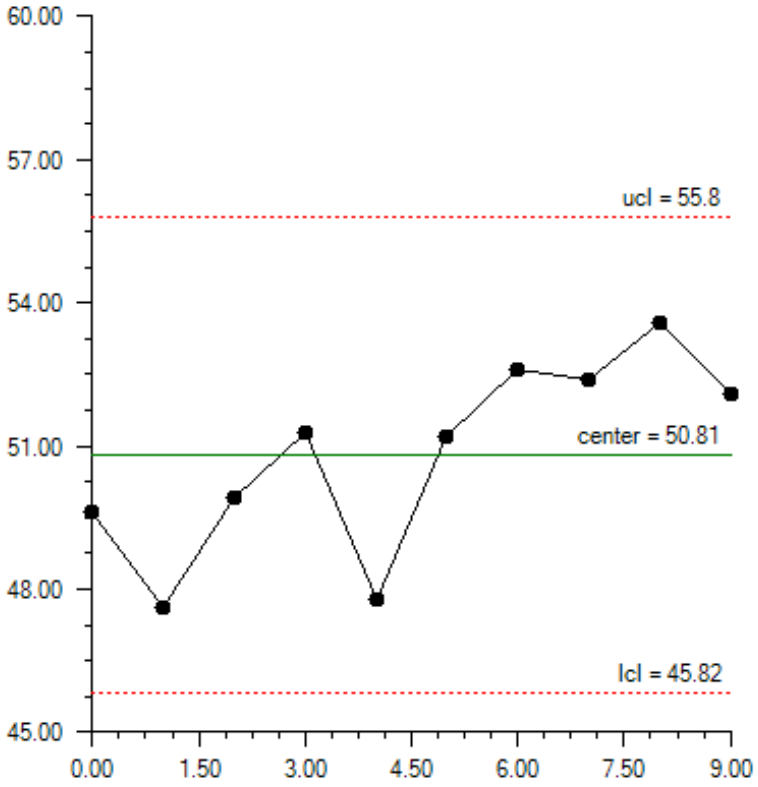
Setup chart with current settings.

Example: Moving Range Chart

This moving range chart plots the flowrate for 10 batches. The data is from NIST Engineering Statistics Handbook: Individuals Control Charts.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class XmREx1 : FrameChart
{
    static private readonly double[] flowrate = {
        49.6, 47.6, 49.9, 51.3, 47.8, 51.2, 52.6, 52.4, 53.6, 52.1
    };

    public XmREx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        XmR mr = new XmR(axis, flowrate);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new XmREx1());
    }
}
```

Output



NpChart Class

`public class Imsl.Chart2D.QC.NpChart : ShewhartControlChart`

`NpChart` is an *np*-chart for monitoring the number of defects when defects are not rare.

Control limits are computed using the binomial distribution. If defects are rare `CChart` (p. 1560) should be used instead.

The control limits are at

$$np + k\sqrt{np(1-p)}$$

where p is the is the proportion of defective items, n is the sample size, and k is the value of the "ControlLimit" attribute for the line. By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line equal to $\bar{c}$. Additional control limits can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

The lower control limit is forced to have a minimum value of zero. The y-axis labels are formatted as integers.

Constructors

NpChart

```
public NpChart(Imsl.Chart2D.AxisXY axis, int sampleSize, int[] numberDefective)
```

Description

Creates an np-chart given the number of defects in a series of samples.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – The number of observations in each sample. It must be at least one.

`numberDefective` – An array containing the number of defects in each of a series of samples. All of its entries must be nonnegative.

NpChart

```
public NpChart(Imsl.Chart2D.AxisXY axis, int[] sampleSize, int[]
numberDefective)
```

Description

Creates a np-chart given the number of defects in a series of samples, where the number of observations per sample is not constant.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – An array containg the number of observations in each sample. Each sample must contain at least one observation.

`numberDefective` – An array containing the number of defects in each of a series of samples. All of its entries must be nonnegative.

Method

PrePaint

```
override public void PrePaint()
```

Description

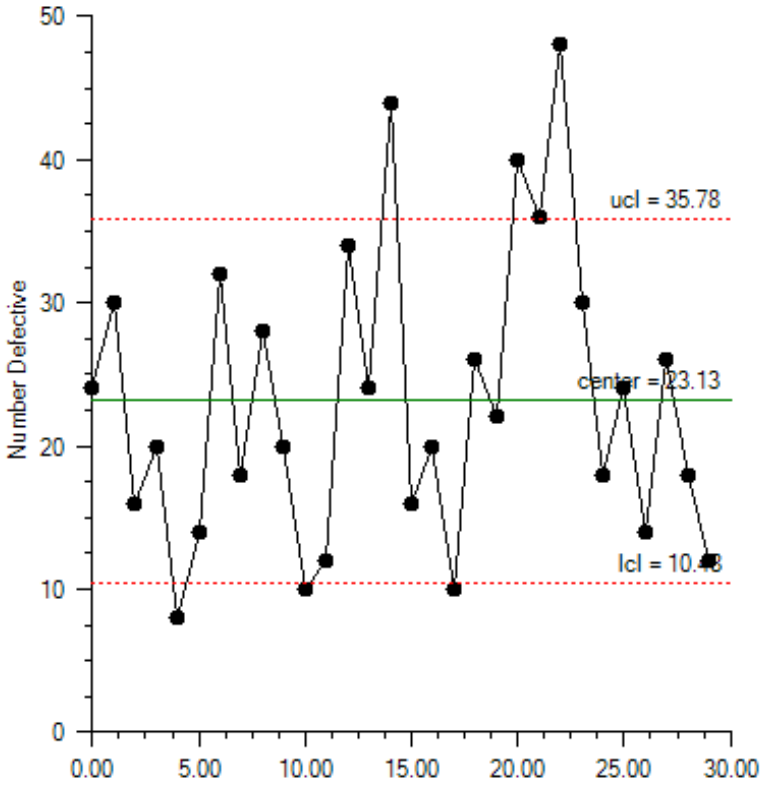
Setup chart with current settings.

Example: *np*-chart

The location of 50 chips on each of 30 successive wafers is measured. The number of defects is the number of horizontal or vertical misregistrations of chips. The data is from NIST Engineering Statistics Handbook: Proportions Control Charts.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class NpChartEx1 : FrameChart
{
    private static readonly int[] numberDefective = {
        24, 30, 16, 20, 8, 14, 32, 18, 28, 20, 10, 12,
        34, 24, 44, 16, 20, 10, 26, 22, 40, 36, 48, 30,
        18, 24, 14, 26, 18, 12
    };
    private const int sampleSize = 100;

    public NpChartEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        axis.AxisY.AxisTitle.SetTitle("Number Defective");
        new NpChart(axis, sampleSize, numberDefective);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new NpChartEx1());
    }
}
```

Output



PChart Class

`public class Imsl.Chart2D.QC.PChart : ShewhartControlChart`

`PChart` is a *p*-chart for monitoring the defect rate when defects are not rare.

The defect rate is the number of defects found divided by the number of samples inspected. The number of defects are not assumed to be rare. Control limits are computed using the binomial distribution. If defects are rare, use `UChart` (p. 1563) instead.

The control limits are at

$$\bar{p} + k\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

where $\bar{p}$ is the mean defect rate, n is the sample size, and k is the value of the "ControlLimit" attribute for the line. By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line equal to $\bar{p}$. Additional control limits can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

The true fraction conforming p can be used by setting the attribute "Center" to p.

The lower control limit is forced to have a minimum value of zero. The upper control limit is forced to have a maximum value of one.

Constructors

PChart

```
public PChart(Imsl.Chart2D.AxisXY axis, int sampleSize, double[] defectRate)
```

Description

Creates a p-Chart given the defect rates for a series of samples with equal sample sizes.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – The number of observations in each sample. It must be at least one.

`defectRate` – An array containing defect rates of the samples. The defect rates must all be in the range [0,1].

PChart

```
public PChart(Imsl.Chart2D.AxisXY axis, int[] sampleSize, double[] defectRate)
```

Description

Creates a p-Chart given the defect rates for a series of samples with varying sample sizes.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – An array containing the number of observations in each sample. It must be at least one.

`defectRate` – An array containing defect rates of the samples. The defect rates must all be in the range [0,1]. The lengths of the arrays `sampleSize` and `defectRate` must be equal.

PChart

```
public PChart(Imsl.Chart2D.AxisXY axis, int sampleSize, int[] numberDefects)
```

Description

Creates a p-Chart given the number of defects for a series of samples with equal sample sizes.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – The number of observations in each sample. Each sample must contain at least one observation.

`numberDefects` – An array containing the number of defects in each of a series of samples. The number of defects should not be less than zero.

PChart

```
public PChart(Imsl.Chart2D.AxisXY axis, int[] sampleSize, int[] numberDefects)
```

Description

Creates a p-Chart given the number of defects for a series of samples with varying sample sizes.

Parameters

`axis` – The `AxisXY` parent of this node.

`sampleSize` – An array containing the number of observations in each sample. It must be at least one.

`numberDefects` – An array of arrays containing the number of defects in each of a series of samples. The number of defects should not be less than zero.

Method

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

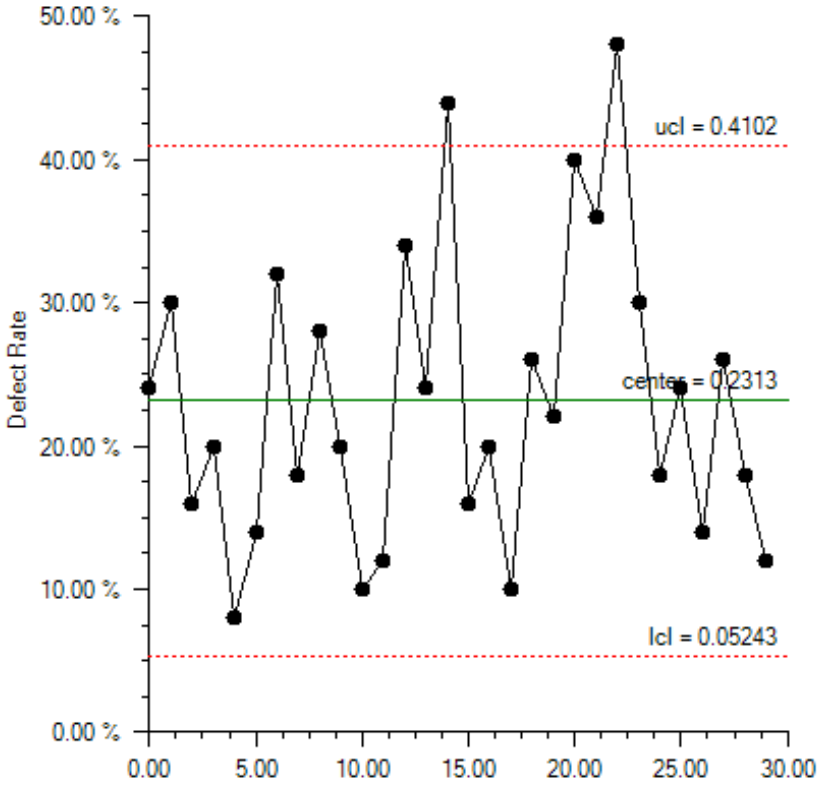
Example: p-Chart

The location of 50 chips on each of 30 successive wafers is measured. The number of defects is the number of horizontal or vertical misregistrations of chips. The data is from NIST Engineering Statistics Handbook: Proportions Control Charts.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class PChartEx1 : FrameChart
{
    static private readonly double[] defectRate = {
        0.24, 0.3, 0.16, 0.2, 0.08, 0.14, 0.32, 0.18, 0.28, 0.2,
        0.1, 0.12, 0.34, 0.24, 0.44, 0.16, 0.2, 0.1, 0.26, 0.22,
        0.4, 0.36, 0.48, 0.3, 0.18, 0.24, 0.14, 0.26, 0.18, 0.12
    };
```

```
    static private readonly int sampleSize = 50;

    public PChartEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        axis.AxisY.AxisTitle.SetTitle("Defect Rate");
        axis.AxisY.AxisLabel.TextFormat = "P";
        new PChart(axis, sampleSize, defectRate);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new PChartEx1());
    }
}
```

Output



CChart Class

```
public class Imsl.Chart2D.QC.CChart :  ShewhartControlChart
```

`CChart` is a *c*-chart for monitoring the count of the number of defects when defects are rare. Control limits are computed using the Poisson distribution. If defects are not rare `NpChart` (p. 1553) should be used instead.

The control limits are at

$$\bar{c} + k\sqrt{\bar{c}}$$

where $\bar{c}$ is the mean number of defects per sample and k is the value of the "ControlLimit" attribute for the line.

By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line equal to $\bar{c}$. Additional control limits can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

The lower control limit is computed using the Poisson distribution. First the probability of the number of defects being less than k (the "ControlLimit" value) standard deviations from the mean in a normal distribution is computed. The number of defects required to produce the same probability assuming a Poisson distribution, with the same mean, is then computed. The lower limit is set to be one more than this number of defects.

Constructor

CChart

```
public CChart(Imsl.Chart2D.AxisXY axis, int[] numberDefects)
```

Description

Creates a C-chart given the number of defects in a series of samples.

Parameters

`axis` – The `AxisXY` parent of this node

`numberDefects` – The number of defects in a series of samples. The number of defects should not be less than zero. There should be the same number of items in each sample.

Method

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

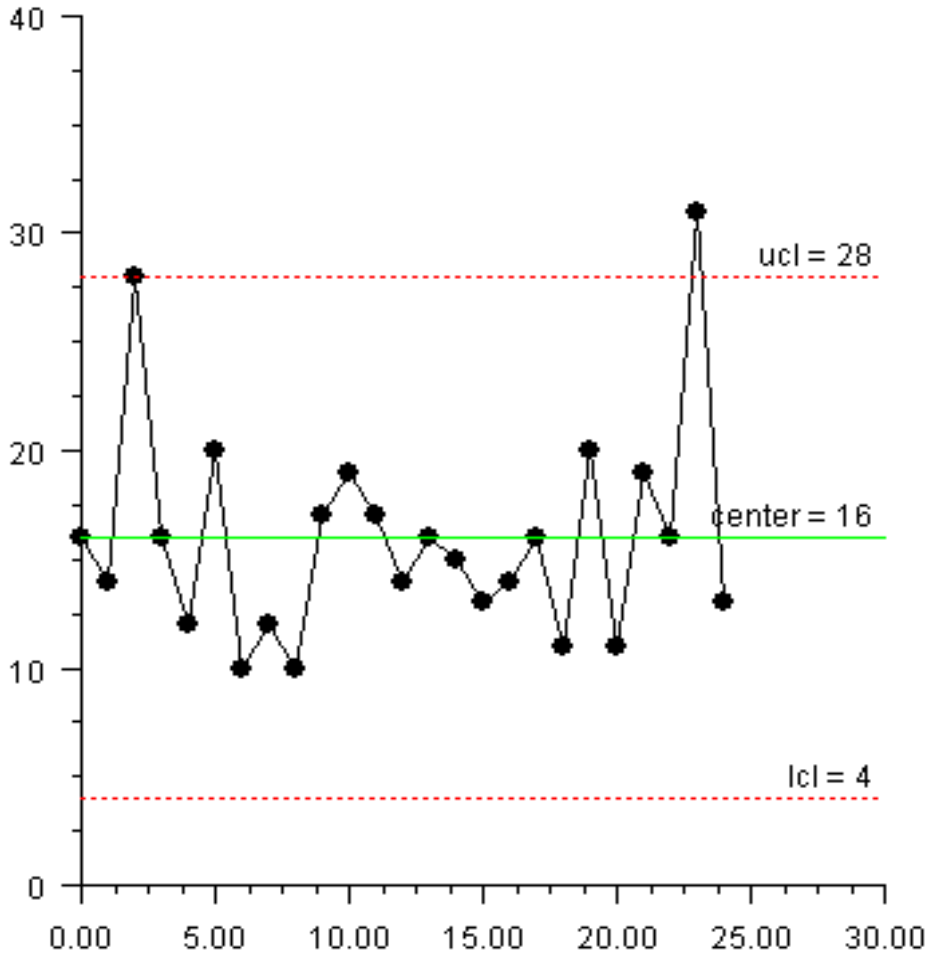
Example: c-chart

The number of defects on each of 25 successive wafers is plotted. The data is from NIST Engineering Statistics Handbook: Counts Control Charts.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
```

```
using System;

public class CChartEx1 : FrameChart
{
    static private readonly int[] numberOfDefects = {
        16, 14, 28, 16, 12, 20, 10, 12, 10, 17, 19, 17, 14, 16, 15, 13,
        14, 16, 11, 20, 11, 19, 16, 31, 13
    };

    public CChartEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        new CChart(axis, numberOfDefects);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new CChartEx1());
    }
}
```

Output



UChart Class

`public class Imsl.Chart2D.QC.UChart : ShewhartControlChart`

`UChart` is a *u*-chart for monitoring the defect rate when defects are rare.

The defect rate is the number of defects found divided by the number of samples inspected. The number of defects are assumed to be rare. Control limits are computed using the Poisson distribution. If defects are not rare, use `PChart` (p. 1556) instead. The sample sizes are not required to be equal.

The control limits are at

$$\bar{u} + k\sqrt{\bar{u}/n}$$

where $\bar{u}$ is the observed average number of defects per unit, n is the number of inspection units, and k is the value of the "ControlLimit" attribute for the line. By default, the chart contains an upper control limit line with k=3, a lower control limit line with k=-3, and a central line equal to $\bar{u}$. Additional control limits can be added. The method `AddWeco` adds control limits with k = -2, -1, 1, 2.

Constructors

UChart

```
public UChart(Imsl.Chart2D.AxisXY axis, double sizeSample, int[] numberDefects)
```

Description

Creates a u-Chart given the number of defects for a series of samples with equal sample sizes.

Parameters

`axis` – The `AxisXY` parent of this node.

`sizeSample` – The size of each sample.

`numberDefects` – An array of arrays containing the number of defects. The number of defects must all be nonnegative.

UChart

```
public UChart(Imsl.Chart2D.AxisXY axis, double[] sizeSample, int[]
numberDefects)
```

Description

Creates a u-Chart given the number of defects rates for a series of samples with varying sample sizes.

Parameters

`axis` – The `AxisXY` parent of this node.

`sizeSample` – An array containing the size of each sample.

`numberDefects` – An array of arrays containing the number of defects. The number of defects must all be nonnegative. The length of the `sizeSample` and `numberDefects` arrays must be equal.

Method

PrePaint

```
override public void PrePaint()
```

Description

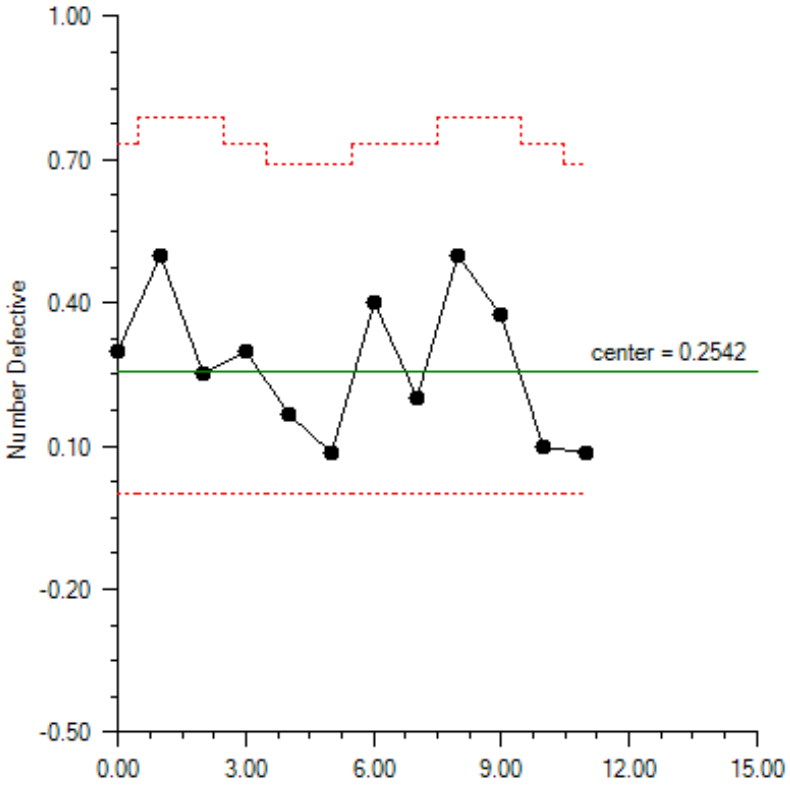
Setup chart with current settings.

Example: u-Chart

The number of defects in each of 12 samples was counted. The number of items in the samples varied from 8 to 12.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class UChartEx1 : FrameChart
{
    static private readonly int[] numberDefects = {
        3, 4, 2, 3, 2, 1, 4, 2, 4, 3, 1, 1
    };
    static private readonly double[] sizeSample = {
        10, 8, 8, 10, 12, 12, 10, 10, 8, 8, 10, 12
    };

    public UChartEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        axis.AxisY.AxisTitle.SetTitle("Number Defective");
        new UChart(axis, sizeSample, numberDefects);
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new UChartEx1());
    }
}
```

Output



EWMA Class

`public class Imsl.Chart2D.QC.EWMA : ShewhartControlChart`

`EWMA` is an *exponentially weighted moving average* control chart.

The EWMA statistic is given by

$$\text{EWMA}_t = \lambda x_t + (1-\lambda)\text{EWMA}_{t-1}$$

where x_t is the observation at time t and $0 < \lambda \leq 1$ is the decay parameter which determines the depth of memory of EWMA.

Property

Lambda

```
virtual public double Lambda {get; set; }
```

Description

The value of the attribute “Lambda”, the decay parameter.

Remarks

Its default value is 0.25.

Constructors

EWMA

```
public EWMA(Imsl.Chart2D.AxisXY axis, double[] x, double lambda)
```

Description

Creates an exponentially weighted moving average chart. The expected mean and standard deviation are computed from the sample data.

Parameters

`axis` – The `AxisXY` parent of this node

`x` – An array containing the data.

`lambda` – The decay parameter. It is usually between 0.2 and 0.3.

Exception

`System.ArgumentException` is thrown if $0 < \lambda \leq 1$ does not hold.

EWMA

```
public EWMA(Imsl.Chart2D.AxisXY axis, double[] data, double lambda, double
expectedMean, double expectedStandardDeviation)
```

Description

Creates an exponentially weighted moving average chart using the given values for the expected mean and standard deviation.

Parameters

`axis` – The `AxisXY` parent of this node.

`data` – An array containing data.

`lambda` – The decay parameter. It is usually between 0.2 and 0.3.

`expectedMean` – The expected mean of the data.

`expectedStandardDeviation` – The expected standard deviation of the data.

Exception

`System.ArgumentException` is thrown if $0 < \lambda \leq 1$ does not hold.

Method

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

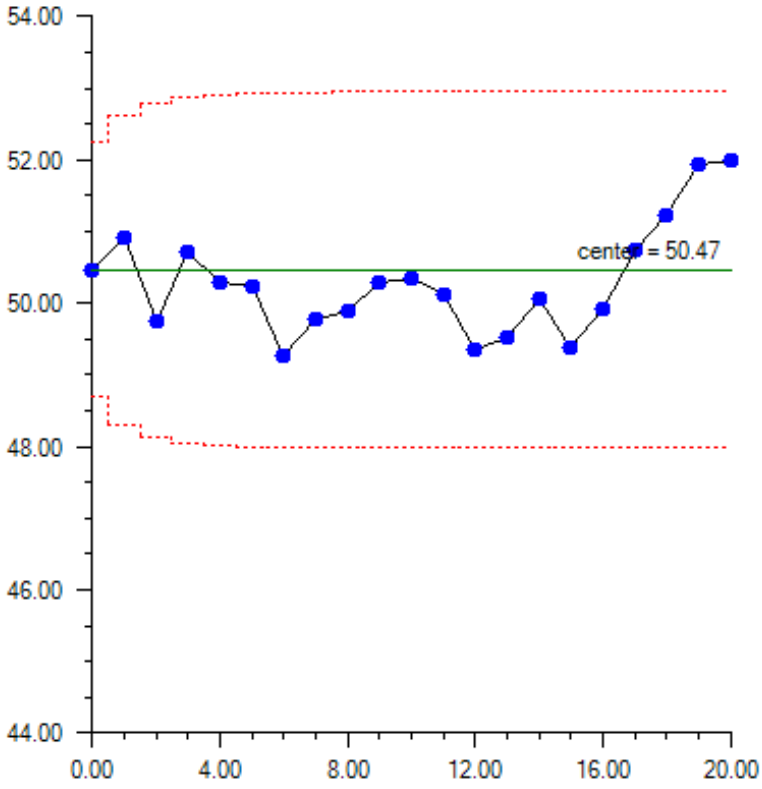
Example: EWMA Chart

A process is monitored using the EWMA control chart. The data is from NIST Engineering Statistics Handbook: EWMA Control Charts.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class EWMAEx1 : FrameChart
{
    static private readonly double[] data = {
        52.0, 47.0, 53.0, 49.3, 50.1, 47.0,
        51.0, 50.1, 51.2, 50.5, 49.6, 47.6,
        49.9, 51.3, 47.8, 51.2, 52.6, 52.4,
        53.6, 52.1
    };

    public EWMAEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        double lambda = 0.3;
        EWMA ewma = new EWMA(axis, data, lambda);
        ewma.ControlData.MarkerColor = System.Drawing.Color.Blue;
    }

    public static void Main(string[] argv)
    {
```

```
        System.Windows.Forms.Application.Run(new EWMAEx1());
    }
}
```

Output



CuSum Class

```
public class Imsl.Chart2D.QC.CuSum :  ShewhartControlChart
```

`CuSum` is a *cumulative sum* chart. It is more efficient than a Shewhart chart for detecting small shifts in the mean of a process.

CuSum plots the cumulative sum of the deviations of the expected value. If μ_0 is the expected mean for a process and $\bar{x_i}$ are the sample means then the cumulative sum is

$$C_i = C_{i-1} + (\bar{x_i} - \mu_0)$$

Property

ExpectedMean

```
virtual public double ExpectedMean {get; set; }
```

Description

The expected mean of all of the data from all of the samples.

Property Value

The value of `ExpectedMean` is either the expected mean value or the target mean. Its default value is computed from the data passed to the constructor.

Constructor

CuSum

```
public CuSum(Imsl.Chart2D.AxisXY axis, double[] xbar)
```

Description

Creates a CuSum chart given the means of a series of samples.

Parameters

`axis` – The `AxisXY` parent of this node

`xbar` – The means of the samples.

Methods

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

SetDataRange

```
override public void SetDataRange(double[] range)
```

Description

Update the data range, `range = {xmin,xmax,ymin,ymax}`

Parameter

`range` – A `double` array which contains the updated range, {xmin,xmax,ymin,ymax}

Remarks

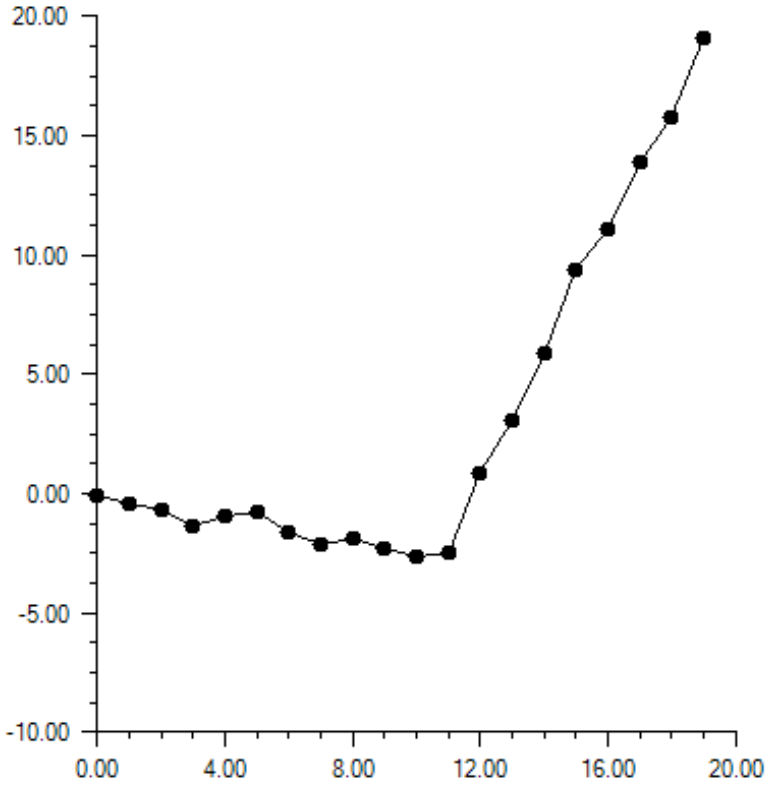
The entries in range are updated to reflect the extent of the data in this node. The argument `range` is an input/output variable. Its value should be updated only if the data in this node is outside the range already in the array.

Example: CuSum Chart

A CuSum chart is constructed from the data at NIST Engineering Statistics Handbook: CuSum Control Charts.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class CuSumEx1 : FrameChart
{
    static private readonly double[] data = {
        324.925, 324.675, 324.725, 324.350, 325.350, 325.225, 324.125,
        324.525, 325.225, 324.600, 324.625, 325.150, 328.325, 327.250,
        327.825, 328.500, 326.675, 327.775, 326.875, 328.350
    };

    public CuSumEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        double mean = 325;
        CuSum cusum = new CuSum(axis, data);
        cusum.ExpectedMean = mean;
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new CuSumEx1());
    }
}
```

Output



CuSumStatus Class

```
public class Imsl.Chart2D.QC.CuSumStatus :  ShewhartControlChart
```

`CuSumStatus` is a *cumulative sum status* chart. CuSumStatus plots the tabular CuSum results.

The one-sided upper and lower cusums, C^+ and C^- are computed. These are plotted as two bar charts. The C^+ are plotted as a bar chart above the x-axis and C^- is below the axis. By default, in-control bars are green and out-of-control bars are red.

Control limit lines are drawn at $\pm h$, where h is the value of the property `AbsoluteH`. This can be set either to an absolute number, using the property `AbsoluteH`, or relative to the standard deviation of the data, using the property `RelativeH`.

Properties

AbsoluteH

```
virtual public double AbsoluteH {get; set; }
```

Description

The value for H used for setting limits.

Property Value

There is no attribute "AbsoluteH". Its value is `Sigma` times `RelativeH`.

BarMinus

```
virtual public Imsl.Chart2D.Bar BarMinus {get; }
```

Description

The value of the attribute "BarMinus" contains the `Bar` (p. 1464) object for the negative CuSum (C^-) bars.

Property Value

To access the C^- bars for where the process is in control, apply the method `GetBarSet(0,0)` to the `Bar` object returned by this method. For the bars for where the process is not in control, apply the method `GetBarSet(1,0)` to the `Bar` object returned by this method.

BarPlus

```
virtual public Imsl.Chart2D.Bar BarPlus {get; }
```

Description

The value of the attribute "BarPlus" contains the `Bar` (p. 1464) object for the positive CuSum (C^+) bars.

Property Value

To access the C^+ bars for where the process is in control, apply the method `GetBarSet(0,0)` to the `Bar` object returned by this method. For the bars for where the process is not in control, apply the method `GetBarSet(1,0)` to the `Bar` object returned by this method.

CMinus

```
virtual public double[] CMinus {get; }
```

Description

The C^- values.

CPlus

```
virtual public double[] CPlus {get; }
```

Description

The C^+ values.

DataMarkers

```
virtual public Imsl.Chart2D.Data DataMarkers {get; }
```

Description

The “DataMarkers” attribute containing the data markers.

DataMarkersAxis

```
virtual public Imsl.Chart2D.AxisXY DataMarkersAxis {get; }
```

Description

The “DataMarkersAxis” attribute containing the axis associated with the data markers. This is normally the right-side axis.

ExpectedMean

```
virtual public double ExpectedMean {get; set; }
```

Description

The expected mean of all of the data from all of the samples.

Property Value

The value of `ExpectedMean` is either the expected mean value or the target mean. Its default value is computed from the data passed to the constructor.

InitialCMinus

```
virtual public double InitialCMinus {get; set; }
```

Description

The initial value of C^-.

Property Value

This is used to initiate a fast initial response (headstart). Typically, this would be set to $H/2$ for a 50% headstart. Its default value is zero.

InitialCPlus

```
virtual public double InitialCPlus {get; set; }
```

Description

The initial value of C^+.

Property Value

This is used to initiate a fast initial response (headstart). Typically, this would be set to $H/2$ for a 50% headstart. Its default value is zero.

NMinus

```
virtual public int[] NMinus {get; }
```

Description

N^-, the number of consecutive periods that the cusums C_i^- have been nonzero.

NPlus

```
virtual public int[] NPlus {get; }
```

Description

N^+, the number of consecutive periods that the cusums C_i^+ have been nonzero.

RelativeH

```
virtual public double RelativeH {get; set; }
```

Description

The relative *h*.

Property Value

This is the value of the attribute “RelativeH”. Its default value is 3.0.

Sigma

```
virtual public double Sigma {get; set; }
```

Description

The standard deviation of the data.

SlackValue

```
virtual public double SlackValue {get; }
```

Description

The slack value. This is also called *K*, or the reference value or the allowance.

Constructor

CuSumStatus

```
public CuSumStatus(Imsl.Chart2D.AxisXY axis, double[] data, double
expectedMean, double slackValue)
```

Description

Creates a CuSumStatus chart.

Parameters

`axis` – The `AxisXY` parent of this node

`data` – An array of measurements of a continuous variable.

`expectedMean` – The expected (or target) mean.

`slackValue` – The slack value, also called *K*, or the reference value, or the allowance.

Methods

AddDataMarkers

```
virtual public Imsl.Chart2D.Data AddDataMarkers()
```

Description

Adds the original data to the chart on a newly created axis. The new axis is stored as attribute “DataMarkersAxis” and the new Data node is stored as attribute “DataMarkers”.

Returns

A `Data` object containing the markers.

AddDataMarkers

```
virtual public Imsl.Chart2D.Data AddDataMarkers(Imsl.Chart2D.AxisXY
axisDataMarkers)
```

Description

Adds the original data to the chart. The axis is stored as attribute “DataMarkersAxis” and the new Data node is stored as attribute “DataMarkers”.

Parameter

`axisDataMarkers` – The axis for the data markers.

Returns

A `Data` object containing the markers.

CreateDataAxis

```
virtual public Imsl.Chart2D.AxisXY CreateDataAxis()
```

Description

Creates a new axis to hold a cummulative line. The created axis is drawn on the right side. Its x-axis is not drawn, since it would overlap with the primary chart axis. The x-axis is set to have the same window size as the orignal axis and its autoscaling attribute is turned off.

Returns

An `AxisXY` containing the newly created axis.

PrePaint

```
override public void PrePaint()
```

Description

Setup chart with current settings.

Print

```
virtual public void Print()
```

Description

Prints the tabular CuSum results. This prints the input data, C^+, N^+, C^-, and N^-.

SetX

```
override public void SetX(double[] x)
```

Description

Sets the x-coordinates of the bars. This also sets the “BarWidth” attribute to (x[1]-x[0])/2.

Parameter

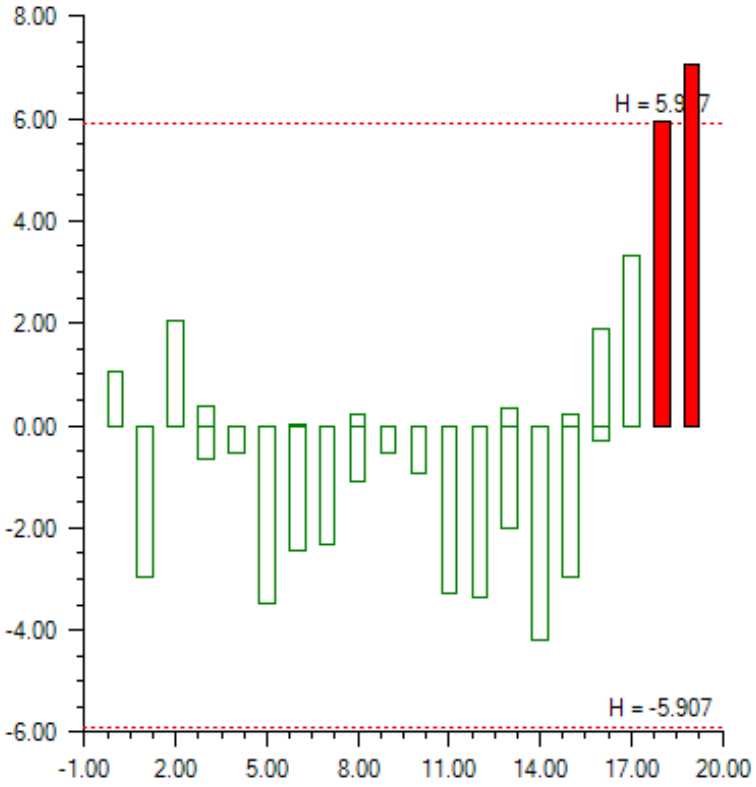
x – An array containing the *x*-coordinates. Its length must equal the length of the data array used to construct this object.

Example: CuSumStatus Chart

A process is monitored using the CuSumStatus control chart. The solid red bars indicate the process is out-of-control. The hollow green bars indicate that the process is in control. The data is from NIST Engineering Statistics Handbook: CuSumStatus Control Charts.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class CuSumStatusEx1 : FrameChart
{
    static private readonly double[] data = {
        52.0, 47.0, 53.0, 49.3, 50.1, 47.0,
        51.0, 50.1, 51.2, 50.5, 49.6, 47.6,
        49.9, 51.3, 47.8, 51.2, 52.6, 52.4,
        53.6, 52.1
    };

    public CuSumStatusEx1()
    {
        AxisXY axis = new AxisXY(this.Chart);
        double mean = Imsl.Stat.Summary.GetMean(data);
        double slackValue = 0.5;
        CuSumStatus cusum = new CuSumStatus(axis, data, mean, slackValue);
        cusum.BarPlus.GetBarSet(0,0).FillType = ChartNode.FILL_TYPE_NONE;
        cusum.BarMinus.GetBarSet(0,0).FillType = ChartNode.FILL_TYPE_NONE;
        cusum.BarPlus.GetBarSet(0,0).FillOutlineColor = System.Drawing.Color.Green;
        cusum.BarMinus.GetBarSet(0,0).FillOutlineColor = System.Drawing.Color.Green;
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new CuSumStatusEx1());
    }
}
```

Output



ParetoChart Class

```
public class Imsl.Chart2D.QC.ParetoChart :  Bar
```

`ParetoChart` is a *Pareto* bar chart.

The bars are sorted into descending order. It is used in quality assurance tracking to identify and prioritize areas of greatest impact. It extends `Bar`.

The method `AddCumulativeLine` adds a cumulative percentage line to the chart. This is the percent of defects accounted for by the current item and items to its left. If the cumulative percentage line is added, a second axis is created on the right. This is required because the units for this line are 0% to 100%. The units of the original axis (on the left) are the number of defects.

Properties

CumulativeAxis

```
virtual public Imsl.Chart2D.AxisXY CumulativeAxis {get; }
```

Description

The “CumulativeAxis” attribute, the axis on which the cumulative line is drawn.

CumulativeLine

```
virtual public Imsl.Chart2D.Data CumulativeLine {get; }
```

Description

The “CumulativeLine” attribute.

Property Value

The `Data` object containing the cumulative line.

Constructors

ParetoChart

```
public ParetoChart(Imsl.Chart2D.AxisXY axisBar, string[] labels, int[]
numberDefects)
```

Description

Constructs a Pareto chart.

Parameters

`axisBar` – The `AxisXY` parent of this node. Its formatting is changed to integer formatting.

`labels` – A `String` array which contains the labels for the data values.

`numberDefects` – An `int` array which contains the number of defects. These data values must be in the same order as the values in `labels`, but they do not need to be sorted.

Exception

`System.ArgumentException` is thrown if the length of the `labels` and `numberDefects` arrays are unequal or if any element of `numberDefects` is negative.

ParetoChart

```
public ParetoChart(Imsl.Chart2D.AxisXY axisBar, string[] labels, int[]
numberDefects, double maximumFractionCategoriesPlotted, string otherLabel)
```

Description

Constructs a Pareto chart showing only the most important bars.

Parameters

`axisBar` – The `AxisXY` parent of this node. Its formatting is changed to integer formatting.

`labels` – A `String` array which contains the labels for the data values.

`numberDefects` – An `int` array which contains the number of defects. These data values must be in the same order as the values in `labels`, but they do not need to be sorted.

`maximumFractionCategoriesPlotted` – The maximum cumulative fraction to be represented in separate categories. The remaining categories are consolidated into a single bar. A typical value for this argument is 0.80. This must be at least 0 and no more than 1.

`otherLabel` – The label of the bar holding total defect count of the categories not plotted.

Exception

`System.ArgumentException` is thrown if the length of the `labels` and `numberDefects` arrays are unequal or if any element of `numberDefects` is negative.

ParetoChart

```
public ParetoChart(Imsl.Chart2D.AxisXY axisBar, string[] labels, int[]
numberDefects, int maximumCategoriesPlotted, string otherLabel)
```

Description

Constructs a Pareto chart showing only a limited number of bars. The numberDefects-axis formatting is changed to integer formatting.

Parameters

`axisBar` – The `AxisXY` parent of this node.

`labels` – A `String` array which contains the labels for the data values.

`numberDefects` – An `int` array which contains the number of defects. These data values must be in the same order as the values in `labels`, but they do not need to be sorted.

`maximumCategoriesPlotted` – The maximum number of categories to be plotted. Categories with smaller number of defects are consolidated into a single bar. The total number of bars will be `maximumCategoriesPlotted`+1. This must be at least 0 and no more than the length of `numberDefects`.

`otherLabel` – The label of the bar holding total defect count of the categories not plotted.

Exception

`System.ArgumentException` is thrown if the length of the `labels` and `numberDefects` arrays are unequal or if any element of `numberDefects` is negative.

Methods

AddCumulativeLine

```
virtual public Imsl.Chart2D.Data AddCumulativeLine(Imsl.Chart2D.AxisXY
axisCumulativeLine)
```

Description

Adds a cumulative line to the specified axis.

Parameter

`axisCumulativeLine` – The axis for the cumulative line.

AddCumulativeLine

```
virtual public Imsl.Chart2D.Data AddCumulativeLine()
```

Description

Creates a new right-side axis and adds a cumulative line to it.

Returns

A `Data` object containing the cumulative percentage line.

CreateCumulativeLineAxis

```
virtual public Imsl.Chart2D.AxisXY CreateCumulativeLineAxis()
```

Description

Creates a new axis to hold a cumulative line.

Returns

An `AxisXY` object containing the newly created axis.

Remarks

The created axis is drawn on the right side. Its x-axis is not drawn, since it would overlap with the primary Pareto chart axis. Its y-axis is labeled using the “P0” (percent) format. Its y-axis is scaled to [0,1], it is not autoscaled.

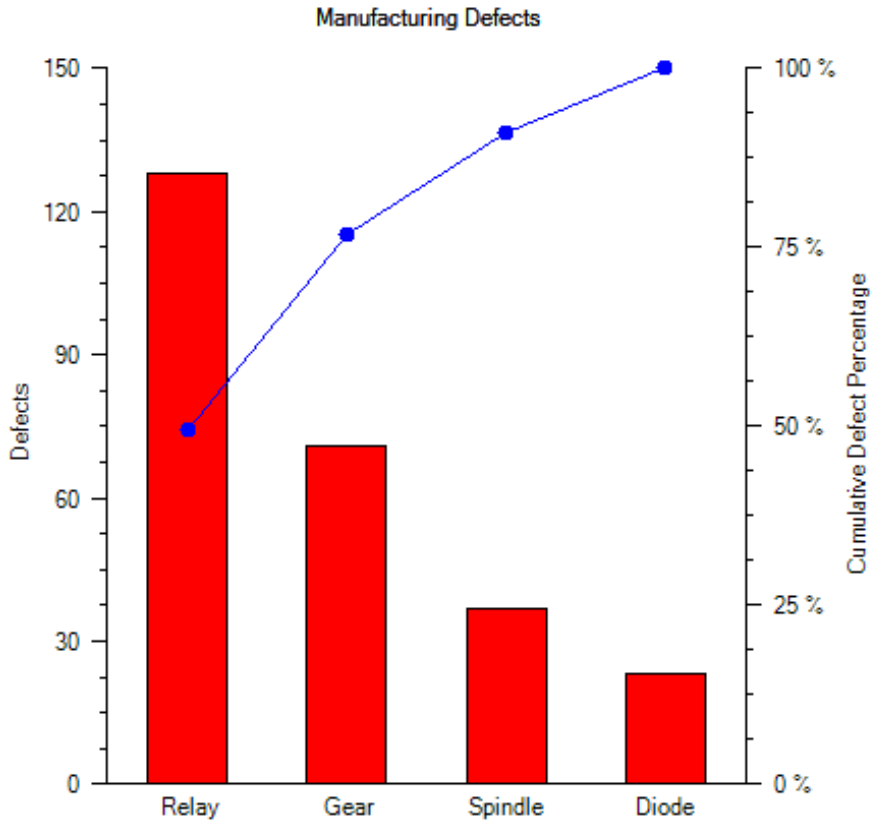
Example: Pareto Chart

The number of defects caused by four different factors were measured and plotted as a Pareto chart. The class `ParetoChart` sorts the factors in order of number of defects.

```
using Imsl.Chart2D;
using Imsl.Chart2D.QC;
using System;

public class ParetoEx1 : FrameChart
{
    public ParetoEx1()
    {
        AxisXY axisBar = new AxisXY(this.Chart);
```

```
        int[] numberDefects = new int[]{71, 23, 128, 37};
        System.String[] labels = new System.String[]{"Gear", "Diode", "Relay", "Spindle"};
        ParetoChart pareto = new ParetoChart(axisBar, labels, numberDefects);
        pareto.FillColor = System.Drawing.Color.Red;
        pareto.AddCumulativeLine();
        Data cumulativeLine = pareto.CumulativeLine;
        cumulativeLine.MarkerType = Data.MARKER_TYPE_FILLED_CIRCLE;
        cumulativeLine.LineColor = System.Drawing.Color.Blue;
        cumulativeLine.MarkerColor = System.Drawing.Color.Blue;

        this.Chart.ChartTitle.SetTitle("Manufacturing Defects");
        axisBar.AxisY.AxisTitle.SetTitle("Defects");
        pareto.CumulativeAxis.AxisY.AxisTitle.SetTitle("Cumulative Defect Percentage");
    }

    public static void Main(string[] argv)
    {
        System.Windows.Forms.Application.Run(new ParetoEx1());
    }
}
```

Output


Manufacturing Defects
150
120
90
60
30
0
Defects
100 %
75 %
50 %
25 %
0 %
Cumulative Defect Percentage
Relay
Gear
Spindle
Diode

Chapter 26: Data Mining

Type

Usage Notes

Data Mining - An Overview

The Data mining process involves the use of a collection of statistical and analytical methods for extracting useful information from large databases. The problem of extracting information from large databases is common to government, industry, engineering and sciences.

Data Filtering

The first step in this process is to filter data from its raw form into formats required by sophisticated analytical algorithms.

Data fall into two major categories: continuous and categorical. Many algorithms, such as neural network forecasting, perform better if continuous data are mapped into a common scale. Class `ScaleFilter` implements several techniques for automatically scaling continuous data, including several variations of z-score scaling. If the continuous data represent a time series, `TimeSeriesFilter` and `TimeSeriesClassFilter` can be used to create a matrix of lagged values required as input to forecasting neural networks.

Categorical data must also be mapped into a corresponding numerical representation before they can be used in solving forecasting and classification problems. There are two types of categorical data: ordinal and nominal. Ordinal data have a natural ordering among the categories, such as a school grade. Nominal data are categories without a natural ordering, such as eye color. The function imsls_unsupervised_ordinal_filter encodes and decodes ordinal data into the range [0, 1] using cumulative percentages. The function imsls_unsupervised_nominal_filter uses binary encoding to map nominal data into a matrix of zeros and ones.

Naive Bayes

Naive Bayes is a classification algorithm. A classifier is trained using `NaiveBayesClassifier` with known classifications. Once the classifier is trained, the classifier can be used to classify patterns with

unknown classifications.

Classification problems can be solved using other algorithms such as discriminant analysis and neural networks. In general these alternatives have smaller classification error rates, but they are too slow for large classification problems. During training, `NaiveBayesClassifier` uses the non-missing training data to estimate two-way correlations among the attributes. Higher order correlations are assumed to be zero. This can increase the classification error rate, but it significantly reduces the time needed to train the classifier.

In addition, the Naive Bayes algorithm is the only classification algorithm that can handle data with missing values. Other algorithms such as discriminant analysis do not allow missing values in the data. This is a significant limitation for applying other techniques to a larger database.

Classification problems are characterized by a need to classify unknown patterns or data into one of m categories based upon the values of k attributes $x_1, x_2, \ldots, x_k$. There are many algorithms for solving classification problems including discriminant analysis, neural networks and Naive Bayes. Each algorithm has its strengths and weaknesses. Discriminant analysis is robust but it requires $x_1, x_2, \ldots, x_k$ to be continuous, and since it uses a simple linear equation for the discriminant function, its error rate can be higher than the other algorithms. See `Imsl.Stat.DiscriminantAnalysis`.

Neural Networks provides a linear or non-linear classification algorithm that accepts both nominal and continuous input attributes. However, network training can be unacceptably slow for problems with a larger number of attributes, typically when $k > 1000$. Naive Bayes, on the other hand, is a simple algorithm that is very fast. A Naive Bayes classifier can be trained to classify patterns involving thousands of attributes and applied to thousands of patterns. As a result, Naive Bayes is a preferred algorithm for text mining and other large classification problems. However, its computational efficiency comes at a price. The error rate for a Naive Bayes classifier is typically higher than the equivalent Neural Network classifier, although it is usually low enough for many applications such as text mining.

If C is the classification attribute and $X^T = x_1, x_2, \ldots, x_k$ is the vector valued array of input attributes, the classification problem simplifies to estimating the conditional probability P(c|X) from a set of training patterns. The Bayes rule states that this probability can be expressed as the ratio:

$$P(\{C = c | X = \{x_1, x_2, \ldots, x_k\}) = \frac{P(C = c)P(X = \{x_1, x_2, \ldots, x_k\}) | C = c)}{P(X = \{x_1, x_2, x, \ldots, x_k\})}$$

where c is equal to one of the target classes $0, 1, \ldots, \text{nClasses} - 1$. In practice, the denominator of this expression is constant across all target classes since it is only a function of the given values of X. As a result, the Naive Bayes algorithm does not expend computational time estimating $P(X = \{x_1, x_2, \ldots, x_k\})$ for every pattern. Instead, a Naive Bayes classifier calculates the numerator $P(C = c)P(X = \{x_1, x_2, \ldots, x_k\}) | C = c)$ for each target class and then classifies X to the target class with the largest value, i.e.,

$$X \xleftarrow[\max(c=0,1,\ldots,\text{nClasses}-1)]{} P(C = c)P(X | C = c)$$

This is equivalent to assuming that the values of the input attributes, given C, are independent of one another, i.e.

$$P(x_i | x_j, C = c) = P(x_i | C = c), \text{ for all } i \neq j$$

In real world data this assumption rarely holds, yet in many cases this approach results in surprisingly low classification error rates. Since, the estimate of $P(C = c | X = \{x_1, x_2, \ldots, x_k\})$ from a Naive Bayes

classifier is generally an approximation, classifying patterns based upon the Naive Bayes algorithm can have acceptably low classification error rates.

Class `NaiveBayesClassifier` is used to train a classifier from a set of training patterns that contains patterns with values for both the input and target attributes. Classifications of new patterns with unknown classifications can be predicted using `NaiveBayesClassifier` methods `PredictClass` and `Probabilities`.

Neural Networks

Neural networks can be used for a variety of problems, some of which have been solved by existing statistical methods, and some of which have not. These applications fall into one of three categories, forecasting, classification, or statistical pattern recognition. More details on Neural Networks can be found in the `Neural Networks` chapter.

NaiveBayesClassifier Class

`public class Imsl.DataMining.NaiveBayesClassifier`

Trains a Naive Bayes Classifier

`NaiveBayesClassifier` trains a Naive Bayes classifier for classifying data into one of `nClasses` target classes. Input attributes can be a combination of both nominal and continuous data. Ordinal data can be treated as either nominal attributes or continuous. If the distribution of the ordinal data is known or can be approximated using one of the continuous distributions, then associating them with continuous attributes allows a user to specify that distribution. Missing values are allowed.

Before training the classifier the input attributes must be specified. For each nominal attribute, use method `CreateNominalAttribute` to specify the number of categories in each `nNominal` attribute. Specify the input attributes in the same column order that they will be supplied to the `Train` method. For example, if the input attribute in the first two columns of the nominal input data, `nominalData`, represent the first two nominal attributes and have two and three categories respectively, then the first call to the `CreateNominalAttribute` method would specify two categories and the second call to `CreateNominalAttribute` would specify three categories.

Likewise, for each continuous attribute, the method `CreateContinuousAttribute` can be used to specify a `IProbabilityDistribution` other than the default `NormalDistribution`. A second `CreateContinuousAttribute` is provided to allow specification of a different distribution for each target class (see Example 3). Create each continuous attribute in the same column order they will be supplied to the `Train` method. If `CreateContinuousAttribute` is not invoked for all `nContinuous` attributes, the `NormalDistribution` probability distribution will be used. For example, if five continuous attributes have been specified in the constructor, but only three calls to `CreateContinuousAttribute` have been invoked, the last two attributes, or columns of `continuousData` in the `Train` method, will use the `NormalDistribution` probability distribution.

Nominal only, continuous only, and a combination of both nominal and continuous input attributes are allowed. The `Train` method allows either nominal or continuous input arrays to be `null`.

Let C be the classification attribute with target categories $0, 1, \ldots,$ nClasses $- 1$, and let $X = \{x_1, x_2, \ldots, x_k\}$ be a vector valued array of k=`nNominal+nContinuous` input attrtibutes, where `nNominal` is the number of nominal attributes and `nContinuous` is the number of continuous attributes. See methods `CreateNominalAttribute` to specify the number of categories for each nominal attribute and `CreateContinuousAttribute` to specify the distribution for each continuous attribute. The classification problem simplifies to estimate the conditional probability *P(C|X)* from a set of training patterns. The Bayes rule states that this probability can be expressed as the ratio:

$$P(C = c|X = \{x_1, x_2, \ldots, x_k\}) = \frac{P(C = c)P(X = \{x_1, x_2, \ldots, x_k\})|C = c)}{P(X = \{x_1, x_2, x, \ldots, x_k\})}$$

where c is equal to one of the target classes $0, 1, \ldots,$ nClasses $- 1$. In practice, the denominator of this expression is constant across all target classes since it is only a function of the given values of X. As a result, the Naive Bayes algorithm does not expend computational time estimating $P(X = \{x_1, x_2, x, \ldots, x_k\})$ for every pattern. Instead, a Naive Bayes classifier calculates the numerator $P(C = c)P(X = \{x_1, x_2, \ldots, x_k\})|C = c)$ for each target class and then classifies X to the target class with the largest value, i.e.,

$$X \xleftarrow[\max(c=0,1,\ldots,\text{nClasses}-1)]{} P(C = c)P(X|C = c)$$

The classifier simplifies this calculation by assuming conditional independence. That is it assumes that:

$$P(X = \{x_1, x_2, \ldots, x_k\}|C = c) = \prod_{j=1}^{k} P(x_j|C = c)$$

This is equivalent to assuming that the values of the input attributes, given C, are independent of one another, i.e.,

$$P(x_i|x_j, C = c) = P(x_i|C = c), \text{ for all } i \neq j$$

In real world data this assumption rarely holds, yet in many cases this approach results in surprisingly low classification error rates. Since, the estimate of $P(C = c|X = \{x_1, x_2, \ldots, x_k\})$ from a Naive Bayes classifier is generally an approximation, classifying patterns based upon the Naive Bayes algorithm can have acceptably low classification error rates.

For nominal attributes, this implementation of the Naive Bayes classifier estimates conditional probabilities using a smoothed estimate:

$$P(x_j|C = c) = \frac{\#N\{x_j \cap C = c\} + \lambda}{\#N\{C = c\} + \lambda j},$$

where $\#N\{Z\}$ is the number of training patterns with attribute Z and j is equal to the number of categories associated with the j-th attribute.

The probability *P(C=c)* is also estimated using a smoothed estimate:

$$P(C = c) = \frac{\#N\{C = c\} + \lambda}{\text{nPatterns} + \lambda\,(\text{nClasses})}\ .$$

These estimates correspond to the maximum a priori (MAP) estimates for a Dirichelet prior assuming equal priors. The smoothing parameter can be any non-negative value. Setting $\lambda = 0$ corresponds to no

smoothing. The default smoothing used in this algorithm, $\lambda = 1$, is commonly referred to as Laplace smoothing. This can be specified using the property `DiscreteSmoothingValue`.

For continuous attributes, the same conditional probability $P(x_j|C = c)$ in the Naive Bayes formula is replaced with the conditional probability density function $f(x_j|C = c)$. By default, the density function for continuous attributes is the normal (Gaussian) probability density function (see `NormalDistribution`):

$$f(x_j|C = c) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(x_j-\mu\right)^2}{2\sigma^2}}$$

where μ and σ are the conditional mean and standard deviation, i.e. the mean and standard deviation of x_j when $C = c$. For convenience, methods `GetMeans` and `GetStandardDeviations` are provided to calculate the conditional mean and standard deviations of the training patterns.

In addition to the default normal pdf, users can select any continuous distribution to model the continuous attribute by providing an implementation of the `Imsl.Stat.IProbabilityDistribution` interface. See `NormalDistribution`, `LogNormalDistribution`, `GammaDistribution`, and `PoissonDistribution` for classes that implement the `IProbabilityDistribution` interface.

Smoothing conditional probability calculations for continuous attributes is controlled by the properties `ContinuousSmoothingValue` and `ZeroCorrection`. By default, conditional probability calculations for continuous attributes are unadjusted for calculations near zero. The value specified in the `ContinuousSmoothingValue` property will be added to each continuous probability calculation. This is similar to the effect of setting the property `DiscreteSmoothingValue` for the corresponding discrete calculations.

The value specified in the `ZeroCorrection` property is used when $(f(x|C = c) + \lambda) = 0$, where λ is the smoothing parameter setting. If this condition occurs, the conditional probability is replaced with the property value set in `ZeroCorrection`.

Methods `GetClassificationErrors`, `GetPredictedClass`, `GetProbabilities`, and `GetTrainingErrors` provide information on how well the trained `NaiveBayesClassifier` predicts the known target classifications of the training patterns.

Methods `Probabilities` and `PredictClass` estimate classification probabilities and predict classification of the input pattern using the trained Naive Bayes Classifier. The predicted classification returned by `PredictClass` is the class with the largest estimated classification probability. Method `ClassError` predicts the classification from the trained Naive Bayes classifier and compares the predicted classifications with the known target classification provided. This allows verification of the classifier with a set of patterns other than the training patterns.

Properties

ContinuousSmoothingValue

```
public double ContinuousSmoothingValue {get; set; }
```

Description

Parameter for calculating smoothed estimates of conditional probabilities for continuous attributes.

Property Value

A `double` containing the smoothing parameter to be used for calculating smoothed estimates of conditional probabilities for continuous attributes. `ContinuousSmoothingValue` must be non-negative.

Default: `ContinuousSmoothingValue`=0, i.e. no smoothing is done.

DiscreteSmoothingValue

```
public double DiscreteSmoothingValue {get; set; }
```

Description

Parameter for calculating smoothed estimates of conditional probabilities for discrete (nominal) attributes.

Property Value

A `double` containing the smoothing parameter to be used for calculating smoothed estimates of conditional probabilities for discrete attributes. `DiscreteSmoothingValue` must be non-negative.

Default: `DiscreteSmoothingValue` = 1.0, i.e. Laplace smoothing of conditional probabilities.

ZeroCorrection

```
public double ZeroCorrection {get; set; }
```

Description

Specifies the replacement value to be used for conditional probabilities equal to zero.

Property Value

A `double` containing the value to replace conditional probabilities equal to zero. `ZeroCorrection` must be non-negative.

Default: No correction will be performed.

Constructor

NaiveBayesClassifier

```
public NaiveBayesClassifier(int nContinuous, int nNominal, int nClasses)
```

Description

Constructs a NaiveBayesClassifier.

Parameters

`nContinuous` – An `int` containing the number of continuous attributes.

`nNominal` – An `int` containing the number of nominal attributes.

`nClasses` – An `int` containing the number of target classifications.

Methods

ClassError

```
public double ClassError(double[] continuous, int[] nominal, int
classification)
```

Description

Returns the classification probability error for the input pattern and known target classification.

Parameters

`continuous` – A `double` array of length `nContinuous` containing an input pattern of continuous attributes. If `nContinuous` = 0, a `null` is allowed.

`nominal` – An `int` array of length `nNominal` containing an input pattern of nominal attributes. If `nNominal` = 0, a `null` is allowed.

`classification` – An `int` containing the target classification.

Returns

A `double` containing the classification probability error for the input pattern. The classification error for the input pattern is equal to 1-*p*, where *p* is the predicted class probability of input `classification`. The predicted class probability of input `classification` can be obtained by the method `Probabilities`. If *p* = `Probabilities` and *k* is equal to `classification`, then the classification error is 1 - *p*[k].

CreateContinuousAttribute

```
public void CreateContinuousAttribute(Imsl.Stat.IProbabilityDistribution pdf)
```

Description

Create a continuous variable and the associated distribution function.

Parameter

`pdf` – A `IProbabilityDistribution` to be applied to the continuous attribute. The distribution function will be applied to all classes.

Default: `NormalDistribution` is used.

CreateContinuousAttribute

```
public void CreateContinuousAttribute(Imsl.Stat.IProbabilityDistribution[] pdf)
```

Description

Create a continuous variable and the associated distribution functions for each target classification.

Parameter

`pdf` – An array of `IProbabilityDistributions` containing `nClasses` distribution functions for a continuous attribute. This allows a different distribution function to be applied to each classification.

Default: `NormalDistribution` is used.

CreateNominalAttribute

```
public void CreateNominalAttribute(int nCategories)
```

Description

Create a nominal attribute and the number of categories

Parameter

`nCategories` – An `int` containing the number of categories in the nominal attribute. The category values are expected to be encoded with integers ranging from 0 to `nCategories` -1. No default is used for `nCategories`. If `nNominal` is not zero, and `CreateNominalAttribute` is not invoked for each `nNominal` attribute, an `InvalidOperationException` will be thrown when the `Train` method is invoked.

GetClassCounts

```
public int[] GetClassCounts(int[] classificationData)
```

Description

Returns the number of patterns for each target classification.

Parameter

`classificationData` – An `int` array containing the target classifications for the training patterns. These must be encoded from zero to `nClasses`-1. Any value outside this range is considered a missing value. In this case, the data in that pattern are not used to train the Naive Bayes classifier. However, any pattern with missing values is still classified after the classifier is trained.

Returns

An `int` array containing the class counts.

GetClassificationErrors

```
public double[] GetClassificationErrors()
```

Description

Returns the classification probability errors for each pattern in the training data.

Returns

A `double` array containing the classification probability errors for each pattern in the training data. The classification error for the *i*-th training pattern is equal to 1-*predictedClassProbability*`[i][k]`, where *predictedClassProbability* is returned from `GetProbabilities` and *k* is equal to `classificationData[i]`.

GetMeans

```
public double[][] GetMeans(double[][] continuousData, int[] classificationData)
```

Description

Returns a table of means for each continuous attribute in `continuousData` segmented by the target classes in `classificationData`.

Parameters

`continuousData` – A `double` matrix containing training values for the continuous attributes.

`classificationData` – An `int` array containing the target classifications for the training patterns.

Returns

A `continuousData[0].Length` by `nClasses` `double` matrix, *means*, containing the means segmented by the target classes. The *i*-th row contains the means of the *i*-th continuous attribute for each value of the target classification. That is, *means*[i][j] is the mean for the *i*-th continuous attribute when the target classification equals *j*, unless there are no training patterns for this condition.

Remarks

This method is provided as a utility, prior training is not necessary.

GetPredictedClass

```
public int[] GetPredictedClass()
```

Description

Returns the predicted classification for each training pattern.

Returns

An `int` array containing the predicted classification for each training pattern.

GetProbabilities

```
public double[][] GetProbabilities()
```

Description

Returns the predicted classification probabilities for each target class.

Returns

A `double` matrix, *prob*, of size *nPatterns* by `nClasses` containing the predicted classification probabilities for each target class, where *nPatterns* is the number of patterns trained. *prob*[i][j] is the estimated probability that the *i*-th pattern belongs to the *j*-th target class.

GetStandardDeviations

```
public double[][] GetStandardDeviations(double[][] continuousData, int[]
classificationData)
```

Description

Returns a table of standard deviations for each continuous attribute in `continuousData` segmented by the target classes in `classificationData`.

Parameters

`continuousData` – A `double` matrix containing training values for the continuous attributes.

`classificationData` – An `int` array containing the target classifications for the training patterns.

Returns

A `continuousData[0].Length` by `nClasses` `double` matrix, *stdev*, containing the standard deviations segmented by the target classes. The *i*-th row contains the standard deviation of the *i*-th continuous attribute for each value of the target classification. That is, *stdev*[i][j] is the standard deviations for the *i*-th continuous attribute when the target classification equals *j*, unless there are no training patterns for this condition.

Remarks

This method is provided as a utility, prior training is not necessary.

GetTrainingErrors

```
public int[][] GetTrainingErrors()
```

Description

Returns a table of classification errors of non-missing classifications for each target classification plus the overall total of classification errors.

Returns

An `int` matrix containing `nClasses` + 1 rows and two columns. The first column contains the number of misclassifications and the second column contains the total number of classifications for the *i*-th row target class. The last row of the matrix contains the total number of misclassifications in column one and the total non-missing classifications in column two.

IgnoreMissingValues

```
public void IgnoreMissingValues(bool ignoreMissing)
```

Description

Specifies whether or not missing values will be ignored during the training process.

Parameter

`ignoreMissing` – A `boolean` specifying whether or not to ignore patterns during training when one or more input attributes are missing. By default, both missing and non-missing values are used to train the classifier. Classification predictions are still returned for all patterns even when set to true.

Default: `ignoreMissing` = `false`.

PredictClass

```
public int PredictClass(double[] continuous, int[] nominal)
```

Description

Predicts the classification for the input pattern using the trained Naive Bayes classifier.

Parameters

`continuous` – A `double` array containing an input pattern of `nContinuous` continuous attributes. If `nContinuous` = 0, a `null` is allowed.

`nominal` – An `int` array of length `nNominal` containing an input pattern of nominal attributes. If `nNominal` = 0, a `null` is allowed.

Returns

An `int` containing the predicted classification for the input pattern using the trained Naive Bayes Classifier. The predicted classification returned is the class with the largest estimated classification probability. The classification probabilities can be predicted using the `Probabilities` method.

Probabilities

```
public double[] Probabilities(double[] continuous, int[] nominal)
```

Description

Predicts the classification probabilities for the input pattern using the trained Naive Bayes classifier.

Parameters

`continuous` – A `double` array containing an input pattern of `nContinuous` continuous attributes. If `nContinuous` = 0, a `null` is allowed.

`nominal` – An `int` array of length `nNominal` containing an input pattern of nominal attributes. If `nNominal` = 0, a `null` is allowed.

Returns

a `double` array of length `nClasses` containing the predicted classification probabilities for each target class.

Train

```
public void Train(double[][] continuousData, int[][] nominalData, int[]
classificationData)
```

Description

Trains a Naive Bayes classifier for classifying data into one of `nClasses` target classifications.

Parameters

`continuousData` – A `double` matrix containing the training values for the `nContinuous` continuous attributes. The *i*-th row contains the input attributes for the *i*-th training pattern. The *j*-th column contains the values for the *j*-th continuous attribute. Missing values should be set to `Double.NaN`. Patterns with both non-missing and missing values are used to train the classifier unless the `IgnoreMissingValues` method has been set to `true`. A `null` is allowed when `nContinuous` is equal to zero.

`nominalData` – An `int` matrix containing the training values for the `nNominal` nominal attributes. The *i*-th row contains the input attributes for the *i*-th training pattern. The *j*-th column contains the classifications for the *j*-th nominal attribute. The values for the *j*-th nominal attribute are expected to be encoded with integers starting from 0 to *nCategories* - 1, where *nCategories* is specified in the `CreateNominalAttribute` method. Any value outside this range is treated as a missing value. Patterns with both non-missing and missing values are used to train the classifier unless the `IgnoreMissingValues` method has been set to `true`. A `null` is allowed when `nNominal` is equal to zero.

`classificationData` – An `int` array containing the target classifications for the training patterns. These must be encoded from zero to `nClasses`-1. Any value outside this range is considered a missing value. In this case, the data in that pattern are not used to train the Naive Bayes classifier. However, any pattern with missing values is still classified after the classifier is trained.

Example 1: Continuous Attribute Example

Fisher's (1936) Iris data is often used for benchmarking classification algorithms. It consists of the following continuous input attributes and a classification target:

- *Continuous Attributes Usage*

  - Sepal Length
  - Sepal Width
  - Petal Length
  - Petal Width
- Classification of Iris Type
  - Setosa
  - Versicolour
  - Virginica

This example trains a Naive Bayes classifier using 140 of the 150 continuous patterns, then classifies ten unknown plants using their sepal and petal measurements.

```
using System;
using Imsl.DataMining;
using NormalDistribution = Imsl.Stat.NormalDistribution;

public class NaiveBayesClassifierEx1
{
   private static double[][] irisFisherData = new double[][]{
        new double[]{1.0, 5.1, 3.5, 1.4, .2},
        new double[]{1.0, 4.9, 3.0, 1.4, .2},
        new double[]{1.0, 4.7, 3.2, 1.3, .2},
        new double[]{1.0, 4.6, 3.1, 1.5, .2},
        new double[]{1.0, 5.0, 3.6, 1.4, .2},
        new double[]{1.0, 5.4, 3.9, 1.7, .4},
        new double[]{1.0, 4.6, 3.4, 1.4, .3},
        new double[]{1.0, 5.0, 3.4, 1.5, .2},
        new double[]{1.0, 4.4, 2.9, 1.4, .2},
        new double[]{1.0, 4.9, 3.1, 1.5, .1},
        new double[]{1.0, 5.4, 3.7, 1.5, .2},
        new double[]{1.0, 4.8, 3.4, 1.6, .2},
        new double[]{1.0, 4.8, 3.0, 1.4, .1},
        new double[]{1.0, 4.3, 3.0, 1.1, .1},
        new double[]{1.0, 5.8, 4.0, 1.2, .2},
        new double[]{1.0, 5.7, 4.4, 1.5, .4},
        new double[]{1.0, 5.4, 3.9, 1.3, .4},
        new double[]{1.0, 5.1, 3.5, 1.4, .3},
        new double[]{1.0, 5.7, 3.8, 1.7, .3},
        new double[]{1.0, 5.1, 3.8, 1.5, .3},
        new double[]{1.0, 5.4, 3.4, 1.7, .2},
        new double[]{1.0, 5.1, 3.7, 1.5, .4},
        new double[]{1.0, 4.6, 3.6, 1.0, .2},
        new double[]{1.0, 5.1, 3.3, 1.7, .5},
        new double[]{1.0, 4.8, 3.4, 1.9, .2},
        new double[]{1.0, 5.0, 3.0, 1.6, .2},
        new double[]{1.0, 5.0, 3.4, 1.6, .4},
        new double[]{1.0, 5.2, 3.5, 1.5, .2},
        new double[]{1.0, 5.2, 3.4, 1.4, .2},
        new double[]{1.0, 4.7, 3.2, 1.6, .2},
        new double[]{1.0, 4.8, 3.1, 1.6, .2},
        new double[]{1.0, 5.4, 3.4, 1.5, .4},
```

```
new double[]{1.0, 5.2, 4.1, 1.5, .1},
new double[]{1.0, 5.5, 4.2, 1.4, .2},
new double[]{1.0, 4.9, 3.1, 1.5, .2},
new double[]{1.0, 5.0, 3.2, 1.2, .2},
new double[]{1.0, 5.5, 3.5, 1.3, .2},
new double[]{1.0, 4.9, 3.6, 1.4, .1},
new double[]{1.0, 4.4, 3.0, 1.3, .2},
new double[]{1.0, 5.1, 3.4, 1.5, .2},
new double[]{1.0, 5.0, 3.5, 1.3, .3},
new double[]{1.0, 4.5, 2.3, 1.3, .3},
new double[]{1.0, 4.4, 3.2, 1.3, .2},
new double[]{1.0, 5.0, 3.5, 1.6, .6},
new double[]{1.0, 5.1, 3.8, 1.9, .4},
new double[]{1.0, 4.8, 3.0, 1.4, .3},
new double[]{1.0, 5.1, 3.8, 1.6, .2},
new double[]{1.0, 4.6, 3.2, 1.4, .2},
new double[]{1.0, 5.3, 3.7, 1.5, .2},
new double[]{1.0, 5.0, 3.3, 1.4, .2},
new double[]{2.0, 7.0, 3.2, 4.7, 1.4},
new double[]{2.0, 6.4, 3.2, 4.5, 1.5},
new double[]{2.0, 6.9, 3.1, 4.9, 1.5},
new double[]{2.0, 5.5, 2.3, 4.0, 1.3},
new double[]{2.0, 6.5, 2.8, 4.6, 1.5},
new double[]{2.0, 5.7, 2.8, 4.5, 1.3},
new double[]{2.0, 6.3, 3.3, 4.7, 1.6},
new double[]{2.0, 4.9, 2.4, 3.3, 1.0},
new double[]{2.0, 6.6, 2.9, 4.6, 1.3},
new double[]{2.0, 5.2, 2.7, 3.9, 1.4},
new double[]{2.0, 5.0, 2.0, 3.5, 1.0},
new double[]{2.0, 5.9, 3.0, 4.2, 1.5},
new double[]{2.0, 6.0, 2.2, 4.0, 1.0},
new double[]{2.0, 6.1, 2.9, 4.7, 1.4},
new double[]{2.0, 5.6, 2.9, 3.6, 1.3},
new double[]{2.0, 6.7, 3.1, 4.4, 1.4},
new double[]{2.0, 5.6, 3.0, 4.5, 1.5},
new double[]{2.0, 5.8, 2.7, 4.1, 1.0},
new double[]{2.0, 6.2, 2.2, 4.5, 1.5},
new double[]{2.0, 5.6, 2.5, 3.9, 1.1},
new double[]{2.0, 5.9, 3.2, 4.8, 1.8},
new double[]{2.0, 6.1, 2.8, 4.0, 1.3},
new double[]{2.0, 6.3, 2.5, 4.9, 1.5},
new double[]{2.0, 6.1, 2.8, 4.7, 1.2},
new double[]{2.0, 6.4, 2.9, 4.3, 1.3},
new double[]{2.0, 6.6, 3.0, 4.4, 1.4},
new double[]{2.0, 6.8, 2.8, 4.8, 1.4},
new double[]{2.0, 6.7, 3.0, 5.0, 1.7},
new double[]{2.0, 6.0, 2.9, 4.5, 1.5},
new double[]{2.0, 5.7, 2.6, 3.5, 1.0},
new double[]{2.0, 5.5, 2.4, 3.8, 1.1},
new double[]{2.0, 5.5, 2.4, 3.7, 1.0},
new double[]{2.0, 5.8, 2.7, 3.9, 1.2},
new double[]{2.0, 6.0, 2.7, 5.1, 1.6},
new double[]{2.0, 5.4, 3.0, 4.5, 1.5},
new double[]{2.0, 6.0, 3.4, 4.5, 1.6},
new double[]{2.0, 6.7, 3.1, 4.7, 1.5},
new double[]{2.0, 6.3, 2.3, 4.4, 1.3},
```

```
new double[]{2.0, 5.6, 3.0, 4.1, 1.3},
new double[]{2.0, 5.5, 2.5, 4.0, 1.3},
new double[]{2.0, 5.5, 2.6, 4.4, 1.2},
new double[]{2.0, 6.1, 3.0, 4.6, 1.4},
new double[]{2.0, 5.8, 2.6, 4.0, 1.2},
new double[]{2.0, 5.0, 2.3, 3.3, 1.0},
new double[]{2.0, 5.6, 2.7, 4.2, 1.3},
new double[]{2.0, 5.7, 3.0, 4.2, 1.2},
new double[]{2.0, 5.7, 2.9, 4.2, 1.3},
new double[]{2.0, 6.2, 2.9, 4.3, 1.3},
new double[]{2.0, 5.1, 2.5, 3.0, 1.1},
new double[]{2.0, 5.7, 2.8, 4.1, 1.3},
new double[]{3.0, 6.3, 3.3, 6.0, 2.5},
new double[]{3.0, 5.8, 2.7, 5.1, 1.9},
new double[]{3.0, 7.1, 3.0, 5.9, 2.1},
new double[]{3.0, 6.3, 2.9, 5.6, 1.8},
new double[]{3.0, 6.5, 3.0, 5.8, 2.2},
new double[]{3.0, 7.6, 3.0, 6.6, 2.1},
new double[]{3.0, 4.9, 2.5, 4.5, 1.7},
new double[]{3.0, 7.3, 2.9, 6.3, 1.8},
new double[]{3.0, 6.7, 2.5, 5.8, 1.8},
new double[]{3.0, 7.2, 3.6, 6.1, 2.5},
new double[]{3.0, 6.5, 3.2, 5.1, 2.0},
new double[]{3.0, 6.4, 2.7, 5.3, 1.9},
new double[]{3.0, 6.8, 3.0, 5.5, 2.1},
new double[]{3.0, 5.7, 2.5, 5.0, 2.0},
new double[]{3.0, 5.8, 2.8, 5.1, 2.4},
new double[]{3.0, 6.4, 3.2, 5.3, 2.3},
new double[]{3.0, 6.5, 3.0, 5.5, 1.8},
new double[]{3.0, 7.7, 3.8, 6.7, 2.2},
new double[]{3.0, 7.7, 2.6, 6.9, 2.3},
new double[]{3.0, 6.0, 2.2, 5.0, 1.5},
new double[]{3.0, 6.9, 3.2, 5.7, 2.3},
new double[]{3.0, 5.6, 2.8, 4.9, 2.0},
new double[]{3.0, 7.7, 2.8, 6.7, 2.0},
new double[]{3.0, 6.3, 2.7, 4.9, 1.8},
new double[]{3.0, 6.7, 3.3, 5.7, 2.1},
new double[]{3.0, 7.2, 3.2, 6.0, 1.8},
new double[]{3.0, 6.2, 2.8, 4.8, 1.8},
new double[]{3.0, 6.1, 3.0, 4.9, 1.8},
new double[]{3.0, 6.4, 2.8, 5.6, 2.1},
new double[]{3.0, 7.2, 3.0, 5.8, 1.6},
new double[]{3.0, 7.4, 2.8, 6.1, 1.9},
new double[]{3.0, 7.9, 3.8, 6.4, 2.0},
new double[]{3.0, 6.4, 2.8, 5.6, 2.2},
new double[]{3.0, 6.3, 2.8, 5.1, 1.5},
new double[]{3.0, 6.1, 2.6, 5.6, 1.4},
new double[]{3.0, 7.7, 3.0, 6.1, 2.3},
new double[]{3.0, 6.3, 3.4, 5.6, 2.4},
new double[]{3.0, 6.4, 3.1, 5.5, 1.8},
new double[]{3.0, 6.0, 3.0, 4.8, 1.8},
new double[]{3.0, 6.9, 3.1, 5.4, 2.1},
new double[]{3.0, 6.7, 3.1, 5.6, 2.4},
new double[]{3.0, 6.9, 3.1, 5.1, 2.3},
new double[]{3.0, 5.8, 2.7, 5.1, 1.9},
new double[]{3.0, 6.8, 3.2, 5.9, 2.3},
```

```
      new double[]{3.0, 6.7, 3.3, 5.7, 2.5},
      new double[]{3.0, 6.7, 3.0, 5.2, 2.3},
      new double[]{3.0, 6.3, 2.5, 5.0, 1.9},
      new double[]{3.0, 6.5, 3.0, 5.2, 2.0},
      new double[]{3.0, 6.2, 3.4, 5.4, 2.3},
      new double[]{3.0, 5.9, 3.0, 5.1, 1.8}};

public static void Main(String[] args)
{

   /* Data corrections described in the KDD data mining archive */
   irisFisherData[34][4] = 0.1;
   irisFisherData[37][2] = 3.1;
   irisFisherData[37][3] = 1.5;

   /*  Train first 140 patterns of the iris Fisher Data */
   int[] irisClassificationData =
         new int[irisFisherData.Length - 10];
   double[][] irisContinuousData =
         new double[irisFisherData.Length - 10][];
   for (int i = 0; i < irisFisherData.Length - 10; i++)
   {
      irisContinuousData[i] =
            new double[irisFisherData[0].Length - 1];
   }

   for (int i = 0; i < irisFisherData.Length - 10; i++)
   {
      irisClassificationData[i] = (int)irisFisherData[i][0] - 1;
      Array.Copy(irisFisherData[i], 1, irisContinuousData[i], 0,
            irisFisherData[0].Length - 1);
   }


   int nNominal = 0; /* no nominal input attributes      */
   int nContinuous = 4; /* four continuous input attributes */
   int nClasses = 3; /* three classification categories  */

   NaiveBayesClassifier nbTrainer =
         new NaiveBayesClassifier(nContinuous, nNominal, nClasses);

   for (int i = 0; i < nContinuous; i++)
      nbTrainer.CreateContinuousAttribute(
            new NormalDistribution());
   nbTrainer.Train(irisContinuousData, null,
         irisClassificationData);

   int[][] classErrors = nbTrainer.GetTrainingErrors();

   Console.Out.WriteLine(
         "     Iris Classification Training Error Rates");
   Console.Out.WriteLine(
         "-----------------------------------------------");
   Console.Out.WriteLine(
         "  Setosa   Versicolour   Virginica    |   Total");
   Console.Out.WriteLine("  " + classErrors[0][0] + "/" +
```

```
      classErrors[0][1] + "         " + classErrors[1][0] + "/" +
      classErrors[1][1] + "        " + classErrors[2][0] + "/" +
      classErrors[2][1] + "       |   " + classErrors[3][0] +
      "/" + classErrors[3][1]);
Console.Out.WriteLine(
      "-------------------------------------------------\n\n\n");


/*  Classify last  10 iris data patterns
   * with the trained classifier
   */
double[] continuousInput =
      new double[(irisFisherData[0].Length - 1)];
double[] classifiedProbabilities = new double[nClasses];

Console.Out.WriteLine(
      "Probabilities for Incorrect Classifications");
Console.Out.WriteLine(" Predicted   ");
Console.Out.WriteLine(
      "   Class     |  Class       |   P(0)     P(1)     P(2) ");
Console.Out.WriteLine(
      "-------------------------------------------------------");
for (int i = 0; i < 10; i++)
{
   int targetClassification =(int)
      irisFisherData[(irisFisherData.Length - 10) + i][0] - 1;
   Array.Copy(irisFisherData[(irisFisherData.Length - 10) + i],
      1, continuousInput, 0, (irisFisherData[0].Length - 1));

   classifiedProbabilities =
      nbTrainer.Probabilities(continuousInput, null);
   int classification =
      nbTrainer.PredictClass(continuousInput, null);
   if (classification == 0)
      Console.Out.Write("Setosa       |");
   else if (classification == 1)
      Console.Out.Write("Versicolour  |");
   else if (classification == 2)
      Console.Out.Write("Virginica    |");
   else
      Console.Out.Write("Missing      |");
   if (targetClassification == 0)
      Console.Out.Write(" Setosa       |");
   else if (targetClassification == 1)
      Console.Out.Write(" Versicolour  |");
   else if (targetClassification == 2)
      Console.Out.Write(" Virginica    |");
   else
      Console.Out.Write(" Missing      |");

   for (int j = 0; j < nClasses; j++)
   {
      Object[] pArgs = new Object[] {
         (double)classifiedProbabilities[j] };
      Console.Out.Write("   {0, 2:f3} ", pArgs);
   }
```

```
            Console.Out.WriteLine();
        }
    }
}
```

Output

The Naive Bayes classifier incorrectly classifies 6 of the 150 training patterns.

```
     Iris Classification Training Error Rates
-----------------------------------------------
  Setosa   Versicolour   Virginica    |   Total
  0/50         0/50        20/40       |   20/140
-----------------------------------------------



Probabilities for Incorrect Classifications
 Predicted
   Class     |  Class       |   P(0)     P(1)     P(2)
-------------------------------------------------------
Virginica    | Virginica    |   0.000    0.436    0.564
Virginica    | Virginica    |   0.000    0.466    0.534
Versicolour  | Virginica    |   0.000    0.542    0.458
Virginica    | Virginica    |   0.000    0.441    0.559
Virginica    | Virginica    |   0.000    0.412    0.588
Virginica    | Virginica    |   0.000    0.466    0.534
Versicolour  | Virginica    |   0.000    0.542    0.458
Versicolour  | Virginica    |   0.000    0.515    0.485
Virginica    | Virginica    |   0.000    0.460    0.540
Versicolour  | Virginica    |   0.000    0.551    0.449
```

Example 2: Nominal Attribute Usage

This example trains a Naive Bayes classifier using 24 training patterns with four nominal input attributes.

The first nominal attribute has three classifications and the others have two. The target classifications are contact lense prescriptions: hard, soft or neither recommended. These data are benchmark data from the Knowledge Discovery Databases archive maintained at the University of California, Irvine:

```
using System;
using Imsl.DataMining;

public class NaiveBayesClassifierEx2
{
    public static void  Main(String[] args)
    {

        int[][] contactLensData = new int[][]{
            new int[]{1, 1, 1, 1}, new int[]{1, 1, 1, 2},
            new int[]{1, 1, 2, 1}, new int[]{1, 1, 2, 2},
            new int[]{1, 2, 1, 1}, new int[]{1, 2, 1, 2},
            new int[]{1, 2, 2, 1}, new int[]{1, 2, 2, 2},
```

```
    new int[]{2, 1, 1, 1}, new int[]{2, 1, 1, 2},
    new int[]{2, 1, 2, 1}, new int[]{2, 1, 2, 2},
    new int[]{2, 2, 1, 1}, new int[]{2, 2, 1, 2},
    new int[]{2, 2, 2, 1}, new int[]{2, 2, 2, 2},
    new int[]{3, 1, 1, 1}, new int[]{3, 1, 1, 2},
    new int[]{3, 1, 2, 1}, new int[]{3, 1, 2, 2},
    new int[]{3, 2, 1, 1}, new int[]{3, 2, 1, 2},
    new int[]{3, 2, 2, 1}, new int[]{3, 2, 2, 2}
};

int[] classificationData = new int[]{
    3, 2, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1,
    3, 2, 3, 3, 3, 3, 3, 1, 3, 2, 3, 3
};
/* classification values must start at 0 */
for (int i = 0; i < classificationData.Length; i++)
{
    classificationData[i] -= 1;
    for (int j = 0; j < contactLensData[0].Length; j++)
    {
        contactLensData[i][j] -= 1;
    }
}
NaiveBayesClassifier nbTrainer =
    new NaiveBayesClassifier(0, 4, 3);

int nNominal = 4;
int[] categories = new int[]{3, 2, 2, 2};
for (int i = 0; i < nNominal; i++)
    nbTrainer.CreateNominalAttribute(categories[i]);
nbTrainer.Train(null, contactLensData, classificationData);

int[][] classErrors = nbTrainer.GetTrainingErrors();

Console.Out.WriteLine("\n    Contact Lense Error Rates");
Console.Out.WriteLine(
    "-----------------------------------------------");
Console.Out.WriteLine(
    "  Hard        Soft       Neither    |   Total");
Console.Out.WriteLine("  " + classErrors[0][0] + "/" +
    classErrors[0][1] + "         " + classErrors[1][0] + "/" +
    classErrors[1][1] + "        " + classErrors[2][0] + "/" +
    classErrors[2][1] + "       |   " + classErrors[3][0] +
    "/" + classErrors[3][1]);
Console.Out.WriteLine(
    "-----------------------------------------------\n\n\n");


/*  Classify all patterns with the trained classifier */
int[] nominalInput = new int[contactLensData[0].Length];
double[] classifiedProbabilities = new double[3];

Console.Out.WriteLine(
    "Probabilities for Incorrect Classifications");
Console.Out.WriteLine(" Predicted   ");
Console.Out.WriteLine("   Class     |  Class       |   "
```

```
            + "" + "P(0)     P(1)     P(2)   |  classification error");
        Console.Out.WriteLine(
            "-------------------------------------" +
            "-----------------------------------------");
        for (int i = 0; i < contactLensData.Length; i++)
        {
            Array.Copy(contactLensData[i], 0, nominalInput, 0,
                contactLensData[0].Length);

            classifiedProbabilities =
                nbTrainer.Probabilities(null, nominalInput);
            int classification =
                nbTrainer.PredictClass(null, nominalInput);
            double error = nbTrainer.ClassError(null, nominalInput,
                classificationData[i]);
            if (classification == 0)
                Console.Out.Write(" Hard        |");
            else if (classification == 1)
                Console.Out.Write(" Soft        |");
            else if (classification == 2)
                Console.Out.Write(" Neither     |");
            else
                Console.Out.Write(" Missing     |");
            if (classificationData[i] == 0)
                Console.Out.Write(" Hard         |");
            else if (classificationData[i] == 1)
                Console.Out.Write(" Soft         |");
            else if (classificationData[i] == 2)
                Console.Out.Write(" Neither      |");
            else
                Console.Out.Write(" Missing      |");

            for (int j = 0; j < 3; j++)
            {
               Object[] pArgs = new Object[] {
                   (double)classifiedProbabilities[j] };
                Console.Out.Write("   {0, 2:f3} ", pArgs);
            }
            Console.Out.WriteLine(" |  " + error);
        }
    }
}
```

Output

```
    Contact Lense Error Rates
-----------------------------------------------
  Hard        Soft       Neither    |   Total
  0/4         0/5        1/15       |   1/24
-----------------------------------------------



Probabilities for Incorrect Classifications
 Predicted
```

```
   Class     |  Class       |   P(0)     P(1)     P(2)   |  classification error
-------------------------------------------------------------------------------
 Neither     | Neither      |   0.044    0.130    0.827  |  0.173282735372243
 Soft        | Soft         |   0.174    0.622    0.203  |  0.377703751596285
 Neither     | Neither      |   0.186    0.018    0.795  |  0.204814844531185
 Hard        | Hard         |   0.724    0.086    0.190  |  0.27622138817044
 Neither     | Neither      |   0.019    0.154    0.827  |  0.173119280942829
 Soft        | Soft         |   0.076    0.724    0.200  |  0.275800758853599
 Neither     | Neither      |   0.092    0.024    0.884  |  0.116366478280659
 Hard        | Hard         |   0.524    0.166    0.310  |  0.475967127191525
 Neither     | Neither      |   0.025    0.113    0.862  |  0.137948891733597
 Soft        | Soft         |   0.118    0.633    0.248  |  0.366677564499419
 Neither     | Neither      |   0.113    0.017    0.870  |  0.1300941614979
 Hard        | Hard         |   0.606    0.108    0.286  |  0.394395320215918
 Neither     | Neither      |   0.011    0.133    0.856  |  0.143807127534487
 Soft        | Soft         |   0.050    0.714    0.236  |  0.286218909502689
 Neither     | Neither      |   0.054    0.021    0.925  |  0.0747706562988114
 Neither     | Neither      |   0.394    0.187    0.419  |  0.58120710821219
 Neither     | Neither      |   0.023    0.068    0.909  |  0.0907529770941407
 Soft        | Neither      |   0.142    0.509    0.349  |  0.650925868235811
 Neither     | Neither      |   0.099    0.010    0.891  |  0.109251850144518
 Hard        | Hard         |   0.599    0.071    0.330  |  0.401383015922963
 Neither     | Neither      |   0.010    0.081    0.909  |  0.090658834477251
 Soft        | Soft         |   0.062    0.594    0.344  |  0.406256189123439
 Neither     | Neither      |   0.047    0.012    0.941  |  0.0590094638695026
 Neither     | Neither      |   0.391    0.124    0.485  |  0.514955474332508
```

Example 3: Naive Bayes Classifier Using User Supplied Probability Function

This example is the same as Example 1, using Fisher's (1936) Iris data to train a Naive Bayes classifier using 140 of the 150 continuous patterns, then classifies ten unknown plants using their sepal and petal measurements.

Instead of using the `NormalDistribution` class from the `Imsl.Stat` namespace, a user supplied normal (Gaussian) distribution is used. Rather than calculating the means and standard deviations from the data, as is done by the `NormalDistribution`'s `Eval(double[])` method, the user supplied class requires the means and standard deviations in the class constructor. The output is the same as in Example 1, since the means and standard deviations in this example are simply rounded means and standard deviations of the actual data subset by target classifications.

```
using System;
using Imsl.DataMining;
using IProbabilityDistribution = Imsl.Stat.IProbabilityDistribution;

public class NaiveBayesClassifierEx3
{

   private static double[][] irisFisherData = new double[][]{
        new double[]{1.0, 5.1, 3.5, 1.4, .2},
        new double[]{1.0, 4.9, 3.0, 1.4, .2},
        new double[]{1.0, 4.7, 3.2, 1.3, .2},
        new double[]{1.0, 4.6, 3.1, 1.5, .2},
```

```
new double[]{1.0, 5.0, 3.6, 1.4, .2},
new double[]{1.0, 5.4, 3.9, 1.7, .4},
new double[]{1.0, 4.6, 3.4, 1.4, .3},
new double[]{1.0, 5.0, 3.4, 1.5, .2},
new double[]{1.0, 4.4, 2.9, 1.4, .2},
new double[]{1.0, 4.9, 3.1, 1.5, .1},
new double[]{1.0, 5.4, 3.7, 1.5, .2},
new double[]{1.0, 4.8, 3.4, 1.6, .2},
new double[]{1.0, 4.8, 3.0, 1.4, .1},
new double[]{1.0, 4.3, 3.0, 1.1, .1},
new double[]{1.0, 5.8, 4.0, 1.2, .2},
new double[]{1.0, 5.7, 4.4, 1.5, .4},
new double[]{1.0, 5.4, 3.9, 1.3, .4},
new double[]{1.0, 5.1, 3.5, 1.4, .3},
new double[]{1.0, 5.7, 3.8, 1.7, .3},
new double[]{1.0, 5.1, 3.8, 1.5, .3},
new double[]{1.0, 5.4, 3.4, 1.7, .2},
new double[]{1.0, 5.1, 3.7, 1.5, .4},
new double[]{1.0, 4.6, 3.6, 1.0, .2},
new double[]{1.0, 5.1, 3.3, 1.7, .5},
new double[]{1.0, 4.8, 3.4, 1.9, .2},
new double[]{1.0, 5.0, 3.0, 1.6, .2},
new double[]{1.0, 5.0, 3.4, 1.6, .4},
new double[]{1.0, 5.2, 3.5, 1.5, .2},
new double[]{1.0, 5.2, 3.4, 1.4, .2},
new double[]{1.0, 4.7, 3.2, 1.6, .2},
new double[]{1.0, 4.8, 3.1, 1.6, .2},
new double[]{1.0, 5.4, 3.4, 1.5, .4},
new double[]{1.0, 5.2, 4.1, 1.5, .1},
new double[]{1.0, 5.5, 4.2, 1.4, .2},
new double[]{1.0, 4.9, 3.1, 1.5, .2},
new double[]{1.0, 5.0, 3.2, 1.2, .2},
new double[]{1.0, 5.5, 3.5, 1.3, .2},
new double[]{1.0, 4.9, 3.6, 1.4, .1},
new double[]{1.0, 4.4, 3.0, 1.3, .2},
new double[]{1.0, 5.1, 3.4, 1.5, .2},
new double[]{1.0, 5.0, 3.5, 1.3, .3},
new double[]{1.0, 4.5, 2.3, 1.3, .3},
new double[]{1.0, 4.4, 3.2, 1.3, .2},
new double[]{1.0, 5.0, 3.5, 1.6, .6},
new double[]{1.0, 5.1, 3.8, 1.9, .4},
new double[]{1.0, 4.8, 3.0, 1.4, .3},
new double[]{1.0, 5.1, 3.8, 1.6, .2},
new double[]{1.0, 4.6, 3.2, 1.4, .2},
new double[]{1.0, 5.3, 3.7, 1.5, .2},
new double[]{1.0, 5.0, 3.3, 1.4, .2},
new double[]{2.0, 7.0, 3.2, 4.7, 1.4},
new double[]{2.0, 6.4, 3.2, 4.5, 1.5},
new double[]{2.0, 6.9, 3.1, 4.9, 1.5},
new double[]{2.0, 5.5, 2.3, 4.0, 1.3},
new double[]{2.0, 6.5, 2.8, 4.6, 1.5},
new double[]{2.0, 5.7, 2.8, 4.5, 1.3},
new double[]{2.0, 6.3, 3.3, 4.7, 1.6},
new double[]{2.0, 4.9, 2.4, 3.3, 1.0},
new double[]{2.0, 6.6, 2.9, 4.6, 1.3},
new double[]{2.0, 5.2, 2.7, 3.9, 1.4},
```

```
new double[]{2.0, 5.0, 2.0, 3.5, 1.0},
new double[]{2.0, 5.9, 3.0, 4.2, 1.5},
new double[]{2.0, 6.0, 2.2, 4.0, 1.0},
new double[]{2.0, 6.1, 2.9, 4.7, 1.4},
new double[]{2.0, 5.6, 2.9, 3.6, 1.3},
new double[]{2.0, 6.7, 3.1, 4.4, 1.4},
new double[]{2.0, 5.6, 3.0, 4.5, 1.5},
new double[]{2.0, 5.8, 2.7, 4.1, 1.0},
new double[]{2.0, 6.2, 2.2, 4.5, 1.5},
new double[]{2.0, 5.6, 2.5, 3.9, 1.1},
new double[]{2.0, 5.9, 3.2, 4.8, 1.8},
new double[]{2.0, 6.1, 2.8, 4.0, 1.3},
new double[]{2.0, 6.3, 2.5, 4.9, 1.5},
new double[]{2.0, 6.1, 2.8, 4.7, 1.2},
new double[]{2.0, 6.4, 2.9, 4.3, 1.3},
new double[]{2.0, 6.6, 3.0, 4.4, 1.4},
new double[]{2.0, 6.8, 2.8, 4.8, 1.4},
new double[]{2.0, 6.7, 3.0, 5.0, 1.7},
new double[]{2.0, 6.0, 2.9, 4.5, 1.5},
new double[]{2.0, 5.7, 2.6, 3.5, 1.0},
new double[]{2.0, 5.5, 2.4, 3.8, 1.1},
new double[]{2.0, 5.5, 2.4, 3.7, 1.0},
new double[]{2.0, 5.8, 2.7, 3.9, 1.2},
new double[]{2.0, 6.0, 2.7, 5.1, 1.6},
new double[]{2.0, 5.4, 3.0, 4.5, 1.5},
new double[]{2.0, 6.0, 3.4, 4.5, 1.6},
new double[]{2.0, 6.7, 3.1, 4.7, 1.5},
new double[]{2.0, 6.3, 2.3, 4.4, 1.3},
new double[]{2.0, 5.6, 3.0, 4.1, 1.3},
new double[]{2.0, 5.5, 2.5, 4.0, 1.3},
new double[]{2.0, 5.5, 2.6, 4.4, 1.2},
new double[]{2.0, 6.1, 3.0, 4.6, 1.4},
new double[]{2.0, 5.8, 2.6, 4.0, 1.2},
new double[]{2.0, 5.0, 2.3, 3.3, 1.0},
new double[]{2.0, 5.6, 2.7, 4.2, 1.3},
new double[]{2.0, 5.7, 3.0, 4.2, 1.2},
new double[]{2.0, 5.7, 2.9, 4.2, 1.3},
new double[]{2.0, 6.2, 2.9, 4.3, 1.3},
new double[]{2.0, 5.1, 2.5, 3.0, 1.1},
new double[]{2.0, 5.7, 2.8, 4.1, 1.3},
new double[]{3.0, 6.3, 3.3, 6.0, 2.5},
new double[]{3.0, 5.8, 2.7, 5.1, 1.9},
new double[]{3.0, 7.1, 3.0, 5.9, 2.1},
new double[]{3.0, 6.3, 2.9, 5.6, 1.8},
new double[]{3.0, 6.5, 3.0, 5.8, 2.2},
new double[]{3.0, 7.6, 3.0, 6.6, 2.1},
new double[]{3.0, 4.9, 2.5, 4.5, 1.7},
new double[]{3.0, 7.3, 2.9, 6.3, 1.8},
new double[]{3.0, 6.7, 2.5, 5.8, 1.8},
new double[]{3.0, 7.2, 3.6, 6.1, 2.5},
new double[]{3.0, 6.5, 3.2, 5.1, 2.0},
new double[]{3.0, 6.4, 2.7, 5.3, 1.9},
new double[]{3.0, 6.8, 3.0, 5.5, 2.1},
new double[]{3.0, 5.7, 2.5, 5.0, 2.0},
new double[]{3.0, 5.8, 2.8, 5.1, 2.4},
new double[]{3.0, 6.4, 3.2, 5.3, 2.3},
```

```
        new double[]{3.0, 6.5, 3.0, 5.5, 1.8},
        new double[]{3.0, 7.7, 3.8, 6.7, 2.2},
        new double[]{3.0, 7.7, 2.6, 6.9, 2.3},
        new double[]{3.0, 6.0, 2.2, 5.0, 1.5},
        new double[]{3.0, 6.9, 3.2, 5.7, 2.3},
        new double[]{3.0, 5.6, 2.8, 4.9, 2.0},
        new double[]{3.0, 7.7, 2.8, 6.7, 2.0},
        new double[]{3.0, 6.3, 2.7, 4.9, 1.8},
        new double[]{3.0, 6.7, 3.3, 5.7, 2.1},
        new double[]{3.0, 7.2, 3.2, 6.0, 1.8},
        new double[]{3.0, 6.2, 2.8, 4.8, 1.8},
        new double[]{3.0, 6.1, 3.0, 4.9, 1.8},
        new double[]{3.0, 6.4, 2.8, 5.6, 2.1},
        new double[]{3.0, 7.2, 3.0, 5.8, 1.6},
        new double[]{3.0, 7.4, 2.8, 6.1, 1.9},
        new double[]{3.0, 7.9, 3.8, 6.4, 2.0},
        new double[]{3.0, 6.4, 2.8, 5.6, 2.2},
        new double[]{3.0, 6.3, 2.8, 5.1, 1.5},
        new double[]{3.0, 6.1, 2.6, 5.6, 1.4},
        new double[]{3.0, 7.7, 3.0, 6.1, 2.3},
        new double[]{3.0, 6.3, 3.4, 5.6, 2.4},
        new double[]{3.0, 6.4, 3.1, 5.5, 1.8},
        new double[]{3.0, 6.0, 3.0, 4.8, 1.8},
        new double[]{3.0, 6.9, 3.1, 5.4, 2.1},
        new double[]{3.0, 6.7, 3.1, 5.6, 2.4},
        new double[]{3.0, 6.9, 3.1, 5.1, 2.3},
        new double[]{3.0, 5.8, 2.7, 5.1, 1.9},
        new double[]{3.0, 6.8, 3.2, 5.9, 2.3},
        new double[]{3.0, 6.7, 3.3, 5.7, 2.5},
        new double[]{3.0, 6.7, 3.0, 5.2, 2.3},
        new double[]{3.0, 6.3, 2.5, 5.0, 1.9},
        new double[]{3.0, 6.5, 3.0, 5.2, 2.0},
        new double[]{3.0, 6.2, 3.4, 5.4, 2.3},
        new double[]{3.0, 5.9, 3.0, 5.1, 1.8}};

    public static void  Main(System.String[] args)
    {

        /* Data corrections described in the KDD data mining archive */
        irisFisherData[34][4] = 0.1;
        irisFisherData[37][2] = 3.1;
        irisFisherData[37][3] = 1.5;

        /*  Train first 140 patterns of the iris Fisher Data */
        int[] irisClassificationData =
            new int[irisFisherData.Length - 10];
        double[][] irisContinuousData =
            new double[irisFisherData.Length - 10][];
        for (int i = 0; i < irisFisherData.Length - 10; i++)
        {
            irisContinuousData[i] =
                new double[irisFisherData[0].Length - 1];
        }

        for (int i = 0; i < irisFisherData.Length - 10; i++)
        {
```

```
    irisClassificationData[i] = (int) irisFisherData[i][0] - 1;
    Array.Copy(irisFisherData[i], 1, irisContinuousData[i], 0,
        irisFisherData[0].Length - 1);
}

int nNominal = 0; /* no nominal input attributes      */
int nContinuous = 4; /* four continuous input attributes */
int nClasses = 3; /* three classification categories  */

NaiveBayesClassifier nbTrainer =
    new NaiveBayesClassifier(nContinuous, nNominal, nClasses);

double[][] means = new double[][]{
    new double[]{5.06, 5.94, 6.58},
    new double[]{3.42, 2.8, 2.97},
    new double[]{1.5, 4.3, 5.6},
    new double[]{0.25, 1.33, 2.1}
};
double[][] stdev = new double[][]{
    new double[]{0.35, 0.52, 0.64},
    new double[]{0.38, 0.3, 0.32},
    new double[]{0.17, 0.47, 0.55},
    new double[]{0.12, 0.198, 0.275}
};

for (int i = 0; i < nContinuous; i++)
{
    IProbabilityDistribution[] pdf =
        new IProbabilityDistribution[nClasses];
    for (int j = 0; j < nClasses; j++)
    {
        pdf[j] = new TestGaussFcn1(means[i][j], stdev[i][j]);
    }
    nbTrainer.CreateContinuousAttribute(pdf);
}
nbTrainer.Train(irisContinuousData, null,
 irisClassificationData);

int[][] classErrors = nbTrainer.GetTrainingErrors();

System.Console.Out.WriteLine(
    "     Iris Classification Error Rates");
System.Console.Out.WriteLine(
    "-----------------------------------------------");
System.Console.Out.WriteLine(
    "  Setosa   Versicolour   Virginica    |   Total");
System.Console.Out.WriteLine("  " + classErrors[0][0] + "/" +
    classErrors[0][1] + "         " + classErrors[1][0] + "/" +
    classErrors[1][1] + "        " + classErrors[2][0] + "/" +
    classErrors[2][1] + "       |   " + classErrors[3][0] +
    "/" + classErrors[3][1]);
System.Console.Out.WriteLine(
    "-----------------------------------------------\n\n\n");


/*  Classify last 10 iris data patterns
```

```
* with the trained classifier
*/
 double[] continuousInput =
  new double[(irisFisherData[0].Length - 1)];
 double[] classifiedProbabilities = new double[nClasses];

 System.Console.Out.WriteLine(
     "Probabilities for Incorrect Classifications");
 System.Console.Out.WriteLine(" Predicted   ");
 System.Console.Out.WriteLine(
     "   Class     |  Class       |   P(0)     P(1)     P(2) ");
 System.Console.Out.WriteLine(
     "-----------------------------------------------------");
 for (int i = 0; i < 10; i++)
 {
     int targetClassification = (int)
         irisFisherData[(irisFisherData.Length - 10) + i][0] - 1;
     Array.Copy(irisFisherData[(irisFisherData.Length - 10) + i],
         1, continuousInput, 0, (irisFisherData[0].Length - 1));

     classifiedProbabilities =
         nbTrainer.Probabilities(continuousInput, null);
     int classification =
         nbTrainer.PredictClass(continuousInput, null);
     if (classification == 0)
     {
         System.Console.Out.Write("Setosa       |");
     }
     else if (classification == 1)
     {
         System.Console.Out.Write("Versicolour  |");
     }
     else if (classification == 2)
     {
         System.Console.Out.Write("Virginica    |");
     }
     else
     {
         System.Console.Out.Write("Missing      |");
     }
     if (targetClassification == 0)
     {
         System.Console.Out.Write(" Setosa       |");
     }
     else if (targetClassification == 1)
     {
         System.Console.Out.Write(" Versicolour  |");
     }
     else if (targetClassification == 2)
     {
         System.Console.Out.Write(" Virginica    |");
     }
     else
     {
         System.Console.Out.Write(" Missing      |");
     }
```

```
        for (int j = 0; j < nClasses; j++)
        {
            System.Object[] pArgs = new System.Object[] {
               (double)classifiedProbabilities[j] };
            System.Console.Out.Write("   {0, 2:f3} ", pArgs);
        }
        System.Console.Out.WriteLine("");
    }
}

public class TestGaussFcn1 : IProbabilityDistribution
{
    virtual public System.Object[] GetParameters()
    {
        System.Object[] parms = new System.Object[2];
        parms[0] = this.mean;
        parms[1] = this.stdev;
        return parms;
    }

    private double mean;
    private double stdev;

    public TestGaussFcn1(double mean, double stdev)
    {
        this.mean = mean;
        this.stdev = stdev;
    }

    public virtual double[] Eval(double[] xData)
    {
        double[] pdf = new double[xData.Length];
        for (int i = 0; i < xData.Length; i++)
        {
            pdf[i] = Eval(xData[i], null);
        }
        return pdf;
    }

    public virtual double[] Eval(double[] xData,
     System.Object[] Params)
    {
        double[] pdf = new double[xData.Length];
        for (int i = 0; i < xData.Length; i++)
        {
            pdf[i] = Eval(xData[i], Params);
        }
        return pdf;
    }

    public virtual double Eval(double xData, System.Object[] Params)
    {
        return GaussianPdf(xData, mean, stdev);
    }

    private double GaussianPdf(double x, double mean, double stdev)
```

```
        {
            double e, phi2, z, s;
            double sqrt_pi2 = 2.506628274631; /* sqrt(2*pi) */
            if (System.Double.IsNaN(x))
            {
                return System.Double.NaN;
            }
            if (System.Double.IsNaN(mean) || System.Double.IsNaN(stdev))
            {
                return System.Double.NaN;
            }
            else
            {
                z = x;
                z -= mean;
                s = stdev;
                phi2 = sqrt_pi2 * s;
                e = (- 0.5) * (z * z) / (s * s);
                return System.Math.Exp(e) / phi2;
            }
        }
    }
}
```

Output

The Naive Bayes classifier incorrectly classifies 6 of the 150 training patterns.

```
     Iris Classification Error Rates
-----------------------------------------------
  Setosa   Versicolour   Virginica    |   Total
  0/50         2/50        4/40       |   6/140
-----------------------------------------------



Probabilities for Incorrect Classifications
 Predicted
   Class     |  Class       |   P(0)     P(1)     P(2)
-------------------------------------------------------
Virginica    | Virginica    |   0.000    0.000    1.000
Virginica    | Virginica    |   0.000    0.000    1.000
Virginica    | Virginica    |   0.000    0.051    0.949
Virginica    | Virginica    |   0.000    0.000    1.000
Virginica    | Virginica    |   0.000    0.000    1.000
Virginica    | Virginica    |   0.000    0.000    1.000
Virginica    | Virginica    |   0.000    0.048    0.952
Virginica    | Virginica    |   0.000    0.001    0.999
Virginica    | Virginica    |   0.000    0.000    1.000
Virginica    | Virginica    |   0.000    0.126    0.874
```

Chapter 27: Neural Nets

Types

Usage Notes

Neural Networks - An Overview

Today, neural networks are used to solve a wide variety of problems, some of which have been solved by existing statistical methods, and some of which have not. These applications fall into one of the following three categories:

- *Forecasting*: predicting one or more quantitative outcomes from both quantitative and categorical input data,
- *Classification*: classifying input data into one of two or more categories, or
- *Statistical pattern recognition*: uncovering patterns, typically spatial or temporal, among a set of variables.

Forecasting, pattern recognition and classification problems are not new. They existed years before the discovery of neural network solutions in the 1980's. What is new is that neural networks provide a single framework for solving so many traditional problems and, in some cases, extend the range of problems that can be solved.

Traditionally, these problems have been solved using a variety of well known statistical methods:

- linear regression and general least squares,
- logistic regression and discrimination,
- principal component analysis,
- discriminant analysis,
- *k*-nearest neighbor classification, and
- ARMA and non-linear ARMA time series forecasts.

In many cases, simple neural network configurations yield the same solution as many traditional statistical applications. For example, a single-layer, feed-forward neural network with linear activation for its output perceptron is equivalent to a general linear regression fit. Neural networks can provide more accurate and robust solutions for problems where traditional methods do not completely apply.

Mandic and Chambers (2001) point out that traditional methods for time series forecasting are unsuitable when a time series:

- is non-stationary,
- has large amounts of noise, such as a biomedical series, or
- is too short.

ARIMA and other traditional time series approaches can produce poor forecasts when one or more of the above conditions exist. The forecasts of ARMA and non-linear ARMA (NARMA) depend heavily upon key assumptions about the model or underlying relationship between the output of the series and its patterns.

Neural networks, on the other hand, adapt to changes in a non-stationary series and can produce reliable forecasts even when the series contains a good deal of noise or when only a short series is available for training. Neural networks provide a single tool for solving many problems traditionally solved using a wide variety of statistical tools and for solving problems when traditional methods fail to provide an acceptable solution.

Although neural network solutions to forecasting, pattern recognition, and classification problems can be very different, they are always the result of computations that proceed from the network inputs to the network outputs. The network inputs are referred to as *patterns*, and outputs are referred to as *classes*. Frequently the flow of these computations is in one direction, from the network input patterns to its outputs. Networks with forward-only flow are referred to as feed-forward networks.

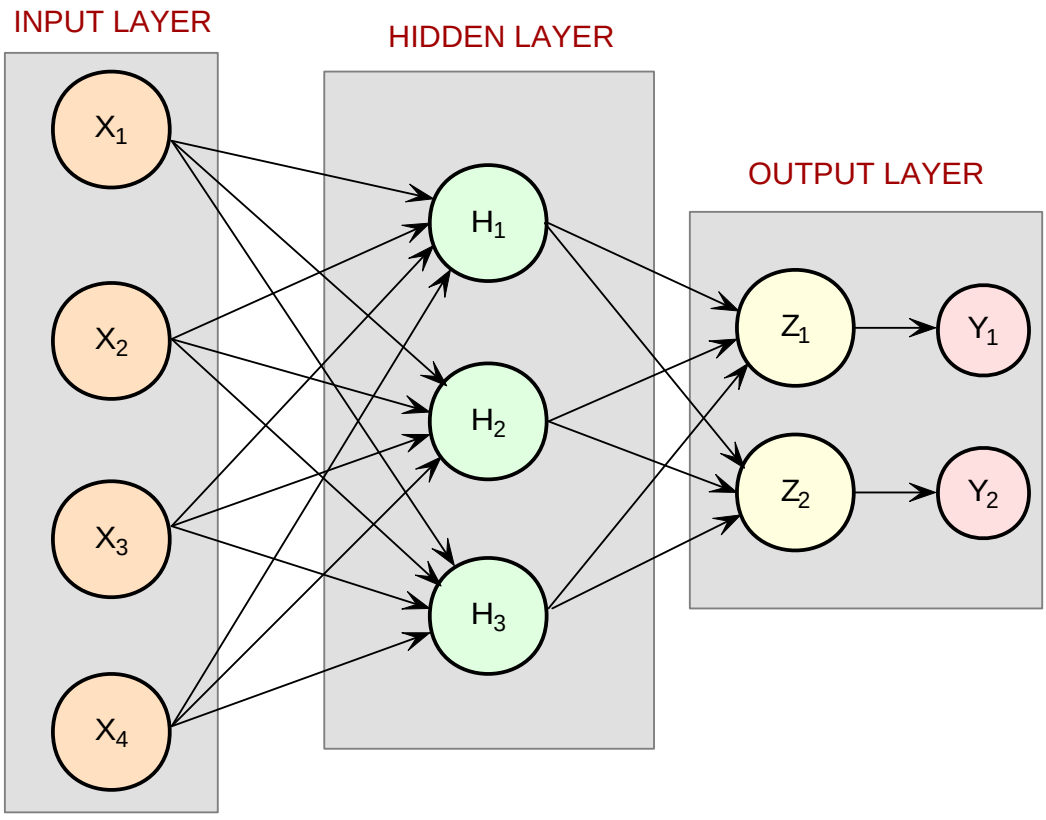


Figure 1. A 2-layer, Feed-Forward Network with 4 Inputs and 2 Outputs

Other networks, such as recurrent neural networks, allow data and information to flow in both directions,

see Mandic and Chambers (2001).

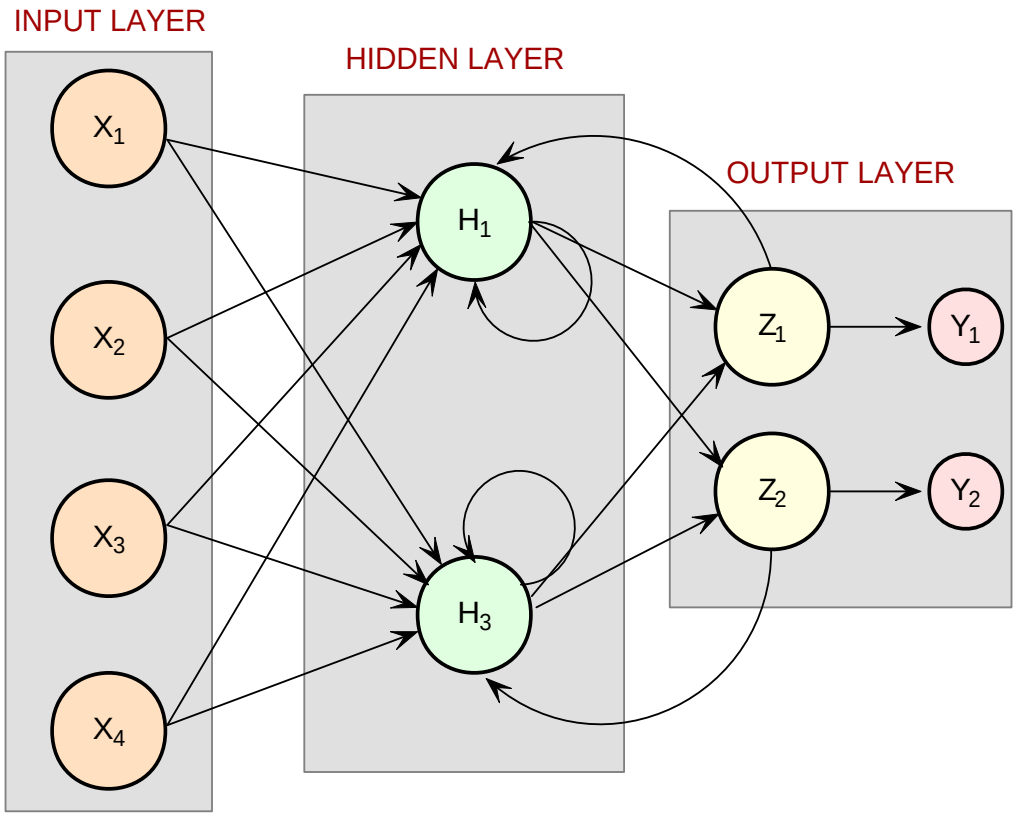


Figure 2. A Recurrent Neural Network with 4 Inputs and 2 Outputs

A neural network is defined not only by its architecture and flow, or interconnections, but also by computations used to transmit information from one node or input to another node. These computations are determined by network weights. The process of fitting a network to existing data to determine these weights is referred to as *training* the network, and the data used in this process are referred to as *patterns*. Individual network inputs are referred to as *attributes* and outputs are referred to as *classes*. Many terms used to describe neural networks are synonymous to common statistical terminology.

Table 1. Synonyms between Neural Network and Common Statistical Terminology

Neural Network Terminology	Traditional Statistical Terminology	Description
Training	Model Fitting	Estimating unknown parameters or coefficients in the analysis.
Patterns	Cases or Observations	A single observation of all input and output variables.
Attributes	Independent variables	Inputs to the network or model.
Classes	Dependent variables	Outputs from the network or model calculations.

Neural Networks – History and Terminology

The Threshold Neuron

McCulloch and Pitts (1943) wrote one of the first published works on neural networks. In their paper, they describe the threshold neuron as a model for how the human brain stores and processes information.

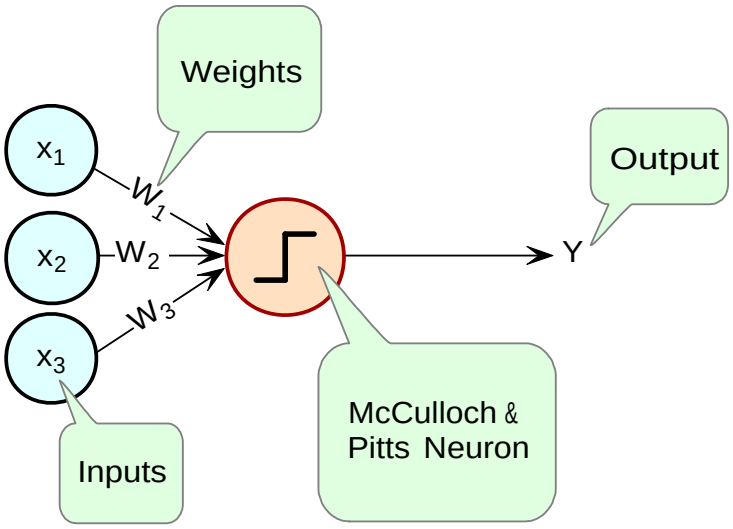


Figure 3. The McCulloch and Pitts Threshold Neuron

All inputs to a threshold neuron are combined into a single number, Z, using the following weighted sum:

$$Z = \sum_{i=1}^{m} w_i x_i - \mu$$

where w_i is the weight associated with the i-th input (attribute) x_i. The term μ in this calculation is referred to as the *bias term*. In traditional statistical terminology, it might be referred to as the *intercept*. The weights and bias terms in this calculation are estimated during network training.

In McCulloch and Pitt's description of the threshold neuron, the neuron does not respond to its inputs unless Z is greater than zero. If Z is greater than zero then the output from this neuron is set to 1. If Z is less than zero the output is zero:

$$Y = \begin{cases} 1 & \text{if} Z > 0 \\ 0 & \text{if} Z \leq 0 \end{cases}$$

where Y is the neuron's output.

For years following their 1943 paper, interest in the McCulloch and Pitts neural network was limited to theoretical discussions, such as those of Hebb (1949), about learning, memory, and the brain's structure.

The Perceptron

The McCulloch and Pitts neuron is also referred to as a threshold neuron since it abruptly changes its output from 0 to 1 when its potential, Z, crosses a threshold. Mathematically, this behavior can be viewed as a step function that maps the neuron's potential, Z, to the neuron's output, Y.

Rosenblatt (1958) extended the McCulloch and Pitts threshold neuron by replacing this step function with a continuous function that maps Z to Y. The Rosenblatt neuron is referred to as the perceptron, and the continuous function mapping Z to Y makes it easier to train a network of perceptrons than a network of threshold neurons.

Unlike the threshold neuron, the perceptron produces analog output rather than the threshold neuron's purely binary output. Carefully selecting the analog function makes Rosenblatt's perceptron differentiable, whereas the threshold neuron is not. This simplifies the training algorithm.

Like the threshold neuron, Rosenblatt's perceptron starts by calculating a weighted sum of its inputs, $Z = \sum_{i=1}^{m} w_i x_i - \mu$. This is referred to as the perceptron's *potential*.

Rosenblatt's perceptron calculates its analog output from its potential. There are many choices for this calculation. The function used for this calculation is referred to as the activation function in Figure 4 below.

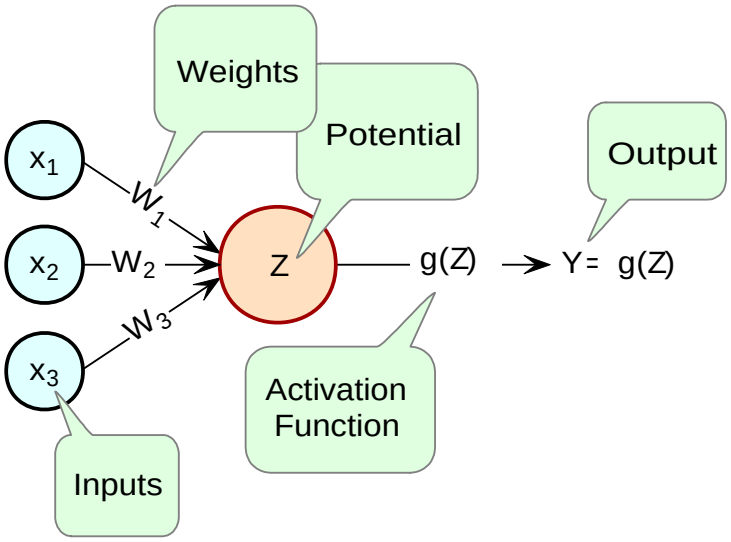


Figure 4. The Perceptron

As shown in Figure 4, perceptrons consist of the following five components:

Component	Example
Inputs	X_1, X_2, X_3
Input Weights	W_1, W_2, W_3
Potential	$Z = \sum_{i=1}^{3} W_i X_i - \mu$, where μ is a bias correction.
Activation Function	$g(Z)$
Output	$g(Z)$

Like threshold neurons, perceptron inputs can be either the initial raw data inputs or the output from another perceptron. The primary purpose of the network training is to estimate the weights associated with each perceptron's potential. The activation function maps this potential to the perceptron's output.

The Activation Function

Although theoretically any differential function can be used as an activation function, the identity and sigmoid functions are the two most commonly used.

The *identity activation* function, also referred to as a *linear activation* function, is a flow-through mapping of the perceptron's potential to its output:

$$g(Z) = Z$$

Output perceptrons in a forecasting network often use the identity activation function.

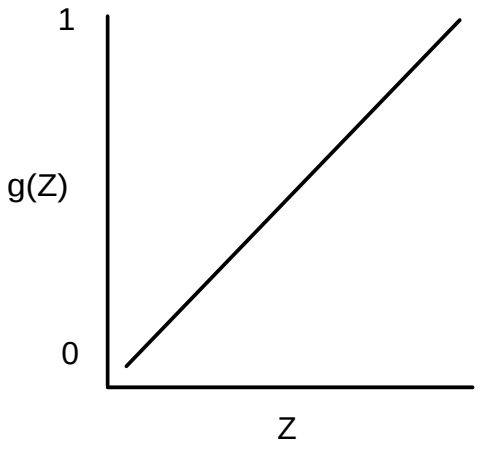


Figure 5. An Identity (Linear) Activation Function

If the identity activation function is used throughout the network, then it is easily shown that the network is equivalent to fitting a linear regression model of the form $Y_i = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$, where $x_1, x_2, \cdots, x_k$ are the k network inputs, Y_i is the *i*-th network output and $\beta_0, \beta_1, \cdots, \beta_k$ are the coefficients in the regression equation. As a result, it is uncommon to find a neural network with identity activation used in all its perceptrons.

Sigmoid activation functions are differentiable functions that map the perceptron's potential to a range of values, such as 0 to 1, i.e., $\mathbb{R}^K \rightarrow \mathbb{R}$ where K is the number of perceptron inputs.

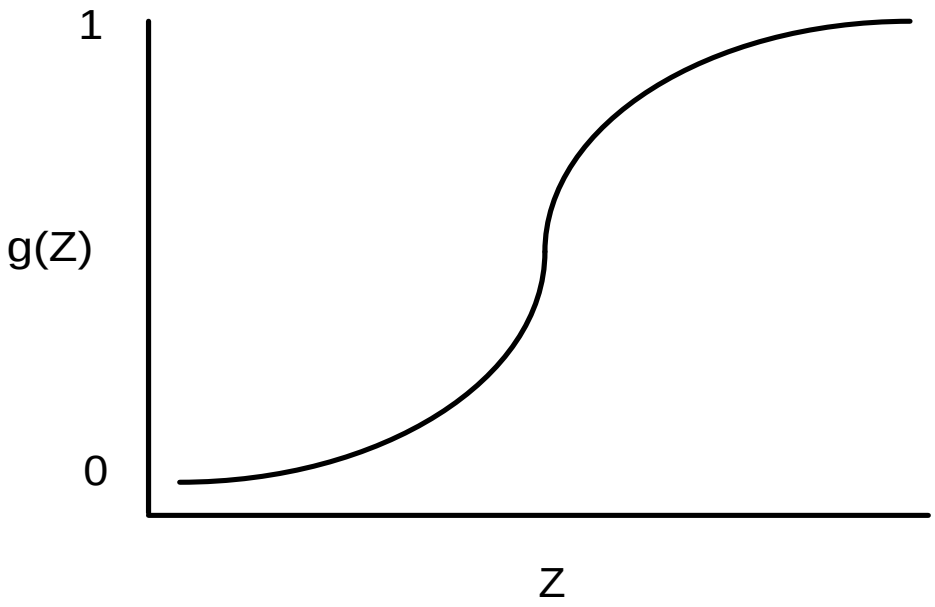


Figure 6. A Sigmoid Activation Function

In practice, the most common sigmoid activation function is the logistic function that maps the potential into the range 0 to 1:

$$g(Z) = \frac{1}{1+e^{-Z}}$$

Since $0 < g(Z) < 1$, the logistic function is very popular for use in networks that output probabilities.

Other popular sigmoid activation functions include:

1. the hyperbolic-tangent $g(Z) = \tanh(Z) = \frac{e^{\alpha Z} - e^{-\alpha Z}}{e^{\alpha Z} + e^{-\alpha Z}}$
2. the arc-tangent $g(Z) = \frac{2}{\pi} \arctan\left(\frac{\pi Z}{2}\right)$, and
3. the squash activation function (Elliott (1993)) $g(Z) = \frac{Z}{1+|Z|}$

It is easy to show that the hyperbolic-tangent and logistic activation functions are linearly related. Consequently, forecasts produced using logistic activation should be close to those produced using hyperbolic-tangent activation. However, one function may be preferred over the other when training performance is a concern. Researchers report that the training time using the hyperbolic-tangent activation function is shorter than using the logistic activation function.

Network Applications

Forecasting using Neural Networks

There are many good statistical forecasting tools. Most require assumptions about the relationship between the variables being forecasted and the variables used to produce the forecast, as well as the distribution of forecast errors. Such statistical tools are referred to as *parametric methods*. ARIMA time series models, for example, assume that the time series is stationary, that the errors in the forecasts follow a particular ARIMA model, and that the probability distribution for the residual errors is Gaussian, see Box and Jenkins (1970). If these assumptions are invalid, then ARIMA time series forecasts can be very poor.

Neural networks, on the other hand, require few assumptions. Since neural networks can approximate highly non-linear functions, they can be applied without an extensive analysis of underlying assumptions.

Another advantage of neural networks over ARIMA modeling is the number of observations needed to produce a reliable forecast. ARIMA models generally require 50 or more equally spaced, sequential observations in time. In many cases, neural networks can also provide adequate forecasts with fewer observations by incorporating exogenous, or external, variables in the network's input.

For example, a company applying ARIMA time series analysis to forecast business expenses would normally require each of its departments, and each sub-group within each department to prepare its own forecast. For large corporations this can require fitting hundreds or even thousands of ARIMA models. With a neural network approach, the department and sub-group information could be incorporated into the network as exogenous variables. Although this can significantly increase the network's training time, the result would be a single model for predicting expenses within all departments and sub-departments.

Linear least squares models are also popular statistical forecasting tools. These methods range from simple linear regression for predicting a single quantitative outcome to logistic regression for estimating probabilities associated with categorical outcomes. It is easy to show that simple linear least squares forecasts and logistic regression forecasts are equivalent to a feed-forward network with a single layer. For this reason, single-layer feed-forward networks are rarely used for forecasting. Instead multilayer networks are used.

Hutchinson (1994) and Masters (1995) describe using multilayer feed-forward neural networks for forecasting. Multilayer feed-forward networks are characterized by the forward-only flow of information in the network. The flow of information and computations in a feed-forward network is always in one direction, mapping an M-dimensional vector of inputs to a C-dimensional vector of outputs, i.e., $\mathbb{R}^M \rightarrow \mathbb{R}^C$.

There are many other types of networks without this feed-forward requirement. Information and computations in a recurrent neural network, for example, flows in both directions. Output from one level of a recurrent neural network can be fed back, with some delay, as input into the same network, see Figure 2. Recurrent networks are very useful for time series prediction, see Mandic and Chambers (2001).

Pattern Recognition using Neural Networks

Neural networks are also extensively used in statistical pattern recognition. Pattern recognition applications that make wide use of neural networks include:

- natural language processing: Manning and Schütze (1999)
- speech and text recognition: Lippmann (1989)
- face recognition: Lawrence, et al. (1997)
- playing backgammon, Tesauro (1990)
- classifying financial news, Calvo (2001).

The interest in pattern recognition using neural networks has stimulated the development of important variations of feed-forward networks. Two of the most popular are:

- Self-Organizing Maps, also called Kohonen Networks, Kohonen (1995),
- and Radial Basis Function Networks, Bishop (1995).

Good mathematical descriptions of the neural network methods underlying these applications are given by Bishop (1995), Ripley (1996), Mandic and Chambers (2001), and Abe (2001). An excellent overview of neural networks, from a statistical viewpoint, is also found in Warner and Misra (1996).

Neural Networks for Classification

Classifying observations using prior concomitant information is possibly the most popular application of neural networks. Data classification problems abound in business and research. When decisions based upon data are needed, they can often be treated as a neural network data classification problem. Decisions to buy, sell, hold or do nothing with a stock, are decisions involving four choices. Classifying loan applicants as good or bad credit risks, based upon their application, is a classification problem involving two choices. Neural networks are powerful tools for making decisions or choices based upon data.

These same tools are ideally suited for automatic selection or decision-making. Incoming email, for example, can be examined to separate spam from important email using a neural network trained for this task. A good overview of solving classification problems using multilayer feed-forward neural networks is found in Abe (2001) and Bishop (1995).

There are two popular methods for solving data classification problems using multilayer feed-forward neural networks, depending upon the number of choices (classes) in the classification problem. If the classification problem involves only two choices, then it can be solved using a neural network with one logistic output. This output estimates the probability that the input data belong to one of the two choices.

For example, a multilayer feed-forward network with a single, logistic output can be used to determine whether a new customer is credit-worthy. The network's input would consist of information on the applicants credit application, such as age, income, etc. If the network output probability is above some threshold value (such as 0.5 or higher) then the applicant's credit application is approved.

This is referred to as binary classification using a multilayer feed-forward neural network. If more than two classes are involved then a different approach is needed. A popular approach is to assign logistic output perceptrons to each class in the classification problem. The network assigns each input pattern to the class associated with the output perceptron that has the highest probability for that input pattern. However, this approach produces invalid probabilities since the sum of the individual class probabilities

for each input is not equal to one, which is a requirement for any valid multivariate probability distribution.

To avoid this problem, the softmax activation function, see Bridle (1990), applied to the network outputs ensures that the outputs conform to the mathematical requirements of multivariate classification probabilities. If the classification problem has C categories, or classes, then each category is modeled by one of the network outputs. If Z_i is the weighted sum of products between its weights and inputs for the i-th output, i.e., $Z_i = \sum_j w_{ji} y_{ji}$, then

$$\text{softmax}_i = \frac{e^{Z_i}}{\sum_{j=1}^{C} e^{Z_j}}$$

The softmax activation function ensures that the outputs all conform to the requirements for multivariate probabilities. That is,

$$0 < \text{softmax}_i < 1, \quad \text{for all } i = 1, 2, \ldots, C$$

and

$$\sum_{i=1}^{C} \text{softmax}_i = 1$$

A pattern is assigned to the i-th classification when softmax_i is the largest among all C classes.

However, multilayer feed-forward neural networks are only one of several popular methods for solving classification problems. Others include:

- Support Vector Machines (SVM Neural Networks), Abe (2001),
- Classification and Regression Trees (CART), Breiman, et al. (1984),
- Quinlan's classification algorithms C4.5 and C5.0, Quinlan (1993), and
- Quick, Unbiased and Efficient Statistical Trees (QUEST), Loh and Shih (1997).

Support Vector Machines are simple modifications of traditional multilayer feed-forward neural networks (MLFF) configured for pattern classification.

Multilayer Feed-Forward Neural Networks

A multilayer feed-forward neural network is an interconnection of perceptrons in which data and calculations flow in a single direction, from the input data to the outputs. The number of layers in a neural network is the number of layers of perceptrons. The simplest neural network is one with a single input layer and an output layer of perceptrons. The network in Figure 7 illustrates this type of network. Technically this is referred to as a one-layer feed-forward network with two outputs because the output layer is the only layer with an activation calculation.

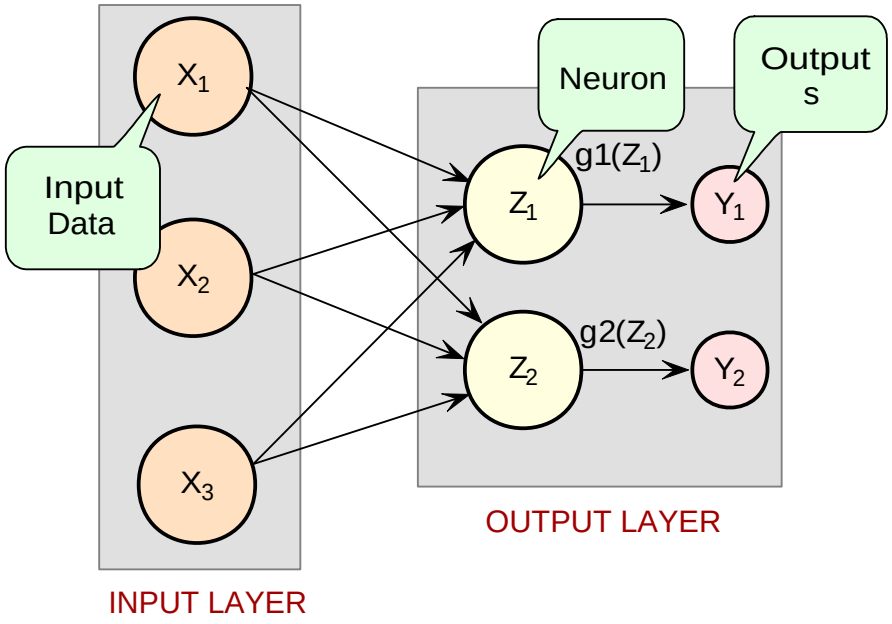


Figure 7. A Single-Layer Feed-Forward Neural Net

In this single-layer feed-forward neural network, the networks inputs are directly connected to the output layer perceptrons, Z_1 and Z_2.

The output perceptrons use activation functions, g_1 and g_2, to produce the outputs Y_1 and Y_2.

Since

$$Z_1 = \sum_{i=1}^{3} W_{1,i} X_i - \mu_1$$

and

$$Z_2 = \sum_{i=1}^{3} W_{2,i} X_i - \mu_2$$

$$Y_1 = g_1(Z_1) = g_1(\sum_{i=1}^{3} W_{1,i} X_i - \mu_1)$$

and

$$Y_2 = g_2(Z_2) = g_2(\sum_{i=1}^{3} W_{2,i} X_i - \mu_2)$$

When the activation functions g_1 and g_2 are identity activation functions, a single-layer neural net is equivalent to a linear regression model. Similarly, if g_1 and g_2 are logistic activation functions, then the single-layer neural net is equivalent to logistic regression. Because of this correspondence between single-layer neural networks and linear and logistic regression, single-layer neural networks are rarely used in place of linear and logistic regression.

The next most complicated neural network is one with two layers. This extra layer is referred to as a hidden layer. In general there is no restriction on the number of hidden layers. However, it has been shown mathematically that a two-layer neural network, such as shown in Figure 1, can accurately reproduce any differentiable function, provided the number of perceptrons in the hidden layer is unlimited.

However, increasing the number of neurons increases the number of weights that must be estimated in the network, which in turn increases the execution time for this network. Instead of increasing the number of perceptrons in the hidden layers to improve accuracy, it is sometimes better to add additional hidden layers, which typically reduces both the total number of network weights and the computational time. However, in practice, it is uncommon to see neural networks with more than two or three hidden layers.

Neural Network Error Calculations

Error Calculations for Forecasting

The error calculations used to train a neural network are very important. Researchers have investigated many error calculations, trying to find a calculation with a short training time that is appropriate for the network's application. Typically error calculations are very different depending primarily on the network's application.

For forecasting, the most popular error function is the sum-of-squared errors, or one of its scaled versions. This is analogous to using the minimum least squares optimization criterion in linear regression. Like least squares, the sum-of-squared errors is calculated by looking at the squared difference between what the network predicts for each training pattern and the target value, or observed value, for that pattern. Formally, the equation is the same as one-half the traditional least squares error:

$$E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{C}(t_{ij} - \hat{t}_{ij})^2$$

where N is the total number of training cases, C is equal to the number of network outputs, t_{ij} is the observed output for the *i*-th training case and the *j*-th network output, and $\hat{t}_{ij}$ is the network's forecast for that case.

Common practice recommends fitting a different network for each forecast variable. That is, the recommended practice is to use C=1 when using a multilayer feed-forward neural network for forecasting. For classification problems with more than two classes, it is common to associate one output with each classification category, i.e., C=number of classes.

Notice that in ordinary least squares, the sum-of-squared errors are not multiplied by one-half. Although this has no impact on the final solution, it significantly reduces the number of computations required during training.

Also note that as the number of training patterns increases, the sum-of-squared errors increases. As a result, it is often useful to use the root-mean-square (RMS) error instead of the unscaled sum-of-squared

errors:

$$E^{RMS} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{C}(t_{ij}-\hat{t}_{ij})^2}{\sum_{i=1}^{N}\sum_{j=1}^{C}(t_{ij}-\bar{t})^2}$$

where $\bar{t}$ is the average output:

$$\bar{t} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{C} t_{ij}}{N \cdot C}$$

Unlike the unscaled sum-of-squared errors, E^{RMS} does not increase as N increases. The smaller the value of E^{RMS} the closer the network is predicting its targets during training. A value of $E^{RMS} = 0$ indicates that the network is able to predict every pattern exactly. A value of $E^{RMS} = 1$ indicates that the network is predicting the training cases only as well as using the mean of the training cases for forecasting.

Notice that the root-mean-squared error is related to the sum-of-squared error by a simple scale factor:

$$E^{RMS} = \frac{2}{\bar{t}} \cdot E$$

Another popular error calculation for forecasting from a neural network is the Minkowski-R error. The sum-of-squared error, E, and the root-mean-squared error, E^{RMS}, are both theoretically motivated by assuming the noise in the target data is Gaussian. In many cases, this assumption is invalid. A generalization of the Gaussian distribution to other distributions gives the following error function, referred to as the Minkowski-R error:

$$E^R = \sum_{i=1}^{N}\sum_{j=1}^{C}\left|t_{ij}-\hat{t}_{ij}\right|^R.$$

Notice that $E^R = 2E$ when R = 2.

A good motivation for using E^R instead of E is to reduce the impact of outliers in the training data. The usual error measures, E and E^{RMS}, emphasize larger differences between the training data and network forecasts since they square those differences. If outliers are expected, then it is better to de-emphasize larger differences. This can be done by using the Minkowski-R error with R=1. When R=1, the Minkowski-R error simplifies to the sum of absolute differences:

$$L = E^1 = \sum_{i=1}^{N}\sum_{j=1}^{C}\left|t_{ij}-\hat{t}_{ij}\right|.$$

L is also referred to as the Laplacian error. Its name is derived from the fact that it can be theoretically justified by assuming the noise in the training data follows a Laplacian rather than Gaussian distribution.

Of course, similar to E, L generally increases when the number of training cases increases. Similar to E^{RMS}, a scaled version of the Laplacian error can be calculated using the following formula:

$$L^{RMS} = \frac{\sum_{i=1}^{N}\sum_{j=1}^{C}\left|t_{ij}-\hat{t}_{ij}\right|}{\sum_{i=1}^{N}\sum_{j=1}^{C}\left|t_{ij}-\bar{t}\right|}$$

Cross-Entropy Error for Binary Classification

As previously mentioned, multilayer feed-forward neural networks can be used for both forecasting and classification applications. Training a forecasting network involves finding the network weights that minimize either the Gaussian or Laplacian distributions, E or L respectively, or equivalently their scaled versions, E^{RMS} or L^{RMS}. Although these error calculations can be adapted for use in classification by setting the target classification variable to zeros and ones, this is not recommended. Use of the sum-of-squared and Laplacian error calculations is based on the assumption that the target variable is continuous. In classification applications, the target variable is a discrete random variable with C possible values, where C=number of classes.

A multilayer feed-forward neural network for classifying patterns into one of only two categories is referred to as a binary classification network. It has a single output: the estimated probability that the input pattern belongs to one of the two categories. The probably that it belongs to the other category is equal to one minus this probability, i.e.,

$$P(C_2) = P(\text{not } C_1) = 1 - P(C_1)$$

Binary classification applications are very common. Any problem requiring *yes*/*no* classification is a binary classification application. For example, deciding to sell or buy a stock is a binary classification problem. Deciding to approve a loan application is also a binary classification problem. Deciding whether to approve a new drug or to provide one of two medical treatments are binary classification problems.

For binary classification problems, only a single output is used, C=1. This output represents the probability that the training case should be classified as *yes*. A common choice for the activation function of the output of a binary classification networks is the logistic activation function, which always results in an output in the range 0 to 1, regardless of the perceptron's potential.

One choice for training a binary classification network is to use sum-of-squared errors with the class value of *yes* patterns coded as a 1 and the *no* classes coded as a 0, i.e.:

$$t_{ij} = \begin{cases} 1 & \text{if training pattern } i\text{=}yes \\ 0 & \text{if the training pattern } i\text{=}no \end{cases}$$

However, using either the sum-of-squared or Laplacian errors for training a network with these target values assumes that the noise in the training data are Gaussian. In binary classification, the zeros and ones are not Gaussian. They follow the Bernoulli distribution:

$$P(t_i = t) = p^t (1-p)^{1-t}$$

where p is equal to the probability that a randomly selected case belongs to the *yes* class.

Modeling the binary classes as Bernoulli observations leads to the cross- entropy error function described by Hopfield (1987) and Bishop (1995):

$$E^C = -\sum_{i=1}^{N} \{t_i \ln(\hat{t}_i) + (1-t_i)\ln(1-\hat{t}_i)\}$$

where N is the number of training patterns, t_i is the target value for the i-th case (either 1 or 0), and $\hat{t}_i$ is the network's output for the i-th case. This is equal to the neural network's estimate of the probability that the i-th case should be classified as *yes*.

For situations in which the target variable is a probability in the range $0 < t_{ij} < 1$, the value of the cross-entropy at the networks optimum is equal to:

$$E^C_{\min} = -\sum_{i=1}^{N} \{t_i \ln(t_i) + (1-t_i)\ln(1-t_i)\}$$

Subtracting this from E^C gives an error term bounded below by zero, i.e., $E^{CE} \geq 0$ where:

$$E^{CE} = E^C - E^C_{\min} = -\sum_{i=1}^{N} \left\{ t_i \ln\left[\frac{\hat{t}_i}{t_i}\right] + (1-t_i)\ln\left[\frac{1-\hat{t}_i}{1-t_i}\right] \right\}$$

This adjusted cross-entropy is normally reported when training a binary classification network where $0 < t_{ij} < 1$. Otherwise E^C, the non-adjusted cross-entropy error, is used. Small values, values near zero, would indicate that the training resulted in a network with a low error rate and that patterns are being classified correctly most of the time.

Back-Propagation in Multilayer Feed-Forward Neural Network

Sometimes a multilayer feed-forward neural network is referred to incorrectly as a back-propagation network. The term back-propagation does not refer to the structure or architecture of a network. Back-propagation refers to the method used during network training. More specifically, back-propagation refers to a simple method for calculating the gradient of the network, that is the first derivative of the weights in the network.

The primary objective of network training is to estimate an appropriate set of network weights based upon a training dataset. Many ways have been researched for estimating these weights, but they all involve minimizing some error function. In forecasting, the most commonly used error function is the sum-of-squared errors:

$$E = \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{C} (t_{ij} - \hat{t}_{ij})^2$$

Training uses one of several possible optimization methods to minimize this error term. Some of the more common are: steepest descent, quasi-Newton, conjugant gradient, and many various modifications of these optimization routines.

Back-propagation is a method for calculating the first derivative, or gradient, of the error function required by some optimization methods. It is certainly not the only method for estimating the gradient. However, it is the most efficient. In fact, some will argue that the development of this method by Werbos (1974), Parket (1985), and Rumelhart, Hinton and Williams (1986) contributed to the popularity of neural network methods by significantly reducing the network training time and making it possible to train networks consisting of a large number of inputs and perceptrons.

Simply stated, back-propagation is a method for calculating the first derivative of the error function with respect to each network weight. Bishop (1995) derives and describes these calculations for the two most common forecasting error functions, the sum of squared errors and Laplacian error functions. Abe (2001) gives the description for the classification error function, the cross-entropy error function. For all of these error functions, the basic formula for the first derivative of the network weight w_{ji} at the i-th

perceptron applied to the output from the *j*-th perceptron:

$$\frac{\partial E}{\partial w_{ji}} = \delta_j Z_i,$$

where $Z_i = g(a_i)$ is the output from the *i*-th perceptron after activation, and

$$\frac{\partial E}{\partial w_{ji}}$$

is the derivative for a single output and a single training pattern. The overall estimate of the first derivative of w_{ji} is obtained by summing this calculation over all N training patterns and C network outputs.

The term back-propagation gets its name from the way the term δ_j in the back-propagation formula is calculated:

$$\delta_j = g'(a_j) \cdot \sum_k w_{kj}\delta_k,$$

where the summation is over all perceptrons that use the activation from the *j*-th perceptron, $g(a_j)$.

The derivative of the activation functions, $g'(a)$, varies among these functions, see the following table:

Table 2.Activation Functions and Their Derivatives

Activation Function	$g(a)$	$g'(a)$
Linear	$g(a) = a$	$g'(a) = 1$ (where a is a constant)
Logistic	$g(a) = \frac{1}{1+e^{-a}}$	$g'(a) = g(a)(1-g(a))$
Hyperbolic-tangent	$g(a) = \tanh(a)$	$g'(a) = \text{sech}^2(a) = 1 - \tanh^2(a)$
Squash	$g(a) = \frac{a}{1+\|a\|}$	$g'(a) = \frac{1}{(1+\|a\|)^2}$

Creating a Feed Forward Network

The following code fragment creates the feed forward neural network shown in the following figure:

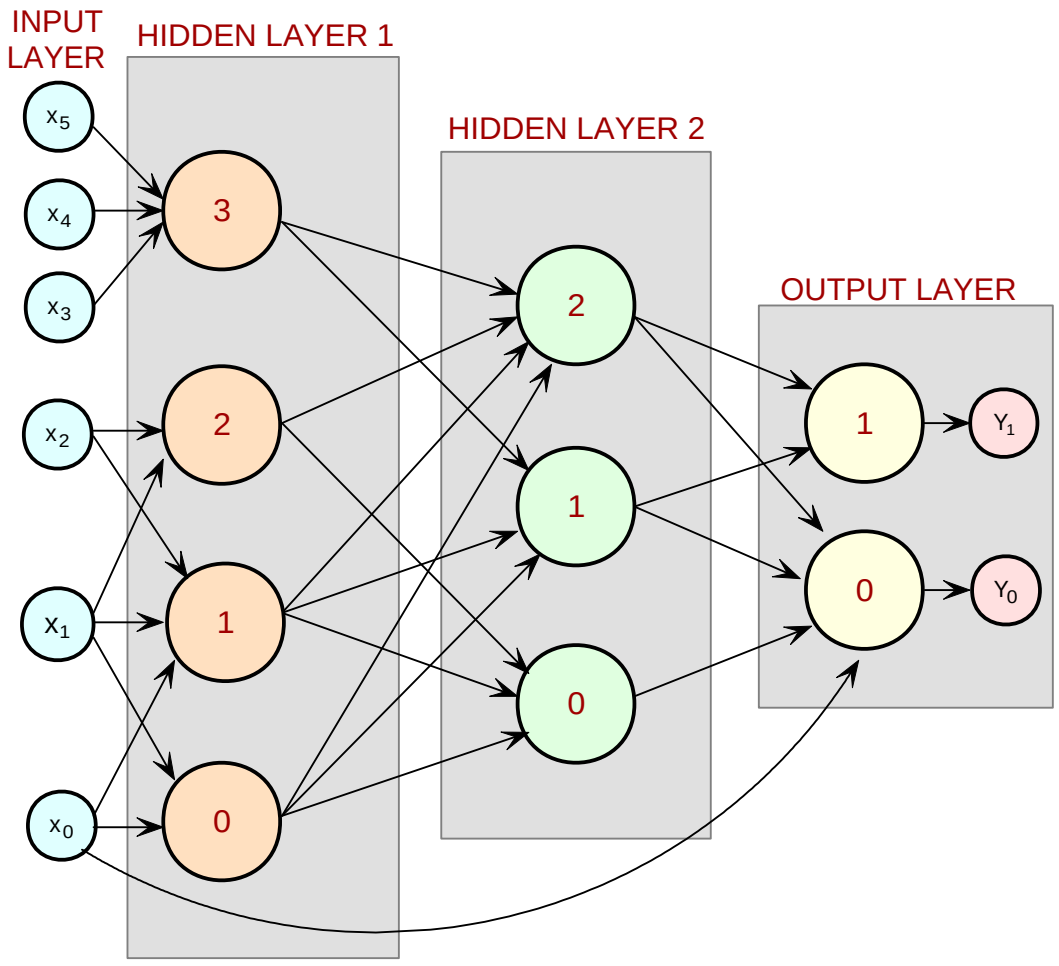


Figure 8. A Three-Layer Feed-Forward Neural Net

Notice that this network is more complex than the typical feed-forward network in which all nodes from each layer are connected to every node in the next layer. This network has 6 input nodes, and they are not all connected to every node in the 1st hidden layer.

Note also that the 4 perceptrons in the 1st hidden layer are not connected to every node in the 2nd hidden layer, and the perceptrons in the 2nd hidden layer are not all connected to the two outputs.

```
// ************************************************************
// EXAMPLE CODE FOR CREATING LINKS AMONG NETWORK NODES
// ************************************************************

   FeedForwardNetwork network = new FeedForwardNetwork();
   network.InputLayer.CreateInputs(6);
   network.CreateHiddenLayer().CreatePerceptrons(4);
```

```
   network.CreateHiddenLayer().CreatePerceptrons(3);
   network.OutputLayer.CreatePerceptrons(2);
   HiddenLayers[] hiddenLayer = network.HiddenLayers;
   Node[] inputNode  = network.InputLayer.Nodes;
   Node[] layer1Node = hiddenLayer[0].Nodes;
   Node[] layer2Node = hiddenLayer[1].Nodes;
   Node[] outputNode = network.OutputLayer.Nodes;
// Create links between input nodes and 1st hidden layer
   network.Link(inputNode[0], layer1Node[0]);
   network.Link(inputNode[0], layer1Node[1]);
   network.Link(inputNode[1], layer1Node[0]);
   network.Link(inputNode[1], layer1Node[1]);
   network.Link(inputNode[1], layer1Node[3]);
   network.Link(inputNode[2], layer1Node[1]);
   network.Link(inputNode[2], layer1Node[2]);
   network.Link(inputNode[3], layer1Node[3]);
   network.Link(inputNode[4], layer1Node[3]);
   network.Link(inputNode[5], layer1Node[3]);
// Create links between 1st and 2nd hidden layers
   network.Link(layer1Node[0], layer2Node[0]);
   network.Link(layer1Node[0], layer2Node[1]);
   network.Link(layer1Node[0], layer2Node[2]);
   network.Link(layer1Node[1], layer2Node[0]);
   network.Link(layer1Node[1], layer2Node[1]);
   network.Link(layer1Node[1], layer2Node[2]);
   network.Link(layer1Node[2], layer2Node[0]);
   network.Link(layer1Node[2], layer2Node[2]);
   network.Link(layer1Node[3], layer2Node[1]);
   network.Link(layer1Node[3], layer2Node[2]);
// Create links between 2nd hidden layer and output layer
   network.Link(layer2Node[0], outputNode[0]);
   network.Link(layer2Node[1], outputNode[0]);
   network.Link(layer2Node[1], outputNode[1]);
   network.Link(layer2Node[2], outputNode[0]);
   network.Link(layer2Node[2], outputNode[1]);
// Create link between input node[0] and ouput node[0]
   network.Link(inputNode[0], outputNode[0]);
// ************************************************************
```

By default, the `FeedForwardNetwork` constructor creates a feed forward network with an empty input layer, no hidden layers and an empty output layer. Input nodes are created by accessing the empty input layer and creating 6 nodes within it. Two hidden layers are then created within the network using the `FeedForwardNetwork.CreateHiddenLayer().CreatePerceptrons()` method. Four perceptrons are created within the first hidden layer and three within the second. Output perceptrons are created by accessing the empty output layer and creating the Perceptrons within it: `FeedForwardNetwork.OutputLayer.CreatePerceptrons()`.

Links among the input nodes and perceptrons can be created using one of several approaches. If all inputs are connected to every perceptron in the first hidden layer, and if all perceptrons are connected to every perceptron in the following layer, which is a standard architecture for feed forward networks, then a call to the `FeedForwardNetwork.LinkAll()` method can be used to create these links.

However, this example does not use that standard configuration. Some links are missing. In this case, the approach used is to construct individual links using the `FeedForwardNetwork.Link()` method. This

requires one call for every link.

An alternate approach is to first create all links and then to remove those that are not needed. The following code illustrates this approach:

```
// ************************************************************
// EXAMPLE CODE FOR REMOVING LINKS AMONG NETWORK NODES
// ************************************************************

   FeedForwardNetwork network = new FeedForwardNetwork();
   InputNode[] inputNode      = network.InputLayer.CreateInputs(6);
   Perceptron[] hiddenLayer1  =
      network.CreateHiddenLayer().CreatePerceptrons(4);
   Perceptron[] hiddenLayer2  =
      network.CreateHiddenLayer().CreatePerceptrons(3);
   Perceptron[] outputLayer   = network.OutputLayer.CreatePerceptrons(2);
   network.LinkAll(); // Creates standard feed forward configuration
// Remove links between input nodes and 1st hidden layer
   network.Remove(network.FindLink(inputNode[0],hiddenLayer1[2]));
   network.Remove(network.FindLink(inputNode[0],hiddenLayer1[3]));
   network.Remove(network.FindLink(inputNode[1],hiddenLayer1[3]));
   network.Remove(network.FindLink(inputNode[2],hiddenLayer1[0]));
   network.Remove(network.FindLink(inputNode[2],hiddenLayer1[3]));
   network.Remove(network.FindLink(inputNode[3],hiddenLayer1[0]));
   network.Remove(network.FindLink(inputNode[3],hiddenLayer1[1]));
   network.Remove(network.FindLink(inputNode[3],hiddenLayer1[2]));
   network.Remove(network.FindLink(inputNode[4],hiddenLayer1[0]));
   network.Remove(network.FindLink(inputNode[4],hiddenLayer1[1]));
   network.Remove(network.FindLink(inputNode[4],hiddenLayer1[2]));
   network.Remove(network.FindLink(inputNode[5],hiddenLayer1[0]));
   network.Remove(network.FindLink(inputNode[5],hiddenLayer1[1]));
   network.Remove(network.FindLink(inputNode[5],hiddenLayer1[2]));
// Remove links between 1st and 2nd hidden layers
   network.Remove(network.FindLink(hiddenLayer1[2],hiddenLayer2[1]));
   network.Remove(network.FindLink(hiddenLayer1[3],hiddenLayer2[0]));
// Remove links between 2nd hidden layer and the output layer
   network.Remove(network.FindLink(hiddenLayer2[0],outputLayer[1]));
// Add link from input node[0] to output node[0]
   network.Link(inputNode[0], outputNode[0]);
// ************************************************************
```

In the above fragment, all links are created using the `FeedForwardNetwork.LinkAll()` method. This creates a total of 6*4+4*3+3*2=42 links, not including the link between the first input node and the first output node. Links that skip layers are not created by the `LinkAll()` method.

Links are then selectively removed starting with the first input node and proceeding to links between the last hidden layer and the output layers. In this case, there are 6*4=24 possible links between the input nodes and first hidden layer. Fourteen of them had to be removed. Between the first hidden layer and second, there are 4*3=12 possible links. Two of them were removed. Between the second hidden layer and output layer there are 3*2=6 possible links, and only one needed to be removed. Finally the skip-layer link between the first input node and first output node is added.

After creating and removing links among layers, the activation function used with each perceptron can be selected. By default, every perceptron in the hidden layers use the logistic activation function and

every perceptron in the output layers uses the linear activation function. The following fragment shows how to change the activation function in the hidden layer perceptrons from logistic to hyperbolic-tangent and the output layer from linear to logistic. It also creates a connection directly from the first input node to the output node.

```
// ************************************************************
// EXAMPLE CODE FOR SETTING NON-DEFAULT ACTIVATION FUNCTIONS
// ************************************************************

   FeedForwardNetwork network = new FeedForwardNetwork();
   InputNode[] inputNode      = network.InputLayer.CreateInputs(6);
   Perceptron[] hiddenLayer1  =
      network.CreateHiddenLayer().CreatePerceptrons(4);
   Perceptron[] hiddenLayer2  =
      network.CreateHiddenLayer().CreatePerceptrons(3);
   Perceptron[] outputLayer   = network.OutputLayer.CreatePerceptrons(2);

// Get Network Perceptrons for Setting Their Activation Functions
   Perceptron[] perceptrons = network.Perceptrons;

   for (int k = 0;  k < hiddenLayer1.Length -1;  k++) {
       perceptrons[k].Activation = Imsl.DataMining.Neural.Activation.Tanh;
   }
   perceptrons[perceptrons.Length - 1].Activation =
       Imsl.DataMining.Neural.Activation.Logistic;
.
.
.
// ************************************************************
```

Training

Trainers are used to find the network weights that produce network outputs matching a set of training targets. The training targets together with their associated network inputs are referred to as training patterns. Training patterns can be historical data relating network inputs to its outputs, or they can be developed from expert opinion or theoretical analysis. In the end, each training pattern relates specific network inputs to its real or desired target outputs.

In IMSL C# Numerical Library all trainers implement the `Imsl.DataMining.Neural.ITrainer` interface. The number of training input attributes must equal the number of input nodes, and the number of training outputs, sometimes called training targets, must equal the number of output perceptrons created for the network.

Single Stage Trainers

`QuasiNewtonTrainer` and `LeastSquaresTrainer` are single stage trainers. They use all available training patterns and a specific optimization method to find optimum network weights. The best set of weights is a set that minimizes the error between the network output and its training targets. The following code fragment illustrates how to use the quasi-Newton method for single stage network training.

```
// ************************************************************
// EXAMPLE CODE FOR ONE-STAGE TRAINER
```

```
// ************************************************************
   double xData[,] = ...
   double yData[,] = ...
   QuasiNewtonTrainer trainer = new QuasiNewtonTrainer();
   trainer.GradientTolerance = 1.0e-7;
   trainer.Train(network, xData, yData);
.
.
.
// ************************************************************
```

In this example, xData and yData are two-dimensional arrays containing the input attributes and output targets respectively. The number of rows in these arrays is equal to the number of training patterns. The number of columns in xData is equal to the number of input attributes, after applying any necessary preprocessing. The number of columns in yData is equal to the number of network outputs. The `GradientTolerance` property is one of several optional settings for tailoring the convergence criteria used with the training optimizer.

`LeastSquaresTrainer` is another single stage trainer. There are two principal differences between this trainer and the quasi-Newton trainer. First their optimization algorithms are different. The least squares trainer uses the Levenberg-Marquardt algorithm to optimize the network. As the name implies, the quasi-Newton trainer uses a modified Newton algorithm for optimization. In some applications, depending upon the data and the network architecture, one method may train the network faster than the other.

Another key difference between these single stage trainers is that the least squares trainer only uses one error function, the sum of squared errors. The quasi-Newton trainer, by default, uses the same error function. However, it also has an interface that accepts a user-supplied error function.

Multistage Trainers

When there are a large number of training patterns, single stage trainers will often take too long to complete network training. For these applications, a multistage trainer could be used to reduce training time. Multistage trainers provide considerably more flexibility in designing an optimum training scheme. All of these trainers break network training into two stages. Stage II is optional. That is, a multistage trainer can be requested to only conduct Stage I training, or it can be requested to conduct both Stage I and II training.

The main difference between Stage I and II training is that Stage I training is conducted multiple times using randomly selected subsets of all available training patterns. Each training session is referred to as an epoch. Although each epoch uses a different set of randomly selected training patterns, the number of patterns is the same for every epoch. Typically, because they are using different data, the solutions vary among epochs.

Stage II training is conducted following the Stage I training using the best set of weights obtained during Stage I. This ensures that the weights developed during Stage II training will always be as good as or better than those determined during Stage I training. The entire set of original training patterns is used during Stage II training, and only one training session is completed.

There is no requirement to use the same trainer for both stages, although there is nothing wrong with that approach. The least squares trainer might be used for Stage I training and the quasi-Newton trainer might be used for Stage II training. In addition, the optimization settings for each trainer can be

different. The multistage trainer is implemented using the `EpochTrainer` class.

The following code fragment illustrates the use of the epoch multistage trainer:

```
// ************************************************************
// EXAMPLE CODE FOR MULTISTAGE EPOCH TRAINER
// ************************************************************
   double xData[,] = ...
   double yData[,] = ...
   QuasiNewtonTrainer stageITrainer   = new QuasiNewtonTrainer();
   LeastSquaresTrainer stageIITrainer = new LeastSquaresTrainer();
   EpochTrainer trainer = new EpochTrainer(stageITrainer, stageIITrainer);
   trainer.NumberOfEpochs = 20;
   trainer.EpochSize = 3000;
.
.
.
// ************************************************************
```

In this example, a quasi-Newton trainer is selected for the Stage I trainer, and the least squares trainer is used for Stage II. Stage I will consists of 20 training epochs. The training of each epoch uses 3,000 randomly selected training patterns with the quasi-Newton trainer. The epoch with the smallest training error supplies the starting values for the Stage II trainer.

Data Preprocessing

Data preprocessing, or filtering, is the term used to describe the process of scaling or transforming input attributes into numerical values suitable for network training. In general it is important to scale all input attributes to a common range, either [0, 1] or [-1, 1]. The algorithm used for obtaining values for the network weights assumes that the inputs are scaled to one of these ranges. If some network inputs have values that cover a much broader range, then the initial weights can be far from optimum causing network training to fail or take an excessively long time.

Network input data are classified into three general categories: continuous, ordinal and nominal. IMSL C# Numerical Library provides methods for preprocessing all three data types. Continuous data are scaled using the `ScaleFilter` class. In addition, lagged versions of continuous time series data can be created using the `TimeSeriesFilter` or `TimeSeriesClassFilter` class.

Categorical data, such as color or preference ratings, are either ordinal and nominal data. `UnsupervisedOrdinalFilter` and `UnsupervisedNominalFilter` are provided to preprocess ordinal and nominal data respectively. `UnsupervisedOrdinalFilter` transforms ordinal data into values between 0 and 1, which allows them to be treated as continuous data.

Nominal data, on the other hand, can be transformed using several methods. `UnsupervisedNominalFilter` converts a single nominal variable with *m* classes into *m* columns containing the values 0 and 1. This is referred to as binary encoding of nominal classification information.

The following code fragment illustrates the use of some of these preprocessing methods:

```
// ************************************************************
// EXAMPLE CODE FOR PREPROCESSING NOMINAL AND CONTINUOUS DATA
// ************************************************************
```

```
   double[,] yData = {....};
   int[] nominalVariable={.....};
   int nClasses = 3;

// Create a nominal filter for binary encoding of a nominal variable
// that has 3 categorical values
   UnsupervisedNominalFilter nominalFilter =
      new UnsupervisedNominalFilter(nClasses);
   int[,] binaryColumns = nominalFilter.Encode(nominalVariable);

// Create a scale filter for scaling continuous data in a range of [0,1]
   ScaleFilter scaleFilter = new ScaleFilter(ScaleFilter.ScalingMethod.Bounded);
// Apply the scale filter to two continuous variables, x1 and x2
   scaleFilter.SetBounds(-200,1000,0,1); // Original values [-200, 1000]
   scaleFilter.Encode(x1);
   scaleFilter.SetBounds(0,5000,0,1);    // Original values [0, 5000]
   scaleFilter.Encode(x2);

// Load the encoded columns into xData
   int n = nominalVariable.Length;
   double[,] xData = new double[n, 3+3];
   for(int i=0; i < n; i++){
      xData[i,0] = x1[i];
      xData[i,1] = x2[i];
      for(int j=0; j < nClasses; j++) xData[i,j+2] = binaryColumns[i,j];
   }
.
.
.
// ****************************************************************
```

In the above example, one nominal variable consisting of values representing 3 different classes, or categories, is encoded into 3 binary columns using `UnsupervisedNominalFilter` class. Two continuous variables are scaled using the `ScaleFilter` class, and these five columns are then loaded into xData in preparation for network training.

Serialization

Neural network training can require a substantial amount of time, so it is often desirable to save a trained network for later use in forecasting. Serialization can be used to save the results of network training.

When an object is serialized, its state is saved. However, the code implementing the class (the class file) is not saved with the serialized file. Hence when the object is deserialized, the code that created the serialized object should be in the classpath. Otherwise deserialization will fail.

For an object to be serialized, the class must use the *Serializable* attribute. The following code fragment serializes key network and training information into four files. One contains the network weights, another contains the training statistics, and two additional files contain the training patterns. This is done using a `write(Object,String)` method that takes a file name and writes the serialized object to that file.

```
// ****************************************************************
// EXAMPLE CODE FOR SAVING TRAINED NETWORK USING SERIALIZATION
// ****************************************************************
```

```
using System.Runtime.Serialization;
using System.Runtime.Serialization.Formatters.Binary;
.
.
.
// ************************************************************
// SAVE A TRAINED NETWORK BY SAVING THE SERIALIZED NETWORK OBJECTS
// ************************************************************
// Saving network weights and structural information
   write(network, "MyNetwork.ser");
// Saving training information available from computeStatistics()
   write(trainer, "MyNetworkTrainer.ser");
// Saving xData training targests
   write(xData, "MyNetworkxData.ser");
// Saving yData training targets
   write(yData, "MyNetworkyData.ser);

// ************************************************************************
// WRITE SERIALIZED NETWORK TO A FILE
// ************************************************************************
static public void  write(System.Object obj, System.String filename)
{
   System.IO.FileStream fos = new System.IO.FileStream(filename,
      System.IO.FileMode.Create);
   IFormatter oos = new BinaryFormatter();
   oos.Serialize(fos, obj);
   fos.Close();
}

// ************************************************************
```

Notice that not only is the network object serialized and saved, the trainer and training patterns, xData and yData, are also saved. This was only done to allow someone to calculate the additional network statistics. If these are not needed, then these training patterns need not be saved. However, for forecasting, it is essential to remember the specific order and nature of the network inputs used during training. This information is not saved in the network serialized file.

When an object is deserialized, the object is reconstructed using the saved serialization file. The following code deserializes the previously saved network information.

```
// ************************************************************
// EXAMPLE CODE FOR READING TRAINED NETWORK FROM SERIALIZED FILES
// ************************************************************
using System.Runtime.Serialization;
using System.Runtime.Serialization.Formatters.Binary;
.
.
.
// ************************************************************
// READ THE TRAINED NETWORK FROM THE SERIALIZED NETWORK OBJECT
   Network network = (Network)read("MyNetwork.ser");
// READ THE SERIALIZED XDATA[,] AND YDATA[,] ARRAYS OF TRAINING
// PATTERNS.
   xData = (double[,])read("MyNetworkxData.ser");
```

```
   yData = (double[,])read("MyNetworkyData.ser");
// READ THE SERIALIZED TRAINER OBJECT
   Trainer trainer = (ITrainer)read("MyNetworkTrainer.ser");
// ************************************************************
// DISPLAY TRAINING STATISTICS
// ************************************************************
   double stats[] = network.computeStatistics(xData, yData);
.
.
.

// ************************************************************
// READ SERIALIZED NETWORK FROM A FILE
// ************************************************************
static public System.Object read(System.String filename)
{
   System.IO.FileStream fis = new System.IO.FileStream(filename,
      System.IO.FileMode.Open, System.IO.FileAccess.Read);
   IFormatter ois = new BinaryFormatter();
   System.Object obj = (System.Object) ois.Deserialize(fis);
   fis.Close();
   return obj;
}
// ************************************************************
```

FeedForwardNetwork Class

`public class Imsl.DataMining.Neural.FeedForwardNetwork : Network`

A representation of a feed forward neural network.

A `Network` contains an InputLayer (p. 1640), an OutputLayer (p. 1641) and zero or more HiddenLayers (p. 1656). The `null` `InputLayer` and `OutputLayer` are automatically created by the Network (p. 1786) constructor. The `InputNodes` are added using the `InputLayer.CreateInputs(nInputs)` method. Output Perceptrons (p. 1661) are added using the `OutputLayer.CreatePerceptrons(nOutputs)` method, and `HiddenLayers` can be created using the `CreateHiddenLayer().CreatePerceptrons(nPerceptrons)` method.

The `InputLayer` contains `InputNodes`. The `HiddenLayers` and `OutputLayers` contain `Perceptron` nodes. These Nodes (p. 1659) are created using factory methods in the Layers (p. 1654).

The `Network` also contains Links (p. 1643) between `Nodes`. `Links` are created by methods in this class.

Each `Link` has a *weight* and *gradient* value. Each `Perceptron` node has a *bias* value. When the `Network` is trained, the *weight* and *bias* values are used as initial guesses. After the `Network` is trained the *weight*, *gradient* and *bias* values are set to the values computed by the training.

A feed forward network is a network in which links are only allowed from one layer to a following layer.

Properties

HiddenLayers

```
virtual public Imsl.DataMining.Neural.HiddenLayer[] HiddenLayers {get; }
```

Description

The HiddenLayers in this Network.

Property Value

An array of HiddenLayers in this Network.

InputLayer

```
override public Imsl.DataMining.Neural.InputLayer InputLayer {get; }
```

Description

The InputLayer in this Network.

Property Value

The neural network InputLayer.

Links

```
override public Imsl.DataMining.Neural.Link[] Links {get; }
```

Description

All the Links in this Network.

Property Value

An array of Links containing all of the Links in this Network.

NumberOfInputs

```
override public int NumberOfInputs {get; }
```

Description

The number of InputNodes to the Network.

Property Value

An int containing the number of InputNodes to the Network.

NumberOfLinks

```
override public int NumberOfLinks {get; }
```

Description

The number of Links in the Network.

Property Value

An int which contains the number of Links in the Network.

NumberOfOutputs

```
override public int NumberOfOutputs {get; }
```

Description

The number of output Perceptrons from the Network.

Property Value

An int containing the number of output Perceptrons from the Network.

NumberOfWeights

```
override public int NumberOfWeights {get; }
```

Description

The number of *weights* in the Network.

Property Value

An int which contains the number of *weights* in the Network.

OutputLayer

```
override public Imsl.DataMining.Neural.OutputLayer OutputLayer {get; }
```

Description

The neural network OutputLayer.

Property Value

Contains the neural network OutputLayer.

Perceptrons

```
override public Imsl.DataMining.Neural.Perceptron[] Perceptrons {get; }
```

Description

The Perceptrons in this Network.

Property Value

An array of Perceptrons in this Network.

Weights

```
override public double[] Weights {get; set; }
```

Description

The *weight* values for the Links (p. 1666) in this Network.

Property Value

An array of doubles containing the *weights*.

Remarks

The array contains the *weights* for each Link followed by the Perceptron (p. 1662) *bias* values. The Link *weights* are the order in which the Links were created. The *weights* values are first, followed by the *bias* values in the HiddenLayer and then the *bias* values in the OutputLayer, and in the order in which the Perceptrons were created.

Constructor

FeedForwardNetwork

```
public FeedForwardNetwork()
```

Description

Creates a new instance of `FeedForwardNetwork`.

Methods

CreateHiddenLayer

```
override public Imsl.DataMining.Neural.HiddenLayer CreateHiddenLayer()
```

Description

Creates a `HiddenLayer`.

Returns

A `HiddenLayer` object which specifies a neural network hidden layer.

FindLink

```
virtual public Imsl.DataMining.Neural.Link FindLink(Imsl.DataMining.Neural.Node
from, Imsl.DataMining.Neural.Node to)
```

Description

Returns the `Link` between two `Nodes`.

Parameters

`from` – The origination `Node`.

`to` – The destination `Node`.

Returns

A `Link` between the two `Nodes`, or `null` if no such `Link` exists.

FindLinks

```
virtual public Imsl.DataMining.Neural.Link[]
FindLinks(Imsl.DataMining.Neural.Node to)
```

Description

Returns all of the `Links` to a given `Node`.

Parameter

`to` – A `Node` whose `Links` are to be determined.

Returns

An array of `Link`s containing all of the `Link`s to the given `Node`.

Forecast

```
override public double[] Forecast(double[] x)
```

Description

Computes a forecast using the `Network`.

Parameter

`x` – A `double` array of values to which the `Node`s in the `InputLayer` are to be set.

Returns

A `double` array containing the values of the `Node`s in the `OutputLayer`.

GetForecastGradient

```
override public double[,] GetForecastGradient(double[] xData)
```

Description

Returns the derivatives of the outputs with respect to the *weights*.

Parameter

`xData` – A `double` array which specifies the input values at which the *gradient* is to be evaluated.

Returns

A `double` array containing the *gradient* values.

Remarks

The value of `gradient[i][j]` is dy_i/dw_j, where y_i is the i-th output and w_j is the j-th weight.

Link

```
virtual public Imsl.DataMining.Neural.Link Link(Imsl.DataMining.Neural.Node
from, Imsl.DataMining.Neural.Node to)
```

Description

Establishes a `Link` between two `Node`s.

Parameters

`from` – The origination `Node`.

`to` – The destination `Node`.

Returns

A `Link` between the two `Node`s.

Remarks

Any existing `Link` between these `Nodes` is removed.

Link

```
virtual public Imsl.DataMining.Neural.Link Link(Imsl.DataMining.Neural.Node
from, Imsl.DataMining.Neural.Node to, double weight)
```

Description

Establishes a `Link` between two `Nodes` with a specified weight (p. 1666).

Parameters

`from` – The origination `Node`.

`to` – The destination `Node`.

`weight` – A `double` which specifies the *weight* to be given the `Link`.

Returns

A `Link` between the two `Nodes`.

LinkAll

```
virtual public void LinkAll(Imsl.DataMining.Neural.Layer from,
Imsl.DataMining.Neural.Layer to)
```

Description

Links all of the `Nodes` in one `Layer` to all of the `Nodes` in another `Layer`.

Parameters

`from` – The origination `Layer`.

`to` – The destination `Layer`.

LinkAll

```
virtual public void LinkAll()
```

Description

For each `Layer` in the `Network`, link each `Node` in the `Layer` to each `Node` in the next `Layer`.

Remove

```
virtual public void Remove(Imsl.DataMining.Neural.Link link)
```

Description

Removes a `Link` from the `Network`.

Parameter

`link` – The `Link` deleted from the `Network`.

SetEqualWeights

```
virtual public void SetEqualWeights(double[,] xData)
```

Description

Initializes network *weights* using equal weighting.

Parameter

`xData` – An input `double` matrix containing training patterns. The number of columns in `xData` must equal the number of nodes in the `InputLayer`.

Remarks

The equal weights approach starts by assigning equal values to the inputs of each `Perceptron`. If a `Perceptron` has 4 inputs, then this method starts by assigning the value 1/4 to each of the `Perceptron`'s input *weights*. The *bias* weight is initially assigned a value of zero.

The *weights* for the first layer of `Perceptrons`, either the first `HiddenLayer` if the number of layers is greater than 1 or the `OutputLayer`, are scaled using the training patterns. Scaling is accomplished by dividing the initial *weights* for the first layer by the standard deviation, *s*, of the potential for that `Perceptron`. The *bias* weight is set to *-avg/s*, where *avg* is the average potential for that `Perceptron`. This makes the average potential for the `Perceptrons` in this first layer approximately 0 and its standard deviation equal to 1.

This reduces the possibility of saturation during network training resulting from very large or small values for the `Perceptrons` potential. During training random noise is added to these intial values at each training stage. If the `EpochTrainer` is used, noise is added to these initial values at the start of each epoch.

SetRandomWeights

```
virtual public void SetRandomWeights(double[,] xData, System.Random random)
```

Description

Initializes network *weights* using random weights.

Parameters

`xData` – An input `double` matrix containing training patterns. The number of columns in `xData` must equal the number of nodes in the `InputLayer`.

`random` – A `Random` object.

Remarks

The random weights algorithm assigns equal weights to all `Perceptrons`, except those in the first layer connected to the `InputLayer`. Like the equal weights algorithm, `Perceptrons` not in the first layer are assigned *weights* 1/*k*, where *k* is the number of `InputNodes` connected to that `Perceptron`.

For the first layer `Perceptron` *weights* are initially assigned values from the uniform random distribution on the interval [-0.5, +0.5]. These are then scaled using the training patterns. The random weights for a perceptron are divided by *s*, the standard deviation of the potential for that `Perceptron` calculated using the initial random values. Its *bias* value is set to *-avg/s*, where *avg* is the average potential for that `Perceptron`. This makes the average potential for the `Perceptrons` in this first layer approximately 0 and its standard deviation equal to 1.

This reduces the possibility of saturation during network training resulting from very large or small values for the `Perceptrons` potential. During training random noise is added to these intial values at

each training stage. If the `EpochTrainer` is used, noise is added to these initial values at the start of each epoch.

ValidateLink

```
virtual protected internal void ValidateLink(Imsl.DataMining.Neural.Node from,
Imsl.DataMining.Neural.Node to)
```

Description

Checks that a `Link` between two `Nodes` is valid.

Parameters

`from` – The origination `Node`.

`to` – The destination `Node`.

Remarks

In a feed forward network a link must be from a node in one layer to a node in a later layer. Intermediate layers can be skipped, but a link cannot go backward.

Exception

`System.ArgumentException` is thrown if the `Link` is not valid

Example: FeedForwardNetwork

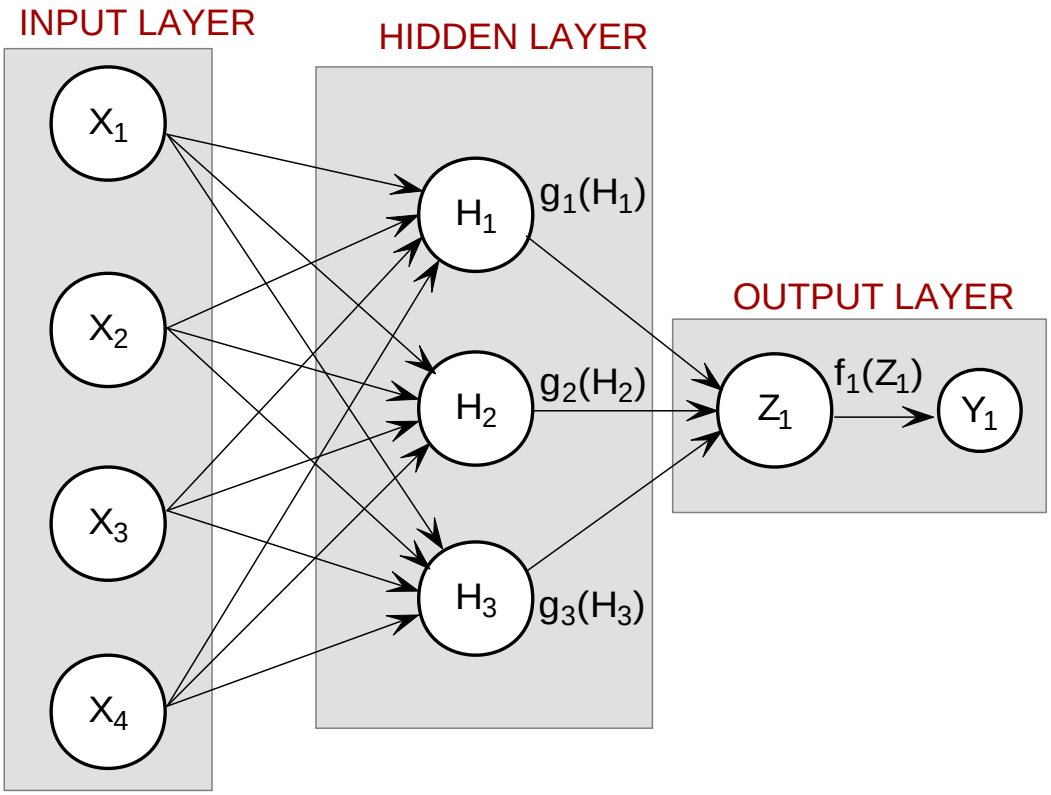
This example trains a 2-layer network using 100 training patterns from one nominal and one continuous input attribute. The nominal attribute has three classifications which are encoded using binary encoding. This results in three binary network input columns. The continuous input attribute is scaled to fall in the interval [0,1].

The network training targets were generated using the relationship:

$y = 10*X_1 + 20*X_2 + 30*X_3 + 2.0*X_4$, where

X_1-X_3 are the three binary columns, corresponding to categories 1-3 of the nominal attribute, and X_4 is the scaled continuous attribute.

The structure of the network consists of four input nodes and two layers, with three perceptrons in the hidden layer and one in the output layer. The following figure illustrates this structure:



There are a total of 19 weights in this network. The activations functions are all linear. Since the target output is a linear function of the input attributes, linear activation functions guarantee that the network forecasts will exactly match their targets. Of course, this same result could have been obtained using linear multiple regression. Training is conducted using the quasi-newton trainer.

```
using System;
using Imsl.DataMining.Neural;
using System.Runtime.Serialization;
using System.Runtime.Serialization.Formatters.Binary;

//***************************************************************************
// Two Layer Feed-Forward Network with 4 inputs: 1 nominal with 3 categories,
// encoded using binary encoding, 1 continuous input attribute, and 1 output
// target (continuous).
// There is a perfect linear relationship between the input and output
// variables:
//
// MODEL:  Y = 10*X1+20*X2+30*X3+2*X4
//
// Variables X1-X3 are the binary encoded nominal variable and X4 is the
// continuous variable.
//***************************************************************************

//[Serializable]
```

```
public class FeedForwardNetworkEx1 //: System.Runtime.Serialization.ISerializable
{

   // Network Settings
   private static int nObs = 100; // number of training patterns
   private static int nInputs = 4; // four inputs
   private static int nCategorical = 3; // three categorical attributes
   private static int nOutputs = 1; // one continuous output
   private static int nPerceptrons = 3; // perceptrons in hidden layer
   private static IActivation hiddenLayerActivation;
   private static IActivation outputLayerActivation;
   private static System.String errorMsg = "";
   // Error Status Messages for the Least Squares Trainer
   private static System.String errorMsg0 =
      "--> Least Squares Training Completed Successfully";
   private static System.String errorMsg1 =
      "--> Scaled step tolerance was satisfied.  The current solution \n" +
      "may be an approximate local solution, or the algorithm is making\n" +
      "slow progress and is not near a solution, or the Step Tolerance\n" +
      "is too big";
   private static System.String errorMsg2 =
      "--> Scaled actual and predicted reductions in the function are\n" +
      "less than or equal to the relative function convergence\n" +
      "tolerance RelativeTolerance";
   private static System.String errorMsg3 =
      "--> Iterates appear to be converging to a noncritical point.\n" +
      "Incorrect gradient information, a discontinuous function,\n" +
      "or stopping tolerances being too tight may be the cause.";
   private static System.String errorMsg4 =
      "--> Five consecutive steps with the maximum stepsize have\n" +
      "been taken.  Either the function is unbounded below, or has\n" +
      "a finite asymptote in some direction, or the maximum stepsize\n" +
      "is too small.";
   private static System.String errorMsg5 =
      "--> Too many iterations required";

   // categoricalAtt[]: A 2D matrix of values for the categorical training
   //                  attribute. In this example, the single categorical
   //                  attribute has 3 categories that are encoded using
   //                  binary encoding for input into the network.
   //                  {1,0,0} = category 1, {0,1,0} = category 2, and
   //                  {0,0,1} = category 3.
   private static double[,] categoricalAtt =
        {{1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0},
         {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0},
         {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0},
         {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0},
         {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0},
         {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {0, 1, 0},
         {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0},
         {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0},
         {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0},
         {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0},
         {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 1, 0}, {0, 0, 1},
         {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1},
         {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1},
```

```
         {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1},
      {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1},
      {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1},
      {0, 0, 1}, {0, 0, 1}, {0, 0, 1}, {0, 0, 1}};
//
// contAtt[]:  A matrix of values for the continuous training attribute
//
private static double[] contAtt = new double[]{4.007054658, 7.10028447,
   4.740350984, 5.714553211, 6.205437459, 2.598930065, 8.65089967,
   5.705787357, 2.513348184, 2.723795955, 4.1829356, 1.93280416,
   0.332941608, 6.745567628, 5.593588463, 7.273544478, 3.162117939,
   4.205381208, 0.16414745, 2.883418275, 0.629342241, 1.082223406,
   8.180324708, 8.004894314, 7.856215418, 7.797143157, 8.350033996,
   3.778254431, 6.964837082, 6.13938006, 0.48610387, 5.686627923,
   8.146173848, 5.879852653, 4.587492779, 0.714028533, 7.56324211,
   8.406012623, 4.225261454, 6.369220241, 4.432772218, 9.52166984,
   7.935791508, 4.557155333, 7.976015058, 4.913538616, 1.473658514,
   2.592338905, 1.386872932, 7.046051685, 1.432128376, 1.153580985,
   5.6561491, 3.31163251, 4.648324851, 5.042514515, 0.657054195,
   7.958308093, 7.557870384, 7.901990083, 5.2363088, 6.95582150,
   8.362167045, 4.875903563, 1.729229471, 4.380370223, 8.527875685,
   2.489198107, 3.711472959, 4.17692681, 5.844828801, 4.825754155,
   5.642267843, 5.339937786, 4.440813223, 1.615143829, 7.542969339,
   8.100542684, 0.98625265, 4.744819569, 8.926039258, 8.813441887,
   7.749383991, 6.551841576, 8.637046998, 4.560281415, 1.386055087,
   0.778869034, 3.883379045, 2.364501589, 9.648737525, 1.21754765,
   3.908879368, 4.253313879, 9.31189696, 3.811953836, 5.78471629,
   3.414486452, 9.345413015, 1.024053777};
//
// outs[]:  A 2D matrix containing the training outputs for this network
// In this case there is an exact linear relationship between these
// outputs and the inputs:  outs = 10*X1+20*X2+30*X3+2*X4, where
// X1-X3 are the categorical variables and X4=contAtt
//
private static double[] outs = new double[]{18.01410932, 24.20056894,
   19.48070197, 21.42910642, 22.41087492, 15.19786013, 27.30179934,
   21.41157471, 15.02669637, 15.44759191, 18.3658712, 13.86560832,
   10.66588322, 23.49113526, 21.18717693, 24.54708896, 16.32423588,
   18.41076242, 10.3282949, 15.76683655, 11.25868448, 12.16444681,
   26.36064942, 26.00978863, 25.71243084, 25.59428631, 26.70006799,
   17.55650886, 23.92967416, 22.27876012, 10.97220774, 21.37325585,
   26.2923477, 21.75970531, 19.17498556, 21.42805707, 35.12648422,
   36.81202525, 28.45052291, 32.73844048, 28.86554444, 39.04333968,
   35.87158302, 29.11431067, 35.95203012, 29.82707723, 22.94731703,
   25.18467781, 22.77374586, 34.09210337, 22.86425675, 22.30716197,
   31.3122982, 26.62326502, 29.2966497, 30.08502903, 21.31410839,
   35.91661619, 35.11574077, 35.80398017, 30.4726176, 33.91164302,
   36.72433409, 29.75180713, 23.45845894, 38.76074045, 47.05575137,
   34.97839621, 37.42294592, 38.35385362, 41.6896576, 39.65150831,
   41.28453569, 40.67987557, 38.88162645, 33.23028766, 45.08593868,
   46.20108537, 31.9725053, 39.48963914, 47.85207852, 47.62688377,
   45.49876798, 43.10368315, 47.274094, 39.1205628, 32.77211017,
   31.55773807, 37.76675809, 34.72900318, 49.29747505, 32.4350953,
   37.81775874, 38.50662776, 48.62379392, 37.62390767, 41.56943258,
   36.8289729, 48.69082603, 32.04810755};
// ************************************************************************
```

```
// MAIN
// ************************************************************************


public static void  Main(System.String[] args)
{

   double[] weight; // network weights
   double[] gradient; // network gradient after training
   double[,] xData; // Input  Attributes for Trainer
   double[,] yData; // Output Attributes for Trainer
   int i, j; // array indicies
   int nWeights = 0; // Number of weights obtained from network
   System.String networkFileName = "FeedForwardNetworkEx1.ser";
   System.String trainerFileName = "FeedForwardTrainerEx1.ser";
   System.String xDataFileName = "FeedForwardxDataEx1.ser";
   System.String yDataFileName = "FeedForwardyDataEx1.ser";
   // ************************************************************************
   // PREPROCESS TRAINING PATTERNS
   // ************************************************************************
   System.Console.Out.WriteLine(
      "--> Starting Preprocessing of Training Patterns");
   xData = new double[nObs,nInputs];
   // for (int i2 = 0; i2 < nObs; i2++)
   // {
   //    xData[i2] = new double[nInputs];
   // }
   yData = new double[nObs,nOutputs];
   // for (int i3 = 0; i3 < nObs; i3++)
   // {
   //    yData[i3] = new double[nOutputs];
   // }
   for (i = 0; i < nObs; i++)
   {
      for (j = 0; j < nCategorical; j++)
      {
         xData[i,j] = categoricalAtt[i,j];
      }
      xData[i,nCategorical] = contAtt[i] / 10.0; // Scale continuous input
      yData[i,0] = outs[i]; // outputs are unscaled
   }
   // ************************************************************************
   // CREATE FEEDFORWARD NETWORK
   // ************************************************************************
   System.Console.Out.WriteLine("--> Creating Feed Forward Network Object");
   FeedForwardNetwork network = new FeedForwardNetwork();
   // setup input layer with number of inputs = nInputs = 4
   network.InputLayer.CreateInputs(nInputs);
   // create a hidden layer with nPerceptrons=3 perceptrons
   network.CreateHiddenLayer().CreatePerceptrons(nPerceptrons);
   // create output layer with nOutputs=1 output perceptron
   network.OutputLayer.CreatePerceptrons(nOutputs);
   // link all inputs and perceptrons to all perceptrons in the next layer
   network.LinkAll();
   // Get Network Perceptrons for Setting Their Activation Functions
   Perceptron[] perceptrons = network.Perceptrons;
```

```
// Set all perceptrons to linear activation
for (i = 0; i < perceptrons.Length - 1; i++)
{
   perceptrons[i].Activation = hiddenLayerActivation;
}
perceptrons[perceptrons.Length - 1].Activation = outputLayerActivation;
System.Console.Out.WriteLine(
   "--> Feed Forward Network Created with 2 Layers");
// ********************************************************************
// TRAIN NETWORK USING QUASI-NEWTON TRAINER
// ********************************************************************
System.Console.Out.WriteLine(
   "--> Training Network using Quasi-Newton Trainer");
// Create Trainer
QuasiNewtonTrainer trainer = new QuasiNewtonTrainer();
// Set Training Parameters
trainer.MaximumTrainingIterations = 1000;
// Train Network
trainer.Train(network, xData, yData);
// Check Training Error Status
switch (trainer.ErrorStatus)
{

   case 0:  errorMsg = errorMsg0;
      break;

   case 1:  errorMsg = errorMsg1;
      break;

   case 2:  errorMsg = errorMsg2;
      break;

   case 3:  errorMsg = errorMsg3;
      break;

   case 4:  errorMsg = errorMsg4;
      break;

   case 5:  errorMsg = errorMsg5;
      break;

   default:  errorMsg = errorMsg0;
      break;

}
System.Console.Out.WriteLine(errorMsg);
// ********************************************************************
// DISPLAY TRAINING STATISTICS
// ********************************************************************
double[] stats = network.ComputeStatistics(xData, yData);
// Display Network Errors
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("--> SSE:                        " +
   (float) stats[0]);
System.Console.Out.WriteLine("--> RMS:                        " +
```

```
      (float) stats[1]);
   System.Console.Out.WriteLine("--> Laplacian Error:           " +
      (float) stats[2]);
   System.Console.Out.WriteLine("--> Scaled Laplacian Error:    " +
      (float) stats[3]);
   System.Console.Out.WriteLine("--> Largest Absolute Residual: " +
      (float) stats[4]);
   System.Console.Out.WriteLine(
      "*******************************************");
   System.Console.Out.WriteLine("");
   // ************************************************************************
   // OBTAIN AND DISPLAY NETWORK WEIGHTS AND GRADIENTS
   // ************************************************************************
   System.Console.Out.WriteLine("--> Getting Network Weights and Gradients");
   // Get weights
   weight = network.Weights;
   // Get number of weights = number of gradients
   nWeights = network.NumberOfWeights;
   // Obtain Gradient Vector
   gradient = trainer.ErrorGradient;
   // Print Network Weights and Gradients
   System.Console.Out.WriteLine(" ");
   System.Console.Out.WriteLine("--> Network Weights and Gradients:");
   System.Console.Out.WriteLine(
      "*******************************************");
   for (i = 0; i < nWeights; i++)
   {
      System.Console.Out.WriteLine("w[" + i + "]=" + (float) weight[i] +
         " g[" + i + "]=" + (float) gradient[i]);
   }
   System.Console.Out.WriteLine(
      "*******************************************");
   // ************************************************************************
   // SAVE THE TRAINED NETWORK BY SAVING THE SERIALIZED NETWORK OBJECT
   // ************************************************************************
   System.Console.Out.WriteLine("\n--> Saving Trained Network into " +
      networkFileName);
   write(network, networkFileName);
   System.Console.Out.WriteLine("--> Saving xData into " + xDataFileName);
   write(xData, xDataFileName);
   System.Console.Out.WriteLine("--> Saving yData into " + yDataFileName);
   write(yData, yDataFileName);
   System.Console.Out.WriteLine("--> Saving Network Trainer into " +
      trainerFileName);
   write(trainer, trainerFileName);
}
// ************************************************************************
// WRITE SERIALIZED NETWORK TO A FILE
// ************************************************************************
static public void  write(System.Object obj, System.String filename)
{
   System.IO.FileStream fos = new System.IO.FileStream(filename,
      System.IO.FileMode.Create);
   IFormatter oos = new BinaryFormatter();
   oos.Serialize(fos, obj);
   fos.Close();
```

```
   }
   static FeedForwardNetworkEx1()
   {
      hiddenLayerActivation = Imsl.DataMining.Neural.Activation.Linear;
      outputLayerActivation = Imsl.DataMining.Neural.Activation.Linear;
   }
}
```

Output

```
--> Starting Preprocessing of Training Patterns
--> Creating Feed Forward Network Object
--> Feed Forward Network Created with 2 Layers
--> Training Network using Quasi-Newton Trainer
--> Least Squares Training Completed Successfully
*********************************************
--> SSE:                       1.013444E-15
--> RMS:                       2.007463E-19
--> Laplacian Error:           3.005804E-07
--> Scaled Laplacian Error:    3.535235E-10
--> Largest Absolute Residual: 2.784275E-08
*********************************************

--> Getting Network Weights and Gradients

--> Network Weights and Gradients:
*********************************************
w[0]=-1.491785 g[0]=-2.611079E-08
w[1]=-1.491785 g[1]=-2.611079E-08
w[2]=-1.491785 g[2]=-2.611079E-08
w[3]=1.616918 g[3]=6.182035E-08
w[4]=1.616918 g[4]=6.182035E-08
w[5]=1.616918 g[5]=6.182035E-08
w[6]=4.725622 g[6]=-5.273856E-08
w[7]=4.725622 g[7]=-5.273856E-08
w[8]=4.725622 g[8]=-5.273856E-08
w[9]=6.217407 g[9]=-8.733E-10
w[10]=6.217407 g[10]=-8.733E-10
w[11]=6.217407 g[11]=-8.733E-10
w[12]=1.072258 g[12]=-1.690978E-07
w[13]=1.072258 g[13]=-1.690978E-07
w[14]=1.072258 g[14]=-1.690978E-07
w[15]=3.850755 g[15]=-1.7029E-08
w[16]=3.850755 g[16]=-1.7029E-08
w[17]=3.850755 g[17]=-1.7029E-08
w[18]=2.411725 g[18]=-1.588144E-08
*********************************************

--> Saving Trained Network into FeedForwardNetworkEx1.ser
--> Saving xData into FeedForwardxDataEx1.ser
--> Saving yData into FeedForwardyDataEx1.ser
--> Saving Network Trainer into FeedForwardTrainerEx1.ser
```

Layer Class

```
public class Imsl.DataMining.Neural.Layer
```

The base class for Layers in a neural network.

See Also

Imsl.DataMining.Neural.InputLayer (p. 1655), Imsl.DataMining.Neural.HiddenLayer (p. 1656)

Properties

Index

```
virtual public int Index {get; set; }
```

Description

The Index of this Layer.

Nodes

```
virtual public Imsl.DataMining.Neural.Node[] Nodes {get; }
```

Description

A list of the Nodes in this Layer.

Property Value

An array containing the Nodes associated with this Layer.

Constructor

Layer

```
protected internal Layer(Imsl.DataMining.Neural.FeedForwardNetwork network)
```

Description

Constructs a Layer.

Parameter

network – The FeedForwardNetwork to which this Layer is associated.

Method

AddNode

```
virtual protected internal void AddNode(Imsl.DataMining.Neural.Node node)
```

Description

Associates a Imsl.DataMining.Neural.Perceptron (p. 1661) with this `Layer`.

Parameter

`node` – A `Node` to associate with this `Layer`.

InputLayer Class

```
public class Imsl.DataMining.Neural.InputLayer :  Layer
```

Input layer in a neural network.

An `InputLayer` is automatically created by `Network`.

See Also

Imsl.DataMining.Neural.Network (p. 1786)

Property

Nodes

```
override public Imsl.DataMining.Neural.Node[] Nodes {get; }
```

Description

The `Perceptrons` in the `InputLayer`.

Property Value

An `InputNode` array containing the `Nodes` in the `InputLayer`.

Methods

CreateInput

```
virtual public Imsl.DataMining.Neural.InputNode CreateInput()
```

Description

Creates an `InputNode` in the `InputLayer` of the neural network.

CreateInputs

```
virtual public Imsl.DataMining.Neural.InputNode[] CreateInputs(int n)
```

Description

Creates a number of `InputNodes` in this `Layer` of the neural network.

Parameter

`n` – An `int` which specifies the number of `InputNodes` to be created in this `Layer`.

Returns

An `InputNode` array containing the created `InputNodes`.

HiddenLayer Class

```
public class Imsl.DataMining.Neural.HiddenLayer :  Layer
```

Hidden layer in a neural network. This is created by a factory method in `Network`.

See Also

Imsl.DataMining.Neural.Network.CreateHiddenLayer (p. 1789)

Methods

CreatePerceptron

```
virtual public Imsl.DataMining.Neural.Perceptron
CreatePerceptron(Imsl.DataMining.Neural.IActivation activation, double bias)
```

Description

Creates a `Perceptron` in this `Layer` with a specified activation function and *bias*.

Parameters

`activation` – The `IActivation` object which specifies the activation function to be used.

`bias` – A `double` which specifies the initial value for the *bias* weight.

CreatePerceptron

```
virtual public Imsl.DataMining.Neural.Perceptron CreatePerceptron()
```

Description

Creates a `Perceptron` in this `Layer` of the neural network.

Remarks

The created `Perceptron` uses the logistic activation function and has an initial *bias* value of zero.

CreatePerceptrons

```
virtual public Imsl.DataMining.Neural.Perceptron[] CreatePerceptrons(int n)
```

Description

Creates a number of `Perceptrons` in this `Layer` of the neural network.

Parameter

`n` – An `int` which specifies the number of `Perceptrons` to be created.

Returns

An array containing the created `Perceptrons`.

Remarks

The created `Perceptrons` use the logistic activation function and have an initial *bias* value of zero.

CreatePerceptrons

```
virtual public Imsl.DataMining.Neural.Perceptron[] CreatePerceptrons(int n,
Imsl.DataMining.Neural.IActivation activation, double bias)
```

Description

Creates a number of `Perceptrons` in this `Layer` with the specified *bias*.

Parameters

`n` – An `int` which specifies the number of `Perceptrons` to be created.

`activation` – The `IActivation` object which specifies the action function to be used.

`bias` – A `double` containing the initial value to be applied as the `Bias` values for the `Perceptrons`.

Returns

An array containing the created `Perceptrons`.

OutputLayer Class

```
public class Imsl.DataMining.Neural.OutputLayer :  Layer
```

Output layer in a neural network.

An empty `OutputLayer` is automatically created by `FeedForwardNetwork`.

See Also

Imsl.DataMining.Neural.Network (p. 1786)

Property

Nodes

```
override public Imsl.DataMining.Neural.Node[] Nodes {get; }
```

Description

The `Perceptrons` in the `OutputLayer`.

Property Value

An `OutputPerceptron` array containing the `Nodes` in the `OutputLayer`.

Remarks

This method overrides the method in Layer (p. 1654) to return the `Perceptrons` in an `OutputPerceptron` array.

Methods

CreatePerceptron

```
virtual public Imsl.DataMining.Neural.Perceptron
CreatePerceptron(Imsl.DataMining.Neural.IActivation activation, double bias)
```

Description

Creates a `Perceptron` in this `Layer` with a specified `activation` and `bias`.

Parameters

`activation` – The `IActivation` object which specifies the action function to be used.

`bias` – A `double` which specifies the initial value for the *bias* for this `Perceptron`.

CreatePerceptron

```
virtual public Imsl.DataMining.Neural.Perceptron CreatePerceptron()
```

Description

Creates a `Perceptron` in this `Layer` of the neural network. By default, the created `Perceptron` uses the linear activation function and has an initial *bias* value of zero.

CreatePerceptrons

```
virtual public Imsl.DataMining.Neural.Perceptron[] CreatePerceptrons(int n)
```

Description

Creates a number of `Perceptrons` in this `Layer` of the neural network. By default, they will use linear activation and a zero initial *bias*.

Parameter

`n` – An `int` which specifies the number of `Perceptrons` to be created in this `Layer`.

Returns

An array containing the created `Perceptrons`.

CreatePerceptrons

```
virtual public Imsl.DataMining.Neural.Perceptron[] CreatePerceptrons(int n,
Imsl.DataMining.Neural.IActivation activation, double bias)
```

Description

Creates a number of `Perceptrons` in this `Layer` with specified `activation` and `bias` Bias (p. 1662).

Parameters

`n` – An `int` which specifies the number of `Perceptrons` to be created.

`activation` – The `IActivation` object which indicates the action function to be used.

`bias` – A `double` which specifies the initial *bias* for the `Perceptrons`.

Returns

An array containing the created `Perceptrons`.

Node Class

```
public class Imsl.DataMining.Neural.Node
```

A `Node` in a neural network.

`Node` is an abstract class that serves as the base class for the concrete classes `InputNode` and `Perceptron`.

See Also

Imsl.DataMining.Neural.InputNode (p. 1660), Imsl.DataMining.Neural.Perceptron (p. 1661)

Property

Layer

```
virtual public Imsl.DataMining.Neural.Layer Layer {get; }
```

Description

The `Layer` in which this `Node` exists.

Property Value

The `Layer` associated with this `Node`.

Methods

GetValue

```
virtual public double GetValue()
```

Description

Returns the value of this `Node`.

Returns

A `double` which contains the value of the `Node`.

SetValue

```
virtual public void SetValue(double node)
```

Description

Sets the value of this `Node`.

Parameter

`node` – A `double` which specifies a value for the `Node`.

InputNode Class

```
public class Imsl.DataMining.Neural.InputNode :  Node
```

A `Node` in the `InputLayer`.

`InputNodes` are not created directly. Instead factory methods in `InputLayer` are used to create `InputNodes` within the `InputLayer`. For example, InputLayer.CreateInput (p. 1655) creates a single `InputNode`.

See Also

Feed Forward Class Example 1

Methods

GetValue

```
override public double GetValue()
```

Description

Returns the value of this Imsl.DataMining.Neural.Node (p. 1659).

Returns

A `double` which contains the value of this `InputNode`.

SetValue

```
override public void SetValue(double node)
```

Description

Sets the value of this Imsl.DataMining.Neural.Node (p. 1659).

Parameter

`node` – A `double` which specifies the new value of this `InputNode`.

Perceptron Class

```
public class Imsl.DataMining.Neural.Perceptron :  Node
```

A `Perceptron` node in a neural network.

`Perceptrons` are created by factory methods in a `Layer`. Each `Perceptron` has an *activation* function (g) (p. 1661) and a *bias* (μ) (p. 1662). The value of a `Perceptron` is given by $g(\sum_i w_i X_i + \mu)$, where X_is are the values of nodes input to this `Perceptron` with *weight* (w_i) (p. 1666).

`Network` training will use existing *bias* values for the starting values for the trainer. Upon completion of `Network` training, the *bias* values are set to the values computed by the trainer.

Properties

Activation

```
virtual public Imsl.DataMining.Neural.IActivation Activation {get; set; }
```

Description

The activation function.

Property Value

An `Activation` object which represents the activation *g* used by this `Perceptron`.

Bias

```
virtual public double Bias {get; set; }
```

Description

The *bias* for this perceptron.

Property Value

A `double` representing the *bias* for this `Perceptron`.

Remarks

The `Bias` has a default value of 0.

OutputPerceptron Class

```
public class Imsl.DataMining.Neural.OutputPerceptron :  Perceptron
```

A `Perceptron` in the `OutputLayer`.

`OutputPerceptrons` are created by factory methods in `Outputlayer`.

`OutputPerceptrons` are not created directly. Instead factory methods in `OutputLayer` are used to create `OutputPerceptrons` within the `OutputLayer`. For example, `OutputLayer.createPerceptron()` creates a single `OutputPerceptron`.

See Also

Imsl.DataMining.Neural.OutputLayer (p. 1657)

Method

GetValue

```
override public double GetValue()
```

Description

Returns the value of the output perceptron determined using the current `Network` state and inputs.

Returns

A `double` value of the `OutputPerceptron` determined using the current `Network` state and inputs.

IActivation Interface

`public interface Imsl.DataMining.Neural.IActivation`

Interface implemented by perceptron activation functions.

Standard activation functions are defined as static members of this interface. New activation functions can be defined by implementing a method, `g(double x)`, returning the value and a method, `derivative(double x, double y)`, returning the derivative of `g` evaluated at `x` where *y = g(x)*.

See Also

Imsl.DataMining.Neural.Perceptron (p. 1661)

Methods

Derivative

`abstract public double Derivative(double x, double y)`

Description

Returns the value of the derivative of the activation function.

Parameters

`x` – A `double` which specifies the point at which the activation function is to be evaluated.

`y` – A `double` which specifies *y = g(x)*, the value of the activation function at `x`.

Returns

A `double` containing the value of the derivative of the activation function at `x`.

Remarks

y is not mathematically required, but can sometimes be used to more quickly compute the derivative.

G

`abstract public double G(double x)`

Description

Returns the value of the activation function.

Parameter

`x` – A `double` is the point at which the activation function is to be evaluated.

Returns

A `double` containing the value of the activation function at `x`.

Activation Structure

```
public structure Imsl.DataMining.Neural.Activation
```

Fields

Linear

```
public Imsl.DataMining.Neural.IActivation Linear
```

Description

The identity activation function, *g(x) = x*.

Logistic

```
public Imsl.DataMining.Neural.IActivation Logistic
```

Description

The logistic activation function, $g(x) = \frac{1}{1+e^{-x}}$.

LogisticTable

```
public Imsl.DataMining.Neural.IActivation LogisticTable
```

Description

The logistic activation function computed using a table.

Remarks

This is an approximation to the logistic function that is faster to compute.

This version of the logistic function differs from the exact version by at most `4.0e-9`.

Networks trained using this activation should not use `Activation.Logistic` for forecasting. Forecasting should be done using the specific function supplied during training.

Softmax

```
public Imsl.DataMining.Neural.IActivation Softmax
```

Description

The softmax activation function.

$$\text{softmax}_i = \frac{e^{Z_i}}{\sum_{j=1}^{C} e^{Z_j}}$$

.

Squash

`public Imsl.DataMining.Neural.IActivation Squash`

Description

The squash activation function, $g(x) = \frac{x}{1+|x|}$

Tanh

`public Imsl.DataMining.Neural.IActivation Tanh`

Description

The hyperbolic tangent activation function, $g(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Link Class

`public class Imsl.DataMining.Neural.Link`

A link in a neural network.

`Link` objects are not created directly. Instead, they are created by factory methods in `FeedForwardNetwork`.

The most useful method is LinkAll() (p. 1644) which creates `Link` objects connecting every `Node` in each `Layer` to every `Node` in the next `Layer`.

The method Link(node,node) (p. 1643) creates a `Link` from a `Node` to any `Node` in a later `Layer`.

The method FindLink(Node,Node) (p. 1642) returns the `Link` connecting two `Nodes` in the Imsl.DataMining.Neural.Network (p. 1786).

The method Remove(Link) (p. 1644) removes a `Link` from the `Network`.

Each `Link` object contains a *weight*, see property Weight (p. 1666). *Weight*s are used in computing `Perceptron` values.

See Also

Imsl.DataMining.Neural.FeedForwardNetwork (p. 1639)

Properties

From

```
virtual public Imsl.DataMining.Neural.Node From {get; }
```

Description

The origination `Node` for this `Link`.

Property Value

A `Node` which is the origination `Node` for this `Link`.

To

```
virtual public Imsl.DataMining.Neural.Node To {get; }
```

Description

The destination `Node` for this `Link`.

Property Value

A `Node` which is the destination `Node` for this `Link`.

Weight

```
virtual public double Weight {get; set; }
```

Description

The *weight* for this `Link`.

Property Value

A `double` which contains the *weight* attributed to this `Link`.

ITrainer Interface

```
public interface Imsl.DataMining.Neural.ITrainer
```

Interface implemented by classes used to train an `Network`.

The method `Train` is used to adjust the Weights (p. 1666) in a network to best fit a set of observed data. After a `Network` is trained, the other methods in this interface can be used to check the quality of the fit.

Properties

ErrorGradient

```
abstract public double[] ErrorGradient {get; }
```

Description

The value of the gradient of the error function with respect to the Weights (p. 1666).

Property Value

A `double` array, the length of the number of *weights*, containing the value of the *gradient* of the error function with respect to the *weights* at the computed optimal point.

Remarks

Before training, `null` is returned.

ErrorStatus

```
abstract public int ErrorStatus {get; }
```

Description

The error status.

Property Value

An `int` specifying the error. If there was no error, zero is returned.

Remarks

A non-zero return indicates a potential problem with the trainer.

ErrorValue

```
abstract public double ErrorValue {get; }
```

Description

The value of the error function minimized by the trainer.

Property Value

A `double` indicating the final value of the error function from the last training.

Remarks

Before training, `NaN` is returned.

Method

Train

```
abstract public void Train(Imsl.DataMining.Neural.Network network, double[,]
xData, double[,] yData)
```

Description

Trains the neural network using supplied training patterns.

Parameters

`network` – A `Network` object, which is the `Network` to be trained.

`xData` – A `double` matrix containing the input training patterns.

`yData` – A `double` matrix containing the output training patterns.

Remarks

The number of columns in `xData` must equal the number of nodes in the `?InputLayer`. Each row of `xData` contains a training pattern.

The number of columns in `yData` must equal the number of `Perceptrons` in the `OutputLayer`. Each row of `yData` contains a training pattern.

QuasiNewtonTrainer Class

```
public class Imsl.DataMining.Neural.QuasiNewtonTrainer :
Imsl.DataMining.Neural.ITrainer, ICloneable
```

Trains a `Network` using the quasi-Newton method, `MinUnconMultiVar`.

See Also

Imsl.Math.MinUnconMultiVar (p. 344)

Field

SUM_OF_SQUARES

```
public Imsl.DataMining.Neural.QuasiNewtonTrainer.IError SUM_OF_SQUARES
```

Description

Compute the sum of squares error.

Remarks

The sum of squares error term is $e(y,\hat{y}) = (y-\hat{y})^2/2$.

This is the default `IError` object used by `QuasiNewtonTrainer`.

Properties

EpochNumber

```
virtual protected internal int EpochNumber {set; }
```

Description

The epoch number for the trainer.

Property Value

An `int` array containing the epoch number.

Error

```
virtual public Imsl.DataMining.Neural.QuasiNewtonTrainer.IError Error {get;
set; }
```

Description

The error function used by the trainer.

Property Value

An `IError` object representing the error function used by the trainer.

ErrorGradient

```
virtual public double[] ErrorGradient {get; }
```

Description

The value of the gradient of the error function with respect to the Weights (p. 1666).

Property Value

A `double` array whose length is equal to the number of network `Weights`, containing the value of the gradient of the error function with respect to the `Weights`.

Remarks

Before training, `null` is returned.

ErrorStatus

```
virtual public int ErrorStatus {get; }
```

Description

The error status from the trainer.

Property Value

An `int` representing the error status from the trainer.

Remarks

Zero indicates that no errors were encountered during training. Any non-zero value indicates that some error condition arose during training. In many cases the trainer is able to recover from these conditions and produce a well-trained network.

Error Status	Condition
0	No error occurred during training.
1	The last global step failed to locate a lower point than the current error value. The current solution may be an approximate solution and no more accuracy is possible, or the step tolerance may be too large.
2	Relative function convergence; both the actual and predicted relative reductions in the error function are less than or equal to the relative function convergence tolerance.
3	Scaled step tolerance satisfied; the current point may be an approximate local solution, or the algorithm is making very slow progress and is not near a solution, or the step tolerance is too big.
4	Optimizer threw a `FalseConvergenceException`.
5	Optimizer threw a `MaxIterationsException`.
6	Optimizer threw a `UnboundedBelowException`.

See Also: Imsl.Math.FalseConvergenceException (p. 1822), Imsl.Math.MaxIterationsException (p. 1831), Imsl.Math.UnboundedBelowException (p. 1873)

ErrorValue

```
virtual public double ErrorValue {get; }
```

Description

The final value of the error function.

Property Value

A `double` representing the final value of the error function from the last training.

Remarks

Before training, `NaN` is returned.

GradientTolerance

```
virtual public double GradientTolerance {get; set; }
```

Description

The gradient tolerance.

Property Value

A `double` specifying the gradient tolerance.

Remarks

By default, `GradientTolerance` equals the cube root of machine precision.

See Also: Imsl.Math.MinUnconMultiVar.GradientTolerance (p. 346)

MaximumStepsize

```
virtual public double MaximumStepsize {get; set; }
```

Description

The maximum step size.

Property Value

A nonnegative `double` value specifying the maximum allowable step size in the optimizer.

Remarks

The value of `MaximumStepsize` will be equal to -999.0 if the default value is to be used and the `Train` method has not been called.

See Also: (p. 347)

MaximumTrainingIterations

```
virtual public int MaximumTrainingIterations {get; set; }
```

Description

The maximum number of iterations to use in a training.

Property Value

An `int` representing the maximum number of training iterations.

Remarks

By default, `MaximumTrainingIterations` = 100.

See Also: (p. 347)

NumberOfProcessors

```
public int NumberOfProcessors {get; set; }
```

Description

Perform the parallel calculations with the maximum possible number of processors set to `NumberOfProcessors`.

Property Value

An `int` indicating the maximum possible number of processors to use.

Remarks

By default, `NumberOfProcessors` = `Environment.ProcessorCount`. If `NumberOfProcessors` is set to a number less than 1 or greater than `Environment.ProcessorCount`, `Environment.ProcessorCount` will be used internally. This property has no effect if the application is used with a scalar version of the IMSL C# Numerical Library.

ParallelMode

```
virtual protected internal System.Collections.ArrayList[] ParallelMode {set; }
```

Description

The trainer to be used in multi-threaded EpochTainer.

Property Value

An `ArrayList` array containing the log records.

StepTolerance

```
virtual public double StepTolerance {get; set; }
```

Description

The scaled step tolerance.

Property Value

A `double` which is the step tolerance.

Remarks

The second stopping criterion for MinUnconMultiVar (p. 344), the optimizer used by this `ITrainer` , is that the scaled distance between the last two steps be less than the step tolerance.

by default, `StepTolerance` = 3.66685e-11.

See Also: Imsl.Math.MinUnconMultiVar.StepTolerance (p. 348)

TrainingIterations

```
virtual public int TrainingIterations {get; }
```

Description

The number of iterations used during training.

Property Value

An `int` representing the number of iterations used during training.

See Also: (p. 347)

UseBackPropagation

```
virtual public bool UseBackPropagation {get; set; }
```

Description

Specify the use of the back propagation algorithm for gradient calculations during network training.

Property Value

A `bool` specifying whether or not back propagation is used for gradient calculations. By default, `UseBackPropagation` = `true`.

Remarks

By default, the quasi-newton algorithm optimizes the network using numerical gradients. This method directs the quasi-newton trainer to use the back propagation algorithm for gradient calculations during network training. Depending upon the data and network architecture, one approach is typically faster than the other, or is less sensitive to finding local network optima.

Constructor

QuasiNewtonTrainer

```
public QuasiNewtonTrainer()
```

Description

Constructs a `QuasiNewtonTrainer` object.

Methods

Clone

```
virtual public object Clone()
```

Description

Clones a copy of the trainer.

GetError

```
virtual public Imsl.DataMining.Neural.QuasiNewtonTrainer.IError GetError()
```

Description

Returns the function used to compute the error to be minimized.

Returns

The `IError` object containing the function to be minimized.

SetError

```
virtual public void SetError(Imsl.DataMining.Neural.QuasiNewtonTrainer.IError
error)
```

Description

Sets the function that computes the network error.

Parameter

`error` – The `IError` object containing the function to be used to compute the network error.

Remarks

The default is to compute the sum of squares error, `SUM_OF_SQUARES`.

Train

```
virtual public void Train(Imsl.DataMining.Neural.Network network, double[,]
xData, double[,] yData)
```

Description

Trains the neural network using supplied training patterns.

Parameters

`network` – The `Network` to be trained.

`xData` – An input `double` matrix containing training patterns.

`yData` – An output `double` matrix containing output training patterns.

Remarks

The number of columns in `xData` must equal the number of `Nodes` in the `InputLayer`.

The number of columns in `yData` must equal the number of `Perceptrons` in the `OutputLayer`.

Each row of `xData` and `yData` contains a training pattern. The number of rows in these two arrays must be at least equal to the number of *weights* in the `Network`.

QuasiNewtonTrainer.IError Interface

`public interface Imsl.DataMining.Neural.QuasiNewtonTrainer.IError`

Error function to be minimized by trainer.

This trainer attempts to solve the problem

$$\min_{w} \sum_{i=0}^{n-1} e(y_i, \hat{y}_i)$$

where w are the weights, n is the number of training patterns, y_i is a training target output and $\hat{y}_i$ is its forecast value.

This interface defines the function $e(y, \hat{y})$ and its derivative with respect to its computed value, $de/d\hat{y}$.

Methods

Error

`abstract public double Error(double[] computed, double[] expected)`

Description

The contribution to the error from a single training output target. This is the function $e(y_i, \hat{y}_i)$.

Parameters

`computed` – A `double` representing the computed value.

`expected` – A `double` representing the expected value.

Returns

A `double` representing the contribution to the error from a single training output target.

ErrorGradient

```
abstract public double[] ErrorGradient(double[] computed, double[] expected)
```

Description

The derivative of the error function with respect to the forecast output.

Parameters

`computed` – A `double` representing the computed value.

`expected` – A `double` representing the expected value.

Returns

A `double` representing the derivative of the error function with respect to the forecast output.

LeastSquaresTrainer Class

```
public class Imsl.DataMining.Neural.LeastSquaresTrainer :
Imsl.DataMining.Neural.ITrainer
```

Trains a `FeedForwardNetwork` using a Levenberg-Marquardt algorithm for minimizing a sum of squares error.

See Also

Feed Forward Class Example 1 , NonlinLeastSquares (p. 353)

Properties

EpochNumber

```
virtual protected internal int EpochNumber {set; }
```

Description

The epoch number for the trainer.

Property Value

An `int` array containing the epoch number.

ErrorGradient

```
virtual public double[] ErrorGradient {get; }
```

Description

The value of the *gradient* of the error function with respect to the Weights (p. 1666).

Property Value

A `double` array whose length is equal to the number of network `Weights`, containing the value of the *gradient* of the error function with respect to the `Weights`.

Remarks

Before training, `null` is returned.

ErrorStatus

```
virtual public int ErrorStatus {get; }
```

Description

The error status from the trainer.

Property Value

An `int` which contains the error status.

Remarks

Zero indicates that no errors were encountered during training. Any non-zero value indicates that some error condition arose during training.

In many cases the trainer is able to recover from these conditions and produce a well-trained network.

Value	Meaning
0	All convergence tests were met.
1	Scaled step tolerance was satisfied. The current point may be an approximate local solution, or the algorithm is making very slow progress and is not near a solution, or `StepTolerance` is too big.
2	Scaled actual and predicted reductions in the function are less than or equal to the relative function convergence tolerance `RelativeTolerance`.
3	Iterates appear to be converging to a noncritical point. Incorrect gradient information, a discontinuous function, or stopping tolerances being too tight may be the cause.
4	Five consecutive steps with the maximum stepsize have been taken. Either the function is unbounded below, or has a finite asymptote in some direction, or the maximum stepsize is too small.
5	Too many iterations required.

ErrorValue

```
virtual public double ErrorValue {get; }
```

Description

The final value of the error function.

Property Value

A `double` containing the final value of the error function from the last training.

Remarks

Before training, `NaN` is returned.

FalseConvergenceTolerance

```
virtual public double FalseConvergenceTolerance {get; set; }
```

Description

The false convergence tolerance.

Property Value

A `double` specifying the false convergence tolerance.

Remarks

by default, `FalseConvergenceTolerance` = 1.0e-14.

See Also: NonlinLeastSquares.FalseConvergenceTolerance (p. 355)

GradientTolerance

```
virtual public double GradientTolerance {get; set; }
```

Description

The *gradient* tolerance.

Property Value

A `double` specifying the *gradient* tolerance.

Remarks

By default, `GradientTolerance` = 2.0e-5.

See Also: NonlinLeastSquares.GradientTolerance (p. 355)

InitialTrustRegion

```
virtual public double InitialTrustRegion {get; set; }
```

Description

The initial trust region.

Property Value

A `double` which specifies the initial trust region radius.

Remarks

By default, `InitialTrustRegion` = -999.0, an unlimited trust region.

The value of `InitialTrustRegion` will be equal to -999.0 if the default value is to be used and the Train method has not been called.

See Also: NonlinLeastSquares.InitialTrustRegion (p. 356)

MaximumStepsize

```
virtual public double MaximumStepsize {get; set; }
```

Description

The maximum step size.

Property Value

A nonnegative `double` value specifying the maximum allowable stepsize in the optimizer.

Remarks

By default, `MaximumStepsize` = $10^3||w||_2$, where w are the values of the Weights (p. 1666) in the network when training starts.

The value of `MaximumStepsize` will be equal to -999.0 if the default value is to be used and the Train method has not been called.

See Also: NonlinLeastSquares.MaximumStepsize (p. 356)

MaximumTrainingIterations

```
virtual public int MaximumTrainingIterations {get; set; }
```

Description

The maximum number of iterations used by the nonlinear least squares solver.

Property Value

An `int` which specifies the maximum number of iterations to be used by the nonlinear least squares solver.

Remarks

Its default value is 1000.

See Also: NonlinLeastSquares.RelativeTolerance (p. 357)

ParallelMode

```
virtual protected internal System.Collections.ArrayList[] ParallelMode {set; }
```

Description

The trainer to be used in multi-threaded EpochTainer.

Property Value

An `ArrayList` array containing the log records.

RelativeTolerance

```
virtual public double RelativeTolerance {get; set; }
```

Description

The relative tolerance.

Property Value

A `double` which specifies the elative error tolerance.

Remarks

It must be in the interval [0,1]. Its default value is 1.0e-20.

See Also: NonlinLeastSquares.RelativeTolerance (p. 357)

StepTolerance

```
virtual public double StepTolerance {get; set; }
```

Description

The step tolerance used to step between Weights (p. 1666).

Property Value

A `double` which specifies the scaled step tolerance to use when changing the `weights`.

Remarks

By default, `StepTolerance` = 1.0e-5.

See Also: NonlinLeastSquares.StepTolerance (p. 358)

Constructor

LeastSquaresTrainer

```
public LeastSquaresTrainer()
```

Description

Creates a `LeastSquaresTrainer`.

Method

Train

```
virtual public void Train(Imsl.DataMining.Neural.Network network, double[,]
xData, double[,] yData)
```

Description

Trains the neural network using supplied training patterns.

Parameters

`network` – The `Network` to be trained.

`xData` – A `double` matrix which contains the input training patterns. The number of columns in `xData` must equal the number of `Nodes` in the `InputLayer`.

`yData` – A `double` matrix which contains the output training patterns. The number of columns in `yData` must equal the number of `Perceptrons` in the `OutputLayer`.

Remarks

Each row of `xData` and `yData` contains a training pattern. These number of rows in two arrays must be equal.

EpochTrainer Class

```
public class Imsl.DataMining.Neural.EpochTrainer :
Imsl.DataMining.Neural.ITrainer
```

Two-stage training using randomly selected training patterns in stage I.

The `EpochTrainer`, is a meta-trainer that combines two trainers. The first trainer is used on a series of randomly selected subsets of the training patterns. For each subset, the Weights (p. 1788) are initialized to their initial values plus a random offset.

Stage II then refines the result found in stage I. The best result from the stage I trainings is used as the initial guess with the second trainer operating on the full set of training patterns. Stage II is optional, if the second trainer is `null` then the best stage I result is returned as the `EpochTrainer`'s result.

Properties

EpochSize

```
virtual public int EpochSize {get; set; }
```

Description

The number of randomly selected training patterns in each stage I epoch.

Property Value

An `int` which contains the number of sample training patterns in each stage I epoch. The default value is the number of observations in the training data.

ErrorGradient

```
virtual public double[] ErrorGradient {get; }
```

Description

The value of the *gradient* of the error function with respect to the Weights (p. 1788).

Property Value

A `double` array whose length is equal to the number of `Network.Weights`, containing the value of the *gradient* of the error function with respect to the *weights*.

Remarks

Before training, `null` is returned.

ErrorStatus

```
virtual public int ErrorStatus {get; }
```

Description

The training error status.

Property Value

An `int` containing the error status from stage II.

Remarks

If there is no stage II then the number of stage I epochs that returned a non-zero error status is returned.

ErrorValue

```
virtual public double ErrorValue {get; }
```

Description

The value of the error function.

Property Value

A `double` containing final value of the error function from the last training. Before training, `NaN` is returned.

NumberOfEpochs

```
virtual public int NumberOfEpochs {get; set; }
```

Description

The number of epochs used during stage I training.

Property Value

An `int` which contains the number of epochs used during stage I training. The default value is 10.

Random

```
virtual public Imsl.Stat.Random Random {get; set; }
```

Description

The random number generator used to perturb the stage I guesses.

Property Value

The `Random` object used to generate stage I perturbations.

Stage1Trainer

```
virtual protected internal Imsl.DataMining.Neural.ITrainer Stage1Trainer {get;
}
```

Description

The stage 1 trainer.

Property Value

A `Trainer` containing the stage 1 trainer.

Stage2Trainer

```
virtual protected internal Imsl.DataMining.Neural.ITrainer Stage2Trainer {get;
}
```

Description

The stage 1 trainer.

Property Value

A `Trainer` containing the stage 2 trainer.

Constructors

EpochTrainer

```
public EpochTrainer(Imsl.DataMining.Neural.ITrainer stage1Trainer)
```

Description

Creates a single stage `EpochTrainer`. Stage II training is bypassed.

Parameter

`stage1Trainer` – The `ITrainer` used in stage I.

EpochTrainer

```
public EpochTrainer(Imsl.DataMining.Neural.ITrainer stage1Trainer,
Imsl.DataMining.Neural.ITrainer stage2Trainer)
```

Description

Creates a two-stage `EpochTrainer`.

Parameters

`stage1Trainer` – The stage I `ITrainer`.

`stage2Trainer` – The stage II `ITrainer`, or `null` if stage II is to be bypassed.

Methods

SetRandomSamples

```
virtual public void SetRandomSamples(Imsl.Stat.Random randomA, Imsl.Stat.Random
randomB)
```

Description

Sets the random number generators used to select random training patterns in stage I.

Parameters

`randomA` – A `Random` object which is the first random number generator.

`randomB` – A `Random` object which is the second random number generator, independent of `randomA`.

Remarks

The two random number generators should be independent.

Train

```
virtual public void Train(Imsl.DataMining.Neural.Network network, double[,]
xData, double[,] yData)
```

Description

Trains the neural network using supplied training patterns.

Parameters

`network` – The `Network` to be trained.

`xData` – A `double` matrix specifying the input training patterns. The number of columns in `xData` must equal the number of `Nodes` in the `InputLayer`.

`yData` – A `double` containing the output training patterns. The number of columns in `yData` must equal the number of `Perceptrons` in the `OutputLayer`.

Remarks

Each row of `xData` and `yData` contains a training pattern. These number of rows in two arrays must be equal.

BinaryClassification Class

```
public class Imsl.DataMining.Neural.BinaryClassification
```

Classifies patterns into two classes.

Uses a `FeedForwardNetwork` to solve binary classification problems. In these problems, the target output for the network is the probability that the pattern falls into one of two classes. The first class,

$P(C_1)$, is usually equal to one and the second class, $P(C_2)$ equal to zero. These probabilities are then used to assign patterns to one of the two classes. Typical applications include determining whether a credit applicant is a good or bad credit risk, and determining whether a person should or should not receive a particular treatment based upon their physical, clinical and laboratory information. This class signals that network training will minimize the binary cross-entropy error, and that network output is the probability that the pattern belongs to the first class, $P(C_1)$. Which is calculated by applying the logistic activation function to the potential of the single output. The probability for the second class is calculated by $P(C_2) = 1 - P(C_1)$.

Properties

Error

```
virtual public Imsl.DataMining.Neural.QuasiNewtonTrainer.IError Error {get; }
```

Description

Returns the error function for use by `QuasiNewtonTrainer` for training a binary classification network.

Property Value

An implementation of the binary-entropy error function.

Network

```
virtual public Imsl.DataMining.Neural.Network Network {get; }
```

Description

The network being used for classification.

Property Value

The network set by the constructor.

Constructor

BinaryClassification

```
public BinaryClassification(Imsl.DataMining.Neural.Network network)
```

Description

Creates a binary classifier.

Parameter

`network` – Is the neural network used for classification. Its `OutputPerceptron` should use the logistic activation function, `Imsl.DataMining.Neural.Activation.Logistic`.

Methods

ComputeStatistics

```
virtual public double[] ComputeStatistics(double[,] xData, int[] yData)
```

Description

Computes the classification error statistics for the supplied network patterns and their associated classifications.

Parameters

`xData` – A `double` matrix specifying the input training patterns. The number of columns in `xData` must equal the number of `Nodes` in the `InputLayer`.

`yData` – An `int` array containing the output classification patterns. The values in `yData` must be 0 or 1.

Returns

A two-element `double` array containing the binary cross-entropy error and the classification error rate.

Remarks

The first element returned is the binary cross-entropy error; the second is the classification error rate. The classification error rate is calculated by comparing the estimated classification probabilities to the target classifications. If the estimated probability for the target class is less than 0.5, then this is tallied as a classification error.

PredictedClass

```
virtual public int PredictedClass(double[] x)
```

Description

Calculates the classification probablities for the input pattern `x`, and returns either 0 or 1 identifying the class with the highest probability.

Parameter

`x` – The `double` array containing the network input patterns to classify. The length of `x` should be equal to the number of inputs in the network.

Returns

The classification predicted by the trained network for `x`. This will be either 0 or 1.

Remarks

This method is used to classify patterns into one of the two target classes based upon the pattern's values. The predicted classification is the class with the largest probability, i.e. greater than 0.5.

Probabilities

```
virtual public double[] Probabilities(double[] x)
```

Description

Returns classification probabilities for the input pattern `x`.

Parameter

x – A `double` array containing the network input pattern to classify. The length of `x` must equal the number of nodes in the input layer.

Returns

The probability of `x` being in class C_1, followed by the probability of `x` being in class C_2.

Remarks

Calculates the two probabilities for the pattern supplied: $P(C_1)$ and $P(C_2)$. The probability that the pattern belongs to the first class, $P(C_1)$, is estimated using the logistic function of the `OutputPerceptron`'s potential. The probability for the second class is calculated as $P(C_2) = 1 - P(C_1)$. The predicted classification is the class with the largest probability, i.e. greater than 0.5.

Train

```
virtual public void Train(Imsl.DataMining.Neural.ITrainer trainer, double[,]
xData, int[] yData)
```

Description

Trains the classification neural network using supplied trainer and patterns.

Parameters

`trainer` – A `Trainer` object, which is used to train the network. The error function in any `QuasiNewton` trainer included in `trainer` should be set to the error function from this class using the `Error` method provided by this class.

`xData` – A `double` matrix containing the input training patterns. The number of columns in `xData` must equal the number of nodes in the input layer. Each row of `xData` contains a training pattern.

`yData` – An `int` array containing the output classification values. These values must be 0 or 1.

Example 1: Binary Classification

This example trains a 3-layer network using 48 training patterns from four nominal input attributes. The first two nominal attributes have two classifications. The third and fourth nominal attributes have three and four classifications respectively. All four attributes are encoded using binary encoding. This results in eleven binary network input columns. The output class is 1 if the first two nominal attributes sum to 1, and 0 otherwise.

The structure of the network consists of eleven input nodes and three layers, with three perceptrons in the first hidden layer, two perceptrons in the second hidden layer, and one perceptron in the output layer.

There are a total of 47 weights in this network, including the six bias weights. The linear activation function is used for both hidden layers. Since the target output is binary classification the logistic activation function is used in the output layer. Training is conducted using the quasi-newton trainer with the binary-entropy error function provided by the `BinaryClassification` class.

```
using System;
using Imsl.DataMining.Neural;
using Random = Imsl.Stat.Random;
using System.Runtime.Serialization;
```

```
using System.Runtime.Serialization.Formatters.Binary;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

//***************************************************************************
// Two Layer Feed-Forward Network with 11 inputs: 4 nominal with 2,2,3,4
// categories, encoded using binary encoding,  and 1 output target (class).
//
//  new classification training_ex1.c
//***************************************************************************

[Serializable]
public class BinaryClassificationEx1
{

   // Network Settings
   private static int nObs = 48; // number of training patterns
   private static int nInputs = 11; // four nominal with 2,2,3,4 categories
   private static int nCategorical = 11; // three categorical attributes
   private static int nOutputs = 1; // one continuous output (nClasses=2)
   private static int nPerceptrons1 = 3; // perceptrons in 1st hidden layer
   private static int nPerceptrons2 = 2; // perceptrons in 2nd hidden layer

   private static IActivation hiddenLayerActivation =
      Imsl.DataMining.Neural.Activation.Linear;
   private static IActivation outputLayerActivation =
      Imsl.DataMining.Neural.Activation.Logistic;

   /* 2 classifications */
   private static int[] x1 = new int[]{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
      1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
      2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2};

   /* 2 classifications */
   private static int[] x2 = new int[]{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2,
      2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2,
      2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2};

   /* 3 classifications */
   private static int[] x3 = new int[]{1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 1,
      1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 1,
      1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3};

   /* 4 classifications */
   private static int[] x4 = new int[]{1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1,
      2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4,
      2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4};

   // ************************************************************************
   // MAIN
   // ************************************************************************


   public static void  Main(System.String[] args)
   {
      double[,] xData; // Input  Attributes for Trainer
```

```
int[] yData; // Output Attributes for Trainer
int i, j; // array indicies


// ****************************************************************
// Binary encode 4 categorical variables.
//         Var x1 contains 2 classes
//         Var x2 contains 2 classes
//         Var x3 contains 3 classes
//         Var x4 contains 4 classes
// ****************************************************************
int[,] z1;
int[,] z2;
int[,] z3;
int[,] z4;
UnsupervisedNominalFilter filter = new UnsupervisedNominalFilter(2);
z1 = filter.Encode(x1);
z2 = filter.Encode(x2);
filter = new UnsupervisedNominalFilter(3);
z3 = filter.Encode(x3);
filter = new UnsupervisedNominalFilter(4);
z4 = filter.Encode(x4);

/* Concatenate binary encoded z's */
xData = new double[nObs,nInputs];
yData = new int[nObs];
for (i = 0; i < (nObs); i++)
{
   for (j = 0; j < nCategorical; j++)
   {
      xData[i,j] = 0;
      if (j < 2)
         xData[i,j] = (double) z1[i,j];
      if (j > 1 && j < 4)
         xData[i,j] = (double) z2[i,j - 2];
      if (j > 3 && j < 7)
         xData[i,j] = (double) z3[i,j - 4];
      if (j > 6)
         xData[i,j] = (double) z4[i,j - 7];
   }
   yData[i] = ((x1[i] + x2[i] == 2)?1:0);
}

// ********************************************************************
// CREATE FEEDFORWARD NETWORK
// ********************************************************************
long t0 = (System.DateTime.Now.Ticks - 621355968000000000) / 10000;

FeedForwardNetwork network = new FeedForwardNetwork();
network.InputLayer.CreateInputs(nInputs);
network.CreateHiddenLayer().CreatePerceptrons(nPerceptrons1);
network.CreateHiddenLayer().CreatePerceptrons(nPerceptrons2);
network.OutputLayer.CreatePerceptrons(nOutputs);

BinaryClassification classification = new BinaryClassification(network);
```

```
network.LinkAll();
System.Random r = new System.Random(123457);
network.SetRandomWeights(xData, r);
Perceptron[] perceptrons = network.Perceptrons;
for (i = 0; i < perceptrons.Length - 1; i++)
{
   perceptrons[i].Activation = hiddenLayerActivation;
}
perceptrons[perceptrons.Length - 1].Activation = outputLayerActivation;


// ************************************************************************
// TRAIN NETWORK USING QUASI-NEWTON TRAINER
// ************************************************************************
QuasiNewtonTrainer trainer = new QuasiNewtonTrainer();
trainer.Error = classification.Error;
trainer.MaximumTrainingIterations = 1000;
trainer.MaximumStepsize = 3.0;
trainer.GradientTolerance = 1.0e-20;
trainer.StepTolerance = 1.0e-20;

classification.Train(trainer, xData, yData);

// ************************************************************************
// DISPLAY TRAINING STATISTICS
// ************************************************************************
double[] stats = classification.ComputeStatistics(xData, yData);
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("--> Cross-entropy error:        " +
   (float)stats[0]);
System.Console.Out.WriteLine("--> Classification error rate:  " +
   (float)stats[1]);
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("");
// ************************************************************************
// OBTAIN AND DISPLAY NETWORK WEIGHTS AND GRADIENTS
// ************************************************************************
double[] weight = network.Weights;
double[] gradient = trainer.ErrorGradient;
double[,] wg = new double[weight.Length,2];
for (i = 0; i < weight.Length; i++)
{
   wg[i,0] = weight[i];
   wg[i,1] = gradient[i];
}
PrintMatrixFormat pmf = new PrintMatrixFormat();
pmf.SetColumnLabels(new System.String[]{"Weights", "Gradients"});
new PrintMatrix().Print(pmf, wg);

//  **************************
//     forecast the network
//  **************************
double[,] report = new double[nObs,6];
for (i = 0; i < nObs; i++)
```

```
      {
         report[i,0] = x1[i];
         report[i,1] = x2[i];
         report[i,2] = x3[i];
         report[i,3] = x4[i];
         report[i,4] = yData[i];
            double[] tmp = new double[xData.GetLength(1)];
            for ( j=0; j<xData.GetLength(1); j++)
                tmp[j] = xData[i,j];
         report[i,5] = classification.PredictedClass(tmp);
      }
      pmf = new PrintMatrixFormat();
      pmf.SetColumnLabels( new System.String[]{"X1", "X2", "X3", "X4",
         "Expected", "Predicted"});
      new PrintMatrix("Forecast").Print(pmf, report);


      // ************************************************************************
      // DISPLAY CLASSIFICATION STATISTICS
      // ************************************************************************
      double[] statsClass = classification.ComputeStatistics(xData, yData);
      // Display Network Errors
      System.Console.Out.WriteLine(
         "*******************************************");
      System.Console.Out.WriteLine("--> Cross-Entropy Error:      " +
         (float)statsClass[0]);
      System.Console.Out.WriteLine("--> Classification Error:     " +
         (float)statsClass[1]);
      System.Console.Out.WriteLine(
         "*******************************************");
      System.Console.Out.WriteLine("");


      long t1 = (System.DateTime.Now.Ticks - 621355968000000000) / 10000;
      double time = t1 - t0;
      time = time / 1000;
      System.Console.Out.WriteLine("***************Time:  " + time);
      System.Console.Out.WriteLine("trainer.getErrorValue = " +
         trainer.ErrorValue);
   }
}
```

Output

```
*********************************************
--> Cross-entropy error:        4.720552E-10
--> Classification error rate:  0
*********************************************

          Weights              Gradients
 0   1.21627824426795    -1.82357413516947E-12
 1  -7.10104582036496     4.00527512119816E-13
 2  -4.48633964224764     9.39571675393506E-13
 3  -2.60725959847449     4.68033472563643E-09
 4   4.29051046748516    -1.0279827879731E-09
```

5	4.48156618132245	-2.41147856550398E-09
6	2.08243325149059	4.67851114994447E-09
7	-6.86920995138462	-1.02758226011905E-09
8	-4.71797133796275	-2.41053899302647E-09
9	-3.15604455817482	1.55679543755327E-18
10	6.63883077984471	-3.41932577051355E-19
11	4.87196888567896	-8.02117593848719E-19
12	-2.30032598691625	-1.82357569033727E-12
13	-1.68298220558949	4.00527853694902E-13
14	1.57459851986335	9.39572476672484E-13
15	-0.264451161585171	5.95159047420712E-10
16	-0.649412990130757	-1.30719978958719E-10
17	-0.124557044208244	-3.06647573315529E-10
18	0.125744106229942	4.08517567977089E-09
19	-0.550795793591294	-8.97262809355956E-10
20	0.213785518241144	-2.10483099298973E-09
21	-0.256460169762842	1.46860577618452E-15
22	0.719269734081326	-3.22562711586887E-16
23	0.181431096608458	-7.56679074905849E-16
24	0.708887793361652	4.98153020814716E-10
25	-0.13780872548447	-1.09413698206104E-10
26	-0.0714270451586269	-2.56666542556397E-10
27	-1.69080392381724	-1.82357569087194E-12
28	-1.05894442754157	4.00527853812335E-13
29	0.916792419032194	9.39572476947963E-13
30	0.8484018617882	4.18218023777164E-09
31	0.809968676185521	-9.18568767545977E-10
32	-0.621014362842385	-2.15481126707006E-09
33	3.92683360366486	-1.53010895369397E-09
34	3.92683360375881	-1.53010895423418E-09
35	-0.862484756532178	6.25095768828527E-10
36	-0.862484756327281	6.2509576904922E-10
37	-2.02324740024013	-1.05620718555491E-09
38	-2.02324740013514	-1.05620718592781E-09
39	1.26293423975155	-5.17748567942779E-09
40	1.26293424019744	-5.17748567954339E-09
41	-2.95381709009598	4.67851115150126E-09
42	3.4767065441968	-1.02758226046098E-09
43	-1.14723052699379	-2.41053899382858E-09
44	-3.57249695785368	5.9571038944113E-10
45	-3.57249695788631	5.95710389651447E-10
46	5.5108803426636	4.71687575402437E-10

					Forecast	
	X1	X2	X3	X4	Expected	Predicted
0	1	1	1	1	1	1
1	1	1	1	2	1	1
2	1	1	1	3	1	1
3	1	1	1	4	1	1
4	1	1	2	1	1	1
5	1	1	2	2	1	1
6	1	1	2	3	1	1
7	1	1	2	4	1	1
8	1	1	3	1	1	1
9	1	1	3	2	1	1
10	1	1	3	3	1	1

```
11  1   1   3   4       1           1
12  1   2   1   1       0           0
13  1   2   1   2       0           0
14  1   2   1   3       0           0
15  1   2   1   4       0           0
16  1   2   2   1       0           0
17  1   2   2   2       0           0
18  1   2   2   3       0           0
19  1   2   2   4       0           0
20  1   2   3   1       0           0
21  1   2   3   2       0           0
22  1   2   3   3       0           0
23  1   2   3   4       0           0
24  2   1   1   1       0           0
25  2   1   1   2       0           0
26  2   1   1   3       0           0
27  2   1   1   4       0           0
28  2   1   2   1       0           0
29  2   1   2   2       0           0
30  2   1   2   3       0           0
31  2   1   2   4       0           0
32  2   1   3   1       0           0
33  2   1   3   2       0           0
34  2   1   3   3       0           0
35  2   1   3   4       0           0
36  2   2   1   1       0           0
37  2   2   1   2       0           0
38  2   2   1   3       0           0
39  2   2   1   4       0           0
40  2   2   2   1       0           0
41  2   2   2   2       0           0
42  2   2   2   3       0           0
43  2   2   2   4       0           0
44  2   2   3   1       0           0
45  2   2   3   2       0           0
46  2   2   3   3       0           0
47  2   2   3   4       0           0

*********************************************
--> Cross-Entropy Error:      4.720552E-10
--> Classification Error:     0
*********************************************

***************Time:  0.343
trainer.getErrorValue = 4.72055172799E-10
```

Example 2: Binary Classification Network

This example uses a database of a complete set of possible board configurations at the end of tic-tac-toe games, where “x” is assumed to have played first. The target concept is “win for x” (i.e., true when “x” has one of 8 possible ways to create a “three-in-a-row”).

There are nine nominal input attributes for each square on the tic-tac-toe board and are encoded such that 0=player x has taken, 1=player o has taken, 2=blank.

Input attributes

1. top-left-square: {x,o,b}
2. top-middle-square: {x,o,b}
3. top-right-square: {x,o,b}
4. middle-left-square: {x,o,b}
5. middle-middle-square: {x,o,b}
6. middle-right-square: {x,o,b}
7. bottom-left-square: {x,o,b}
8. bottom-middle-square: {x,o,b}
9. bottom-right-square: {x,o,b}

The predicted atribute is a win or lose at tic-tac-toe. For this example the first 626 observations are a win and the next 332 are loss.

The structure of the network consists of 27 input nodes and three layers, with five perceptrons in the first hidden layer, three perceptrons in the second hidden layer, and one perceptron in the output layer.

There are a total of 162 weights in this network. The activations functions are logistic for all layers. Since the target output is binary classification the logistic activation function must be used in the output layer. Training is conducted using the quasi-newton trainer using the binary entropy error function provided by the `BinaryClassification` class.

```
using System;
using Imsl.DataMining.Neural;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;
using Random = Imsl.Stat.Random;

//*****************************************************************************
// Three Layer Feed-Forward Network with 4 inputs, all
// continuous, and 2 classification categories.
//
//  new classification training_ex4.c
//
// Three Layer Feed-Forward Network with 4 inputs, all
// continuous, and 2 classification categories.
//
//  This database encodes the complete set of possible board configurations
//   at the end of tic-tac-toe games, where "x" is assumed to have played
//   first.  The target concept is "win for x" (i.e., true when "x" has one
//   of 8 possible ways to create a "three-in-a-row").
//
//  Predicted attribute: win or loose at tic-tac-toe
//     First 626 obs are positive (win) and the next 332 are negative (loss)
//
```

```
//  Input Attributes (10 categorical Attributes)
//     Attribute Information: (0=player x has taken, 1=player o has taken, 2=blank)
//
//    1. top-left-square: {x,o,b}
//    2. top-middle-square: {x,o,b}
//    3. top-right-square: {x,o,b}
//    4. middle-left-square: {x,o,b}
//    5. middle-middle-square: {x,o,b}
//    6. middle-right-square: {x,o,b}
//    7. bottom-left-square: {x,o,b}
//    8. bottom-middle-square: {x,o,b}
//    9. bottom-right-square: {x,o,b}
//   10. Class: {positive,negative}
//
//*****************************************************************************

[Serializable]
public class BinaryClassificationEx2
{
   private static int nObs = 958; // number of training patterns
   private static int nInputs = 27; // 9 nominal coded as 0=x, 1=O, 2=blank
   private static int nOutputs = 1; // one continuous output (nClasses=2)
   private static int nPerceptrons1 = 5; // perceptrons in 1st hidden layer
   private static int nPerceptrons2 = 3; // perceptrons in 2nd hidden layer

   private static IActivation hiddenLayerActivation =
      Imsl.DataMining.Neural.Activation.Logistic;
   private static IActivation outputLayerActivation =
      Imsl.DataMining.Neural.Activation.Logistic;

   private static int[][] data = new int[][]{new int[]{0, 0, 0, 0, 1, 1, 0, 1, 1},
      new int[]{0, 0, 0, 0, 1, 1, 1, 0, 1}, new int[]{0, 0, 0, 0, 1, 1, 1, 1, 0},
      new int[]{0, 0, 0, 0, 1, 1, 1, 2, 2}, new int[]{0, 0, 0, 0, 1, 1, 2, 1, 2},
      new int[]{0, 0, 0, 0, 1, 1, 2, 2, 1}, new int[]{0, 0, 0, 0, 1, 2, 1, 1, 2},
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   0.00000000000000000000, 0.00000000000000000000};

// ************************************************************************
// MAIN
// ************************************************************************


public static void  Main(System.String[] args)
{
   double[,] xData; // Input  Attributes for Trainer
   int[] yData; // Output Attributes for Trainer
   int i, j; // array indicies
   int[,] z;

   // ************************************************************************
   // PREPROCESS TRAINING PATTERNS
   // ************************************************************************
   long t0 = (System.DateTime.Now.Ticks - 621355968000000000) / 10000;

   xData = new double[nObs,nInputs];
   yData = new int[nObs];

   /* Perform Binary Filtering. */
   for (i = 0; i < data.Length; i++)
   {
      for (j = 0; j < data[0].Length; j++)
      {
         data[i][j]++;
      }
   }
   int[] xx = new int[nObs];
   UnsupervisedNominalFilter filter = new UnsupervisedNominalFilter(3);
   for (i = 0; i < 9; i++)
   {
      // Copy each variable to a temp var
      for (j = 0; j < nObs; j++)
      {
         xx[j] = data[j][i];
      }
      //  Perform binary filter on temp var
      z = filter.Encode(xx);
      //  Copy binary encoded var to xData
      for (j = 0; j < nObs; j++)
      {
         for (int k = 0; k < 3; k++)
         {
            xData[j,k + (i * 3)] = (double) z[j,k];
         }
      }
   }
```

```
for (i = 0; i < nObs; i++)
{
   yData[i] = (i >= 626?0:1);
}

// ********************************************************************
// CREATE FEEDFORWARD NETWORK
// ********************************************************************
FeedForwardNetwork network = new FeedForwardNetwork();
network.InputLayer.CreateInputs(nInputs);
network.CreateHiddenLayer().CreatePerceptrons(nPerceptrons1);
network.CreateHiddenLayer().CreatePerceptrons(nPerceptrons2);
network.OutputLayer.CreatePerceptrons(nOutputs);
network.LinkAll();
network.Weights = weights;
Perceptron[] perceptrons = network.Perceptrons;
for (i = 0; i < perceptrons.Length - 1; i++)
{
   perceptrons[i].Activation = hiddenLayerActivation;
}
// ********************************************************************
// SET OUTPUT LAYER ACTIVATION FUNCTION TO LOGISTIC FOR BINARY CLASSIFICATION
// ********************************************************************
perceptrons[perceptrons.Length - 1].Activation = outputLayerActivation;

BinaryClassification classification = new BinaryClassification(network);

QuasiNewtonTrainer stageITrainer = new QuasiNewtonTrainer();
QuasiNewtonTrainer stageIITrainer = new QuasiNewtonTrainer();
stageITrainer.SetError(classification.Error);
stageIITrainer.SetError(classification.Error);
stageITrainer.MaximumTrainingIterations = 8000;
stageITrainer.MaximumStepsize = 10.0;
stageIITrainer.MaximumStepsize = 10.0;
stageIITrainer.MaximumTrainingIterations = 8000;
EpochTrainer trainer = new EpochTrainer(stageITrainer, stageIITrainer);

// Set Training Parameters
trainer.NumberOfEpochs = 20;
trainer.EpochSize = nObs;

// Set random number seeds to produce repeatable output
trainer.Random = new Random(5555);
trainer.SetRandomSamples(new Random(5555), new Random(5555));
classification.Train(trainer, xData, yData);
System.Console.Out.WriteLine("trainer.getErrorValue = " +
   trainer.ErrorValue);
System.Console.Out.WriteLine("StageITrainer.getErrorValue = " +
   stageITrainer.ErrorValue);
System.Console.Out.WriteLine("StageIITrainer.getErrorValue = " +
   stageIITrainer.ErrorValue);

// ********************************************************************
// DISPLAY TRAINING STATISTICS
// ********************************************************************
double[] stats = classification.ComputeStatistics(xData, yData);
```

```
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("--> Cross-entropy error:        " +
   (float)stats[0]);
System.Console.Out.WriteLine("--> Classification error rate:  " +
(float)stats[1]);
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("");


// ************************************************************************
// OBTAIN AND DISPLAY NETWORK WEIGHTS AND GRADIENTS
// ************************************************************************
double[] weight = network.Weights;
double[] gradient = trainer.ErrorGradient;
double[][] wg = new double[weight.Length][];
for (int i3 = 0; i3 < weight.Length; i3++)
{
   wg[i3] = new double[2];
}
for (i = 0; i < weight.Length; i++)
{
   wg[i][0] = weight[i];
   wg[i][1] = gradient[i];
}
PrintMatrixFormat pmf = new PrintMatrixFormat();
pmf.SetColumnLabels(new System.String[]{"Weights", "Gradients"});
new PrintMatrix().Print(pmf, wg);

// ************************************************************************
//     forecast the network
// ************************************************************************
double[][] report = new double[nObs][];
for (int i4 = 0; i4 < nObs; i4++)
{
   report[i4] = new double[2];
}
for (i = 0; i < 50; i++)
{
   report[i][0] = yData[i];
      double[] tmp = new double[xData.GetLength(1)];
      for (j=0; j<xData.GetLength(1); j++)
          tmp[j] = xData[i,j];
   report[i][1] = classification.PredictedClass(tmp);
}

pmf = new PrintMatrixFormat();
pmf.SetColumnLabels( new System.String[]{"Expected", "Predicted"});
new PrintMatrix("Forecast").Print(pmf, report);

long t1 = (System.DateTime.Now.Ticks - 621355968000000000) / 10000;
double time = t1 - t0; //Math.max(small, (double)(t1-t0)/(double)iters);
time = time / 1000;
System.Console.Out.WriteLine("***************Time:  " + time);
System.Console.Out.WriteLine("trainer.getErrorValue = " +
```

```
         trainer.ErrorValue);
      System.Console.Out.WriteLine("StageITrainer.getErrorValue = " +
         stageITrainer.ErrorValue);
      System.Console.Out.WriteLine("StageIITrainer.getErrorValue = " +
         stageIITrainer.ErrorValue);
   }
}
```

Output

```
trainer.getErrorValue = 1.38629447264277
StageITrainer.getErrorValue = 341.896192465936
StageIITrainer.getErrorValue = 1.38629447264277
*********************************************
--> Cross-entropy error:        1.386294
--> Classification error rate:  0.002087683
*********************************************

           Weights              Gradients
  0   -26.6097976573308    -1.76013093157232E-07
  1    79.2470900942513    -3.9259607432252E-09
  2   -33.5433683376609    -9.95507902332743E-09
  3     2.21041535535485    6.47606304741283E-12
  4    15.9227299370867    -7.92672053349369E-08
  5    32.2932177476722     2.16769316126848E-07
  6  -110.229528537971      1.60009076640888E-09
  7    43.0115071508866    -9.56560614455885E-08
  8    -6.72341801061297   -1.99042706374008E-08
  9     3.82202932248916   -3.44338957093567E-09
 10    -1.52763776833912   -4.67416809899007E-34
 11    19.096318145894      1.06581268595036E-37
 12    -6.20159756258499   -3.75471657439398E-37
 13     2.30629260428251    1.77102057507573E-37
 14   -26.3461568303238    -1.45338905465151E-42
 15   -27.912063156244     -3.23151405837895E-08
 16    36.3210304224857     1.55178803567368E-09
 17   -40.5700810470139    -1.00405026567488E-08
 18     5.94185679788313   -2.10634338816164E-08
 19    12.8231358563635    -8.13118613514597E-08
 20    32.2113802373171     7.3071363553406E-08
 21   -60.2258886605208    -3.87765801249E-09
 22    42.8334049469       -9.55706378121672E-08
 23     0.253218038120759   1.16563930726296E-09
 24   -10.4192312451491    -1.39873355441285E-09
 25     1.10132050954968    3.30048882191631E-27
 26    12.199990492126      1.00794238510197E-27
 27     3.15699338840009   -1.27487754292101E-36
 28    -7.20259442259759   -4.97300892576816E-27
 29    -9.99851831420775    2.66695912315823E-29
 30   -26.3385484724221     1.52366554789366E-07
 31    65.1653516520099    -2.59304404823118E-09
 32   -40.2337421647594    -9.28867802769452E-08
 33    12.7546549371301     1.16549218783095E-09
 34   -36.9912885381589    -3.44339063945978E-09
 35    31.8719580928069    -1.11610331819749E-07
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36   -75.4669369173807     2.67174071414864E-10
37    38.3417469286413    -1.27243601919707E-08
38   -17.2123954014748    -2.10632867621844E-08
39    18.4014308721567    -7.92672042664128E-08
40    -1.43098018584262   -3.80193732612446E-34
41    -0.265020137782834   8.49527031810026E-29
42     6.4678253527918    -2.4129919645073E-38
43     4.0087366295492    -4.97308613230902E-27
44    10.4385823414027     3.27914964551473E-30
45   -28.1024762680355    -4.21249368544629E-07
46    62.8100726278876    -2.27685589155942E-09
47   -23.9829618085034    -9.56658507793321E-08
48   -18.4761944978989    -1.98977918245327E-08
49    30.645136501639     -1.39871670591387E-09
50    32.4353708210909     4.62005591514245E-07
51   -70.1218158127842    -4.90140852568933E-11
52    36.1916462883953    -9.94528968958386E-09
53     1.03638734536457   -2.74982069124107E-15
54   -13.5594811206575    -8.13118781999587E-08
55     0.18700276514072   -7.06279899438387E-43
56    -3.77324466133111    7.25858300892723E-38
57    -8.383749634121     -1.73858570142904E-36
58    17.5982428451326     7.54095767597564E-37
59   -24.6173879312637    -1.37785999442333E-40
60   -20.258351942152      2.8862622792285E-07
61    56.7905424920708     2.6635105374416E-10
62   -53.7883610055096    -9.5570637836146E-08
63    27.9844966690905    -1.98977923264668E-08
64   -54.4900399816996     2.04467393347846E-09
65    37.0571465811016    -2.47870004953234E-07
66   -73.9187059285026    -2.59222103056048E-09
67    57.7528833255766    -1.00405026327699E-08
68   -19.1914450829349    -2.24788663240107E-15
69    34.4658886111249    -8.4755268839351E-08
70   -13.1090289774173     1.76210736423409E-24
71     5.49900711796567    1.00424771675945E-27
72     1.70369231662346   -5.32757842089505E-23
73    -9.09289929443768   -3.52040697402979E-26
74    12.2953916658469     2.3390441586071E-29
75   -27.4818129360831     4.43281192065308E-07
76    48.1316658031404    -4.98371251655421E-11
77   -34.0761614176618    -7.63854604777945E-09
78   -10.6698732282012     3.71928374478061E-10
79   -20.7852963033874    -8.13118787175936E-08
80    32.5577122725433    -4.02524969095691E-07
81   -61.1887835024785    -2.27603285165078E-09
82    34.8170651512053    -9.79725944211365E-08
83     6.63722445362796   -2.02697229488315E-08
84    16.5220845779096    -1.39871618827893E-09
85    -0.160895088640468   3.30048890011806E-27
86     1.85827799132845    1.0935461721458E-27
87     3.78269514701215   -2.03606355425952E-37
88     2.97974747746492   -7.72068070714235E-32
89    -3.60575967549916    2.45076692281205E-27
90   -27.6593797606404     4.15232805139475E-07
91    72.3342855580823    -2.32585476757858E-09
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 92   -24.7298826022964    -7.56358115079602E-09
 93     5.24227794358502   -2.06913578857583E-08
 94   -16.9304351865014    -8.47552677708944E-08
 95    30.7407379285903    -3.74476582169859E-07
 96   -71.9479456279238    -1.52092377331025E-14
 97    45.8003226113044    -9.80475593181199E-08
 98   -24.6378777857398     7.93563311404888E-10
 99     6.23728601645476    2.0446728650218E-09
100     0.533861406588414   8.69622769614886E-35
101   -11.5890280314343     1.39685762306099E-28
102   -16.1548714189138    -6.28455381082345E-34
103    18.9583573897183    -1.28719086134143E-25
104     3.56568118146885    3.65618430530922E-34
105   -28.465938502988      7.00620144653407E-08
106    21.3448850345718    -3.87683496658173E-09
107   -19.30027174882      -9.79620254277453E-08
108    -2.77292404194946    7.87230245936389E-10
109     1.83277246414745   -1.39873355441285E-09
110    32.3382786494488    -2.93057914957237E-08
111   -49.5952082003002     1.55096498976541E-09
112    29.9349835508889    -7.64911504117054E-09
113    -5.47912940998204   -2.06850248202898E-08
114    -5.64894665504052   -8.13118613514597E-08
115     0.260715006887127   1.75880687601373E-24
116    16.6667759427184     3.69466844883976E-30
117    -7.04065602320258   -5.32757842089507E-23
118     7.02117525437252   -4.01770786660928E-26
119    -3.11151101069225    3.27914964537858E-30
120   -28.4825481958908    -5.36210077248606E-07
121    71.1272184809125     1.60089856884223E-09
122   -32.4116685301335    -1.10772699145759E-07
123    -1.70847780496737   -2.02697238246857E-08
124   -16.2557204309972    -8.27105943881702E-08
125    30.3385841713894     5.76966300218224E-07
126   -87.9377967090049    -3.92676854565855E-09
127    50.246529583999      5.16155867684338E-09
128   -10.096654805183      3.71929250332289E-10
129    10.2633449323559    -5.17702391593651E-16
130     1.96938482868274    1.75880687595947E-24
131     5.34996503177684    1.59734938184303E-37
132   -14.713205212991     -5.32757842089508E-23
133    11.716720259857     -3.52040697399322E-26
134    -2.0119500580206     2.76081718492693E-42
135  -165.787101153353     -3.25241613346471E-08
136  -153.126623517672     -2.18413261959866E-09
137   -21.4407461371651     2.56421775779744E-09
138    19.5580189313542    -1.05973313874125E-08
139    44.9089494067269    -7.19785691192466E-09
140   -49.083352621682      4.30604139136833E-09
141   -18.9381369969533     1.76292638468803E-09
142   -83.3719106138484     7.19716170610117E-09
143   137.944766899566      4.36581809057208E-09
144    39.0313514342063    -4.51667695917841E-08
145    75.599341597783     -3.06933546775251E-11
146   195.169995384174     -1.23667681444798E-10
147   -83.0199447688769     1.86542026484299E-08
```

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148    25.8184987265264    -1.62741417975303E-10
149   -97.6355018936802     4.35877811974283E-09
150   160.889205710213     -2.45893958744307E-09
151    84.4502033182748    -3.00133311935682E-09
152  -112.732145115534     -3.33652618216474E-10
153     4.22767450727702    4.07562229696174E-08
154   -10.8342582573892    -2.32586997681632E-09
155     4.39655805882639   -1.05611140468916E-07
156    -0.553850439419576  -1.98977945743534E-08
157    -8.20433040373114   -8.27105949058726E-08
158   114.83608474643      -2.48317911575925E-08
159   128.51819165101      -2.63457735094495E-09
160    13.7239100272455    -1.99531479597656E-09
161   -48.157060604333     -4.11525980456149E-09
```

	Forecast	
	Expected	Predicted
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2	1	1
3	1	1
4	1	1
5	1	1
6	1	1
7	1	1
8	1	1
9	1	1
10	1	1
11	1	1
12	1	1
13	1	1
14	1	1
15	1	1
16	1	1
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28	1	1
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34	1	1
35	1	1
36	1	1
37	1	1
38	1	1

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39      1           1
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**********Time:  101.567
trainer.getErrorValue = 1.38629447264277
StageITrainer.getErrorValue = 341.896192465936
StageIITrainer.getErrorValue = 1.38629447264277
```

MultiClassification Class

`public class Imsl.DataMining.Neural.MultiClassification`

Classifies patterns into three or more classes.

Extends neural network analysis to solving multi-classification problems. In these problems, the target output for the network is the probability that the pattern falls into each of several classes, where the number of classes is 3 or greater. These probabilities are then used to assign patterns to one of the target classes. Typical applications include determining the credit classification for a business (excellent, good, fair or poor), and determining which of three or more treatments a patient should receive based upon their physical, clinical and laboratory information. This class signals that network training will minimize the multi-classification cross-entropy error, and that network outputs are the probabilities that the pattern belongs to each of the target classes. These probabilities are scaled to sum to 1.0 using softmax activation.

Properties

Error

```
virtual public Imsl.DataMining.Neural.QuasiNewtonTrainer.IError Error {get; }
```

Description

The error function for use by `QuasiNewtonTrainer` for training a classification network.

Property Value

An implementation of the multi-classification cross-entropy error function.

Remarks

This error function combines the softmax activation function and the cross-entropy error function.

Network

```
virtual public Imsl.DataMining.Neural.Network Network {get; }
```

Description

Returns the network being used for classification.

Constructor

MultiClassification

```
public MultiClassification(Imsl.DataMining.Neural.Network network)
```

Description

Creates a classifier.

Parameter

`network` – Is the neural network used for classification. Its `OutputPerceptrons` should use linear activation functions, `Activation.Linear`. The number of output `Perceptrons` should equal the number of classes.

Methods

ComputeStatistics

```
virtual public double[] ComputeStatistics(double[,] xData, int[] yData)
```

Description

Computes classification statistics for the supplied network patterns and their associated classifications.

Parameters

xData – A double matrix specifying the input training patterns. The number of columns in xData must equal the number of Nodes in the InputLayer.

yData – An int[] containing the output classification patterns. The values in yData must be in the range of one to the number of OutputPerceptrons in the network.

Returns

A double[] containing the cross-entropy error and the classification error rate.

Remarks

Method ComputeStatistics returns a two element array where the first element returned is the cross-entropy error; the second is the classification error rate. The classification error rate is calculated by comparing the estimated classification probabilities to the target classifications. If the estimated probability for the target class is not the largest among the target classes, then the pattern is tallied as a classification error.

PredictedClass

```
virtual public int PredictedClass(double[] x)
```

Description

Calculates the classification probablities for the input pattern x, and returns the class with the highest probability.

Parameter

x – The double array containing the network input patterns to classify. The length of x should equal the number of inputs in the network.

Returns

The classification predicted by the trained network for x. This will be one of the integers 1,2,...,*nClasses*, where *nClasses* is equal to nOuptuts. nOuptuts is the number of outputs in the network representing the number of classes.

Remarks

This method classifies patterns into one of the target classes based upon the patterns values.

Probabilities

```
virtual public double[] Probabilities(double[] x)
```

Description

Returns classification probabilities for the input pattern x.

Parameter

x – A double array containing the input patterns to classify. The length of x must be equal to the number of input nodes.

Returns

A double containing the scaled probabilities.

Remarks

The number of probabilities is equal to the number of target classes, which is the number of outputs in the `FeedForwardNetwork`. Each are calculated using the softmax activation for each of the `OutputPerceptrons`. The softmax function transforms the outputs potential z to the probability y by

$$y_i = \text{softmax}_\text{i} = \frac{e^{z_i}}{\sum_{j=1}^{C} e^{z_j}}$$

Train

```
virtual public void Train(Imsl.DataMining.Neural.ITrainer trainer, double[,]
xData, int[] yData)
```

Description

Trains the classification neural network using supplied training patterns.

Parameters

`trainer` – A `Trainer` object, which is used to train the network. The error function in any `QuasiNewton` trainer included in `trainer` should be set to the error function from this class using the `Error` (p. 1728) method.

`xData` – A `double` matrix containing the input training patterns. The number of columns in `xData` must equal the number of nodes in the input layer. Each row of `xData` contains a training pattern.

`yData` – An `int` array containing the output classification patterns. These values must be in the range of one to the number of output perceptrons in the network.

Example 1: MultiClassification

This example trains a 3-layer network using Fisher's Iris data with four continuous input attributes and three output classifications. This is perhaps the best known database to be found in the pattern recognition literature. Fisher's paper is a classic in the field. The data set contains 3 classes of 50 instances each, where each class refers to a type of iris plant.

The structure of the network consists of four input nodes and three layers, with four perceptrons in the first hidden layer, three perceptrons in the second hidden layer and three in the output layer.

The four input attributes represent

1. Sepal length
2. Sepal width
3. Petal length
4. Petal width

The output attribute represents the class of the iris plant and are encoded using binary encoding.

1. Iris Setosa

2. Iris Versicolour
3. Iris Virginica

There are a total of 46 weights in this network, including the bias weights. All hidden layers use the logistic activation function. Since the target output is multi-classification the softmax activation function is used in the output layer and the `MultiClassification` error function class is used by the trainer. The error class `MultiClassification` combines the cross-entropy error calculations and the softmax function.

```
using System;
using Imsl.DataMining.Neural;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

//*************************************************************************
// Three Layer Feed-Forward Network with 4 inputs, all
// continuous, and 3 classification categories.
//
//  new classification training_ex5.c
//
// This is perhaps the best known database to be found in the pattern
//     recognition literature.  Fisher's paper is a classic in the field.
//     The data set contains 3 classes of 50 instances each,
//     where each class refers to a type of iris plant.  One class is
//     linearly separable from the other 2; the latter are NOT linearly
//     separable from each other.
//
//  Predicted attribute: class of iris plant.
//     1=Iris Setosa, 2=Iris Versicolour, and 3=Iris Virginica
//
//  Input Attributes (4 Continuous Attributes)
//     X1: Sepal length, X2: Sepal width, X3: Petal length, and X4: Petal width
//*************************************************************************

[Serializable]
public class MultiClassificationEx1
{
   private static int nObs = 150; // number of training patterns
   private static int nInputs = 4; // 9 nominal coded as 0=x, 1=o, 2=blank
   private static int nOutputs = 3; // one continuous output (nClasses=2)


   // irisData[]:  The raw data matrix.  This is a 2-D matrix with 150 rows and
   //              5 columns. The first 4 columns are the continuous input
   //              attributes and the 5th column is the classification category
   //              (1-3).  These data contain no categorical input attributes.

   private static double[][] irisData = new double[][]{
      new double[]{5.1, 3.5, 1.4, 0.2, 1}, new double[]{4.9, 3.0, 1.4, 0.2, 1},
      new double[]{4.7, 3.2, 1.3, 0.2, 1}, new double[]{4.6, 3.1, 1.5, 0.2, 1},
      new double[]{5.0, 3.6, 1.4, 0.2, 1}, new double[]{5.4, 3.9, 1.7, 0.4, 1},
      new double[]{4.6, 3.4, 1.4, 0.3, 1}, new double[]{5.0, 3.4, 1.5, 0.2, 1},
      new double[]{4.4, 2.9, 1.4, 0.2, 1}, new double[]{4.9, 3.1, 1.5, 0.1, 1},
```

```
new double[]{5.4, 3.7, 1.5, 0.2, 1}, new double[]{4.8, 3.4, 1.6, 0.2, 1},
new double[]{4.8, 3.0, 1.4, 0.1, 1}, new double[]{4.3, 3.0, 1.1, 0.1, 1},
new double[]{5.8, 4.0, 1.2, 0.2, 1}, new double[]{5.7, 4.4, 1.5, 0.4, 1},
new double[]{5.4, 3.9, 1.3, 0.4, 1}, new double[]{5.1, 3.5, 1.4, 0.3, 1},
new double[]{5.7, 3.8, 1.7, 0.3, 1}, new double[]{5.1, 3.8, 1.5, 0.3, 1},
new double[]{5.4, 3.4, 1.7, 0.2, 1}, new double[]{5.1, 3.7, 1.5, 0.4, 1},
new double[]{4.6, 3.6, 1.0, 0.2, 1}, new double[]{5.1, 3.3, 1.7, 0.5, 1},
new double[]{4.8, 3.4, 1.9, 0.2, 1}, new double[]{5.0, 3.0, 1.6, 0.2, 1},
new double[]{5.0, 3.4, 1.6, 0.4, 1}, new double[]{5.2, 3.5, 1.5, 0.2, 1},
new double[]{5.2, 3.4, 1.4, 0.2, 1}, new double[]{4.7, 3.2, 1.6, 0.2, 1},
new double[]{4.8, 3.1, 1.6, 0.2, 1}, new double[]{5.4, 3.4, 1.5, 0.4, 1},
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new double[]{4.9, 3.1, 1.5, 0.1, 1}, new double[]{5.0, 3.2, 1.2, 0.2, 1},
new double[]{5.5, 3.5, 1.3, 0.2, 1}, new double[]{4.9, 3.1, 1.5, 0.1, 1},
new double[]{4.4, 3.0, 1.3, 0.2, 1}, new double[]{5.1, 3.4, 1.5, 0.2, 1},
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new double[]{4.4, 3.2, 1.3, 0.2, 1}, new double[]{5.0, 3.5, 1.6, 0.6, 1},
new double[]{5.1, 3.8, 1.9, 0.4, 1}, new double[]{4.8, 3.0, 1.4, 0.3, 1},
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new double[]{6.9, 3.1, 4.9, 1.5, 2}, new double[]{5.5, 2.3, 4.0, 1.3, 2},
new double[]{6.5, 2.8, 4.6, 1.5, 2}, new double[]{5.7, 2.8, 4.5, 1.3, 2},
new double[]{6.3, 3.3, 4.7, 1.6, 2}, new double[]{4.9, 2.4, 3.3, 1.0, 2},
new double[]{6.6, 2.9, 4.6, 1.3, 2}, new double[]{5.2, 2.7, 3.9, 1.4, 2},
new double[]{5.0, 2.0, 3.5, 1.0, 2}, new double[]{5.9, 3.0, 4.2, 1.5, 2},
new double[]{6.0, 2.2, 4.0, 1.0, 2}, new double[]{6.1, 2.9, 4.7, 1.4, 2},
new double[]{5.6, 2.9, 3.6, 1.3, 2}, new double[]{6.7, 3.1, 4.4, 1.4, 2},
new double[]{5.6, 3.0, 4.5, 1.5, 2}, new double[]{5.8, 2.7, 4.1, 1.0, 2},
new double[]{6.2, 2.2, 4.5, 1.5, 2}, new double[]{5.6, 2.5, 3.9, 1.1, 2},
new double[]{5.9, 3.2, 4.8, 1.8, 2}, new double[]{6.1, 2.8, 4.0, 1.3, 2},
new double[]{6.3, 2.5, 4.9, 1.5, 2}, new double[]{6.1, 2.8, 4.7, 1.2, 2},
new double[]{6.4, 2.9, 4.3, 1.3, 2}, new double[]{6.6, 3.0, 4.4, 1.4, 2},
new double[]{6.8, 2.8, 4.8, 1.4, 2}, new double[]{6.7, 3.0, 5.0, 1.7, 2},
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new double[]{5.6, 3.0, 4.1, 1.3, 2}, new double[]{5.5, 2.5, 4.0, 1.3, 2},
new double[]{5.5, 2.6, 4.4, 1.2, 2}, new double[]{6.1, 3.0, 4.6, 1.4, 2},
new double[]{5.8, 2.6, 4.0, 1.2, 2}, new double[]{5.0, 2.3, 3.3, 1.0, 2},
new double[]{5.6, 2.7, 4.2, 1.3, 2}, new double[]{5.7, 3.0, 4.2, 1.2, 2},
new double[]{5.7, 2.9, 4.2, 1.3, 2}, new double[]{6.2, 2.9, 4.3, 1.3, 2},
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new double[]{4.9, 2.5, 4.5, 1.7, 3}, new double[]{7.3, 2.9, 6.3, 1.8, 3},
new double[]{6.7, 2.5, 5.8, 1.8, 3}, new double[]{7.2, 3.6, 6.1, 2.5, 3},
new double[]{6.5, 3.2, 5.1, 2.0, 3}, new double[]{6.4, 2.7, 5.3, 1.9, 3},
new double[]{6.8, 3.0, 5.5, 2.1, 3}, new double[]{5.7, 2.5, 5.0, 2.0, 3},
new double[]{5.8, 2.8, 5.1, 2.4, 3}, new double[]{6.4, 3.2, 5.3, 2.3, 3},
new double[]{6.5, 3.0, 5.5, 1.8, 3}, new double[]{7.7, 3.8, 6.7, 2.2, 3},
new double[]{7.7, 2.6, 6.9, 2.3, 3}, new double[]{6.0, 2.2, 5.0, 1.5, 3},
new double[]{6.9, 3.2, 5.7, 2.3, 3}, new double[]{5.6, 2.8, 4.9, 2.0, 3},
```

```
   new double[]{7.7, 2.8, 6.7, 2.0, 3}, new double[]{6.3, 2.7, 4.9, 1.8, 3},
   new double[]{6.7, 3.3, 5.7, 2.1, 3}, new double[]{7.2, 3.2, 6.0, 1.8, 3},
   new double[]{6.2, 2.8, 4.8, 1.8, 3}, new double[]{6.1, 3.0, 4.9, 1.8, 3},
   new double[]{6.4, 2.8, 5.6, 2.1, 3}, new double[]{7.2, 3.0, 5.8, 1.6, 3},
   new double[]{7.4, 2.8, 6.1, 1.9, 3}, new double[]{7.9, 3.8, 6.4, 2.0, 3},
   new double[]{6.4, 2.8, 5.6, 2.2, 3}, new double[]{6.3, 2.8, 5.1, 1.5, 3},
   new double[]{6.1, 2.6, 5.6, 1.4, 3}, new double[]{7.7, 3.0, 6.1, 2.3, 3},
   new double[]{6.3, 3.4, 5.6, 2.4, 3}, new double[]{6.4, 3.1, 5.5, 1.8, 3},
   new double[]{6.0, 3.0, 4.8, 1.8, 3}, new double[]{6.9, 3.1, 5.4, 2.1, 3},
   new double[]{6.7, 3.1, 5.6, 2.4, 3}, new double[]{6.9, 3.1, 5.1, 2.3, 3},
   new double[]{5.8, 2.7, 5.1, 1.9, 3}, new double[]{6.8, 3.2, 5.9, 2.3, 3},
   new double[]{6.7, 3.3, 5.7, 2.5, 3}, new double[]{6.7, 3.0, 5.2, 2.3, 3},
   new double[]{6.3, 2.5, 5.0, 1.9, 3}, new double[]{6.5, 3.0, 5.2, 2.0, 3},
   new double[]{6.2, 3.4, 5.4, 2.3, 3}, new double[]{5.9, 3.0, 5.1, 1.8, 3}};


public static void  Main(System.String[] args)
{
   double[,] xData = new double[nObs,nInputs];

   int[] yData = new int[nObs];

   for (int i = 0; i < nObs; i++)
   {
      for (int j = 0; j < nInputs; j++)
      {
         xData[i,j] = irisData[i][j];
      }
      yData[i] = (int) irisData[i][4];
   }

   // Create network
   FeedForwardNetwork network = new FeedForwardNetwork();
   network.InputLayer.CreateInputs(nInputs);
   network.CreateHiddenLayer().CreatePerceptrons(4,
      Imsl.DataMining.Neural.Activation.Logistic, 0.0);
   network.CreateHiddenLayer().CreatePerceptrons(3,
      Imsl.DataMining.Neural.Activation.Logistic, 0.0);
   network.OutputLayer.CreatePerceptrons(nOutputs,
      Imsl.DataMining.Neural.Activation.Softmax, 0.0);
   network.LinkAll();

   MultiClassification classification = new MultiClassification(network);

   // Create trainer
   QuasiNewtonTrainer trainer = new QuasiNewtonTrainer();
   trainer.SetError(classification.Error);
   trainer.MaximumTrainingIterations = 1000;

   // Train Network
   long t0 = (System.DateTime.Now.Ticks - 621355968000000000) / 10000;
   classification.Train(trainer, xData, yData);

   // Display Network Errors
   double[] stats = classification.ComputeStatistics(xData, yData);
   System.Console.Out.WriteLine(
```

```
   "*********************************************");
System.Console.Out.WriteLine(
   "--> Cross-entropy error:        " + (float) stats[0]);
System.Console.Out.WriteLine(
   "--> Classification error rate:  " + (float) stats[1]);
System.Console.Out.WriteLine(
   "*********************************************");
System.Console.Out.WriteLine("");

double[] weight = network.Weights;
double[] gradient = trainer.ErrorGradient;
double[][] wg = new double[weight.Length][];
for (int i2 = 0; i2 < weight.Length; i2++)
{
   wg[i2] = new double[2];
}
for (int i = 0; i < weight.Length; i++)
{
   wg[i][0] = weight[i];
   wg[i][1] = gradient[i];
}
PrintMatrixFormat pmf = new PrintMatrixFormat();
pmf.SetColumnLabels(new System.String[]{"Weights", "Gradients"});
new PrintMatrix().Print(pmf, wg);

double[][] report = new double[nObs][];
for (int i3 = 0; i3 < nObs; i3++)
{
   report[i3] = new double[nInputs + 2];
}
for (int i = 0; i < nObs; i++)
{
   for (int j = 0; j < nInputs; j++)
   {
      report[i][j] = xData[i,j];
   }
   report[i][nInputs] = irisData[i][4];
      double[] xTmp = new double[xData.GetLength(1)];
      for (int j=0; j<xData.GetLength(1); j++)
          xTmp[j] = xData[i,j];
   report[i][nInputs + 1] = classification.PredictedClass(xTmp);
}
pmf = new PrintMatrixFormat();
pmf.SetColumnLabels( new System.String[]{"Sepal Length", "Sepal Width",
   "Petal Length", "Petal Width", "Expected", "Predicted"});
new PrintMatrix("Forecast").Print(pmf, report);


// ********************************************************************
// DISPLAY CLASSIFICATION STATISTICS
// ********************************************************************
double[] statsClass = classification.ComputeStatistics(xData, yData);
// Display Network Errors
System.Console.Out.WriteLine(
   "*********************************************");
System.Console.Out.WriteLine("--> Cross-Entropy Error:      " +
```

```
         (float)statsClass[0]);
      System.Console.Out.WriteLine("--> Classification Error:     " +
         (float)statsClass[1]);
      System.Console.Out.WriteLine(
         "*********************************************");
      System.Console.Out.WriteLine("");
      long t1 = (System.DateTime.Now.Ticks - 621355968000000000) / 10000;
      double time = t1 - t0;
      time = time / 1000;
      System.Console.Out.WriteLine("***************Time:  " + time);

      System.Console.Out.WriteLine("Cross-Entropy Error Value = " +
         trainer.ErrorValue);
   }
}
```

Output

```
*********************************************
--> Cross-entropy error:        5.922989
--> Classification error rate:  0.01333333
*********************************************

           Weights                Gradients
 0    -362.135397018074      0
 1    -560.715494616693      0
 2     527.72406779352       2.08315681881952E-198
 3      -0.195010974908058   0.0264624672168392
 4    -685.159411754574      0
 5    -803.111185471252      0
 6    -195.048774132049      9.50211882268555E-199
 7      -0.649536948895873   0.0122051719495706
 8    1271.3576681191        0
 9     495.146969161579      0
10    -599.774609230401      1.2791313799769E-198
11       0.807201787624182   0.0211017819573165
12    1143.16439557087       0
13     950.665441902952      0
14    -501.328120816785      3.65466108564829E-199
15       1.82418587547183    0.0071872906296363
16     188.142460610324      0.606686791122197
17      99.9378596561514     4.43416403183572E-05
18      48.0430167242241     0
19     103.388473027949      0
20     -28.0558983107429     0
21    -122.247432161144      0
22     674.932369179828      5.41468289698623E-202
23     295.103891118948      3.9419889529911E-206
24    -885.621857738157      0
25      31.424095865498      0.000136908693839501
26     -11.7156988030889     9.32263272488116E-09
27     250.928447008145      0
28   -9490.85347121632       0
29    1392.60893930402      -0.000263660213410609
30    8099.24453182847       0.000263660213411595
```

```
31   -9687.90452156921       0
32    2423.00583945991      -0.000401325677808519
33    7265.89868208763       0.000401325677809067
34   -9908.74662211761       0
35    3132.8896497266       -0.00040133483404004
36    6776.85697236237       0.000401334834041256
37    -251.58662719581       0
38    -395.299386555686      0
39    -361.151672607701      3.65466108564829E-199
40     -12.3192632922507     0.00431460331538189
41    -187.499672307873      0.606686791122197
42     -89.2472537574958     4.43416403183572E-05
43      34.2625870237822     0
44   22537.1579103508        0
45   -4821.83723997329      -0.00040133483404004
46  -17715.3206704603        0.000401334834041256
```

			Forecast			
	Sepal Length	Sepal Width	Petal Length	Petal Width	Expected	Predicted
0	5.1	3.5	1.4	0.2	1	1
1	4.9	3	1.4	0.2	1	1
2	4.7	3.2	1.3	0.2	1	1
3	4.6	3.1	1.5	0.2	1	1
4	5	3.6	1.4	0.2	1	1
5	5.4	3.9	1.7	0.4	1	1
6	4.6	3.4	1.4	0.3	1	1
7	5	3.4	1.5	0.2	1	1
8	4.4	2.9	1.4	0.2	1	1
9	4.9	3.1	1.5	0.1	1	1
10	5.4	3.7	1.5	0.2	1	1
11	4.8	3.4	1.6	0.2	1	1
12	4.8	3	1.4	0.1	1	1
13	4.3	3	1.1	0.1	1	1
14	5.8	4	1.2	0.2	1	1
15	5.7	4.4	1.5	0.4	1	1
16	5.4	3.9	1.3	0.4	1	1
17	5.1	3.5	1.4	0.3	1	1
18	5.7	3.8	1.7	0.3	1	1
19	5.1	3.8	1.5	0.3	1	1
20	5.4	3.4	1.7	0.2	1	1
21	5.1	3.7	1.5	0.4	1	1
22	4.6	3.6	1	0.2	1	1
23	5.1	3.3	1.7	0.5	1	1
24	4.8	3.4	1.9	0.2	1	1
25	5	3	1.6	0.2	1	1
26	5	3.4	1.6	0.4	1	1
27	5.2	3.5	1.5	0.2	1	1
28	5.2	3.4	1.4	0.2	1	1
29	4.7	3.2	1.6	0.2	1	1
30	4.8	3.1	1.6	0.2	1	1
31	5.4	3.4	1.5	0.4	1	1
32	5.2	4.1	1.5	0.1	1	1
33	5.5	4.2	1.4	0.2	1	1
34	4.9	3.1	1.5	0.1	1	1
35	5	3.2	1.2	0.2	1	1
36	5.5	3.5	1.3	0.2	1	1

37	4.9	3.1	1.5	0.1	1	1
38	4.4	3	1.3	0.2	1	1
39	5.1	3.4	1.5	0.2	1	1
40	5	3.5	1.3	0.3	1	1
41	4.5	2.3	1.3	0.3	1	1
42	4.4	3.2	1.3	0.2	1	1
43	5	3.5	1.6	0.6	1	1
44	5.1	3.8	1.9	0.4	1	1
45	4.8	3	1.4	0.3	1	1
46	5.1	3.8	1.6	0.2	1	1
47	4.6	3.2	1.4	0.2	1	1
48	5.3	3.7	1.5	0.2	1	1
49	5	3.3	1.4	0.2	1	1
50	7	3.2	4.7	1.4	2	2
51	6.4	3.2	4.5	1.5	2	2
52	6.9	3.1	4.9	1.5	2	2
53	5.5	2.3	4	1.3	2	2
54	6.5	2.8	4.6	1.5	2	2
55	5.7	2.8	4.5	1.3	2	2
56	6.3	3.3	4.7	1.6	2	2
57	4.9	2.4	3.3	1	2	2
58	6.6	2.9	4.6	1.3	2	2
59	5.2	2.7	3.9	1.4	2	2
60	5	2	3.5	1	2	2
61	5.9	3	4.2	1.5	2	2
62	6	2.2	4	1	2	2
63	6.1	2.9	4.7	1.4	2	2
64	5.6	2.9	3.6	1.3	2	2
65	6.7	3.1	4.4	1.4	2	2
66	5.6	3	4.5	1.5	2	2
67	5.8	2.7	4.1	1	2	2
68	6.2	2.2	4.5	1.5	2	2
69	5.6	2.5	3.9	1.1	2	2
70	5.9	3.2	4.8	1.8	2	2
71	6.1	2.8	4	1.3	2	2
72	6.3	2.5	4.9	1.5	2	2
73	6.1	2.8	4.7	1.2	2	2
74	6.4	2.9	4.3	1.3	2	2
75	6.6	3	4.4	1.4	2	2
76	6.8	2.8	4.8	1.4	2	2
77	6.7	3	5	1.7	2	2
78	6	2.9	4.5	1.5	2	2
79	5.7	2.6	3.5	1	2	2
80	5.5	2.4	3.8	1.1	2	2
81	5.5	2.4	3.7	1	2	2
82	5.8	2.7	3.9	1.2	2	2
83	6	2.7	5.1	1.6	2	3
84	5.4	3	4.5	1.5	2	2
85	6	3.4	4.5	1.6	2	2
86	6.7	3.1	4.7	1.5	2	2
87	6.3	2.3	4.4	1.3	2	2
88	5.6	3	4.1	1.3	2	2
89	5.5	2.5	4	1.3	2	2
90	5.5	2.6	4.4	1.2	2	2
91	6.1	3	4.6	1.4	2	2
92	5.8	2.6	4	1.2	2	2

93	5	2.3	3.3	1	2	2
94	5.6	2.7	4.2	1.3	2	2
95	5.7	3	4.2	1.2	2	2
96	5.7	2.9	4.2	1.3	2	2
97	6.2	2.9	4.3	1.3	2	2
98	5.1	2.5	3	1.1	2	2
99	5.7	2.8	4.1	1.3	2	2
100	6.3	3.3	6	2.5	3	3
101	5.8	2.7	5.1	1.9	3	3
102	7.1	3	5.9	2.1	3	3
103	6.3	2.9	5.6	1.8	3	3
104	6.5	3	5.8	2.2	3	3
105	7.6	3	6.6	2.1	3	3
106	4.9	2.5	4.5	1.7	3	3
107	7.3	2.9	6.3	1.8	3	3
108	6.7	2.5	5.8	1.8	3	3
109	7.2	3.6	6.1	2.5	3	3
110	6.5	3.2	5.1	2	3	3
111	6.4	2.7	5.3	1.9	3	3
112	6.8	3	5.5	2.1	3	3
113	5.7	2.5	5	2	3	3
114	5.8	2.8	5.1	2.4	3	3
115	6.4	3.2	5.3	2.3	3	3
116	6.5	3	5.5	1.8	3	3
117	7.7	3.8	6.7	2.2	3	3
118	7.7	2.6	6.9	2.3	3	3
119	6	2.2	5	1.5	3	3
120	6.9	3.2	5.7	2.3	3	3
121	5.6	2.8	4.9	2	3	3
122	7.7	2.8	6.7	2	3	3
123	6.3	2.7	4.9	1.8	3	3
124	6.7	3.3	5.7	2.1	3	3
125	7.2	3.2	6	1.8	3	3
126	6.2	2.8	4.8	1.8	3	3
127	6.1	3	4.9	1.8	3	3
128	6.4	2.8	5.6	2.1	3	3
129	7.2	3	5.8	1.6	3	3
130	7.4	2.8	6.1	1.9	3	3
131	7.9	3.8	6.4	2	3	3
132	6.4	2.8	5.6	2.2	3	3
133	6.3	2.8	5.1	1.5	3	2
134	6.1	2.6	5.6	1.4	3	3
135	7.7	3	6.1	2.3	3	3
136	6.3	3.4	5.6	2.4	3	3
137	6.4	3.1	5.5	1.8	3	3
138	6	3	4.8	1.8	3	3
139	6.9	3.1	5.4	2.1	3	3
140	6.7	3.1	5.6	2.4	3	3
141	6.9	3.1	5.1	2.3	3	3
142	5.8	2.7	5.1	1.9	3	3
143	6.8	3.2	5.9	2.3	3	3
144	6.7	3.3	5.7	2.5	3	3
145	6.7	3	5.2	2.3	3	3
146	6.3	2.5	5	1.9	3	3
147	6.5	3	5.2	2	3	3
148	6.2	3.4	5.4	2.3	3	3

```
149      5.9           3            5.1           1.8         3          3

*********************************************
--> Cross-Entropy Error:      5.922989
--> Classification Error:     0.01333333
*********************************************

****************Time:  5.891
Cross-Entropy Error Value = 17.7689670987518
```

Example 2: MultiClassification

This example trains a 2-layer network using three binary inputs (X0, X1, X2) and one three-level classification (Y). Where

Y = 0 if X1 = 1

Y = 1 if X2 = 1

Y = 2 if X3 = 1

```
using System;
using Imsl.DataMining.Neural;
using PrintMatrix = Imsl.Math.PrintMatrix;
using PrintMatrixFormat = Imsl.Math.PrintMatrixFormat;

//*************************************************************************
// Two-Layer FFN with 3 binary inputs (X0, X1, X2) and one three-level
// classification variable (Y)
// Y = 0 if X1 = 1
// Y = 1 if X2 = 1
// Y = 2 if X3 = 1
//  (training_ex6)
//*************************************************************************

[Serializable]
public class MultiClassificationEx2
{
   private static int nObs = 6; // number of training patterns
   private static int nInputs = 3; // 3 inputs, all categorical
   private static int nOutputs = 3; //
   private static double[,] xData = {{1, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 1, 0},
                                     {0, 0, 1}, {0, 0, 1}};
   private static int[] yData = new int[]{1, 1, 2, 2, 3, 3};

   private static double[] weights = new double[]{1.29099444873580580000,
      -0.64549722436790280000, -0.64549722436790291000, 0.00000000000000000000,
      1.11803398874989490000, -1.11803398874989470000, 0.57735026918962584000,
      0.57735026918962584000, 0.57735026918962584000, 0.33333333333333331000,
      0.33333333333333331000, 0.33333333333333331000, 0.33333333333333331000,
      0.33333333333333331000, 0.33333333333333331000, 0.33333333333333331000,
      0.33333333333333331000, 0.33333333333333331000, -0.00000000000000005851,
      -0.00000000000000005851, -0.57735026918962573000, 0.00000000000000000000,
      0.00000000000000000000, 0.00000000000000000000};
```

```
public static void  Main(System.String[] args)
{
   FeedForwardNetwork network = new FeedForwardNetwork();
   network.InputLayer.CreateInputs(nInputs);
   network.CreateHiddenLayer().CreatePerceptrons(3,
      Imsl.DataMining.Neural.Activation.Linear, 0.0);
   network.OutputLayer.CreatePerceptrons(nOutputs,
      Imsl.DataMining.Neural.Activation.Softmax, 0.0);
   network.LinkAll();
   network.Weights = weights;

   MultiClassification classification = new MultiClassification(network);

   QuasiNewtonTrainer trainer = new QuasiNewtonTrainer();
   trainer.SetError(classification.Error);
   trainer.MaximumTrainingIterations = 1000;
   trainer.GradientTolerance = 1.0e-20;
   trainer.StepTolerance = 1.0e-20;

   // Train Network
   classification.Train(trainer, xData, yData);


   // Display Network Errors
   double[] stats = classification.ComputeStatistics(xData, yData);
   System.Console.Out.WriteLine(
      "*********************************************");
   System.Console.Out.WriteLine(
      "--> Cross-Entropy Error:      " + (float) stats[0]);
   System.Console.Out.WriteLine(
      "--> Classification Error:     " + (float) stats[1]);
   System.Console.Out.WriteLine(
      "*********************************************");
   System.Console.Out.WriteLine();

   double[] weight = network.Weights;
   double[] gradient = trainer.ErrorGradient;
   double[][] wg = new double[weight.Length][];
   for (int i = 0; i < weight.Length; i++)
   {
      wg[i] = new double[2];
   }
   for (int i = 0; i < weight.Length; i++)
   {
      wg[i][0] = weight[i];
      wg[i][1] = gradient[i];
   }
   PrintMatrixFormat pmf = new PrintMatrixFormat();
   pmf.SetColumnLabels(new System.String[]{"Weights", "Gradients"});
   new PrintMatrix().Print(pmf, wg);

   double[][] report = new double[nObs][];
   for (int i2 = 0; i2 < nObs; i2++)
   {
      report[i2] = new double[nInputs + nOutputs + 2];
```

```
      }
      for (int i = 0; i < nObs; i++)
      {
         for (int j = 0; j < nInputs; j++)
         {
            report[i][j] = xData[i,j];
         }
         report[i][nInputs] = yData[i];
            double[] xTmp = new double[xData.GetLength(1)];
            for (int j=0; j<xData.GetLength(1); j++)
                xTmp[j] = xData[i,j];
         double[] p = classification.Probabilities(xTmp);
         for (int j = 0; j < nOutputs; j++)
         {
            report[i][nInputs + 1 + j] = p[j];
         }
         report[i][nInputs + nOutputs + 1] =
            classification.PredictedClass(xTmp);
      }
      pmf = new PrintMatrixFormat();
      pmf.SetColumnLabels(new System.String[]{"X1", "X2", "X3", "Y", "P(C1)",
         "P(C2)", "P(C3)", "Predicted"});
      new PrintMatrix("Forecast").Print(pmf, report);
      System.Console.Out.WriteLine("Cross-Entropy Error Value = " +
         trainer.ErrorValue);

      // ************************************************************************
      // DISPLAY CLASSIFICATION STATISTICS
      // ************************************************************************
      double[] statsClass = classification.ComputeStatistics(xData, yData);
      // Display Network Errors
      System.Console.Out.WriteLine(
         "*******************************************");
      System.Console.Out.WriteLine("--> Cross-Entropy Error:      " +
         (float)statsClass[0]);
      System.Console.Out.WriteLine("--> Classification Error:     " +
         (float)statsClass[1]);
      System.Console.Out.WriteLine(
         "*******************************************");
      System.Console.Out.WriteLine("");
   }
}
```

Output

```
*******************************************
--> Cross-Entropy Error:      0
--> Classification Error:     0
*******************************************

         Weights              Gradients
 0   3.2662441281062    -1.07718767511655E-21
 1  -3.65785075719586    9.08961951958712E-21
 2  -1.34287873934408    4.06084823246345E-20
 3  -1.35223399551086    1.06265766864622E-14
```

```
 4   2.68535529295218   -7.71219393879981E-15
 5  -2.87468867482566    6.14442307941349E-14
 6   0.763542892743849   3.50677840927428E-42
 7   2.27362473750162   -4.55519260305946E-42
 8   5.01937720316383   -2.23318295615209E-42
 9   4.54576678825492   -3.19544164040846E-18
10  -3.2703465073713     1.92921479004263E-15
11  -0.275420280883618  -2.00863231270801E-15
12  -5.9048518354049     1.72804459210703E-17
13   4.58077622000282   -1.04328902233367E-14
14   2.32407561540204    1.08624405719788E-14
15  -2.89362789116603   -8.61521051367963E-18
16  -6.4893352039981     5.20134410597363E-15
17  10.3829630951641    -5.41550390842253E-15
18   0.809208307413763   1.06265756092745E-14
19   0.251242239686325  -7.71218484918029E-15
20   1.41064046373263    6.14442714026172E-14
21   0.278456324867293   5.88451285169608E-18
22   0.345102584752059  -3.55271367880049E-15
23  -0.623558909619325   3.69899697182332E-15

                                        Forecast
   X1  X2  X3  Y         P(C1)                 P(C2)                 P(C3)          Predicted
0  1   0   0   1  1                     3.76564781867374E-30  1.95553436920243E-21      1
1  1   0   0   1  1                     3.76564781867374E-30  1.95553436920243E-21      1
2  0   1   0   2  2.94225642584804E-18  0.999999999999998     1.84949653037729E-15      2
3  0   1   0   2  2.94225642584804E-18  0.999999999999998     1.84949653037729E-15      2
4  0   0   1   3  3.85758062517221E-43  5.41178908162658E-47  1                         3
5  0   0   1   3  3.85758062517221E-43  5.41178908162658E-47  1                         3

Cross-Entropy Error Value = 0
*********************************************
--> Cross-Entropy Error:      0
--> Classification Error:     0
*********************************************
```

ScaleFilter Class

```
public class Imsl.DataMining.Neural.ScaleFilter
```

Scales or unscales continuous data prior to its use in neural network training, testing, or forecasting.

Bounded scaling is used to ensure that the values in the scaled array fall between a lower and upper bound. The scale limits have the following interpretation:

Argument	Interpretation
realMin	The lowest value expected in x.
realMax	The largest value expected in x.
targetMin	The lower bound for the values in the scaled data.
targetMax	The upper bound for the values in the scaled data.

The scale limits are set using the method SetBounds (p. 1747).

The specific scaling used is controlled by the argument scalingMethod used when constructing the filter object. If scalingMethod is ScalingMethod.None, then no scaling is performed on the data.

If the input parameter scalingMethod is ScaleMethod.Bounded then the bounded method of scaling and unscaling is applied to *x*. The scaling operation is conducted using the scale limits set in method SetBounds, using the following calculation:

$$z = r(x - \text{realMin}) + \text{targetMin},$$

where

$$r = \frac{\text{targetMax} - \text{targetMin}}{\text{realMax} - \text{realMin}}.$$

If scalingMethod is one of UnboundedZScoreMeanStdev, UnboundedZScoreMedianMAD, BoundedZScoreMeanStdev, or BoundedZScoreMedianMAD, then the z-score method of scaling is used. These calculations are based upon the following scaling calculation:

$$z = \frac{(x - a)}{b},$$

where *a* is a measure of center for *x*, and *b* is a measure of the spread of *x*.

If scalingMethod is UnboundedZScoreMeanStdev, or BoundedZScoreMeanStdev, then *a* and *b* are the arithmetic average and sample standard deviation of the training data.

If scalingMethod is UnboundedZScoreMedianMAD or BoundedZScoreMedianMAD, then *a* and *b* are the median and $\tilde{s}$, where $\tilde{s}$ is a robust estimate of the population standard deviation:

$$\tilde{s} = \frac{\text{MAD}}{0.6745}$$

where MAD is the Mean Absolute Deviation

$$\text{MAD} = median\{\mid x - median\{x\} \mid\}$$

The Mean Absolute Deviation is a robust measure of spread calculated by finding the median of the absolute value of differences between each non-missing value for the *i*-th variable and the median of those values.

If the method `Decode` (p. 1745) is called then an unscaling operation is conducted by inverting using:

$$x = \frac{(z - \text{targetMin})}{r} + \text{realMin}.$$

Unbounded z-score Scaling

If `scalingMethod` is `UnboundedZScoreMeanStdev` or `UnboundedZScoreMedianMAD`, then a scaling operation is conducted using the z-score calculation:

$$z = \frac{(x - center)}{spread},$$

If `scalingMethod` is `UnboundedZScoreMeanStdev` then `Center` (p. 1744) is set equal to the arithmetic average $\bar{x}$ of x, and `Spread` (p. 1745) is set equal to the sample standard deviation of x. If `scalingMethod` is `UnboundedZScoreMedianMAD` then `Center` is set equal to the median $\tilde{m}$ of x, and `Spread` is set equal to the Mean Absolute Difference (MAD).

The method `Decode` can be used to unfilter data using the inverse calculation for the above equation:

$$x = spread \cdot z + center.$$

Bounded z-score Scaling

This method is essentially the same as the z-score calculation described above with additional scaling or unscaling using the scale limits set in method `SetBounds`. The scaling operation is conducted using the well known z-score calculation:

$$z = \frac{r \cdot (x - center)}{spread} - r \cdot \text{realMin} + \text{targetMin}.$$

If `scalingMethod` is `UnboundedZScoreMeanStdev` then `Center` is set equal to the arithmetic average $\bar{x}$ of x, and `Spread` is set equal to the sample standard deviation of x. If `scalingMethod` is `UnboundedZScoreMedianMAD` then `Center` is set equal to the median $\tilde{m}$ of x, and `Spread` is set equal to the Mean Absolute Difference (MAD).

The method `Decode` can be used to unfilter data using the inverse calculation for the above equation:

$$x = \frac{spread \cdot (z - \text{targetMin})}{r} + spread \cdot \text{realMin} + center$$

Properties

Center

```
virtual public double Center {get; set; }
```

Description

The measure of center to be used during z-score scaling.

Property Value

A `double` containing the measure of center to be used during scaling.

Remarks

If this property is not set then the measure of center is computed from the data.

Spread

```
virtual public double Spread {get; set; }
```

Description

The measure of spread to be used during z-score scaling.

Property Value

A `double` containing the measure of spread to be used during z-score scaling.

Remarks

If this property is not set then the measure of spread is computed from the data.

Constructor

ScaleFilter

```
public ScaleFilter(Imsl.DataMining.Neural.ScaleFilter.ScalingMethod
scalingMethod)
```

Description

Constructor for `ScaleFilter`.

Parameter

`scalingMethod` – An `int` specifying the scaling method to be applied.

Remarks

`scalingMethod` is specified by: ScalingMethod.None (p. 1752), ScalingMethod.Bounded (p. 1751), ScalingMethod.UnboundedZScoreMeanStdev (p. 1752), ScalingMethod.UnboundedZScoreMedianMAD (p. 1752), ScalingMethod.BoundedZScoreMeanStdev (p. 1752), or ScalingMethod.BoundedZScoreMedianMAD (p. 1752).

Methods

Decode

```
virtual public double Decode(double z)
```

Description

Unscales a value.

Parameter

`z` – A `double` containing the value to be unscaled.

Returns

A `double` containing the filtered data.

Decode

```
virtual public double[] Decode(double[] z)
```

Description

Unscales an array of values.

Parameter

`z` – A `double` array of values to be unscaled.

Returns

A `double` array containing the filtered data.

Decode

```
virtual public void Decode(int columnIndex, double[,] z)
```

Description

Unscales a single column of a two dimensional array of values.

Parameters

`columnIndex` – An `int` specifying the index of the column of `z` to unscale.

`z` – A `double` matrix containing the values to be unscaled.

Remarks

Indexing is zero-based. Its `columnIndex`-th column is modified in place.

Encode

```
virtual public double Encode(double x)
```

Description

Scales a value.

Parameter

`x` – A `double` containing the value to be scaled.

Returns

A `double` containing the scaled value.

Encode

```
virtual public double[] Encode(double[] x)
```

Description

Scales an array of values.

Parameter

`x` – A `double` array containing the data to be scaled.

Returns

A `double` array containing the scaled data.

Encode

```
virtual public void Encode(int columnIndex, double[,] x)
```

Description

Scales a single column of a two dimensional array of values.

Parameters

`columnIndex` – An `int` specifying the index of the column of `x` to scale.

`x` – A `double` matrix containing the value to be scaled.

Remarks

Indexing is zero-based. Its `columnIndex`-th column is modified in place.

GetBounds

```
virtual public double[] GetBounds()
```

Description

Retrieves bounds used during bounded scaling.

Returns

A `double` array of length 4 containing the bounds.

Remarks

i	result[b]
0	`realMin`. Lowest expected value in the data to be filtered.
1	`realMax`. Largest expected value in the data to be filtered.
2	`targetMin`. Lowest allowed value in the filtered data.
3	`targetMax`. Largest allowed value in the filtered data.

SetBounds

```
virtual public void SetBounds(double realMin, double realMax, double targetMin,
double targetMax)
```

Description

Sets bounds to be used during bounded scaling and unscaling.

Parameters

`realMin` – A `double` containing the lowest expected value in the data to be filtered.

`realMax` – A `double` containing the largest expected value in the data to be filtered.

`targetMin` – A `double` containing the lowest allowed value in the filtered data.

`targetMax` – A `double` containing the largest allowed value in the filtered data.

Remarks

This method is normally called prior to calls to `Encode` or `Decode`. Otherwise the default bounds are `realMin` = 0, `realMax` = 1, `targetMin` = 0, and `targetMax` = 1. These bounds are ignored for unbounded scaling.

Example: ScaleFilter

In this example three sets of data, X_0, X_1, and X_2 are scaled using the methods described in the following table:

Variables and Scaling Methods

Variable	Method	Description
X_0	0	No Scaling
X_1	4	Bounded Z-score scaling using the mean and standard deviation of X_1
X_2	5	Bounded Z-score scaling using the median and MAD of X_2

The bounds, measuresof center and spreadfor $\mathbf{X_1}$ and $\mathbf{X_2}$ are:

Scaling Limits and Measures of Center and Spread

Variable	Real Limits	Target Limits	Measure of Center	Measure of Spread
X_1	(-6, +6)	(-3, +3)	3.4 (Mean)	1.7421 (Std. Dev.)
X_2	(-3, +3)	(-3, +3)	2.4 (Median)	1.3343(MAD/0.6745)

The real and target limits are used for bounded scaling. The measures of center and spread are used to calculate z-scores. Using these values for $\mathbf{x_1[0]=3.5}$ yields the following calculations:

For $\mathbf{x_1[0]}$, the scale factor is calculated using the real and target limits in the above table:

r = (3-(-3))/(6-(-6)) = 0.5

The z-score for $\mathbf{x_1[0]}$ is calculated using the measures of center and spread:

$\mathbf{z_1[0]}$ = (3.5 - 3.4)/1.7421 = 0.057402

Since method=4 is used for $\mathbf{x_1}$, this z-score is bounded (scaled) using the real and target limits:

$\mathbf{z_1(bounded)}$ = **r**($\mathbf{z_1[0]}$) - r(realMin) + (targetMin)
= 0.5(0.057402) - 0.5(-6) + (-3) = 0.029

Thecalculations for $\mathbf{x_2[0]}$ are nearly identical, except that since method=5 for $\mathbf{x_2}$, the median and MAD replace the mean and standard deviation used to calculate $\mathbf{z_1(bounded)}$:

r = (3-(-3))/(3-(-3)) = 1,

$\mathbf{z_2[0]}$ = (3.1 - 2.4)/1.3343 = 0.525, and

$\mathbf{z_2(bounded)}$ = **r**($\mathbf{z_2[0]}$) - r(realMin) + (targetMin)
= 1(0.525) - 1(-3) + (-3) = 0.525

```
using System;
using Imsl.Stat;
using Imsl.Math;
using Imsl.DataMining.Neural;

public class ScaleFilterEx1
{


   public static void  Main(System.String[] args)
   {
      ScaleFilter[] scaleFilter = new ScaleFilter[3];
      scaleFilter[0] = new ScaleFilter(ScaleFilter.ScalingMethod.None);
      scaleFilter[1] = new ScaleFilter(
         ScaleFilter.ScalingMethod.BoundedZScoreMeanStdev);
      scaleFilter[1].SetBounds(- 6.0, 6.0, - 3.0, 3.0);
      scaleFilter[2] = new ScaleFilter(
         ScaleFilter.ScalingMethod.BoundedZScoreMedianMAD);
      scaleFilter[2].SetBounds(- 3.0, 3.0, - 3.0, 3.0);
      double[] y0, y1, y2;
      double[] x0 = new double[]{1.2, 0.0, - 1.4, 1.5, 3.2};
      double[] x1 = new double[]{3.5, 2.4, 4.4, 5.6, 1.1};
      double[] x2 = new double[]{3.1, 1.5, - 1.5, 2.4, 4.2};

      // Perform forward filtering
      y0 = scaleFilter[0].Encode(x0);
      y1 = scaleFilter[1].Encode(x1);
      y2 = scaleFilter[2].Encode(x2);
      // Display x0
      System.Console.Out.Write("X0 = {");
      for (int i = 0; i < 4; i++)
         System.Console.Out.Write(x0[i] + ", ");
      System.Console.Out.WriteLine(x0[4] + "}");
      // Display summary statistics for X1
      System.Console.Out.Write("\nX1 = {");
      for (int i = 0; i < 4; i++)
         System.Console.Out.Write(x1[i] + ", ");
      System.Console.Out.WriteLine(x1[4] + "}");
      System.Console.Out.WriteLine("X1 Mean:       " + scaleFilter[1].Center);
      System.Console.Out.WriteLine("X1 Std. Dev.:  " + scaleFilter[1].Spread);
      // Display summary statistics for X2
      System.Console.Out.Write("\nX2 = {");
      for (int i = 0; i < 4; i++)
         System.Console.Out.Write(x2[i] + ", ");
      System.Console.Out.WriteLine(x2[4] + "}");
      System.Console.Out.WriteLine("X2 Median:     " + scaleFilter[2].Center);
      System.Console.Out.WriteLine("X2 MAD/0.6745: " + scaleFilter[2].Spread);
      System.Console.Out.WriteLine("");
      PrintMatrix pm = new PrintMatrix();
```

```
      pm.SetTitle("Filtered X0 Using Method=0 (no scaling)");
      pm.Print(y0);
      pm.SetTitle("Filtered X1 Using Bounded Z-score Scaling\n" +
         "with Center=Mean and Spread=Std. Dev.");
      pm.Print(y1);
      pm.SetTitle("Filtered X2 Using Bounded Z-score Scaling\n" +
         "with Center=Median and Spread=MAD/0.6745");
      pm.Print(y2);

      // Perform inverse filtering
      double[] z0, z1, z2;
      z0 = scaleFilter[0].Decode(y0);
      z1 = scaleFilter[1].Decode(y1);
      z2 = scaleFilter[2].Decode(y2);
      pm.SetTitle("Decoded Z0");
      pm.Print(z0);
      pm.SetTitle("Decoded Z1");
      pm.Print(z1);
      pm.SetTitle("Decoded Z2");
      pm.Print(z2);
   }
}
```

Output

```
X0 = {1.2, 0, -1.4, 1.5, 3.2}

X1 = {3.5, 2.4, 4.4, 5.6, 1.1}
X1 Mean:       3.4
X1 Std. Dev.:  1.74212513901843

X2 = {3.1, 1.5, -1.5, 2.4, 4.2}
X2 Median:     2.4
X2 MAD/0.6745: 1.33434199665504

Filtered X0 Using Method=0 (no scaling)
    0
0   1.2
1   0
2  -1.4
3   1.5
4   3.2

Filtered X1 Using Bounded Z-score Scaling
with Center=Mean and Spread=Std. Dev.
            0
0   0.0287005788965145
1  -0.287005788965146
2   0.287005788965146
3   0.631412735723321
4  -0.660113314619835

Filtered X2 Using Bounded Z-score Scaling
with Center=Median and Spread=MAD/0.6745
           0
```

```
0   0.524603139041397
1  -0.674489750196082
2  -2.92278891751635
3   0
4   1.34897950039216

Decoded Z0
    0
0   1.2
1   0
2  -1.4
3   1.5
4   3.2

Decoded Z1
    0
0  3.5
1  2.4
2  4.4
3  5.6
4  1.1

Decoded Z2
    0
0   3.1
1   1.5
2  -1.5
3   2.4
4   4.2
```

ScaleFilter.ScalingMethod Enumeration

`public enumeration Imsl.DataMining.Neural.ScaleFilter.ScalingMethod`

Scaling Method

Fields

Bounded

`public Imsl.DataMining.Neural.ScaleFilter.ScalingMethod Bounded`

Description

Flag to indicate bounded scaling.

BoundedZScoreMeanStdev

```
public Imsl.DataMining.Neural.ScaleFilter.ScalingMethod BoundedZScoreMeanStdev
```

Description

Flag to indicate bounded z-score scaling using the mean and standard deviation.

BoundedZScoreMedianMAD

```
public Imsl.DataMining.Neural.ScaleFilter.ScalingMethod BoundedZScoreMedianMAD
```

Description

Flag to indicate bounded z-score scaling using the median and mean absolute difference.

None

```
public Imsl.DataMining.Neural.ScaleFilter.ScalingMethod None
```

Description

Flag to indicate no scaling.

UnboundedZScoreMeanStdev

```
public Imsl.DataMining.Neural.ScaleFilter.ScalingMethod
UnboundedZScoreMeanStdev
```

Description

Flag to indicate unbounded z-score scaling using the mean and standard deviation.

UnboundedZScoreMedianMAD

```
public Imsl.DataMining.Neural.ScaleFilter.ScalingMethod
UnboundedZScoreMedianMAD
```

Description

Flag to indicate unbounded z-score scaling using the median and mean absolute difference.

UnsupervisedNominalFilter Class

```
public class Imsl.DataMining.Neural.UnsupervisedNominalFilter
```

Converts nominal data into a series of binary encoded columns for input to a neural network. It also reverses the aforementioned encoding, accepting binary encoded data and returns an array of integers representing the classes for a nominal variable.

Binary Encoding

Method `Encode` (p. 1754) can be used to apply binary encoding. Referring to the result as *z*, binary encoding takes each category in the nominal variable `x`, and creates a column in *z* containing all zeros and ones. A value of zero indicates that this category was not present and a value of one indicates that it is present.

For example, if `x[]={2,1,3,4,2,4}` then `nClasses=4`, and

$$z = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Notice that the number of columns in the result, *z*, is equal to the number of distinct classes in `x`. The number of rows in *z* is equal to the length of `x`.

Binary Decoding

Unfiltering can be performed using the method `Decode` (p. 1754). In this case, `z` is the input, and we refer to *x* as the output. Binary unfiltering takes binary representation in `z`, and returns the appropriate class in *x*.

For example, if a row in `z` equals {0, 1, 0, 0}, then the return value from `Decode` would be 2 for that row. If a row in `z` equals {1,0,0,0}, then the return value from `Decode` would be 1 for that row. Notice these are the same values as the first two elements of the original *x* because classes are numbered sequentially from 1 to `nClasses`. This ensures that the results of `Decode` are associated with the *i*-th class in *x*.

Property

NumberOfClasses

```
virtual public int NumberOfClasses {get; }
```

Description

The number of classes in the nominal variable.

Property Value

An `int` containing the number of classes in the nominal variable.

Constructor

UnsupervisedNominalFilter

```
public UnsupervisedNominalFilter(int nClasses)
```

Description

Constructor for `UnsupervisedNominalFilter`.

Parameter

`nClasses` – An `int` specifying the number of categories in the nominal variable to be filtered.

Methods

Decode

```
virtual public int Decode(int[] z)
```

Description

Decodes a binary encoded array into its nominal category.

Parameter

`z` – An `int` array containing the data to be decoded.

Returns

An `int` containing the number associated with the category encoded in `z`.

Remarks

This is the inverse of the `Encode` (p. 1754) method.

Decode

```
virtual public int[] Decode(int[,] z)
```

Description

Decodes a matrix representing the binary encoded columns of the nominal variable.

Parameter

`z` – An `int` matrix containing the data to be decoded.

Returns

An `int` array containing the decoded data.

Remarks

This is the inverse of the `Encode` (p. 1754) method.

Encode

```
virtual public int[,] Encode(int[] x)
```

Description

Encodes class data prior to its use in neural network training.

Parameter

`x` – An `int` array containing the data to be encoded.

Returns

An `int` matrix containing the encoded data.

Remarks

Class number must be in the range 1 to `nClasses`.

Encode

```
virtual public int[] Encode(int x)
```

Description

Apply forward encoding to a value.

Parameter

`x` – An `int` containing the value to be encoding.

Returns

An `int` array containing the encoded data.

Remarks

Class number must be in the range 1 to `nClasses`.

Example: UnsupervisedNominalFilter

In this example a data set with 7 observations and 3 classes is filtered.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using Imsl.DataMining.Neural;

public class UnsupervisedNominalFilterEx1
{


   public static void  Main(System.String[] args)
   {
      int nClasses = 3;
      UnsupervisedNominalFilter filter = new UnsupervisedNominalFilter(nClasses);
      int nObs = 7;
      int[] x = new int[]{3, 3, 1, 2, 2, 1, 2};
      int[] xBack = new int[nObs];
      int[,] z;

      // Perform Binary Filtering.
      z = filter.Encode(x);
      PrintMatrix pm = new PrintMatrix();
      pm.SetTitle("Filtered x");
      pm.Print(z);

      //  Perform Binary Un-filtering.
      int[] tmp = new int[z.GetLength(1)];
```

```
      for (int i = 0; i < nObs; i++)
      {
         for (int j=0; j< z.GetLength(1); j++)
            tmp[j] = z[i,j];
         xBack[i] = filter.Decode(tmp);
      }
      pm.SetTitle("Result of inverse filtering");
      pm.Print(xBack);
   }
}
```

Output

```
 Filtered x
   0  1  2
0  0  0  1
1  0  0  1
2  1  0  0
3  0  1  0
4  0  1  0
5  1  0  0
6  0  1  0

Result of inverse filtering
   0
0  3
1  3
2  1
3  2
4  2
5  1
6  2
```

UnsupervisedOrdinalFilter Class

`public class Imsl.DataMining.Neural.UnsupervisedOrdinalFilter`

Encodes ordinal data into percentages for input to a neural network. It also allows decoding, accepting a percentage and converting it into an ordinal value.

Class `UnsupervisedOrdinalFilter` is designed to either encode or decode ordinal variables. Encoding consists of transforming the ordinal classes into percentages, with each percentage being equal to the percentage of the data at or below this class.

Ordinal Encoding

In this case, `x` is input to the method `Encode` (p. 1759) and is filtered by converting each ordinal class value into a cumulative percentage.

For example, if `x[]={2,1,3,4,2,4,1,1,3,3}` then `nClasses =4`, and `Encode` returns the ordinal class designation with the cumulative percentages displayed in the following table. Cumulative percentages are equal to the percent of the data in this class or a lower class.

Ordinal Class	Frequency	Cumulative Percentage
1	3	30%
2	2	50%
3	3	80%
4	2	100%

Classes in `x` must be numbered from 1 to `nClasses`.

The values returned from encoding or decoding depend upon the setting of `Transform` (p. 1758). In this example, if the filter was constructed with `Transform = TransformMethod.None`, then the method `Encode` will return

$$z[] = \{50, 30, 80, 100, 50, 100, 30, 30, 80, 80\}.$$

If the filter was constructed with `Transform = TransformMethod.Sqrt`, then the square root of these values is returned; i.e.,

$$z[i] = \sqrt{\frac{z[i]}{100}}$$

$$z[] = \{0.71, 0.55, 0.89, 1.0, 0.71, 1.0, 0.55, 0.55, 0.89, 0.89\};$$

If the filter was constructed with `Transform = TransformMethod.AsinSqrt`, then the arcsin square root of these values is returned using the following calculation:

$$z[i] = \arcsin\left(\sqrt{\frac{z[i]}{100}}\right)$$

Ordinal Decoding

Ordinal decoding takes a transformed cumulative proportion and converts it into an ordinal class value.

Properties

NumberOfClasses

```
virtual public int NumberOfClasses {get; }
```

Description

The number of categories associated with this ordinal variable.

Property Value

An `int` containing the number of categories associated with this ordinal variable.

Percentages

```
virtual public double[] Percentages {get; set; }
```

Description

The cumulative percentages used during encoding and decoding.

Property Value

A `double` array of length `nClasses` containing the cumulative percentages associated with the ordinal categories.

Remarks

If a transform has been applied to the percentages then the transformed percentages are returned. Setting untransformed cumulative percentages with this method bypasses calculating cumulative percentages based on the data being encoded. The percentages must be nondecreasing in the interval [0, 100], with the last element equal to 100. If this method is used it must be called prior to any calls to the encoding and decoding methods.

Transform

```
virtual public int Transform {get; }
```

Description

The transform flag used for encoding and decoding.

Property Value

An `int` containing the transform flag used for encoding and decoding.

Constructor

UnsupervisedOrdinalFilter

```
public UnsupervisedOrdinalFilter(int nClasses,
Imsl.DataMining.Neural.UnsupervisedOrdinalFilter.TransformMethod transform)
```

Description

Constructor for `UnsupervisedOrdinalFilter`.

Parameters

`nClasses` – An `int` specifying the number of classes in the data to be filtered.

`transform` – An `TransformMethod` specifying the transform to be applied to the percentages.

Remarks

Values for `Transform` (p. 1758) are: `TransformMethod.None` (p. 1762), `TransformMethod.Sqrt` (p. 1762), `TransformMethod.AsinSqrt` (p. 1762)

Methods

Decode

```
virtual public int Decode(double y)
```

Description

Decodes an encoded ordinal variable.

Parameter

`y` – A `double` containing the encoded value to be decoded.

Returns

An `int` containing the ordinal category associated with `y`.

Decode

```
virtual public int[] Decode(double[] y)
```

Description

Decodes an array of encoded ordinal values.

Parameter

`y` – A `double` array containing the encoded ordinal data to be decoded.

Returns

An `int` array containing the decoded ordinal classifications.

Encode

```
virtual public double[] Encode(int[] x)
```

Description

Encodes an array of ordinal categories into an array of transformed percentages.

Parameter

`x` – An `int` array containing the categories for the ordinal variable.

Returns

A `double` array of the transformed percentages.

Remarks

Categories must be numbered from 1 to `nClasses`.

Encode

```
virtual public double Encode(int x)
```

Description

Encodes an ordinal category.

Parameter

`x` – An `int` containing the ordinal category.

Returns

A `double` containing the encoded value, a transformed cumulative percentage.

Remarks

`x` must be an integer between 1 and `nClasses`.

Example: UnsupervisedOrdinalFilter

In this example a data set with 10 observations and 4 classes is filtered.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using Imsl.DataMining.Neural;

public class UnsupervisedOrdinalFilterEx1
{


   public static void  Main(System.String[] args)
   {
      int nClasses = 4;
      UnsupervisedOrdinalFilter filter = new UnsupervisedOrdinalFilter(nClasses,
         UnsupervisedOrdinalFilter.TransformMethod.AsinSqrt);
      int[] x = new int[]{2, 1, 3, 4, 2, 4, 1, 1, 3, 3};
      int nObs = x.Length;
      int[] xBack;
      double[] z;
      // Ordinal Filtering.
      z = filter.Encode(x);
      // Print result without row/column labels.
      PrintMatrix pm = new PrintMatrix();
      PrintMatrixFormat mf;
      mf = new PrintMatrixFormat();
      mf.SetNoRowLabels();
      mf.SetNoColumnLabels();
      pm.SetTitle("Filtered data");
      pm.Print(mf, z);

      // Ordinal Un-filtering.
```

```
      pm.SetTitle("Un-filtered data");
      xBack = filter.Decode(z);

      // Print results of Un-filtering.
      pm.Print(mf, xBack);
   }
}
```

Output

```
    Filtered data

0.785398163397448
0.579639740363704
1.10714871779409
1.5707963267949
0.785398163397448
1.5707963267949
0.579639740363704
0.579639740363704
1.10714871779409
1.10714871779409

Un-filtered data

2
1
3
4
2
4
1
1
3
3
```

UnsupervisedOrdinalFilter.TransformMethod Enumeration

```
public enumeration
Imsl.DataMining.Neural.UnsupervisedOrdinalFilter.TransformMethod
```

Transform type

Fields

AsinSqrt

```
public Imsl.DataMining.Neural.UnsupervisedOrdinalFilter.TransformMethod
AsinSqrt
```

Description

Flag to indicate the arcsine square root transform will be applied to the percentages.

None

```
public Imsl.DataMining.Neural.UnsupervisedOrdinalFilter.TransformMethod None
```

Description

Flag to indicate no transformation of percentages.

Sqrt

```
public Imsl.DataMining.Neural.UnsupervisedOrdinalFilter.TransformMethod Sqrt
```

Description

Flag to indicate the square root transform will be applied to the percentages.

TimeSeriesFilter Class

```
public class Imsl.DataMining.Neural.TimeSeriesFilter
```

Converts time series data to a lagged format used as input to a neural network.

Class `TimeSeriesFilter` can be used to operate on a data matrix and lags every column to form a new data matrix. Using the method `ComputeLags` (p. 1763), each column of the input matrix, `x`, is transformed into (`nLags`+1) columns by creating a column for $lags = 0, 1, \ldots$ nLags.

The output data array, *z*, can be symbolically represented as:

$$z = |x[0] : x[1] : x[2] : \ldots : x[nLags - 1]|,$$

where `x[i]` is a lagged column of the incoming data matrix, `x`.

Consider, an example in which `x` has five rows and two columns with all variables continuous input attributes. Using *nObs* and *nVar* to represent the number of rows and columns in `x`, let

$$x = \begin{bmatrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \\ 5 & 10 \end{bmatrix}$$

If `nLags=1`, then the number of columns in *z[,]* is *nVar**(`nLags`+1) = 2*2 = 4, and the number of rows is (*nObs*-`nLags`) = 5-1 = 4:

$$z = \begin{bmatrix} 1 & 6 & 2 & 7 \\ 2 & 7 & 3 & 8 \\ 3 & 8 & 4 & 9 \\ 4 & 9 & 5 & 10 \end{bmatrix}$$

If `nLags=2`, then the number of rows in *z* will be (*nObs*-`nLags`) = (5-2) = 3 and the number of columns will be *nVar**(`nLags`+1) = 2*3 = 6:

$$z = \begin{bmatrix} 1 & 6 & 2 & 7 & 3 & 8 \\ 2 & 7 & 3 & 8 & 4 & 9 \\ 3 & 8 & 4 & 9 & 5 & 10 \end{bmatrix}$$

Constructor

TimeSeriesFilter

```
public TimeSeriesFilter()
```

Description

Constructor for `TimeSeriesClassFilter`.

Method

ComputeLags

```
virtual public double[,] ComputeLags(int nLags, double[,] x)
```

Description

Lags time series data to a format used for input to a neural network.

Parameters

`nLags` – An `int` containing the requested number of lags.

`x` – A `double` matrix, *nObs* by *nVar*, containing the time series data to be lagged.

Returns

A `double` matrix with (*nObs*-`nLags`) rows and (*nVar*(`nLags`+*1*)) columns. The columns 0 through (*nVar*-1) contain the columns of `x`. The next *nVar* columns contain the first lag of the columns in `x`, etc.

Remarks

`nLags` must be greater than 0. It is assumed that `x` is sorted in descending chronological order.

Example: TimeSeriesFilter

In this example a matrix with 5 rows and 2 columns is lagged twice. This produces a two-dimensional matrix with 5 rows, but 2*3=6 columns. The first two columns correspond to lag=0, which just places the original data into these columns. The 3rd and 4th columns contain the first lags of the original 2 columns and the 5th and 6th columns contain the second lags.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using Imsl.DataMining.Neural;

public class TimeSeriesFilterEx1
{


   public static void  Main(System.String[] args)
   {
      TimeSeriesFilter filter = new TimeSeriesFilter();
      int nLag = 2;
      double[,] x = {{1, 6}, {2, 7}, {3, 8}, {4, 9}, {5, 10}};
      double[,] z = filter.ComputeLags(nLag, x);
      // Print result without row/column labels.
      PrintMatrix pm = new PrintMatrix();
      PrintMatrixFormat mf;
      mf = new PrintMatrixFormat();
      mf.SetNoRowLabels();
      mf.SetNoColumnLabels();
      pm.SetTitle("Lagged data");
      pm.Print(mf, z);
   }
}
```

Output

```
     Lagged data

1  6  2  7  3   8
2  7  3  8  4   9
3  8  4  9  5  10
```

TimeSeriesClassFilter Class

```
public class Imsl.DataMining.Neural.TimeSeriesClassFilter
```

Converts time series data contained within nominal categories to a lagged format for processing by a neural network. Lagging is done within the nominal categories associated with the time series.

Class `TimeSeriesClassFilter` can be used with a data array, `x` to compute a new data array, *z[,]*, containing lagged columns of `x`.

When using the method `ComputeLags` (p. 1766), the output array, *z[,]* of lagged columns, can be symbolically represented as:

$$z = |x[0] : x[1] : x[2] : \ldots : x[nLags - 1]|,$$

where *x[i]* is a lagged column of the incoming data array `x`, and *nLags* is the number of computed lags. The lag associated with *x[i]* is equal to the value in `lags[i]`, and lagging is done within the nominal categories given in `iClass`. This requires the time series data in `x[]` be sorted in time order within each category `iClass`.

Consider an example in which the number of observations in `x[]` is 10. There are two lags requested in `lags`. If

$$x^T = \{1,2,3,4,5,6,7,8,9,10\},$$
$$iClass^T = \{1,1,1,1,1,1,1,1,1,1\},$$

and

$$lag^T = \{0,2\}$$

then, all the time series data fall into a single category, i.e. `nClasses` = 1, and *z* would contain 2 columns and 10 rows. The first column reproduces the values in `x[]` because `lags[0]` = 0, and the second column is the 2nd lag because `lags[1]` = 2.

$$z = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 6 \\ 5 & 7 \\ 6 & 8 \\ 7 & 9 \\ 8 & 10 \\ 9 & NaN \\ 10 & NaN \end{bmatrix}$$

On the other hand, if the data were organized into two classes with

$$iClass^T = \{1,1,1,1,1,2,2,2,2,2\},$$

then `nClasses` is 2, and *z* is still a 2 by 10 matrix, but with the following values:

$$z = \left[\begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & NaN \\ 5 & NaN \\ \hline 6 & 8 \\ 7 & 9 \\ 8 & 10 \\ 9 & NaN \\ 10 & NaN \end{array}\right]$$

The first 5 rows of *z* are the lagged columns for the first category, and the last five are the lagged columns for the second category.

Constructor

TimeSeriesClassFilter

```
public TimeSeriesClassFilter(int nClasses)
```

Description

Constructor for `TimeSeriesClassFilter`.

Parameter

`nClasses` – An `int` specifying the number of nominal categories associated with the time series.

Method

ComputeLags

```
virtual public double[,] ComputeLags(int[] lags, int[] iClass, double[] x)
```

Description

Computes `lags` of an array sorted first by class designations and then descending chronological order.

Parameters

`lags` – An `int` array containing the requested `lags`.

`iClass` – An `int` array containing class number associated with each element of `x`, sorted in ascending order.

`x` – A `double` array containing the time series data to be lagged.

Returns

A `double` matrix containing the lagged data. The *i*-th column of this array is the lagged values of `x` for a lag equal to `lags[i]`. The number of rows is equal to the length of `x`.

Remarks

Every lag must be non-negative.

The *i*-th element of `iClass` is equal to the class associated with the *i*-th element of `x`. `iClass` and `x` must be the same length.

`x` is assumed to be sorted first by class designations and then descending chronological order; i.e., most recent observations appear first within a class.

Example: TimeSeriesClassFilter

For illustration purposes, the time series in this example consists of the integers 1, 2, ..., 10, organized into two classes. Of course, it is assumed that these data are sorted in chronologically descending order. That is for each class, the first number is the latest value and the last number in that class is the earliest.

The values 1-4 are in class 1, and the values 5-10 are in class 2. These values represent two separate time series, one for each class. If you were to list them in chronologically ascending order, starting with time=T_0, the values would be:

Class 1: T_0=4, T_1=3, T_2=2, T_3=1
Class 2: T_0=10, T_1=9, T_2=8, T_3=7, T_4=6, T_5=5

This example requests lag calculations for lags 0, 1, 2, 3. For lag=0, no lagging is performed. For lag=1, the value at time = t replaced with the value at time = t-1, the previous value in that class. If $t - 1 < 0$, then a missing value is placed in that position.

For example, the first lag of a time series at time=t are the values at time=t-1. For the time series values of Class 1 (lag=1), these values are:

Class 1, lag 1: T_0=NaN, T_1=4, T_2=3, T_3=2

The second lag for time=t consists of the values at time=t-2:

Class 1, lag 2: T_0=NaN, T_1=NaN, T_2=4, T_3=3

Notice that the second lag now has two missing observations. In general, lag=n will have n missing values. In some cases this can result in all missing values for classes with few observations. A class will have all missing values in any of its lag columns that have a lag value larger than or equal to the number of observations in that class.

```
using System;
using Imsl.Stat;
using Imsl.Math;
using Imsl.DataMining.Neural;

public class TimeSeriesClassFilterEx1
{
   private static int nClasses = 2;
```

```
   private static int nObs = 10;
   private static int nLags = 4;


   public static void  Main(System.String[] args)
   {

      double[] x = new double[]{1, 2, 3, 4, 5, 6, 7, 8, 9, 10};
      double[] time = new double[]{3, 2, 1, 0, 5, 4, 3, 2, 1, 0};
      int[] iClass = new int[]{1, 1, 1, 1, 2, 2, 2, 2, 2, 2};
      int[] lag = new int[]{0, 1, 2, 3};
      System.String[] colLabels = new System.String[]{"Class", "Time", "Lag=0",
         "Lag=1", "Lag=2", "Lag=3"};

      // Filter Classified Time Series Data
      TimeSeriesClassFilter filter = new TimeSeriesClassFilter(nClasses);
      double[,] y = filter.ComputeLags(lag, iClass, x);
      double[,] z = new double[nObs, (nLags + 2)];
      // for (int i = 0; i < nObs; i++)
      // {
      //    z[i] = new double[nLags + 2];
      // }
      for (int i = 0; i < nObs; i++)
      {
         z[i,0] = (double) iClass[i];
         z[i,1] = time[i];
         for (int j = 0; j < nLags; j++)
         {
            z[i,j + 2] = y[i,j];
         }
      }

      // Print result without row/column labels.
      PrintMatrix pm = new PrintMatrix();
      PrintMatrixFormat mf;
      mf = new PrintMatrixFormat();
      mf.SetNoRowLabels();
      mf.SetColumnLabels(colLabels);
      pm.SetTitle("Lagged data");

      pm.Print(mf, z);
   }
}
```

Output

```
                Lagged data
  Class  Time  Lag=0  Lag=1  Lag=2  Lag=3
  1      3      1      2      3      4
  1      2      2      3      4      NaN
  1      1      3      4      NaN    NaN
  1      0      4      NaN    NaN    NaN
  2      5      5      6      7      8
  2      4      6      7      8      9
  2      3      7      8      9     10
```

```
 2     2      8     9     10      NaN
 2     1      9    10      NaN    NaN
 2     0     10     NaN    NaN    NaN
```

Example: Neural Network Application

This application illustrates one common approach to time series prediction using a neural network. In this case, the output target for this network is a single time series. In general, the inputs to this network consist of lagged values of the time series together with other concomitant variables, both continuous and categorical. In this application, however, only the first three lags of the time series are used as network inputs.

The objective is to train a neural network for forecasting the series Y_t, $t = 0, 1, 2, \ldots$, from the first three lags of Y_t, i.e.

$$Y_t = f(Y_{t-1}, Y_{t-2}, Y_{t-3})$$

Since this series consists of data from several company departments, lagging of the series must be done within departments. This creates many missing values. The original data contains 118,519 training patterns. After lagging, 16,507 are identified as missing and are removed, leaving a total of 102,012 usable training patterns. Missing values are denoted using a number not in the training patterns, the value -9,999,999,999.0.

The structure of the network consists of three input nodes and two layers, with three perceptrons in the hidden layer and one in the output layer. The following figure depicts this structure:

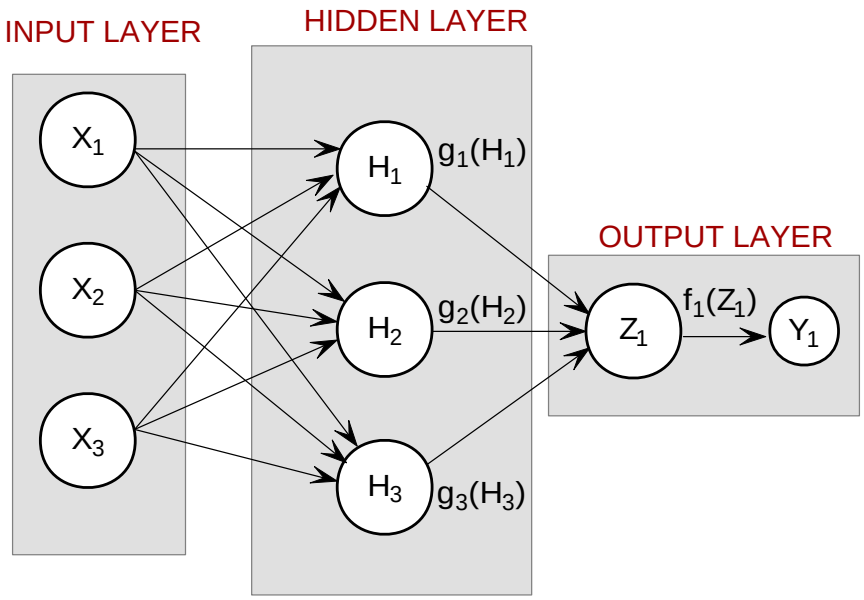

Figure 9. An example 2-layer Feed Forward Neural Network

There are a total of 16 weights in this network, including the 4 bias weights. All perceptrons in the hidden layer use logistic activation, and the output perceptron uses linear activation. Because of the large number of training patterns, the `Activation.LogisticTable` activation funtion is used instead of `Activation.Logistic`. `Activation.LogisticTable` uses a table lookup for calculating the logistic activation function, which significantly reduces training time. However, these are not completely interchangable. If a network is trained using `Activation.LogisticTable`, then it is important to use the same activation function for forecasting.

All input nodes are linked to every perceptron in the hidden layer, which are in turn linked to the output perceptron. Then all inputs and the output target are scaled using the `ScaleFilter` class to ensure that all input values and outputs are in the range [0, 1]. This requires forecasts to be unscaled using the `Decode()` method of the `ScaleFilter` class.

Training is conducted using the epoch trainer. This trainer allows users to customize training into two stages. Typically this is necessary when training using a large number of training patterns. Stage I training uses randomly selected subsets of training patterns to search for network solutions. Stage II training is optional, and uses the entire set of training patterns. For larger sets of training patterns, training could take many hours, or even days. In that case, Stage II training might be bypassed.

In this example, Stage I training is conducted using the quasi-Newton trainer applied to 20 epochs, each consisting of 5,000 randomly selected observations. Stage II training also uses the quasi-Newton trainer.

The training patterns are contained in two data files: `continuous.txt` and `output.txt`. The formats of these files are identical. The first line of the file contains the number of columns or variables in that file. The second contains a line of tab-delimited integer values. These are the column indices associated with the incoming data. The remaining lines contain tab-delimited, floating point values, one for each of the incoming variables.

For example, the first four lines of the `continuous.txt` file consists of the following lines:

```
3
1 2 3
0 0 0
0 0 0
```

There are 3 continuous input variables which are numbered, or labeled, as 1, 2, and 3.

Source Code

```
using System;
using Imsl.DataMining.Neural;
using Imsl.Math;
using System.Runtime.Serialization;
using System.Runtime.Serialization.Formatters.Binary;
//*************************************************************************
// NeuralNetworkEx1.java                                                  *
// Two Layer Feed-Forward Network Complete Example for Simple Time Series *
//*************************************************************************
// Synopsis:   This example illustrates how to use a Feed-Forward Neural  *
//             Network to forecast time series data.  The network target is a *
//             time series and the three inputs are the 1st, 2nd, and 3rd lag *
//             for the target series.                                     *
```

```
// Activation: Logistic_Table in Hidden Layer, Linear in Output Layer         *
// Trainer:    Epoch Trainer: Stage I - Quasi-Newton, Stage II - Quasi-Newton *
// Inputs:     Lags 1-3 of the time series                                    *
// Output:     A Time Series sorted chronologically in descending order,      *
//             i.e., the most recent observations occur before the earliest,  *
//             within each department                                         *
//*****************************************************************************

//[Serializable]
public class NeuralNetworkEx1 //: System.Runtime.Serialization.ISerializable
{

   private static System.String QuasiNewton = "quasi-newton";
   private static System.String LeastSquares = "least-squares";
   // ************************************************************************
   // Network Architecture                                                   *
   // ************************************************************************
   private static int nObs = 118519; // number of training patterns
   private static int nInputs = 3; // four inputs
   private static int nContinuous = 3; // one continuous input attribute
   private static int nOutputs = 1; // one continuous output
   private static int nPerceptrons = 3; // perceptrons in hidden layer
   private static int[] perceptrons = new int[]{3}; // # of perceptrons in each
   // hidden layer
   // PERCEPTRON ACTIVATION
   private static IActivation hiddenLayerActivation;
   private static IActivation outputLayerActivation;
   // ************************************************************************
   // Epoch Training Optimization Settings                                   *
   // ************************************************************************
   private static bool trace = true; //trainer logging    *
   private static int nEpochs = 20; //number of epochs   *
   private static int epochSize = 5000; //samples per epoch  *
   // Stage I Trainer - Quasi-Newton Trainer *********************************
   private static int stage1Iterations = 5000; //max. iterations  *
   private static double stage1StepTolerance = 1e-09; //step tolerance  *
   private static double stage1RelativeTolerance = 1e-11; //rel. tolerance *
   // Stage II Trainer - Quasi-Newton Trainer ********************************
   private static int stage2Iterations = 5000; //max. iterations    *
   private static double stage2StepTolerance = 1e-09; //step tolerance     *
   private static double stage2RelativeTolerance = 1e-11; //rel. tolerance    *
   // ************************************************************************
   // FILE NAMES AND FILE READER DEFINITIONS                                 *
   // ************************************************************************
   // READERS
   private static System.IO.StreamReader contFileInputStream;
   private static System.IO.StreamReader outputFileInputStream;
   // OUTPUT FILES
   // File Name for Serialized Network
   private static System.String networkFileName = "NeuralNetworkEx1.ser";
   // File Name for Serialized Trainer
   private static System.String trainerFileName = "NeuralNetworkTrainerEx1.ser";
   // File Name for Serialized xData File (training input attributes)
   private static System.String xDataFileName = "NeuralNetworkxDataEx1.ser";
   // File Name for Serialized yData File (training output targets)
   private static System.String yDataFileName = "NeuralNetworkyDataEx1.ser";
```

```
// INPUT FILES
// Continuous input attributes file.  File contains Lags 1-3 of series
private static System.String contFileName = "continuous.txt";
// Continuous network targets file.  File contains the original series
private static System.String outputFileName = "output.txt";
// ************************************************************************
// Data Preprocessing Settings                                            *
// ************************************************************************
private static double lowerDataLimit = - 105000; // lower scale limit
private static double upperDataLimit = 25000000; // upper scale limit
// indicator
// ************************************************************************
// Time Parameters for Tracking Training Time                             *
// ************************************************************************
private static int startTime;
// ************************************************************************
// Error Message Encoding for Stage II Trainer - Quasi-Newton Trainer      *
// ************************************************************************
// Note: For the Epoch Trainer, the error status returned is the status for*
// the Stage II trainer, unless Stage II training is not used.            *
// ************************************************************************
private static System.String errorMsg = "";
// Error Status Messages for the Quasi-Newton Trainer
private static System.String errorMsg0 = "--> Network Training";
private static System.String errorMsg1 =
   "--> The last global step failed to locate a lower point than the\n" +
   "current error value.  The current solution may be an approximate\n" +
   "solution and no more accuracy is possible, or the step tolerance\n" +
   "may be too large.";
private static System.String errorMsg2 =
   "--> Relative function convergence;  both both the actual and \n" +
   "predicted relative reductions in the error function are less than\n" +
   "or equal to the relative fu nction convergence tolerance.";
private static System.String errorMsg3 =
   "--> Scaled step tolerance satisfied;  the current solution may be\n" +
   "an approximate local solution, or the algorithm is making very slow\n" +
   "progress and is not near a solution, or the step tolerance is too big.";
private static System.String errorMsg4 =
   "--> Quasi-Newton Trainer threw a \n" +
   "MinUnconMultiVar.FalseConvergenceException.";
private static System.String errorMsg5 =
   "--> Quasi-Newton Trainer threw a \n" +
   "MinUnconMultiVar.MaxIterationsException.";
private static System.String errorMsg6 =
   "--> Quasi-Newton Trainer threw a \n" +
      "MinUnconMultiVar.UnboundedBelowException.";
// ************************************************************************
// MAIN                                                                   *
// ************************************************************************


public static void  Main(System.String[] args)
{

   double[] weight; // Network weights
   double[] gradient; // Network gradient after training
```

```
double[,] xData; // Training Patterns Input Attributes
double[,] yData; // Training Targets Output Attributes
double[,] contAtt; // A 2D matrix for the continuous training attributes
double[,] outs; // A matrix containing the training output tragets
int i, j, m = 0; // Array indicies
int nWeights = 0; // Number of network weights
int nCol = 0; // Number of data columns in input file
int[] ignore; // Array of 0's and 1's (0=missing value)
int[] cont_col, outs_col, isMissing = new int[]{0};
//System.String inputLine = "", temp;
//System.String[] dataElement;
// ************************************************************************
// Initialize timers                                                      *
// ************************************************************************
NeuralNetworkEx1.startTime =
   DateTime.Now.Hour * 60 * 60 * 1000 +
   DateTime.Now.Minute * 60 * 1000 +
   DateTime.Now.Second * 1000 +
   DateTime.Now.Millisecond;
System.Console.Out.WriteLine("--> Starting Data Preprocessing at: " +
   startTime.ToString());

// ************************************************************************
// Read continuous attribute data                                         *
// ************************************************************************
// Initialize ignore[] for identifying missing observations
ignore = new int[nObs];
isMissing = new int[1];
openInputFiles();

nCol = readFirstLine(contFileInputStream);

nContinuous = nCol;
System.Console.Out.WriteLine("--> Number of continuous variables:     " +
   nContinuous);
// If the number of continuous variables is greater than zero then read
// the remainder of this file (contFile)
if (nContinuous > 0)
{
   // contFile contains continuous attribute data
   contAtt = new double[nObs, nContinuous];
   double[] _contAttRow = new double[nContinuous];
   // for (int i2 = 0; i2 < nObs; i2++)
   // {
   //     contAtt[i2] = new double[nContinuous];
   // }
   cont_col = readColumnLabels(contFileInputStream, nContinuous);
   for (i = 0; i < nObs; i++)
   {
      isMissing[0] = - 1;
      _contAttRow = readDataLine(contFileInputStream, nContinuous,
         isMissing);
      for (int jj=0; jj < nContinuous; jj++)
      {
         contAtt[i,jj] = _contAttRow[jj];
      }
```

```
      ignore[i] = isMissing[0];
      if (isMissing[0] >= 0)
         m++;
   }
}
else
{
   nContinuous = 0;
   contAtt = new double[1,1];
   // for (int i3 = 0; i3 < 1; i3++)
   // {
   //     contAtt[i3] = new double[1];
   // }
   contAtt[0,0] = 0;
}
closeFile(contFileInputStream);
// ************************************************************************
// Read continuous output targets                                         *
// ************************************************************************
nCol = readFirstLine(outputFileInputStream);
nOutputs = nCol;
System.Console.Out.WriteLine("--> Number of output variables:         " +
   nOutputs);
outs = new double[nObs, nOutputs];
double[] _outsRow = new double[nOutputs];
// for (int i4 = 0; i4 < nObs; i4++)
// {
//     outs[i4] = new double[nOutputs];
// }
// Read numeric labels for continuous input attributes
outs_col = readColumnLabels(outputFileInputStream, nOutputs);

m = 0;
for (i = 0; i < nObs; i++)
{
   isMissing[0] = ignore[i];
   _outsRow = readDataLine(outputFileInputStream, nOutputs, isMissing);
   for (int jj =0; jj < nOutputs; jj++)
   {
      outs[i, jj] = _outsRow[jj];
   }
   ignore[i] = isMissing[0];
   if (isMissing[0] >= 0)
      m++;
}
System.Console.Out.WriteLine("--> Number of Missing Observations:     "
   + m);
closeFile(outputFileInputStream);
// Remove missing observations using the ignore[] array
m = removeMissingData(nObs, nContinuous, ignore, contAtt);
m = removeMissingData(nObs, nOutputs, ignore, outs);

System.Console.Out.WriteLine("--> Total Number of Training Patterns:  "
   + nObs);
nObs = nObs - m;
System.Console.Out.WriteLine("--> Number of Usable Training Patterns: "
```

```
   + nObs);

// ***********************************************************************
// Setup Method and Bounds for Scale Filter                              *
// ***********************************************************************
ScaleFilter scaleFilter = new ScaleFilter(
   ScaleFilter.ScalingMethod.Bounded);
scaleFilter.SetBounds(lowerDataLimit, upperDataLimit, 0, 1);
// ***********************************************************************
// PREPROCESS TRAINING PATTERNS                                          *
// ***********************************************************************
System.Console.Out.WriteLine(
   "--> Starting Preprocessing of Training Patterns");
xData = new double[nObs, nContinuous];
// for (int i5 = 0; i5 < nObs; i5++)
// {
//    xData[i5] = new double[nContinuous];
// }
yData = new double[nObs, nOutputs];
// for (int i6 = 0; i6 < nObs; i6++)
// {
//    yData[i6] = new double[nOutputs];
// }
for (i = 0; i < nObs; i++)
{
   for (j = 0; j < nContinuous; j++)
   {
      xData[i,j] = contAtt[i,j];
   }
   yData[i,0] = outs[i,0];
}
scaleFilter.Encode(0, xData);
scaleFilter.Encode(1, xData);
scaleFilter.Encode(2, xData);
scaleFilter.Encode(0, yData);
// ***********************************************************************
// CREATE FEEDFORWARD NETWORK                                            *
// ***********************************************************************
System.Console.Out.WriteLine("--> Creating Feed Forward Network Object");
FeedForwardNetwork network = new FeedForwardNetwork();
// setup input layer with number of inputs = nInputs = 3
network.InputLayer.CreateInputs(nInputs);
// create a hidden layer with nPerceptrons=3 perceptrons
network.CreateHiddenLayer().CreatePerceptrons(nPerceptrons);
// create output layer with nOutputs=1 output perceptron
network.OutputLayer.CreatePerceptrons(nOutputs);
// link all inputs and perceptrons to all perceptrons in the next layer
network.LinkAll();
// Get Network Perceptrons for Setting Their Activation Functions
Perceptron[] perceptrons = network.Perceptrons;
// Set all hidden layer perceptrons to logistic_table activation
for (i = 0; i < perceptrons.Length - 1; i++)
{
   perceptrons[i].Activation = hiddenLayerActivation;
}
perceptrons[perceptrons.Length - 1].Activation = outputLayerActivation;
```

```
System.Console.Out.WriteLine(
   "--> Feed Forward Network Created with 2 Layers");
// ********************************************************************
// TRAIN NETWORK USING EPOCH TRAINER                                  *
// ********************************************************************
System.Console.Out.WriteLine("--> Training Network using Epoch Trainer");
ITrainer trainer = createTrainer(QuasiNewton, QuasiNewton);
startTime =
   DateTime.Now.Hour * 60 * 60 * 1000 +
   DateTime.Now.Minute * 60 * 1000 +
   DateTime.Now.Second * 1000 +
   DateTime.Now.Millisecond;
// Train Network
trainer.Train(network, xData, yData);

// Check Training Error Status
switch (trainer.ErrorStatus)
{

   case 0:  errorMsg = errorMsg0;
      break;

   case 1:  errorMsg = errorMsg1;
      break;

   case 2:  errorMsg = errorMsg2;
      break;

   case 3:  errorMsg = errorMsg3;
      break;

   case 4:  errorMsg = errorMsg4;
      break;

   case 5:  errorMsg = errorMsg5;
      break;

   case 6:  errorMsg = errorMsg6;
      break;

   default:  errorMsg = "--> Unknown Error Status Returned from Trainer";
      break;

}
System.Console.Out.WriteLine(errorMsg);
int currentTimeNow =
   DateTime.Now.Hour * 60 * 60 * 1000 +
   DateTime.Now.Minute * 60 * 1000 +
   DateTime.Now.Second * 1000 +
   DateTime.Now.Millisecond;
System.Console.Out.WriteLine("--> Network Training Completed at: " +
   currentTimeNow.ToString());
double duration = (double) (currentTimeNow - startTime) / 1000.0;
System.Console.Out.WriteLine("--> Training Time: " + duration +
   " seconds");
```

```
// ***********************************************************************
// DISPLAY TRAINING STATISTICS                                           *
// ***********************************************************************
double[] stats = network.ComputeStatistics(xData, yData);
// Display Network Errors
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("--> SSE:                       " +
   (float)stats[0]);
System.Console.Out.WriteLine("--> RMS:                       " +
   (float)stats[1]);
System.Console.Out.WriteLine("--> Laplacian Error:           " +
   (float)stats[2]);
System.Console.Out.WriteLine("--> Scaled Laplacian Error:    " +
   (float)stats[3]);
System.Console.Out.WriteLine("--> Largest Absolute Residual: " +
   (float)stats[4]);
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("");
// ***********************************************************************
// OBTAIN AND DISPLAY NETWORK WEIGHTS AND GRADIENTS                      *
// ***********************************************************************
System.Console.Out.WriteLine("--> Getting Network Weights and Gradients");
// Get weights
weight = network.Weights;
// Get number of weights = number of gradients
nWeights = network.NumberOfWeights;
// Obtain Gradient Vector
gradient = trainer.ErrorGradient;
// Print Network Weights and Gradients
System.Console.Out.WriteLine(" ");
System.Console.Out.WriteLine("--> Network Weights and Gradients:");
System.Console.Out.WriteLine(
   "*******************************************");
double[,] printMatrix = new double[nWeights,2];
// for (int i7 = 0; i7 < nWeights; i7++)
// {
//    printMatrix[i7] = new double[2];
// }
for (i = 0; i < nWeights; i++)
{
   printMatrix[i,0] = weight[i];
   printMatrix[i,1] = gradient[i];
}
// Print result without row/column labels.
System.String[] colLabels = new System.String[]{"Weight", "Gradient"};
PrintMatrix pm = new PrintMatrix();
PrintMatrixFormat mf;
mf = new PrintMatrixFormat();
mf.SetNoRowLabels();
mf.SetColumnLabels(colLabels);
pm.SetTitle("Weights and Gradients");
pm.Print(mf, printMatrix);

System.Console.Out.WriteLine(
```

```
      "*****************************************");
   // ************************************************************************
   // SAVE THE TRAINED NETWORK BY SAVING THE SERIALIZED NETWORK OBJECT     *
   // ************************************************************************
   System.Console.Out.WriteLine("\n--> Saving Trained Network into " +
      networkFileName);
   write(network, networkFileName);
   System.Console.Out.WriteLine("--> Saving Network Trainer into " +
      trainerFileName);
   write(trainer, trainerFileName);
   System.Console.Out.WriteLine("--> Saving xData into " + xDataFileName);
   write(xData, xDataFileName);
   System.Console.Out.WriteLine("--> Saving yData into " + yDataFileName);
   write(yData, yDataFileName);
}
// ************************************************************************
// OPEN DATA FILES                                                        *
// ************************************************************************
static public void  openInputFiles()
{
   try
   {
      // Continuous  Input Attributes
      System.IO.Stream contInputStream = new System.IO.FileStream(
         contFileName, System.IO.FileMode.Open, System.IO.FileAccess.Read);
      contFileInputStream = new System.IO.StreamReader(new
         System.IO.StreamReader(contInputStream).BaseStream,
         System.Text.Encoding.UTF7);
      // Continuous Output Targets
      System.IO.Stream outputInputStream = new System.IO.FileStream(
         outputFileName, System.IO.FileMode.Open, System.IO.FileAccess.Read);
      outputFileInputStream = new System.IO.StreamReader(
         new System.IO.StreamReader(outputInputStream).BaseStream,
         System.Text.Encoding.UTF7);
   }
   catch (System.Exception e)
   {
      System.Console.Out.WriteLine("-->ERROR: " + e);
      System.Environment.Exit(0);
   }
}
// ************************************************************************
// READ FIRST LINE OF DATA FILE AND RETURN NUMBER OF COLUMNS IN FILE      *
// ************************************************************************
static public int readFirstLine(System.IO.StreamReader inputFile)
{
   System.String inputLine = "", temp;
   int nCol = 0;
   try
   {
      temp = inputFile.ReadLine();
      inputLine = temp.Trim();
      nCol = System.Int32.Parse(inputLine);
   }
   catch (System.Exception e)
   {
```

```
      System.Console.Out.WriteLine("--> ERROR READING 1st LINE OF File" + e);
      System.Environment.Exit(0);
   }
   return nCol;
}
// ************************************************************************
// READ COLUMN LABELS (2ND LINE IN FILE)                                  *
// ************************************************************************
static public int[] readColumnLabels(System.IO.StreamReader inputFile,
   int nCol)
{
   int[] contCol = new int[nCol];
   System.String inputLine = "", temp;
   System.String[] dataElement;
   // Read numeric labels for continuous input attributes
   try
   {
      temp = inputFile.ReadLine();
      inputLine = temp.Trim();
   }
   catch (System.Exception e)
   {
      System.Console.Out.WriteLine("--> ERROR READING 2nd LINE OF FILE: "
         + e);
      System.Environment.Exit(0);
   }
   dataElement = inputLine.Split(new Char[] {' '});
   for (int i = 0; i < nCol; i++)
   {
      contCol[i] = System.Int32.Parse(dataElement[i]);
   }
   return contCol;
}
// ************************************************************************
// READ DATA ROW                                                          *
// ************************************************************************
static public double[] readDataLine(System.IO.StreamReader inputFile,
   int nCol, int[] isMissing)
{
   double missingValueIndicator = - 9999999999.0;
   double[] dataLine = new double[nCol];
   double[] contCol = new double[nCol];
   System.String inputLine = "", temp;
   System.String[] dataElement;
   try
   {
      temp = inputFile.ReadLine();
      inputLine = temp.Trim();
   }
   catch (System.Exception e)
   {
      System.Console.Out.WriteLine("-->ERROR READING LINE: " + e);
      System.Environment.Exit(0);
   }
   dataElement = inputLine.Split(new Char[] {' '});
   for (int j = 0; j < nCol; j++)
```

```
   {
      dataLine[j] = System.Double.Parse(dataElement[j]);
      if (dataLine[j] == missingValueIndicator)
         isMissing[0] = 1;
   }
   return dataLine;
}
// ************************************************************************
// CLOSE FILE                                                             *
// ************************************************************************
static public void  closeFile(System.IO.StreamReader inputFile)
{
   try
   {
      inputFile.Close();
   }
   catch (System.Exception e)
   {
      System.Console.Out.WriteLine("ERROR: Unable to close file: " + e);
      System.Environment.Exit(0);
   }
}
// ************************************************************************
// REMOVE MISSING DATA                                                    *
// ************************************************************************
// Now remove all missing data using the ignore[] array
// and recalculate the number of usable observations, nObs
// This method is inefficient, but it works.  It removes one case at a
// time, starting from the bottom.  As a case (row) is removed, the cases
// below are pushed up to take it's place.
// ************************************************************************
static public int removeMissingData(int nObs, int nCol, int[] ignore,
   double[,] inputArray)
{
   int m = 0;
   for (int i = nObs - 1; i >= 0; i--)
   {
      if (ignore[i] >= 0)
      {
         // the ith row contains a missing value
         // remove the ith row by shifting all rows below the
         // ith row up by one position, e.g. row i+1  -> row i
         m++;
         if (nCol > 0)
         {
            for (int j = i; j < nObs - m; j++)
            {
               for (int k = 0; k < nCol; k++)
               {
                  inputArray[j,k] = inputArray[j + 1,k];
               }
            }
         }
      }
   }
   return m;
```

```
}
// ************************************************************************
// Create Stage I/Stage II Trainer                                        *
// ************************************************************************
static public ITrainer createTrainer(System.String s1, System.String s2)
{
   EpochTrainer epoch = null; // Epoch Trainer (returned by this method)
   QuasiNewtonTrainer stage1Trainer; // Stage I  Quasi-Newton Trainer
   QuasiNewtonTrainer stage2Trainer; // Stage II Quasi-Newton Trainer
   LeastSquaresTrainer stage1LS; // Stage I  Least Squares Trainer
   LeastSquaresTrainer stage2LS; // Stage II Least Squares Trainer
   int currentTimeNow; // Calendar time tracker

   // Create Epoch (Stage I/Stage II) trainer from above trainers.
   System.Console.Out.WriteLine("    --> Creating Epoch Trainer");
   if (s1.Equals(QuasiNewton))
   {
      // Setup stage I quasi-newton trainer
      stage1Trainer = new QuasiNewtonTrainer();
      //stage1Trainer.setMaximumStepsize(maxStepSize);
      stage1Trainer.MaximumTrainingIterations = stage1Iterations;
      stage1Trainer.StepTolerance = stage1StepTolerance;
      if (s2.Equals(QuasiNewton))
      {
         stage2Trainer = new QuasiNewtonTrainer();
         //stage2Trainer.setMaximumStepsize(maxStepSize);
         stage2Trainer.MaximumTrainingIterations = stage2Iterations;
         epoch = new EpochTrainer(stage1Trainer, stage2Trainer);
      }
      else
      {
         if (s2.Equals(LeastSquares))
         {
            stage2LS = new LeastSquaresTrainer();
            stage2LS.InitialTrustRegion = 1.0e-3;
            //stage2LS.setMaximumStepsize(maxStepSize);
            stage2LS.MaximumTrainingIterations = stage2Iterations;
            epoch = new EpochTrainer(stage1Trainer, stage2LS);
         }
         else
         {
            epoch = new EpochTrainer(stage1Trainer);
         }
      }
   }
   else
   {
      // Setup stage I least squares trainer
      stage1LS = new LeastSquaresTrainer();
      stage1LS.InitialTrustRegion = 1.0e-3;
      stage1LS.MaximumTrainingIterations = stage1Iterations;
      //stage1LS.setMaximumStepsize(maxStepSize);
      if (s2.Equals(QuasiNewton))
      {
         stage2Trainer = new QuasiNewtonTrainer();
         //stage2Trainer.setMaximumStepsize(maxStepSize);
```

```
      stage2Trainer.MaximumTrainingIterations = stage2Iterations;
      epoch = new EpochTrainer(stage1LS, stage2Trainer);
   }
   else
   {
      if (s2.Equals(LeastSquares))
      {
         stage2LS = new LeastSquaresTrainer();
         stage2LS.InitialTrustRegion = 1.0e-3;
         //stage2LS.setMaximumStepsize(maxStepSize);
         stage2LS.MaximumTrainingIterations = stage2Iterations;
         epoch = new EpochTrainer(stage1LS, stage2LS);
      }
      else
      {
         epoch = new EpochTrainer(stage1LS);
      }
   }
}
epoch.NumberOfEpochs = nEpochs;
epoch.EpochSize = epochSize;
epoch.Random = new Imsl.Stat.Random(1234567);
epoch.SetRandomSamples(new Imsl.Stat.Random(12345),
   new Imsl.Stat.Random(67891));
System.Console.Out.WriteLine("    --> Trainer:  Stage I - " + s1 +
   " Stage II " + s2);
System.Console.Out.WriteLine("    --> Number of Epochs:   " + nEpochs);
System.Console.Out.WriteLine("    --> Epoch Size:         " + epochSize);
// Describe optimization setup for Stage I training
System.Console.Out.WriteLine("    --> Creating Stage I  Trainer");
System.Console.Out.WriteLine("    --> Stage I Iterations:          " +
   stage1Iterations);
System.Console.Out.WriteLine("    --> Stage I Step Tolerance:      " +
   stage1StepTolerance);
System.Console.Out.WriteLine("    --> Stage I Relative Tolerance:  " +
   stage1RelativeTolerance);
System.Console.Out.WriteLine("    --> Stage I Step Size:           " +
   "DEFAULT");
System.Console.Out.WriteLine("    --> Stage I Trace:               " +
   trace);
if (s2.Equals(QuasiNewton) || s2.Equals(LeastSquares))
{
   // Describe optimization setup for Stage II training
   System.Console.Out.WriteLine("    --> Creating Stage II Trainer");
   System.Console.Out.WriteLine("    --> Stage II Iterations:         " +
      stage2Iterations);
   System.Console.Out.WriteLine("    --> Stage II Step Tolerance:     " +
      stage2StepTolerance);
   System.Console.Out.WriteLine("    --> Stage II Relative Tolerance: " +
      stage2RelativeTolerance);
   System.Console.Out.WriteLine("    --> Stage II Step Size:          " +
      "DEFAULT");
   System.Console.Out.WriteLine("    --> Stage II Trace:              " +
      trace);
}
currentTimeNow =
```

```
         DateTime.Now.Hour * 60 * 60 * 1000 +
         DateTime.Now.Minute * 60 * 1000 +
         DateTime.Now.Second * 1000 +
         DateTime.Now.Millisecond;
      System.Console.Out.WriteLine("--> Starting Network Training at " +
         currentTimeNow.ToString());
      // Return Stage I/Stage II trainer
      return epoch;
   }

   // ************************************************************************
   // WRITE SERIALIZED OBJECT TO A FILE                                      *
   // ************************************************************************
   static public void  write(System.Object obj, System.String filename)
   {
      System.IO.FileStream fos = new System.IO.FileStream(filename,
         System.IO.FileMode.Create);
      IFormatter oos = new BinaryFormatter();
      oos.Serialize(fos, obj);
      fos.Close();

   }
   static NeuralNetworkEx1()
   {
      hiddenLayerActivation = Imsl.DataMining.Neural.Activation.LogisticTable;
      outputLayerActivation = Imsl.DataMining.Neural.Activation.Linear;
   }
}
// ****************************************************************************
```

Output

```
--> Starting Data Preprocessing at: 44821683
--> Number of continuous variables:     3
--> Number of output variables:         1
--> Number of Missing Observations:     16507
--> Total Number of Training Patterns:  118519
--> Number of Usable Training Patterns: 102012
--> Starting Preprocessing of Training Patterns
--> Creating Feed Forward Network Object
--> Feed Forward Network Created with 2 Layers
--> Training Network using Epoch Trainer
    --> Creating Epoch Trainer
    --> Trainer:  Stage I - quasi-newton Stage II quasi-newton
    --> Number of Epochs:   20
    --> Epoch Size:         5000
    --> Creating Stage I  Trainer
    --> Stage I Iterations:          5000
    --> Stage I Step Tolerance:      1E-09
    --> Stage I Relative Tolerance:  1E-11
    --> Stage I Step Size:           DEFAULT
    --> Stage I Trace:               True
    --> Creating Stage II Trainer
    --> Stage II Iterations:         5000
    --> Stage II Step Tolerance:     1E-09
```

```
    --> Stage II Relative Tolerance: 1E-11
    --> Stage II Step Size:          DEFAULT
    --> Stage II Trace:              True
--> Starting Network Training at 45070408
--> The last global step failed to locate a lower point than the
current error value.  The current solution may be an approximate
solution and no more accuracy is possible, or the step tolerance
may be too large.
--> Network Training Completed at: 52311842
--> Training Time: 7241.434 seconds
*********************************************
--> SSE:                       4.49772
--> RMS:                       0.1423779
--> Laplacian Error:           103.4631
--> Scaled Laplacian Error:    0.1707173
--> Largest Absolute Residual: 0.4921748
*********************************************

--> Getting Network Weights and Gradients

--> Network Weights and Gradients:
*********************************************
            Weights and Gradients
         Weight               Gradient
-248.425149158357     -9.50818419128144E-05
  -4.01301691047852   -9.08459022567118E-07
 248.602873209042     -2.84623837579401E-05
 258.622104579914     -8.49451049786515E-05
   0.125785905718184  -7.51083204612989E-07
-258.811023180973     -2.81816574426092E-05
-394.380943852438     -0.000125916731945308
  -0.356726621727131  -5.25467092773031E-07
 394.428311058654     -2.70798222353788E-05
 422.855858784789     -1.40339989032276E-06
  -1.01024906891467   -8.54119524733673E-07
 422.854960914701      3.37315953950526E-08
  91.0301743864326    -0.000555459860183764
   0.672279284955327  -3.11957565142863E-06
 -91.0431760187523    -0.000120208750794691
-422.186774012951     -1.36686903761535E-06

*********************************************

--> Saving Trained Network into NeuralNetworkEx1.ser
--> Saving Network Trainer into NeuralNetworkTrainerEx1.ser
--> Saving xData into NeuralNetworkxDataEx1.ser
--> Saving yData into NeuralNetworkyDataEx1.ser
```

Results

The above output indicates that the network successfully completed its training. The final sum of squared errors was 3.88, and the RMS (the scaled version of the sum of squared errors) was 0.12. All of the gradients at this solution are nearly zero, which is expected if network training found a local or global optima. Non-zero gradients usually indicate there was a problem with network training.

```
--> Starting Data Preprocessing at: 57217338
--> Number of continuous variables:     3
--> Number of output variables:         1
--> Number of Missing Observations:     16507
--> Total Number of Training Patterns:  118519
--> Number of Usable Training Patterns: 102012
--> Starting Preprocessing of Training Patterns
--> Creating Feed Forward Network Object
--> Feed Forward Network Created with 2 Layers
--> Training Network using Epoch Trainer
    --> Creating Epoch Trainer
    --> Trainer:  Stage I - quasi-newton Stage II quasi-newton
    --> Number of Epochs:   20
    --> Epoch Size:         5000
    --> Creating Stage I  Trainer
    --> Stage I Iterations:          5000
    --> Stage I Step Tolerance:      1E-09
    --> Stage I Relative Tolerance:  1E-11
    --> Stage I Step Size:           DEFAULT
    --> Stage I Trace:               True
    --> Creating Stage II Trainer
    --> Stage II Iterations:         5000
    --> Stage II Step Tolerance:     1E-09
    --> Stage II Relative Tolerance: 1E-11
    --> Stage II Step Size:          DEFAULT
    --> Stage II Trace:              True
--> Starting Network Training at 57233745
--> The last global step failed to locate a lower point than the
current error value.  The current solution may be an approximate
solution and no more accuracy is possible, or the step tolerance
may be too large.
--> Network Training Completed at: 58750179
--> Training Time: 1516.434 seconds
*********************************************
--> SSE:                       4.39503
--> RMS:                       0.1391272
--> Laplacian Error:           115.7665
--> Scaled Laplacian Error:    0.1910183
--> Largest Absolute Residual: 0.510379
*********************************************

--> Getting Network Weights and Gradients

--> Network Weights and Gradients:
*********************************************
             Weights and Gradients
          Weight              Gradient
    6.49062717385272    -7.35781305464078E-07
   -6.59528808872046     4.80847787053182E-05
 4401.51680674985       -1.85429409555815E-08
    0.704338219368497   -8.32580200754921E-07
   -0.725623033199978    5.70308208281333E-05
 2805.39692819902       -1.44534276811825E-08
   -3.93198626371718    -2.13822710907307E-06
    4.04824502335685     6.44781336574968E-05
-1787.37723908109       -1.7603927485418E-09
```

```
   38.3945130188417     -4.36277172935969E-05
   37.6336974274678     -2.57936093318423E-05
   -0.0378671989334698   7.43008547201402E-07
    0.4447658217999     -0.000427699567365199
   -0.495282381760965   -0.000323163187876674
-1037.62096685008        1.57344076332679E-08
  -37.6465155277179     -7.36849253095175E-05

*********************************************

--> Saving Trained Network into NeuralNetworkEx1.ser
--> Saving Network Trainer into NeuralNetworkTrainerEx1.ser
--> Saving xData into NeuralNetworkxDataEx1.ser
--> Saving yData into NeuralNetworkyDataEx1.ser
```

Links to Input Data Files Used in this Example and the Training Log:

Network Class

`public class Imsl.DataMining.Neural.Network`

Neural network base class.

Properties

InputLayer

`abstract public Imsl.DataMining.Neural.InputLayer InputLayer {get; }`

Description

The `InputLayer` object.

Property Value

The `Network` `InputLayer`.

Links

`abstract public Imsl.DataMining.Neural.Link[] Links {get; }`

Description

An array containing the `Link` objects in the `Network`.

Property Value

An array of Links associated with this Network.

NumberOfInputs

```
abstract public int NumberOfInputs {get; }
```

Description

The number of Network inputs.

Property Value

An int which contains the number of inputs.

NumberOfLinks

```
abstract public int NumberOfLinks {get; }
```

Description

The number of Network Links among the Nodes.

Property Value

An int which contains the number of Links (p. 1665) in the Network.

NumberOfOutputs

```
abstract public int NumberOfOutputs {get; }
```

Description

The number of Network output Perceptrons (p. 1661).

Property Value

An int which contains the number of outputs.

NumberOfWeights

```
abstract public int NumberOfWeights {get; }
```

Description

The number of Weights (p. 1666) in the Network.

Property Value

An int which contains the number of Weights associated with this Network.

OutputLayer

```
abstract public Imsl.DataMining.Neural.OutputLayer OutputLayer {get; }
```

Description

The OutputLayer.

Property Value

The Network OutputLayer.

Perceptrons

```
abstract public Imsl.DataMining.Neural.Perceptron[] Perceptrons {get; }
```

Description

An array containing the Perceptrons in the Network.

Property Value

An array of Perceptrons associated with this Network.

Weights

```
abstract public double[] Weights {get; set; }
```

Description

The Weights (p. 1666).

Property Value

A double array containing the Weights associated with Network Links (p. 1665).

Constructor

Network

```
public Network()
```

Description

Default constructor for Network.

Remarks

Since this class is abstract, it cannot be instantiated directly; this constructor is used by constructors in classes derived from Network.

Methods

ComputeStatistics

```
virtual public double[] ComputeStatistics(double[,] xData, double[,] yData)
```

Description

Computes error statistics.

Parameters

xData – A double matrix containing the input values.

yData – A double array containing the observed values.

Returns

A double array containing the above described statistics.

Remarks

This is a static method that can be used to compute the statistics regardless of the training class used to train the `Network`.

Computes statistics related to the error. In this table, the observed values are y_i. The forecasted values are $\hat{y}_i$. The mean observed value is $\bar{y} = \sum_i y_i / NC$, where N is the number of observations and C is the number of classes per observation.

Index	Name	Formula
0	SSE	$\frac{1}{2}\sum_i (y_i - \hat{y}_i)^2$
1	RMS	$\frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y}_i)}$
2	Laplacian	$\sum_i \lvert y_i - \hat{y}_i \rvert$
3	Scaled Laplacian	$\frac{\sum_i \lvert y_i - \hat{y}_i \rvert}{\sum_i \lvert y_i - \bar{y}_i \rvert}$
4	Max residual	$\max_i \lvert y_i - \hat{y}_i \rvert$

CreateHiddenLayer

```
abstract public Imsl.DataMining.Neural.HiddenLayer CreateHiddenLayer()
```

Description

Creates the next `HiddenLayer` in the `Network`.

Returns

The new `HiddenLayer`.

Forecast

```
abstract public double[] Forecast(double[] x)
```

Description

Returns a forecast for each of the `Network`'s outputs computed from the trained `Network`.

Parameter

`x` – A `double` array of values with the same length and order as the training patterns used to train the `Network`.

Returns

A `double` array containing the forecasts for the output Perceptrons (p. 1661). Its length is equal to the number of output `Perceptrons`.

GetForecastGradient

```
abstract public double[,] GetForecastGradient(double[] x)
```

Description

Returns the derivatives of the outputs with respect to the Weights (p. 1666) evaluated at `x`.

Parameter

`x` – A `double` array which specifies the input values at which the *gradient* is to be evaluated.

Returns

A `double` array containing the gradient values. The value of `gradient[i][j]` is dy_i/dw_j, where y_i is the i-th output and w_j is the j-th weight.

Example: Network

This example uses a network previously trained and serialized into four files to obtain information about the network and forecasts. Training was done using the code for the FeedForwardNetwork Example 1.

The network training targets were generated using the relationship:

$y = 10*X_1 + 20*X_2 + 30*X_3 + 2.0*X_4$, where

X_1-X_3 are the three binary columns, corresponding to categories 1 to 3 of the nominal attribute, and X_4 is the scaled continuous attribute.

The structure of the network consists of four input nodes and two layers, with three perceptrons in the hidden layer and one in the output layer. The following figure illustrates this structure:

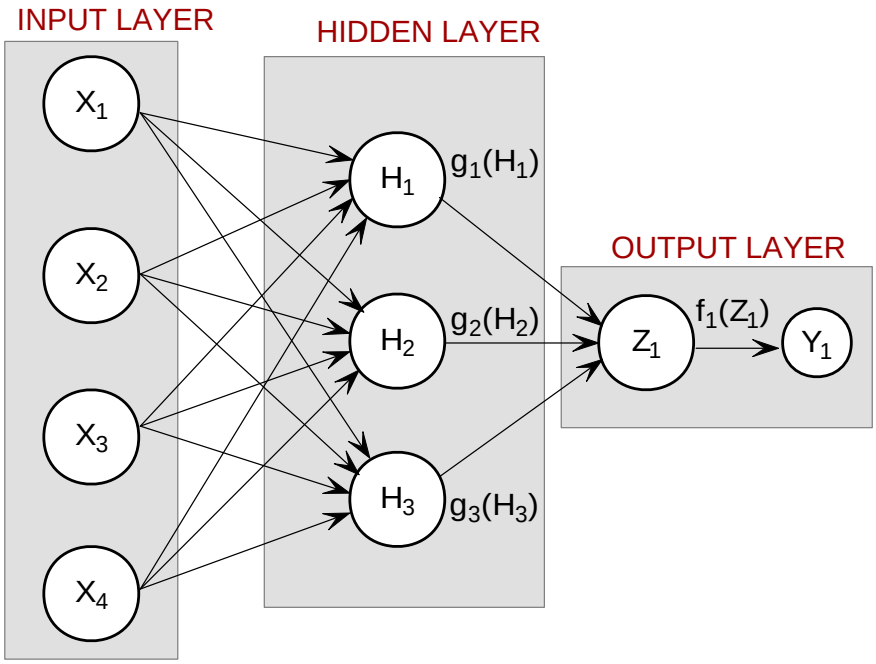

Figure 10. An example 2-layer Feed Forward Neural Network with 4 Inputs

All perceptrons were trained using a Linear Activation Function. Forecasts are generated for 9 conditions, corresponding to the following conditions:
Nominal Class 1-3 with the Continuous Input Attribute = 0
Nominal Class 1-3 with the Continuous Input Attribute = 5.0
Nominal Class 1-3 with the Continuous Input Attribute = 10.0

Note that the network training statistics retrieved from the serialized network confirm that this is the same network used in the previous example. Obtaining these statistics requires retrieval of the training patterns which were serialized and stored into separate files. This information is not serialized with the network, nor with the trainer.

```
using System;
using Imsl.DataMining.Neural;
using System.Runtime.Serialization;
using System.Runtime.Serialization.Formatters.Binary;
//*************************************************************************
// Two Layer Feed-Forward Network with 4 inputs: 1 categorical with 3 classes
// encoded using binary encoding and 1 continuous input, and 1 output
// target (continuous).  There is a perfect linear relationship between
// the input and output variables:
//
// MODEL:  Y = 10*X1 + 20*X2 + 30*X3 + 2*X4
//
// Variables X1-X3 are the binary encoded nominal variable and X4 is the
// continuous variable.
//
// This example uses Linear Activation in both the hidden and output layers
// The network uses a 2-layer configuration, one hidden layer and one
// output layer.  The hidden layer consists of 3 perceptrons.  The output
// layer consists of a single output perceptron.
// The input from the continuous variable is scaled to [0,1] before training
// the network.  Training is done using the Quasi-Newton Trainer.
// The network has a total of 19 weights.
// Since the network target is a linear combination of the network inputs, and
// since all perceptrons use linear activation, the network is able to forecast
// the every training target exactly.  The largest residual is 2.78E-08.
//*************************************************************************

[Serializable]
public class NetworkEx1
{
   // ***********************************************************************
   // MAIN
   // ***********************************************************************


   public static void  Main(System.String[] args)
   {
      double[,] xData; // Input  Attributes for Training Patterns
      double[,] yData; // Output Attributes for Training Patterns
      double[] weight; // network weights
      double[] gradient; // network gradient after training
      // Input Attributes for Forecasting
      double[,] x = {{1, 0, 0, 0.0}, {0, 1, 0, 0.0}, {0, 0, 1, 0.0},
                     {1, 0, 0, 5.0}, {0, 1, 0, 5.0}, {0, 0, 1, 5.0},
                     {1, 0, 0, 10.0}, {0, 1, 0, 10.0}, {0, 0, 1, 10.0}};
      double[] xTemp, y; // Temporary areas for storing forecasts
      int i, j; // loop counters
      // Names of Serialized Files
      System.String networkFileName = "FeedForwardNetworkEx1.ser"; // the network
      System.String trainerFileName = "FeedForwardTrainerEx1.ser"; // the trainer
```

```
System.String xDataFileName = "FeedForwardxDataEx1.ser"; // xData
System.String yDataFileName = "FeedForwardyDataEx1.ser"; // yData
// ************************************************************************
// READ THE TRAINED NETWORK FROM THE SERIALIZED NETWORK OBJECT
// ************************************************************************
System.Console.Out.WriteLine("--> Reading Trained Network from " +
   networkFileName);
Network network = (Network) read(networkFileName);
// ************************************************************************
// READ THE SERIALIZED XDATA[,] AND YDATA[,] ARRAYS OF TRAINING
// PATTERNS.
// ************************************************************************
System.Console.Out.WriteLine("--> Reading xData from " + xDataFileName);
xData = (double[,]) read(xDataFileName);
System.Console.Out.WriteLine("--> Reading yData from " + yDataFileName);
yData = (double[,]) read(yDataFileName);
// ************************************************************************
// READ THE SERIALIZED TRAINER OBJECT
// ************************************************************************
System.Console.Out.WriteLine("--> Reading Network Trainer from " +
   trainerFileName);
ITrainer trainer = (ITrainer) read(trainerFileName);
// ************************************************************************
// DISPLAY TRAINING STATISTICS
// ************************************************************************
double[] stats = network.ComputeStatistics(xData, yData);
// Display Network Errors
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("--> SSE:                       " +
   (float)stats[0]);
System.Console.Out.WriteLine("--> RMS:                       " +
   (float)stats[1]);
System.Console.Out.WriteLine("--> Laplacian Error:           " +
   (float)stats[2]);
System.Console.Out.WriteLine("--> Scaled Laplacian Error:    " +
   (float)stats[3]);
System.Console.Out.WriteLine("--> Largest Absolute Residual: " +
   (float)stats[4]);
System.Console.Out.WriteLine(
   "*******************************************");
System.Console.Out.WriteLine("");
// ************************************************************************
// OBTAIN AND DISPLAY NETWORK WEIGHTS AND GRADIENTS
// ************************************************************************
System.Console.Out.WriteLine("--> Getting Network Information");
// Get weights
weight = network.Weights;
// Get number of weights = number of gradients
int nWeights = network.NumberOfWeights;
// Obtain Gradient Vector
gradient = trainer.ErrorGradient;
// Print Network Weights and Gradients
System.Console.Out.WriteLine(" ");
System.Console.Out.WriteLine("--> Network Weights and Gradients:");
for (i = 0; i < nWeights; i++)
```

```
      {
         System.Console.Out.WriteLine("w[" + i + "]=" + (float) weight[i] +
            " g[" + i + "]=" + (float) gradient[i]);
      }
      // ********************************************************************
      // OBTAIN AND DISPLAY FORECASTS FOR THE LAST 10 TRAINING TARGETS
      // ********************************************************************
      // Get number of network inputs
      int nInputs = network.NumberOfInputs;
      // Get number of network outputs
      int nOutputs = network.NumberOfOutputs;
      xTemp = new double[nInputs]; // temporary x space for forecast inputs
      y = new double[nOutputs]; // temporary y space for forecast output
      System.Console.Out.WriteLine(" ");
      // Obtain example forecasts for input attributes = x[]
      // X1-X3 are binary encoded for one nominal variable with 3 classes
      // X4 is a continuous input attribute ranging from 0-10.  During
      // training, X4 was scaled to [0,1] by dividing by 10.
      for (i = 0; i < 9; i++)
      {
         for (j = 0; j < nInputs; j++)
            xTemp[j] = x[i,j];
         xTemp[nInputs - 1] = xTemp[nInputs - 1] / 10.0;
         y = network.Forecast(xTemp);
         System.Console.Out.Write("--> X1=" + (int) x[i,0] + " X2=" +
            (int) x[i,1] + " X3=" + (int) x[i,2] + " | X4=" + x[i,3]);
         System.Console.Out.WriteLine(" | y=" + (float) (10.0 * x[i,0] + 20.0 *
            x[i,1] + 30.0 * x[i,2] + 2.0 * x[i,3]) + "| Forecast=" +
            (float) y[0]);
      }
   }
   // ********************************************************************
   // READ SERIALIZED NETWORK FROM A FILE
   // ********************************************************************
   static public System.Object read(System.String filename)
   {
      System.IO.FileStream fis = new System.IO.FileStream(filename,
         System.IO.FileMode.Open, System.IO.FileAccess.Read);
      IFormatter ois = new BinaryFormatter();
      System.Object obj = (System.Object) ois.Deserialize(fis);
      fis.Close();
      return obj;
   }
}
```

Output

```
--> Reading Trained Network from FeedForwardNetworkEx1.ser
--> Reading xData from FeedForwardxDataEx1.ser
--> Reading yData from FeedForwardyDataEx1.ser
--> Reading Network Trainer from FeedForwardTrainerEx1.ser
*********************************************
--> SSE:                       1.013444E-15
--> RMS:                       2.007463E-19
--> Laplacian Error:           3.005804E-07
```

```
--> Scaled Laplacian Error:    3.535235E-10
--> Largest Absolute Residual: 2.784275E-08
*********************************************

--> Getting Network Information

--> Network Weights and Gradients:
w[0]=-1.491785 g[0]=-2.611079E-08
w[1]=-1.491785 g[1]=-2.611079E-08
w[2]=-1.491785 g[2]=-2.611079E-08
w[3]=1.616918 g[3]=6.182035E-08
w[4]=1.616918 g[4]=6.182035E-08
w[5]=1.616918 g[5]=6.182035E-08
w[6]=4.725622 g[6]=-5.273856E-08
w[7]=4.725622 g[7]=-5.273856E-08
w[8]=4.725622 g[8]=-5.273856E-08
w[9]=6.217407 g[9]=-8.733E-10
w[10]=6.217407 g[10]=-8.733E-10
w[11]=6.217407 g[11]=-8.733E-10
w[12]=1.072258 g[12]=-1.690978E-07
w[13]=1.072258 g[13]=-1.690978E-07
w[14]=1.072258 g[14]=-1.690978E-07
w[15]=3.850755 g[15]=-1.7029E-08
w[16]=3.850755 g[16]=-1.7029E-08
w[17]=3.850755 g[17]=-1.7029E-08
w[18]=2.411725 g[18]=-1.588144E-08

--> X1=1 X2=0 X3=0 | X4=0 | y=10| Forecast=10
--> X1=0 X2=1 X3=0 | X4=0 | y=20| Forecast=20
--> X1=0 X2=0 X3=1 | X4=0 | y=30| Forecast=30
--> X1=1 X2=0 X3=0 | X4=5 | y=20| Forecast=20
--> X1=0 X2=1 X3=0 | X4=5 | y=30| Forecast=30
--> X1=0 X2=0 X3=1 | X4=5 | y=40| Forecast=40
--> X1=1 X2=0 X3=0 | X4=10 | y=30| Forecast=30
--> X1=0 X2=1 X3=0 | X4=10 | y=40| Forecast=40
--> X1=0 X2=0 X3=1 | X4=10 | y=50| Forecast=50
```

Chapter 28: Error Handling

Types

Warning Class

`public class Imsl.Warning`

Handles warning messages.

This class maintains a single, private, `WarningObject` that actually displays the warning messages.

Properties

WarningObject

`static public Imsl.WarningObject WarningObject {get; set; }`

Description

The `WarningObject` allows warning errors to be handled in a more custom fashion.

Property Value

The current `WarningObject`.

Remarks

`WarningObject` may be set to null, in which case error messages will be ignored.

Writer

```
static public System.IO.TextWriter Writer {get; set; }
```

Description

The stream to which warning messages are to be written.

Remarks

The input may be null, in which case warnings are not written.

Constructor

Warning

```
public Warning()
```

Description

Initializes a new instance of the Imsl.Warning (p. 1795) class.

Method

Print

```
static public void Print(object source, string bundleName, string key, object[]
arg)
```

Description

Issues a warning message.

Parameters

`source` – The `Object` that is the source of the warning.

`bundleName` – A `String` which specifies the base name of the resource. The actual name is formed by appending ".ErrorMessages".

`key` – A `String` which specifies the warning message in the resource.

`arg` – A `Object` which specifies arguments used to format the message.

Remarks

Warning messages are stored as MessageFormat patterns in a ResourceBundle. This method retrieves the pattern from the bundle, formats the message with the supplied arguments, and prints the message to the warning stream.

WarningObject Class

```
public class Imsl.WarningObject
```

Handles warning messages.

Property

Writer

```
public System.IO.TextWriter Writer {get; set; }
```

Description

Reassigns the writer.

Remarks

The new warning writer may be set to null, in which case warnings are not printed.

Constructor

WarningObject

```
public WarningObject()
```

Description

Handle warning messages.

Method

Print

```
virtual public void Print(object source, string baseName, string key, object[]
arg)
```

Description

Issue a warning message.

Parameters

`source` – The `Object` that is the source of the warning.

`baseName` – A `String` which specifies the base name of the resource. The actual name is formed by appending ".ErrorMessages".

`key` – A `String` which specifies the warning message in the resource.

`arg` – A `Object` which specifies arguments used to format the message.

Remarks

Warning messages are stored as string format items in a resource. This method retrieves the format from the resource, formats the message with the supplied arguments, and prints the message to the warning stream.

IMSLException Class

```
public class Imsl.IMSLException :  ApplicationException :  ISerializable
```

Signals that a mathematical exception has occurred.

Constructors

IMSLException

```
IMSLException()
```

Description

Constructs an `IMSLException` with no detail message.

Remarks

A detail message is a `String` that describes this particular exception.

IMSLException

```
IMSLException(string s)
```

Description

Constructs an `IMSLException` with the specified detail message.

Parameter

`s` – A `String` which specifies the detail message.

Remarks

A detail message is a `String` that describes this particular exception.

IMSLException

```
IMSLException(string namespaceName, string key, object[] arguments)
```

Description

Constructs an `IMSLException` with the specified detail message.

Parameters

`namespaceName` – A `String` which specifies the namespace containing the ErrorMessages resource bundle.

`key` – A `String` which specifies the key of the error message in the resource bundle.

`arguments` – An array of `Objects` containing arguments used within the error message string.

Remarks

The error message `String` is in a resource bundle, ErrorMessages.

IMSLException

```
IMSLException(string message, System.Exception exception)
```

Description

Constructs an `IMSLException` with the specified detail message.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

IMSLException

```
IMSLException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Constructs an `IMSLException` with the serialized data.

Parameters

`info` – The `Object` that holds the serialized object data.

`context` – The contextual information about the source or destination.

Logger Class

```
public class Imsl.Logger
```

This class is used to implement logging in some IMSL algorithms. Its design mimics the Java `java.util.Logging` class, but is simpler, meeting only the requirements of logging intermediate results and notes from IMSL classes.

Properties

LogLevel

```
public Imsl.Logger.Level LogLevel {get; set; }
```

Description

The `Logger.Level` assocaited with this `Logger`. The finer grained levels provide more detailed output and are inclusive of the levels before it. Not all levels are implemented for all classes.

Writer

```
public System.IO.TextWriter Writer {get; set; }
```

Description

The `TextWriter` associated with this `Logger` instance. The default is `System.Console.Out`.

Constructor

Logger

```
public Logger()
```

Description

Create a new Logger object using the default TextWriter (System.Console.Out) and the default level (Level.Fine)

Methods

Log

```
public void Log(Imsl.Logger.Level logLevel, string className, string
methodName, string message)
```

Description

Log a message.

Parameters

`logLevel` – The `Logger.Level` of the message. If the logger is set to a lower log level, the message is not logged.

`className` – The name of the class issuing the log message.

`methodName` – The name of the method issuing the log message.

`message` – The `string` message to log.

Remarks

The output of a successful log entry is of the form “ClassName: message”

Log

```
public void Log(Imsl.Logger.Level logLevel, string className, string
methodName, string message, object[] arg)
```

Description

Log a message with parameters.

Parameters

`logLevel` – The `Logger.Level` of the message. If the logger is set to a lower log level, the message is not logged.

`className` – The name of the class issuing the log message.

`methodName` – The name of the method issuing the log message.

`message` – The `string` message to log that contains placeholders of the form `{0}` to contain each element in the parameter list.

`arg` – An `Object` array of parameter values.

Remarks

The output of a successful log entry is of the form “ClassName: formatted message”

Log

```
public void Log(Imsl.Logger.Level logLevel, string className, string
methodName, string name, string key, object[] arg)
```

Description

Log a message with parameters using a resource file.

Parameters

`logLevel` – The `Logger.Level` of the message. If the logger is set to a lower log level, the message is not logged.

`className` – The name of the class issuing the log message.

`methodName` – The name of the method issuing the log message.

`name` – The name of the resource bundle within the IMSL C# Library.

`key` – The key for the message string in the resource bundle.

`arg` – An `Object` array of parameter values.

Remarks

The output of a successful log entry is of the form “ClassName: formatted message”

Chapter 29: Exceptions

Types

IMSLUnexpectedErrorException Class

```
public class Imsl.IMSLUnexpectedErrorException :  IMSLException :
ISerializable
```

Signals that an unexpected error has occurred.

Please contact your IMSL technical support representative to report this problem.

Constructor

IMSLUnexpectedErrorException

```
public IMSLUnexpectedErrorException()
```

Description

Constructs an `IMSLUnexpectedErrorException`.

AllConstraintsNotSatisfiedException Class

```
public class Imsl.Math.AllConstraintsNotSatisfiedException :  IMSLException :
ISerializable
```

All constraints are not satisfied.

L1 minimization was applied to all constraints (including bounds and simple variables) except the equalities to approximate violated non-equalities as well as possible. If a feasible solution is possible then try using refinement.

Constructors

AllConstraintsNotSatisfiedException

```
public AllConstraintsNotSatisfiedException()
```

Description

All constraints are not satisfied.

AllConstraintsNotSatisfiedException

```
public AllConstraintsNotSatisfiedException(string message)
```

Description

All constraints are not satisfied.

Parameter

`message` – The error message that explains the reason for the exception.

AllConstraintsNotSatisfiedException

```
public AllConstraintsNotSatisfiedException(string message, System.Exception
exception)
```

Description

All constraints are not satisfied.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

AllConstraintsNotSatisfiedException

```
AllConstraintsNotSatisfiedException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

All constraints are not satisfied.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

BadInitialGuessException Class

```
public class Imsl.Math.BadInitialGuessException :  IMSLException :
ISerializable
```

Penalty function point infeasible for original problem. Try new initial guess.

Constructors

BadInitialGuessException

```
public BadInitialGuessException()
```

Description

Penalty function point infeasible for original problem. Try new initial guess.

BadInitialGuessException

```
public BadInitialGuessException(string message)
```

Description

Penalty function point infeasible for original problem. Try new initial guess.

Parameter

`message` – The error message that explains the reason for the exception.

BadInitialGuessException

```
public BadInitialGuessException(string message, System.Exception exception)
```

Description

Penalty function point infeasible for original problem. Try new initial guess.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

BadInitialGuessException

```
BadInitialGuessException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Penalty function point infeasible for original problem. Try new initial guess.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

BoundaryInconsistentException Class

```
public class Imsl.Math.BoundaryInconsistentException :  IMSLException :
ISerializable
```

The boundary conditions are inconsistent.

Constructors

BoundaryInconsistentException

```
public BoundaryInconsistentException(string side, string gridVal)
```

Description

The boundary conditions are inconsistent.

Parameters

`side` – A `String`, indicates if the left or right boundary conditions are inconsistent.

`gridVal` – A `String`, the grid value for which the boundary conditions are inconsistent, i.e. “ xMin ” for the left and “ xMax ” for the right boundaries.

BoundaryInconsistentException

```
public BoundaryInconsistentException()
```

Description

The boundary conditions are inconsistent.

BoundaryInconsistentException

```
public BoundaryInconsistentException(string s, object[] args)
```

Description

The boundary conditions are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

BoundaryInconsistentException

```
public BoundaryInconsistentException(string message)
```

Description

The boundary conditions are inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

BoundaryInconsistentException

```
public BoundaryInconsistentException(string s, System.Exception exception)
```

Description

The boundary conditions are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

BoundaryInconsistentException

```
BoundaryInconsistentException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The boundary conditions are inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

BoundsInconsistentException Class

```
public class Imsl.Math.BoundsInconsistentException :  IMSLException :
ISerializable
```

The bounds given are inconsistent.

Constructors

BoundsInconsistentException

```
public BoundsInconsistentException(string nameVariable, string nameLowerBound,
string nameUpperBound, int index, double lowerBound, double upperBound)
```

Description

The bounds given are inconsistent.

Parameters

`nameVariable` – Name of the variable being bounded.

`nameLowerBound` – Name of the lower bound.

`nameUpperBound` – Name of the upper bound.

`index` – The index of the inconsistent bound.

`lowerBound` – Value of the lower bound.

`upperBound` – Value of the upper bound.

BoundsInconsistentException

```
public BoundsInconsistentException()
```

Description

The bounds given are inconsistent.

BoundsInconsistentException

```
public BoundsInconsistentException(string message)
```

Description

The bounds given are inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

BoundsInconsistentException

```
public BoundsInconsistentException(string s, System.Exception exception)
```

Description

The bounds given are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

BoundsInconsistentException

```
BoundsInconsistentException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The bounds given are inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ConstraintEvaluationException Class

```
public class Imsl.Math.ConstraintEvaluationException :  IMSLException :
ISerializable
```

Constraint evaluation returns an error with current point.

Constructors

ConstraintEvaluationException

```
public ConstraintEvaluationException()
```

Description

Constraint evaluation returns an error with current point.

ConstraintEvaluationException

```
public ConstraintEvaluationException(string message)
```

Description

Constraint evaluation returns an error with current point.

Parameter

`message` – The error message that explains the reason for the exception.

ConstraintEvaluationException

```
public ConstraintEvaluationException(string s, System.Exception exception)
```

Description

Constraint evaluation returns an error with current point.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ConstraintEvaluationException

```
ConstraintEvaluationException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Constraint evaluation returns an error with current point.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ConstraintsInconsistentException Class

```
public class Imsl.Math.ConstraintsInconsistentException :  IMSLException :
ISerializable
```

The equality constraints are inconsistent.

Constructors

ConstraintsInconsistentException

```
public ConstraintsInconsistentException()
```

Description

The equality constraints are inconsistent.

ConstraintsInconsistentException

```
public ConstraintsInconsistentException(string s, object[] args)
```

Description

The constraints are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

ConstraintsInconsistentException

```
public ConstraintsInconsistentException(string message)
```

Description

The equality constraints are inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

ConstraintsInconsistentException

```
public ConstraintsInconsistentException(string s, System.Exception exception)
```

Description

The equality constraints are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ConstraintsInconsistentException

```
ConstraintsInconsistentException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The equality constraints are inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ConstraintsNotSatisfiedException Class

```
public class Imsl.Math.ConstraintsNotSatisfiedException :  IMSLException :
ISerializable
```

No vector *x* satisfies all of the constraints.

Constructors

ConstraintsNotSatisfiedException

```
public ConstraintsNotSatisfiedException()
```

Description

No vector *x* satisfies all of the constraints.

ConstraintsNotSatisfiedException

```
public ConstraintsNotSatisfiedException(string message)
```

Description

No vector *x* satisfies all of the constraints.

Parameter

`message` – The error message that explains the reason for the exception.

ConstraintsNotSatisfiedException

```
public ConstraintsNotSatisfiedException(string s, System.Exception exception)
```

Description

No vector *x* satisfies all of the constraints.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ConstraintsNotSatisfiedException

```
ConstraintsNotSatisfiedException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

No vector *x* satisfies all of the constraints.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

CorrectorConvergenceException Class

```
public class Imsl.Math.CorrectorConvergenceException :  IMSLException :
ISerializable
```

Corrector failed to converge.

Constructors

CorrectorConvergenceException

```
public CorrectorConvergenceException(double time)
```

Description

Corrector failed to converge.

Parameter

`time` – A `double` scalar, the time point at which the convergence exception occured.

CorrectorConvergenceException

```
public CorrectorConvergenceException()
```

Description

Corrector failed to converge.

CorrectorConvergenceException

```
public CorrectorConvergenceException(string s, object[] args)
```

Description

Corrector failed to converge.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

CorrectorConvergenceException

```
public CorrectorConvergenceException(string message)
```

Description

Corrector failed to converge.

Parameter

`message` – The error message that explains the reason for the exception.

CorrectorConvergenceException

```
public CorrectorConvergenceException(string s, System.Exception exception)
```

Description

Corrector failed to converge.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

CorrectorConvergenceException

```
CorrectorConvergenceException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Corrector failed to converge.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

CyclingOccurringException Class

```
public class Imsl.Math.CyclingOccurringException :  IMSLException :
ISerializable
```

The algorithm appears to be cycling. Using refinement may help.

Constructors

CyclingOccurringException

```
public CyclingOccurringException()
```

Description

The algorithm appears to be cycling. Using refinement may help.

CyclingOccurringException

```
public CyclingOccurringException(string message)
```

Description

The algorithm appears to be cycling. Using refinement may help.

Parameter

`message` – The error message that explains the reason for the exception.

CyclingOccurringException

```
public CyclingOccurringException(string message, System.Exception exception)
```

Description

The algorithm appears to be cycling. Using refinement may help.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

CyclingOccurringException

```
CyclingOccurringException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The algorithm appears to be cycling. Using refinement may help.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

DidNotConvergeException Class

```
public class Imsl.Math.DidNotConvergeException :  IMSLException :
ISerializable
```

Maximum number of iterations exceeded.

Constructors

DidNotConvergeException

```
public DidNotConvergeException()
```

Description

Maximum number of iterations exceeded.

DidNotConvergeException

```
public DidNotConvergeException(int maximumNumberOfIterations)
```

Description

Maximum number of iterations exceeded.

Parameter

`maximumNumberOfIterations` – Maximum number of iterations allowed exceeded argument.

DidNotConvergeException

```
public DidNotConvergeException(int info, int min)
```

Description

Maximum number of iterations exceeded.

Parameters

`info` – First argument for SVD.DidNotConverge string.

`min` – Second argument for SVD.DidNotConverge string.

DidNotConvergeException

```
public DidNotConvergeException(string s, object[] args)
```

Description

Maximum function evaluations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – An `Object` array which specifies arguments used to format the message.

DidNotConvergeException

```
public DidNotConvergeException(string message)
```

Description

Maximum number of iterations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

DidNotConvergeException

```
public DidNotConvergeException(string message, int maximumNumberOfIterations)
```

Description

Maximum number of iterations exceeded.

Parameters

`message` – The error message that explains the reason for the exception.

`maximumNumberOfIterations` – Maximum number of iterations allowed.

DidNotConvergeException

```
public DidNotConvergeException(string s, System.Exception exception)
```

Description

Maximum number of iterations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

DidNotConvergeException

```
DidNotConvergeException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of iterations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

EqualityConstraintsException Class

```
public class Imsl.Math.EqualityConstraintsException :  IMSLException :
ISerializable
```

The variables are determined by the equality constraints.

Constructors

EqualityConstraintsException

```
public EqualityConstraintsException()
```

Description

The variables are determined by the equality constraints.

EqualityConstraintsException

```
public EqualityConstraintsException(string message)
```

Description

The variables are determined by the equality constraints.

Parameter

`message` – The error message that explains the reason for the exception.

EqualityConstraintsException

```
public EqualityConstraintsException(string s, System.Exception exception)
```

Description

The variables are determined by the equality constraints.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

EqualityConstraintsException

```
EqualityConstraintsException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The variables are determined by the equality constraints.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ErrorTestException Class

```
public class Imsl.Math.ErrorTestException :  IMSLException :  ISerializable
```

Error test failure detected.

Constructors

ErrorTestException

```
public ErrorTestException(double time, double stepsize)
```

Description

Error test failure detected.

Parameters

`time` – A `double` scalar, the time point at which the error test failure occured.

`stepsize` – A `double` scalar, the stepsize used in the integration.

ErrorTestException

```
public ErrorTestException()
```

Description

Error test failure detected.

ErrorTestException

```
public ErrorTestException(string s, object[] args)
```

Description

Error test failure detected.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

ErrorTestException

```
public ErrorTestException(string message)
```

Description

Error test failure detected.

Parameter

`message` – The error message that explains the reason for the exception.

ErrorTestException

```
public ErrorTestException(string s, System.Exception exception)
```

Description

Error test failure detected.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ErrorTestException

```
ErrorTestException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Error test failure detected.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

FalseConvergenceException Class

```
public class Imsl.Math.FalseConvergenceException :  IMSLException :
ISerializable
```

False convergence, the iterates appear to be converging to a noncritical point.

Constructors

FalseConvergenceException

```
public FalseConvergenceException()
```

Description

False convergence, the iterates appear to be converging to a noncritical point.

FalseConvergenceException

```
public FalseConvergenceException(string message)
```

Description

False convergence, the iterates appear to be converging to a noncritical point.

Parameter

`message` – The error message that explains the reason for the exception.

FalseConvergenceException

```
public FalseConvergenceException(string s, System.Exception exception)
```

Description

False convergence, the iterates appear to be converging to a noncritical point.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

FalseConvergenceException

```
FalseConvergenceException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

False convergence, the iterates appear to be converging to a noncritical point.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

IllConditionedException Class

```
public class Imsl.Math.IllConditionedException :  IMSLException :
ISerializable
```

Problem is singular or ill-conditioned.

Constructors

IllConditionedException

```
public IllConditionedException()
```

Description

Problem is singular or ill-conditioned.

IllConditionedException

```
public IllConditionedException(string message)
```

Description

Problem is singular or ill-conditioned.

Parameter

`message` – The error message that explains the reason for the exception.

IllConditionedException

```
public IllConditionedException(string s, System.Exception exception)
```

Description

Problem is singular or ill-conditioned.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

IllConditionedException

```
IllConditionedException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Problem is singular or ill-conditioned.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

InconsistentSystemException Class

```
public class Imsl.Math.InconsistentSystemException :  IMSLException :
ISerializable
```

Inconsistent system.

Constructors

InconsistentSystemException

```
public InconsistentSystemException()
```

Description

Inconsistent system.

InconsistentSystemException

```
public InconsistentSystemException(string message)
```

Description

Inconsistent system.

Parameter

`message` – The error message that explains the reason for the exception.

InconsistentSystemException

```
public InconsistentSystemException(string s, System.Exception exception)
```

Description

Inconsistent system.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

InconsistentSystemException

```
InconsistentSystemException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Inconsistent system.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

InitialConstraintsException Class

```
public class Imsl.Math.InitialConstraintsException :  IMSLException :
ISerializable
```

The constraints at the initial point are inconsistent.

Constructors

InitialConstraintsException

```
public InitialConstraintsException()
```

Description

The constraints at the initial point are inconsistent.

InitialConstraintsException

```
public InitialConstraintsException(string message)
```

Description

The constraints at the initial point are inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

InitialConstraintsException

```
public InitialConstraintsException(string s, object[] args)
```

Description

The constraints at the initial point are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

InitialConstraintsException

```
public InitialConstraintsException(string s, System.Exception exception)
```

Description

The constraints at the initial point are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

InitialConstraintsException

```
InitialConstraintsException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The constraints at the initial point are inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

InvalidMPSFileException Class

```
public class Imsl.Math.InvalidMPSFileException :  IMSLException :
ISerializable
```

Invalid MPS file.

Constructors

InvalidMPSFileException

```
public InvalidMPSFileException(string message)
```

Description

Invalid MPS file.

Parameter

`message` – The error message that explains the reason for the exception.

InvalidMPSFileException

```
public InvalidMPSFileException(string s, object[] args)
```

Description

Invalid MPS file.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

InvalidMPSFileException

```
InvalidMPSFileException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Invalid MPS file.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

IterationMatrixSingularException Class

```
public class Imsl.Math.IterationMatrixSingularException :  IMSLException :
ISerializable
```

Iteration matrix is singular.

Constructors

IterationMatrixSingularException

```
public IterationMatrixSingularException(double time, double stepsize)
```

Description

Iteration matrix is singular.

Parameters

`time` – A `double` scalar, the time point at which the iteration matrix is singular.

`stepsize` – A `double` scalar, the stepsize used in the integration.

IterationMatrixSingularException

```
public IterationMatrixSingularException()
```

Description

Iteration matrix is singular.

IterationMatrixSingularException

```
public IterationMatrixSingularException(string s, object[] args)
```

Description

Iteration matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

IterationMatrixSingularException

```
public IterationMatrixSingularException(string message)
```

Description

Iteration matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

IterationMatrixSingularException

```
public IterationMatrixSingularException(string s, System.Exception exception)
```

Description

Iteration matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

IterationMatrixSingularException

```
IterationMatrixSingularException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Iteration matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

LimitingAccuracyException Class

```
public class Imsl.Math.LimitingAccuracyException :  IMSLException :
ISerializable
```

Limiting accuracy reached for a singular problem.

Constructors

LimitingAccuracyException

```
public LimitingAccuracyException()
```

Description

Limiting accuracy reached for a singular problem.

LimitingAccuracyException

```
public LimitingAccuracyException(string message)
```

Description

Limiting accuracy reached for a singular problem.

Parameter

`message` – The error message that explains the reason for the exception.

LimitingAccuracyException

```
public LimitingAccuracyException(string s, System.Exception exception)
```

Description

Limiting accuracy reached for a singular problem.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

LimitingAccuracyException

```
LimitingAccuracyException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Limiting accuracy reached for a singular problem.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

LinearlyDependentGradientsException Class

```
public class Imsl.Math.LinearlyDependentGradientsException :  IMSLException :
ISerializable
```

Working set gradients are linearly dependent.

Constructors

LinearlyDependentGradientsException

```
public LinearlyDependentGradientsException()
```

Description

Working set gradients are linearly dependent.

LinearlyDependentGradientsException

```
public LinearlyDependentGradientsException(string message)
```

Description

Working set gradients are linearly dependent.

Parameter

`message` – The error message that explains the reason for the exception.

LinearlyDependentGradientsException

```
public LinearlyDependentGradientsException(string s, System.Exception
exception)
```

Description

Working set gradients are linearly dependent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

LinearlyDependentGradientsException

```
LinearlyDependentGradientsException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Working set gradients are linearly dependent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

MaxIterationsException Class

```
public class Imsl.Math.MaxIterationsException :  IMSLException :  ISerializable
```

Maximum number of iterations exceeded.

Constructors

MaxIterationsException

```
public MaxIterationsException()
```

Description

Maximum number of iterations exceeded.

MaxIterationsException

```
public MaxIterationsException(string message)
```

Description

Maximum number of iterations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

MaxIterationsException

```
public MaxIterationsException(string s, System.Exception exception)
```

Description

Maximum number of iterations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

MaxIterationsException

```
MaxIterationsException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of iterations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

MaxNumberStepsAllowedException Class

```
public class Imsl.Math.MaxNumberStepsAllowedException :  IMSLException :
ISerializable
```

Maximum number of steps allowed exceeded.

Constructors

MaxNumberStepsAllowedException

```
public MaxNumberStepsAllowedException()
```

Description

Maximum number of steps allowed exceeded.

MaxNumberStepsAllowedException

```
public MaxNumberStepsAllowedException(int maxSteps)
```

Description

Maximum number of steps allowed exceeded.

Parameter

`maxSteps` – Maximum number of steps allowed.

MaxNumberStepsAllowedException

```
public MaxNumberStepsAllowedException(string message)
```

Description

Maximum number of steps allowed exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

MaxNumberStepsAllowedException

```
public MaxNumberStepsAllowedException(string s, System.Exception exception)
```

Description

Maximum number of steps allowed exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

MaxNumberStepsAllowedException

```
MaxNumberStepsAllowedException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of steps allowed exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

MaxFcnEvalsExceededException Class

```
public class Imsl.Math.MaxFcnEvalsExceededException :  IMSLException :
ISerializable
```

Maximum function evaluations exceeded.

Constructors

MaxFcnEvalsExceededException

```
public MaxFcnEvalsExceededException()
```

Description

Maximum function evaluations exceeded.

MaxFcnEvalsExceededException

```
public MaxFcnEvalsExceededException(string message)
```

Description

Maximum function evaluations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

MaxFcnEvalsExceededException

```
public MaxFcnEvalsExceededException(string s, System.Exception exception)
```

Description

Maximum function evaluations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

MaxFcnEvalsExceededException

```
public MaxFcnEvalsExceededException(string s, object[] args)
```

Description

Maximum function evaluations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – An `Object` array which specifies arguments used to format the message.

MaxFcnEvalsExceededException

```
MaxFcnEvalsExceededException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum function evaluations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

MultipleSolutionsException Class

```
public class Imsl.Math.MultipleSolutionsException :  IMSLException :
ISerializable
```

The problem has multiple solutions producing essentially the same minimum.

Constructors

MultipleSolutionsException

```
public MultipleSolutionsException()
```

Description

The problem has multiple solutions producing essentially the same minimum.

MultipleSolutionsException

```
public MultipleSolutionsException(string message)
```

Description

The problem has multiple solutions producing essentially the same minimum.

Parameter

`message` – The error message that explains the reason for the exception.

MultipleSolutionsException

```
public MultipleSolutionsException(string message, System.Exception exception)
```

Description

The problem has multiple solutions producing essentially the same minimum.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

MultipleSolutionsException

```
MultipleSolutionsException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The problem has multiple solutions producing essentially the same minimum.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoAcceptablePivotException Class

```
public class Imsl.Math.NoAcceptablePivotException :  IMSLException :
ISerializable
```

An acceptable pivot could not be found.

Constructors

NoAcceptablePivotException

```
public NoAcceptablePivotException()
```

Description

An acceptable pivot could not be found.

NoAcceptablePivotException

```
public NoAcceptablePivotException(string message)
```

Description

An acceptable pivot could not be found.

Parameter

`message` – The error message that explains the reason for the exception.

NoAcceptablePivotException

```
public NoAcceptablePivotException(string s, System.Exception exception)
```

Description

An acceptable pivot could not be found.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoAcceptablePivotException

```
NoAcceptablePivotException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

An acceptable pivot could not be found.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoAcceptableStepsizeException Class

```
public class Imsl.Math.NoAcceptableStepsizeException :  IMSLException :
ISerializable
```

No acceptable stepsize in [SIGMA,SIGLA].

Constructors

NoAcceptableStepsizeException

```
public NoAcceptableStepsizeException(double sigma, double sigla)
```

Description

No acceptable stepsize in [SIGMA,SIGLA].

Parameters

`sigma` – A `double` containing the first messages argument SIGMA.

`sigla` – A `double` containing the second messages argument SIGLA.

NoAcceptableStepsizeException

```
public NoAcceptableStepsizeException()
```

Description

No acceptable stepsize in [SIGMA,SIGLA].

NoAcceptableStepsizeException

```
public NoAcceptableStepsizeException(string message)
```

Description

No acceptable stepsize in [SIGMA,SIGLA].

Parameter

`message` – The error message that explains the reason for the exception.

NoAcceptableStepsizeException

```
public NoAcceptableStepsizeException(string s, System.Exception exception)
```

Description

No acceptable stepsize in [SIGMA,SIGLA].

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoAcceptableStepsizeException

```
NoAcceptableStepsizeException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

No acceptable stepsize in [SIGMA,SIGLA].

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoConvergenceException Class

```
public class Imsl.Math.NoConvergenceException :  IMSLException :  ISerializable
```

The conjugate gradient method did not converge within the allowed maximum number of iterations.

Constructors

NoConvergenceException

```
public NoConvergenceException(int maxIterations)
```

Description

The conjugate gradient method did not converge within the allowed maximum number of iterations.

Parameter

`maxIterations` – Maximum number of iterations exceeded.

NoConvergenceException

```
public NoConvergenceException(string message)
```

Description

The conjugate gradient method did not converge within the allowed maximum number of iterations.

Parameter

`message` – The error message that explains the reason for the exception.

NoConvergenceException

```
public NoConvergenceException(string s, System.Exception exception)
```

Description

The conjugate gradient method did not converge within the allowed maximum number of iterations.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoConvergenceException

```
NoConvergenceException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The conjugate gradient method did not converge within the allowed maximum number of iterations.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoLPSolutionException Class

```
public class Imsl.Math.NoLPSolutionException :  IMSLException :  ISerializable
```

No solution for the LP problem with h = 0 was found by `DenseLP`.

Constructors

NoLPSolutionException

```
public NoLPSolutionException(object[] args)
```

Description

No solution for the LP problem with h = 0 was found by `DenseLP`.

Parameter

`args` – An `Object` array containing the input arguments to error string.

NoLPSolutionException

```
public NoLPSolutionException(string message)
```

Description

No solution for the LP problem with h = 0 was found by `DenseLP`.

Parameter

`message` – The error message that explains the reason for the exception.

NoLPSolutionException

```
public NoLPSolutionException(string message, System.Exception exception)
```

Description

No solution for the LP problem with h = 0 was found by `DenseLP`.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoLPSolutionException

```
NoLPSolutionException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

No solution for the LP problem with h = 0 was found by `DenseLP`.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoProgressException Class

```
public class Imsl.Math.NoProgressException :  IMSLException :  ISerializable
```

The algorithm is not making any progress. Try a new initial guess.

Constructors

NoProgressException

```
public NoProgressException()
```

Description

The algorithm is not making any progress. Try a new initial guess.

NoProgressException

```
public NoProgressException(string message)
```

Description

The algorithm is not making any progress. Try a new initial guess.

Parameter

`message` – The error message that explains the reason for the exception.

NoProgressException

```
public NoProgressException(string s, System.Exception exception)
```

Description

The algorithm is not making any progress. Try a new initial guess.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoProgressException

```
NoProgressException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The algorithm is not making any progress. Try a new initial guess.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotDefiniteAMatrixException Class

```
public class Imsl.Math.NotDefiniteAMatrixException :  IMSLException :
ISerializable
```

The input matrix A is indefinite, that is the matrix is not positive or negative definite.

Constructors

NotDefiniteAMatrixException

```
public NotDefiniteAMatrixException()
```

Description

The input matrix A is indefinite, that is the matrix is not positive or negative definite.

NotDefiniteAMatrixException

```
public NotDefiniteAMatrixException(string message)
```

Description

The input matrix A is indefinite, that is the matrix is not positive or negative definite.

Parameter

`message` – The error message that explains the reason for the exception.

NotDefiniteAMatrixException

```
public NotDefiniteAMatrixException(string s, System.Exception exception)
```

Description

The input matrix A is indefinite, that is the matrix is not positive or negative definite.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotDefiniteAMatrixException

```
NotDefiniteAMatrixException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The input matrix A is indefinite, that is the matrix is not positive or negative definite.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotDefiniteJacobiPreconditionerException Class

```
public class Imsl.Math.NotDefiniteJacobiPreconditionerException :
IMSLException :  ISerializable
```

The Jacobi preconditioner is not strictly positive or negative definite.

Constructors

NotDefiniteJacobiPreconditionerException

```
public NotDefiniteJacobiPreconditionerException()
```

Description

The Jacobi preconditioner is not strictly positive or negative definite.

NotDefiniteJacobiPreconditionerException

```
public NotDefiniteJacobiPreconditionerException(string message)
```

Description

The Jacobi preconditioner is not strictly positive or negative definite.

Parameter

`message` – The error message that explains the reason for the exception.

NotDefiniteJacobiPreconditionerException

```
public NotDefiniteJacobiPreconditionerException(string s, System.Exception
exception)
```

Description

The Jacobi preconditioner is not strictly positive or negative definite.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotDefiniteJacobiPreconditionerException

```
NotDefiniteJacobiPreconditionerException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The Jacobi preconditioner is not strictly positive or negative definite.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotDefinitePreconditionMatrixException Class

```
public class Imsl.Math.NotDefinitePreconditionMatrixException :  IMSLException
:  ISerializable
```

The Precondition matrix is indefinite.

Constructors

NotDefinitePreconditionMatrixException

```
public NotDefinitePreconditionMatrixException()
```

Description

The Precondition matrix is indefinite.

NotDefinitePreconditionMatrixException

```
public NotDefinitePreconditionMatrixException(string message)
```

Description

The Precondition matrix is indefinite.

Parameter

`message` – The error message that explains the reason for the exception.

NotDefinitePreconditionMatrixException

```
public NotDefinitePreconditionMatrixException(string s, System.Exception
exception)
```

Description

The Precondition matrix is indefinite.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotDefinitePreconditionMatrixException

```
NotDefinitePreconditionMatrixException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The Precondition matrix is indefinite.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotSPDException Class

```
public class Imsl.Math.NotSPDException :  IMSLException :  ISerializable
```

The matrix is not symmetric, positive definite.

Constructors

NotSPDException

```
public NotSPDException()
```

Description

The matrix is not symmetric, positive definite.

NotSPDException

```
public NotSPDException(string message)
```

Description

The matrix is not symmetric, positive definite.

Parameter

`message` – The error message that explains the reason for the exception.

NotSPDException

```
public NotSPDException(string s, System.Exception exception)
```

Description

The matrix is not symmetric, positive definite.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotSPDException

```
NotSPDException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The matrix is not symmetric, positive definite.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NumericDifficultyException Class

```
public class Imsl.Math.NumericDifficultyException :  IMSLException :
ISerializable
```

Numerical difficulty occurred.

Constructors

NumericDifficultyException

```
public NumericDifficultyException()
```

Description

Numerical difficulty occurred.

NumericDifficultyException

```
public NumericDifficultyException(string message)
```

Description

Numerical difficulty occurred.

Parameter

`message` – The error message that explains the reason for the exception.

NumericDifficultyException

```
public NumericDifficultyException(string s, System.Exception exception)
```

Description

Numerical difficulty occurred.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NumericDifficultyException

```
NumericDifficultyException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Numerical difficulty occurred.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ObjectiveEvaluationException Class

```
public class Imsl.Math.ObjectiveEvaluationException :  IMSLException :
ISerializable
```

Objective evaluation returns an error with current point.

Constructors

ObjectiveEvaluationException

```
public ObjectiveEvaluationException()
```

Description

Objective evaluation returns an error with current point.

ObjectiveEvaluationException

```
public ObjectiveEvaluationException(string message)
```

Description

Objective evaluation returns an error with current point.

Parameter

`message` – The error message that explains the reason for the exception.

ObjectiveEvaluationException

```
public ObjectiveEvaluationException(string s, System.Exception exception)
```

Description

Objective evaluation returns an error with current point.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ObjectiveEvaluationException

```
ObjectiveEvaluationException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Objective evaluation returns an error with current point.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

PenaltyFunctionPointInfeasibleException Class

```
public class Imsl.Math.PenaltyFunctionPointInfeasibleException :  IMSLException
:  ISerializable
```

Penalty function point infeasible.

Constructors

PenaltyFunctionPointInfeasibleException

```
public PenaltyFunctionPointInfeasibleException()
```

Description

Penalty function point infeasible.

PenaltyFunctionPointInfeasibleException

```
public PenaltyFunctionPointInfeasibleException(string message)
```

Description

Penalty function point infeasible.

Parameter

`message` – The error message that explains the reason for the exception.

PenaltyFunctionPointInfeasibleException

```
public PenaltyFunctionPointInfeasibleException(string s, System.Exception
exception)
```

Description

Penalty function point infeasible.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

PenaltyFunctionPointInfeasibleException

```
PenaltyFunctionPointInfeasibleException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Penalty function point infeasible.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ProblemInfeasibleException Class

```
public class Imsl.Math.ProblemInfeasibleException :  IMSLException :
ISerializable
```

The problem is not feasible. The constraints are inconsistent.

Constructors

ProblemInfeasibleException

```
public ProblemInfeasibleException()
```

Description

The problem is not feasible. The constraints are inconsistent.

ProblemInfeasibleException

```
public ProblemInfeasibleException(string message)
```

Description

The problem is not feasible. The constraints are inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

ProblemInfeasibleException

```
public ProblemInfeasibleException(string s, System.Exception exception)
```

Description

The problem is not feasible. The constraints are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ProblemInfeasibleException

```
ProblemInfeasibleException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The problem is not feasible. The constraints are inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ProblemUnboundedException Class

```
public class Imsl.Math.ProblemUnboundedException :  IMSLException :
ISerializable
```

The problem is unbounded.

Constructors

ProblemUnboundedException

```
public ProblemUnboundedException()
```

Description

The problem is unbounded.

ProblemUnboundedException

```
public ProblemUnboundedException(string message)
```

Description

The problem is unbounded.

Parameter

`message` – The error message that explains the reason for the exception.

ProblemUnboundedException

```
public ProblemUnboundedException(string s, System.Exception exception)
```

Description

The problem is unbounded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ProblemUnboundedException

```
ProblemUnboundedException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The problem is unbounded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ProblemVacuousException Class

```
public class Imsl.Math.ProblemVacuousException :  IMSLException :
ISerializable
```

The problem is vacuous.

Constructors

ProblemVacuousException

```
public ProblemVacuousException()
```

Description

The problem is vacuous.

ProblemVacuousException

```
public ProblemVacuousException(string message)
```

Description

The problem is vacuous.

Parameter

`message` – The error message that explains the reason for the exception.

ProblemVacuousException

```
public ProblemVacuousException(string s, System.Exception exception)
```

Description

The problem is vacuous.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ProblemVacuousException

```
ProblemVacuousException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The problem is vacuous.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

QPInfeasibleException Class

```
public class Imsl.Math.QPInfeasibleException :  IMSLException :  ISerializable
```

QP problem seemingly infeasible.

Constructors

QPInfeasibleException

```
public QPInfeasibleException()
```

Description

QP problem seemingly infeasible.

QPInfeasibleException

```
public QPInfeasibleException(string message)
```

Description

QP problem seemingly infeasible.

Parameter

`message` – The error message that explains the reason for the exception.

QPInfeasibleException

```
public QPInfeasibleException(string s, System.Exception exception)
```

Description

QP problem seemingly infeasible.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

QPInfeasibleException

```
QPInfeasibleException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

QP problem seemingly infeasible.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

QPProblemUnboundedException Class

```
public class Imsl.Math.QPProblemUnboundedException :  IMSLException :
ISerializable
```

The object value for the problem is unbounded.

Constructors

QPProblemUnboundedException

```
public QPProblemUnboundedException()
```

Description

The object value for the problem is unbounded.

QPProblemUnboundedException

```
public QPProblemUnboundedException(string message)
```

Description

The object value for the problem is unbounded.

Parameter

`message` – The error message that explains the reason for the exception.

QPProblemUnboundedException

```
public QPProblemUnboundedException(string message, System.Exception exception)
```

Description

The object value for the problem is unbounded.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

QPProblemUnboundedException

```
QPProblemUnboundedException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The object value for the problem is unbounded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SingularException Class

```
public class Imsl.Math.SingularException :  IMSLException :  ISerializable
```

Problem is singular.

Constructors

SingularException

```
public SingularException()
```

Description

Problem is singular.

SingularException

```
public SingularException(string message)
```

Description

Problem is singular.

Parameter

`message` – The error message that explains the reason for the exception.

SingularException

```
public SingularException(string s, System.Exception exception)
```

Description

Problem is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SingularException

```
SingularException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Problem is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SingularMatrixException Class

```
public class Imsl.Math.SingularMatrixException :  IMSLException :
ISerializable
```

The matrix is singular.

Constructors

SingularMatrixException

```
public SingularMatrixException()
```

Description

The matrix is singular.

SingularMatrixException

```
public SingularMatrixException(string message)
```

Description

The matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

SingularMatrixException

```
public SingularMatrixException(string s, System.Exception exception)
```

Description

The matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SingularMatrixException

```
SingularMatrixException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SingularPreconditionMatrixException Class

```
public class Imsl.Math.SingularPreconditionMatrixException :  IMSLException :
ISerializable
```

The Precondition matrix is singular.

Constructors

SingularPreconditionMatrixException

```
public SingularPreconditionMatrixException()
```

Description

PThe Precondition matrix is singular.

SingularPreconditionMatrixException

```
public SingularPreconditionMatrixException(string message)
```

Description

The Precondition matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

SingularPreconditionMatrixException

```
public SingularPreconditionMatrixException(string s, System.Exception
exception)
```

Description

The Precondition matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SingularPreconditionMatrixException

```
SingularPreconditionMatrixException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The Precondition matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SolutionNotFoundException Class

```
public class Imsl.Math.SolutionNotFoundException :  IMSLException :
ISerializable
```

A solution was not found. Try using DenseLP.

Constructors

SolutionNotFoundException

```
public SolutionNotFoundException()
```

Description

A solution was not found. Try using DenseLP.

SolutionNotFoundException

```
public SolutionNotFoundException(string message)
```

Description

A solution was not found. Try using DenseLP.

Parameter

`message` – The error message that explains the reason for the exception.

SolutionNotFoundException

```
public SolutionNotFoundException(string message, System.Exception exception)
```

Description

A solution was not found. Try using DenseLP.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SolutionNotFoundException

```
SolutionNotFoundException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

A solution was not found. Try using DenseLP.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SomeConstraintsDiscardedException Class

```
public class Imsl.Math.SomeConstraintsDiscardedException :  IMSLException :
ISerializable
```

Some constraints were discarded because they were too linearly dependent on other active constraints.

Constructors

SomeConstraintsDiscardedException

```
public SomeConstraintsDiscardedException()
```

Description

Some constraints were discarded because they were too linearly dependent on other active constraints.

SomeConstraintsDiscardedException

```
public SomeConstraintsDiscardedException(string message)
```

Description

Some constraints were discarded because they were too linearly dependent on other active constraints.

Parameter

`message` – The error message that explains the reason for the exception.

SomeConstraintsDiscardedException

```
public SomeConstraintsDiscardedException(string message, System.Exception
exception)
```

Description

Some constraints were discarded because they were too linearly dependent on other active constraints.

Parameters

`message` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SomeConstraintsDiscardedException

```
SomeConstraintsDiscardedException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Some constraints were discarded because they were too linearly dependent on other active constraints.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TcurrentTstopInconsistentException Class

```
public class Imsl.Math.TcurrentTstopInconsistentException :  IMSLException :
ISerializable
```

The end value for the integration in time, *tout*, is not consistent with the current time value, *t*.

Constructors

TcurrentTstopInconsistentException

```
public TcurrentTstopInconsistentException(double time, double stepsize, double
tstop)
```

Description

The end value for the integration in time, *tout*, is not consistent with the current time value, *t*.

Parameters

`time` – A `double` scalar, the current time value.

`stepsize` – A `double` scalar, the stepsize used in the integration.

`tstop` – A `double` scalar, the integration endpoint.

TcurrentTstopInconsistentException

```
public TcurrentTstopInconsistentException()
```

Description

The end value for the integration in time, *tout*, is not consistent with the current time value, *t*.

TcurrentTstopInconsistentException

```
public TcurrentTstopInconsistentException(string s, object[] args)
```

Description

The end value for the integration in time, *tout*, is not consistent with the current time value, *t*.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

TcurrentTstopInconsistentException

```
public TcurrentTstopInconsistentException(string message)
```

Description

The end value for the integration in time, *tout*, is not consistent with the current time value, *t*.

Parameter

`message` – The error message that explains the reason for the exception.

TcurrentTstopInconsistentException

```
public TcurrentTstopInconsistentException(string s, System.Exception exception)
```

Description

The end value for the integration in time, *tout*, is not consistent with the current time value, *t*.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TcurrentTstopInconsistentException

```
TcurrentTstopInconsistentException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The end value for the integration in time, *tout*, is not consistent with the current time value, *t*.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TEqualsToutException Class

```
public class Imsl.Math.TEqualsToutException :  IMSLException :  ISerializable
```

The current integration point in time and the end point are equal.

Constructors

TEqualsToutException

```
public TEqualsToutException(double time)
```

Description

The current integration point in time and the end point are equal.

Parameter

`time` – A `double` scalar, the current time point which is equal to the end point.

TEqualsToutException

```
public TEqualsToutException()
```

Description

The current integration point in time and the end point are equal.

TEqualsToutException

```
public TEqualsToutException(string s, object[] args)
```

Description

The current integration point in time and the end point are equal.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

TEqualsToutException

```
public TEqualsToutException(string message)
```

Description

The current integration point in time and the end point are equal.

Parameter

`message` – The error message that explains the reason for the exception.

TEqualsToutException

```
public TEqualsToutException(string s, System.Exception exception)
```

Description

The current integration point in time and the end point are equal.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TEqualsToutException

```
TEqualsToutException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The current integration point in time and the end point are equal.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TerminationCriteriaNotSatisfiedException Class

```
public class Imsl.Math.TerminationCriteriaNotSatisfiedException :
IMSLException :  ISerializable
```

Termination criteria are not satisfied.

Constructors

TerminationCriteriaNotSatisfiedException

```
public TerminationCriteriaNotSatisfiedException(int numsm)
```

Description

Termination criteria are not satisfied.

Parameter

`numsm` – An `int`containing the criteria value.

TerminationCriteriaNotSatisfiedException

```
public TerminationCriteriaNotSatisfiedException()
```

Description

Termination criteria are not satisfied.

TerminationCriteriaNotSatisfiedException

```
public TerminationCriteriaNotSatisfiedException(string message)
```

Description

Termination criteria are not satisfied.

Parameter

`message` – The error message that explains the reason for the exception.

TerminationCriteriaNotSatisfiedException

```
public TerminationCriteriaNotSatisfiedException(string s, System.Exception
exception)
```

Description

Termination criteria are not satisfied.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TerminationCriteriaNotSatisfiedException

```
TerminationCriteriaNotSatisfiedException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Termination criteria are not satisfied.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TimeIntervalTooSmallException Class

```
public class Imsl.Math.TimeIntervalTooSmallException :  IMSLException :
ISerializable
```

Distance between starting time point and end point for the integration is too small.

Constructors

TimeIntervalTooSmallException

```
public TimeIntervalTooSmallException(double tend, double tstart)
```

Description

Distance between starting time point and end point for the integration is too small.

Parameters

`tend` – A `double` scalar, the end point of the integration.

`tstart` – A `double` scalar, the starting point for the integration.

TimeIntervalTooSmallException

```
public TimeIntervalTooSmallException()
```

Description

Distance between starting time point and end point for the integration is too small.

TimeIntervalTooSmallException

```
public TimeIntervalTooSmallException(string s, object[] args)
```

Description

Distance between starting time point and end point for the integration is too small.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

TimeIntervalTooSmallException

```
public TimeIntervalTooSmallException(string message)
```

Description

Distance between starting time point and end point for the integration is too small.

Parameter

`message` – The error message that explains the reason for the exception.

TimeIntervalTooSmallException

```
public TimeIntervalTooSmallException(string s, System.Exception exception)
```

Description

Distance between starting time point and end point for the integration is too small.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TimeIntervalTooSmallException

```
TimeIntervalTooSmallException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Distance between starting time point and end point for the integration is too small.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ToleranceTooSmallException Class

```
public class Imsl.Math.ToleranceTooSmallException :  IMSLException :
ISerializable
```

Tolerance is too small.

Constructors

ToleranceTooSmallException

```
public ToleranceTooSmallException()
```

Description

Tolerance is too small.

ToleranceTooSmallException

```
public ToleranceTooSmallException(double tol)
```

Description

Tolerance is too small.

Parameter

`tol` – A `double` containing the tolerance value.

ToleranceTooSmallException

```
public ToleranceTooSmallException(string message)
```

Description

Tolerance is too small.

Parameter

`message` – The error message that explains the reason for the exception.

ToleranceTooSmallException

```
public ToleranceTooSmallException(string s, System.Exception exception)
```

Description

Tolerance is too small.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ToleranceTooSmallException

```
public ToleranceTooSmallException(string s, object[] args)
```

Description

Tolerance is too small.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – An `Object` array which specifies arguments used to format the message.

ToleranceTooSmallException

```
ToleranceTooSmallException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Tolerance is too small.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyIterationsException Class

```
public class Imsl.Math.TooManyIterationsException :  IMSLException :
ISerializable
```

Maximum number of iterations exceeded.

Constructors

TooManyIterationsException

```
public TooManyIterationsException(int maximumNumberOfIterations)
```

Description

Maximum number of iterations exceeded.

Parameter

`maximumNumberOfIterations` – Maximum number of iterations allowed.

TooManyIterationsException

```
public TooManyIterationsException()
```

Description

Maximum number of iterations exceeded.

TooManyIterationsException

```
public TooManyIterationsException(string message)
```

Description

Maximum number of iterations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

TooManyIterationsException

```
public TooManyIterationsException(string s, object[] args)
```

Description

Maximum number of iterations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – Arguments for the error message.

TooManyIterationsException

```
public TooManyIterationsException(string s, System.Exception exception)
```

Description

Maximum number of iterations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyIterationsException

```
TooManyIterationsException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of iterations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyStepsException Class

```
public class Imsl.Math.TooManyStepsException :  IMSLException :  ISerializable
```

Too many steps were taken between two consecutive time steps.

Constructors

TooManyStepsException

```
public TooManyStepsException(double t0, int steps, double t1)
```

Description

Too many steps were taken between two consecutive time steps.

Parameters

`t0` – The current time value.

`steps` – The maximum number of iterations allowed.

`t1` – The end point of the internal integration in time.

TooManyStepsException

```
public TooManyStepsException()
```

Description

Too many steps were taken between two consecutive time steps.

TooManyStepsException

```
public TooManyStepsException(string message)
```

Description

Too many steps were taken between two consecutive time steps.

Parameter

`message` – The error message that explains the reason for the exception.

TooManyStepsException

```
public TooManyStepsException(string s, System.Exception exception)
```

Description

Too many steps were taken between two consecutive time steps.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyStepsException

```
TooManyStepsException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Too many steps were taken between two consecutive time steps.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooMuchTimeException Class

```
public class Imsl.Math.TooMuchTimeException :  IMSLException :  ISerializable
```

Maximum time allowed for solve is exceeded.

Constructors

TooMuchTimeException

```
public TooMuchTimeException(long maximumTime)
```

Description

Maximum time allowed for solve is exceeded.

Parameter

`maximumTime` – The maximum time allowed for the solve step.

TooMuchTimeException

```
public TooMuchTimeException()
```

Description

Maximum time allowed for solve is exceeded.

TooMuchTimeException

```
public TooMuchTimeException(string message)
```

Description

Maximum time allowed for solve is exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

TooMuchTimeException

```
public TooMuchTimeException(string s, System.Exception exception)
```

Description

Maximum time allowed for solve is exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooMuchTimeException

```
TooMuchTimeException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum time allowed for solve is exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

UnboundedBelowException Class

```
public class Imsl.Math.UnboundedBelowException :  IMSLException :
ISerializable
```

Five consecutive steps of the maximum allowable stepsize have been taken, either the function is unbounded below, or has a finite asymptote in some directionor the maximum allowable step size is too small.

Constructors

UnboundedBelowException

```
public UnboundedBelowException()
```

Description

Five consecutive steps of the maximum allowable stepsize have been taken, either the function is unbounded below, or has a finite asymptote in some directionor the maximum allowable step size is too small.

UnboundedBelowException

```
public UnboundedBelowException(string message)
```

Description

Five consecutive steps of the maximum allowable stepsize have been taken, either the function is unbounded below, or has a finite asymptote in some directionor the maximum allowable step size is too small.

Parameter

`message` – The error message that explains the reason for the exception.

UnboundedBelowException

```
public UnboundedBelowException(string s, System.Exception exception)
```

Description

Five consecutive steps of the maximum allowable stepsize have been taken, either the function is unbounded below, or has a finite asymptote in some directionor the maximum allowable step size is too small.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

UnboundedBelowException

```
UnboundedBelowException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Five consecutive steps of the maximum allowable stepsize have been taken, either the function is unbounded below, or has a finite asymptote in some directionor the maximum allowable step size is too small.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

VarBoundsInconsistentException Class

```
public class Imsl.Math.VarBoundsInconsistentException :  IMSLException :
ISerializable
```

The equality constraints and the bounds on the variables are found to be inconsistent.

Constructors

VarBoundsInconsistentException

```
public VarBoundsInconsistentException()
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

VarBoundsInconsistentException

```
public VarBoundsInconsistentException(string message)
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

VarBoundsInconsistentException

```
public VarBoundsInconsistentException(string s, System.Exception exception)
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

VarBoundsInconsistentException

```
VarBoundsInconsistentException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

WorkingSetSingularException Class

```
public class Imsl.Math.WorkingSetSingularException :  IMSLException :
ISerializable
```

Working set is singular in dual extended QP.

Constructors

WorkingSetSingularException

```
public WorkingSetSingularException()
```

Description

Working set is singular in dual extended QP.

WorkingSetSingularException

```
public WorkingSetSingularException(string message)
```

Description

Working set is singular in dual extended QP.

Parameter

`message` – The error message that explains the reason for the exception.

WorkingSetSingularException

```
public WorkingSetSingularException(string s, System.Exception exception)
```

Description

Working set is singular in dual extended QP.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

WorkingSetSingularException

```
WorkingSetSingularException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Working set is singular in dual extended QP.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

AllDeletedException Class

```
public class Imsl.Stat.AllDeletedException :  IMSLException :  ISerializable
```

There are no observations.

Constructors

AllDeletedException

```
public AllDeletedException()
```

Description

There are no observations.

AllDeletedException

```
public AllDeletedException(string message)
```

Description

There are no observations.

Parameter

`message` – The error message that explains the reason for the exception.

AllDeletedException

```
public AllDeletedException(string s, System.Exception exception)
```

Description

There are no observations.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

AllDeletedException

```
AllDeletedException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

There are no observations.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

AllMissingException Class

```
public class Imsl.Stat.AllMissingException :  IMSLException :  ISerializable
```

There are no observations.

Constructors

AllMissingException

```
public AllMissingException()
```

Description

There are no observations.

AllMissingException

```
public AllMissingException(string message)
```

Description

There are no observations.

Parameter

`message` – The error message that explains the reason for the exception.

AllMissingException

```
public AllMissingException(string s, System.Exception exception)
```

Description

There are no observations.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

AllMissingException

```
AllMissingException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

There are no observations.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

AltSeriesAccuracyLossException Class

```
public class Imsl.Stat.AltSeriesAccuracyLossException :  IMSLException :
ISerializable
```

The magnitude of alternating series sum is too small relative to the sum of positive terms to permit a reliable accuracy.

Constructors

AltSeriesAccuracyLossException

```
public AltSeriesAccuracyLossException()
```

Description

The magnitude of alternating series sum is too small relative to the sum of positive terms to permit a reliable accuracy.

AltSeriesAccuracyLossException

```
public AltSeriesAccuracyLossException(string message)
```

Description

The magnitude of alternating series sum is too small relative to the sum of positive terms to permit a reliable accuracy.

Parameter

`message` – The error message that explains the reason for the exception.

AltSeriesAccuracyLossException

```
public AltSeriesAccuracyLossException(string s, System.Exception exception)
```

Description

The magnitude of alternating series sum is too small relative to the sum of positive terms to permit a reliable accuracy.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

AltSeriesAccuracyLossException

```
AltSeriesAccuracyLossException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The magnitude of alternating series sum is too small relative to the sum of positive terms to permit a reliable accuracy.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

BadVarianceException Class

```
public class Imsl.Stat.BadVarianceException :  IMSLException :  ISerializable
```

The input variance is not in the allowed range.

Constructors

BadVarianceException

```
public BadVarianceException(int i, double cov, double uniq)
```

Description

The input variance is not in the allowed range.

Parameters

`i` – A `int` specifying the index of variable uniq, causing the error.

`cov` – A `double` specifying the value of cov[i,i].

`uniq` – A `double` specifying the input variance.

BadVarianceException

```
public BadVarianceException()
```

Description

Maximum number of iterations exceeded.

BadVarianceException

```
public BadVarianceException(string message)
```

Description

Maximum number of iterations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

BadVarianceException

```
public BadVarianceException(string s, System.Exception exception)
```

Description

Maximum number of iterations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

BadVarianceException

```
BadVarianceException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of iterations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ClassificationVariableException Class

```
public class Imsl.Stat.ClassificationVariableException :  IMSLException :
ISerializable
```

The ClassificationVariable vector has not been initialized.

Constructors

ClassificationVariableException

```
public ClassificationVariableException()
```

Description

The ClassificationVariable vector has not been initialized.

ClassificationVariableException

```
public ClassificationVariableException(string message)
```

Description

The ClassificationVariable vector has not been initialized.

Parameter

`message` – The error message that explains the reason for the exception.

ClassificationVariableException

```
public ClassificationVariableException(string s, System.Exception exception)
```

Description

The ClassificationVariable vector has not been initialized.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ClassificationVariableException

```
ClassificationVariableException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The ClassificationVariable vector has not been initialized.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ClassificationVariableLimitException Class

```
public class Imsl.Stat.ClassificationVariableLimitException :  IMSLException :
ISerializable
```

The Classification Variable limit set by the user through `setUpperBound` has been exceeded.

Constructors

ClassificationVariableLimitException

```
public ClassificationVariableLimitException(int maxcl)
```

Description

The Classification Variable limit set by the user through `setUpperBound` has been exceeded.

Parameter

`maxcl` – An `int` which specifies the upper bound.

ClassificationVariableLimitException

```
public ClassificationVariableLimitException()
```

Description

The Classification Variable limit set by the user through `setUpperBound` has been exceeded.

ClassificationVariableLimitException

```
public ClassificationVariableLimitException(string message)
```

Description

The Classification Variable limit set by the user through `setUpperBound` has been exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

ClassificationVariableLimitException

```
public ClassificationVariableLimitException(string s, System.Exception
exception)
```

Description

The Classification Variable limit set by the user through `setUpperBound` has been exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ClassificationVariableLimitException

```
public ClassificationVariableLimitException(string s, object[] args)
```

Description

The Classification Variable limit set by the user through `setUpperBound` has been exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`args` – An `Object` array which specifies arguments used to format the message.

ClassificationVariableLimitException

```
ClassificationVariableLimitException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The Classification Variable limit set by the user through `setUpperBound` has been exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ClassificationVariableValueException Class

```
public class Imsl.Stat.ClassificationVariableValueException :  IMSLException :
ISerializable
```

The number of distinct values for each Classification Variable must be greater than 1.

Constructors

ClassificationVariableValueException

```
public ClassificationVariableValueException(int index, int val)
```

Description

The number of distinct values for each Classification Variable must be greater than 1.

Parameters

`index` – An `int` which specifies the index of a classification variable.

`val` – An `int` which specifies the number of distinct values that can be taken by this classification variable.

ClassificationVariableValueException

```
public ClassificationVariableValueException()
```

Description

The number of distinct values for each Classification Variable must be greater than 1.

ClassificationVariableValueException

```
public ClassificationVariableValueException(string message)
```

Description

The number of distinct values for each Classification Variable must be greater than 1.

Parameter

`message` – The error message that explains the reason for the exception.

ClassificationVariableValueException

```
public ClassificationVariableValueException(string s, System.Exception
exception)
```

Description

The number of distinct values for each Classification Variable must be greater than 1.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ClassificationVariableValueException

```
ClassificationVariableValueException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The number of distinct values for each Classification Variable must be greater than 1.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ClusterNoPointsException Class

```
public class Imsl.Stat.ClusterNoPointsException :  IMSLException :
ISerializable
```

There is a cluster with no points.

Constructors

ClusterNoPointsException

```
public ClusterNoPointsException()
```

Description

There is a cluster with no points.

ClusterNoPointsException

```
public ClusterNoPointsException(int clusterNumber)
```

Description

There is a cluster with no points.

Parameter

`clusterNumber` – Number of the cluster with no points.

ClusterNoPointsException

```
public ClusterNoPointsException(string message)
```

Description

There is a cluster with no points.

Parameter

`message` – The error message that explains the reason for the exception.

ClusterNoPointsException

```
public ClusterNoPointsException(string s, System.Exception exception)
```

Description

There is a cluster with no points.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ClusterNoPointsException

```
ClusterNoPointsException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

There is a cluster with no points.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ConstrInconsistentException Class

```
public class Imsl.Stat.ConstrInconsistentException :  IMSLException :
ISerializable
```

The equality constraints are inconsistent.

Constructors

ConstrInconsistentException

```
public ConstrInconsistentException()
```

Description

The equality constraints are inconsistent.

ConstrInconsistentException

```
public ConstrInconsistentException(string message)
```

Description

The equality constraints are inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

ConstrInconsistentException

```
public ConstrInconsistentException(string s, System.Exception exception)
```

Description

The equality constraints are inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ConstrInconsistentException

```
ConstrInconsistentException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The equality constraints are inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

CovarianceSingularException Class

```
public class Imsl.Stat.CovarianceSingularException :  IMSLException :
ISerializable
```

The variance-Covariance matrix is singular.

Constructors

CovarianceSingularException

```
public CovarianceSingularException()
```

Description

The variance-Covariance matrix is singular.

CovarianceSingularException

```
public CovarianceSingularException(int l)
```

Description

The variance-Covariance matrix is singular.

Parameter

`l` – An `int` which specifies the population number.

CovarianceSingularException

```
public CovarianceSingularException(string message)
```

Description

The variance-Covariance matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

CovarianceSingularException

```
public CovarianceSingularException(string s, System.Exception exception)
```

Description

The variance-Covariance matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

CovarianceSingularException

```
CovarianceSingularException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The variance-Covariance matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

CovarianceSingular1Exception Class

```
public class Imsl.Stat.CovarianceSingular1Exception :
CovarianceSingularException :  ISerializable
```

The variance-Covariance matrix is singular.

Constructor

CovarianceSingular1Exception

```
public CovarianceSingular1Exception(int l)
```

Description

The variance-Covariance matrix is singular.

Parameter

`l` – An `int` which specifies the population number.

CovarianceSingular2Exception Class

```
public class Imsl.Stat.CovarianceSingular2Exception :
CovarianceSingularException :  ISerializable
```

The variance-Covariance matrix is singular.

Constructor

CovarianceSingular2Exception

```
public CovarianceSingular2Exception()
```

Description

The variance-Covariance matrix is singular.

CyclingIsOccurringException Class

```
public class Imsl.Stat.CyclingIsOccurringException :  IMSLException :
ISerializable
```

Cycling is occurring.

Constructors

CyclingIsOccurringException

```
public CyclingIsOccurringException(int nStep)
```

Description

Cycling is occurring.

Parameter

`nStep` – An `int` which specifies the number of steps taken.

CyclingIsOccurringException

```
public CyclingIsOccurringException()
```

Description

Cycling is occurring.

CyclingIsOccurringException

```
public CyclingIsOccurringException(string message)
```

Description

Cycling is occurring.

Parameter

`message` – The error message that explains the reason for the exception.

CyclingIsOccurringException

```
public CyclingIsOccurringException(string s, System.Exception exception)
```

Description

Cycling is occurring.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

CyclingIsOccurringException

```
CyclingIsOccurringException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Cycling is occurring.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

DeleteObservationsException Class

```
public class Imsl.Stat.DeleteObservationsException :  IMSLException :
ISerializable
```

The number of observations to be deleted (set by `setObservationMax`) has grown too large.

Constructors

DeleteObservationsException

```
public DeleteObservationsException(int nmax)
```

Description

The number of observations to be deleted (set with `ObservationMax`) has grown too large.

Parameter

`nmax` – An `int` which specifies the maximum number of observations that can be handled in the linear programming.

DeleteObservationsException

```
public DeleteObservationsException()
```

Description

The number of observations to be deleted (set by `setObservationMax`) has grown too large.

DeleteObservationsException

```
public DeleteObservationsException(string message)
```

Description

The number of observations to be deleted (set by `setObservationMax`) has grown too large.

Parameter

`message` – The error message that explains the reason for the exception.

DeleteObservationsException

```
public DeleteObservationsException(string s, System.Exception exception)
```

Description

The number of observations to be deleted (set by `setObservationMax`) has grown too large.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

DeleteObservationsException

```
DeleteObservationsException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The number of observations to be deleted (set by `setObservationMax`) has grown too large.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

DidNotConvergeException Class

```
public class Imsl.Stat.DidNotConvergeException :  IMSLException :
ISerializable
```

The iteration did not converge.

Constructors

DidNotConvergeException

```
public DidNotConvergeException()
```

Description

The iteration did not converge.

DidNotConvergeException

```
public DidNotConvergeException(string message)
```

Description

The iteration did not converge.

Parameter

`message` – The error message that explains the reason for the exception.

DidNotConvergeException

```
public DidNotConvergeException(string s, System.Exception exception)
```

Description

The iteration did not converge.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

DidNotConvergeException

```
DidNotConvergeException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The iteration did not converge.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

DiffObsDeletedException Class

```
public class Imsl.Stat.DiffObsDeletedException :  IMSLException :
ISerializable
```

Different observations are being deleted from return matrix than were originally entered.

Constructors

DiffObsDeletedException

```
public DiffObsDeletedException()
```

Description

Different observations are being deleted from return matrix than were originally entered.

DiffObsDeletedException

```
public DiffObsDeletedException(int i)
```

Description

Different observations are being deleted from return matrix than were originally entered.

Parameter

`i` – An `int` which specifies the index of Variance-Covariance matrix.

DiffObsDeletedException

```
public DiffObsDeletedException(string message)
```

Description

Different observations are being deleted from return matrix than were originally entered.

Parameter

`message` – The error message that explains the reason for the exception.

DiffObsDeletedException

```
public DiffObsDeletedException(string s, System.Exception exception)
```

Description

Different observations are being deleted from return matrix than were originally entered.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

DiffObsDeletedException

```
DiffObsDeletedException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Different observations are being deleted from return matrix than were originally entered.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

EigenvalueException Class

```
public class Imsl.Stat.EigenvalueException :  IMSLException :  ISerializable
```

An error occured in calculating the eigenvalues of the adjusted (inverse) covariance matrix. Check the input covariance matrix.

Constructors

EigenvalueException

```
public EigenvalueException()
```

Description

Eigenvalue error.

EigenvalueException

```
public EigenvalueException(string message)
```

Description

Eigenvalue error.

Parameter

`message` – The error message that explains the reason for the exception.

EigenvalueException

```
public EigenvalueException(string s, System.Exception exception)
```

Description

Eigenvalue error.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

EigenvalueException

```
EigenvalueException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Eigenvalue error.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

EmptyGroupException Class

```
public class Imsl.Stat.EmptyGroupException :  IMSLException :  ISerializable
```

There are no observations in a group. Cannot compute statistics.

Constructors

EmptyGroupException

```
public EmptyGroupException(int group)
```

Description

There are no observations in a group. Cannot compute statistics.

Parameter

`group` – A `int` which specifies the index of empty group.

EmptyGroupException

```
public EmptyGroupException()
```

Description

There are no observations in a group. Cannot compute statistics.

EmptyGroupException

```
public EmptyGroupException(string message)
```

Description

There are no observations in a group. Cannot compute statistics.

Parameter

`message` – The error message that explains the reason for the exception.

EmptyGroupException

```
public EmptyGroupException(string s, System.Exception exception)
```

Description

There are no observations in a group. Cannot compute statistics.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

EmptyGroupException

```
EmptyGroupException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

There are no observations in a group. Cannot compute statistics.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

EqConstrInconsistentException Class

```
public class Imsl.Stat.EqConstrInconsistentException :  IMSLException :
ISerializable
```

The equality constraints and the bounds on the variables are found to be inconsistent.

Constructors

EqConstrInconsistentException

```
public EqConstrInconsistentException()
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

EqConstrInconsistentException

```
public EqConstrInconsistentException(string message)
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

Parameter

`message` – The error message that explains the reason for the exception.

EqConstrInconsistentException

```
public EqConstrInconsistentException(string s, System.Exception exception)
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

EqConstrInconsistentException

```
EqConstrInconsistentException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The equality constraints and the bounds on the variables are found to be inconsistent.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

IllConditionedException Class

```
public class Imsl.Stat.IllConditionedException :  IMSLException :
ISerializable
```

The problem is ill-conditioned.

Constructors

IllConditionedException

```
public IllConditionedException()
```

Description

The problem is ill-conditioned.

IllConditionedException

```
public IllConditionedException(string message)
```

Description

The problem is ill-conditioned.

Parameter

`message` – The error message that explains the reason for the exception.

IllConditionedException

```
public IllConditionedException(string s, System.Exception exception)
```

Description

The problem is ill-conditioned.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

IllConditionedException

```
IllConditionedException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The problem is ill-conditioned.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

IncreaseErrRelException Class

```
public class Imsl.Stat.IncreaseErrRelException :  IMSLException :
ISerializable
```

The bound for the relative error is too small.

Constructors

IncreaseErrRelException

```
public IncreaseErrRelException(double relativeError)
```

Description

The bound for the relative error is too small.

Parameter

`relativeError` – A `double` which specifies the bound for relative error.

IncreaseErrRelException

```
public IncreaseErrRelException()
```

Description

The bound for the relative error is too small.

IncreaseErrRelException

```
public IncreaseErrRelException(string message)
```

Description

The bound for the relative error is too small.

Parameter

`message` – The error message that explains the reason for the exception.

IncreaseErrRelException

```
public IncreaseErrRelException(string s, System.Exception exception)
```

Description

The bound for the relative error is too small.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

IncreaseErrRelException

```
IncreaseErrRelException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The bound for the relative error is too small.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

InitialMAException Class

```
public class Imsl.Stat.InitialMAException :  IMSLException :  ISerializable
```

The initial values for the moving average parameters are not invertable. Execution is halted.

Constructors

InitialMAException

```
public InitialMAException()
```

Description

The initial values for the moving average parameters are not invertable. Execution is halted.

InitialMAException

```
public InitialMAException(string message)
```

Description

The initial values for the moving average parameters are not invertable. Execution is halted.

Parameter

`message` – The error message that explains the reason for the exception.

InitialMAException

```
public InitialMAException(string s, System.Exception exception)
```

Description

The initial values for the moving average parameters are not invertable. Execution is halted.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

InitialMAException

```
InitialMAException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The initial values for the moving average parameters are not invertable. Execution is halted.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

InvalidMatrixException Class

```
public class Imsl.Stat.InvalidMatrixException :  IMSLException :  ISerializable
```

Exception thrown if a computed correlation is greater than one for some pair of variables.

Constructor

InvalidMatrixException

```
public InvalidMatrixException(int var1, int var2)
```

Description

Creates an InvalidMatrixException thrown if a computed correlation is greater than one for some pair of variables.

Parameters

`var1` – is the index of the first variable in the pair.

`var2` – is the index of the second variable in the pair.

InvalidPartialCorrelationException Class

```
public class Imsl.Stat.InvalidPartialCorrelationException :  IMSLException :
ISerializable
```

Exception thrown if a computed partial correlation is greater than one for some pair of variables.

Constructor

InvalidPartialCorrelationException

```
public InvalidPartialCorrelationException(int var1, int var2)
```

Description

Creates an InvalidPartialCorrelationException thrown if a computed partial correlation is greater than one for some pair of variables.

Parameters

`var1` – is the index of the first variable in the pair.

`var2` – is the index of the second variable in the pair.

MatrixSingularException Class

```
public class Imsl.Stat.MatrixSingularException :  IMSLException :
ISerializable
```

The input matrix is singular.

Constructors

MatrixSingularException

```
public MatrixSingularException()
```

Description

The input matrix is singular.

MatrixSingularException

```
public MatrixSingularException(string message)
```

Description

The input matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

MatrixSingularException

```
public MatrixSingularException(string s, System.Exception exception)
```

Description

The input matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

MatrixSingularException

```
MatrixSingularException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The input matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

MoreObsDelThanEnteredException Class

```
public class Imsl.Stat.MoreObsDelThanEnteredException :  IMSLException :
ISerializable
```

More observations are being deleted from the output covariance matrix than were originally entered (the corresponding row, column of the incidence matrix is less than zero).

Constructors

MoreObsDelThanEnteredException

```
public MoreObsDelThanEnteredException(int j, int k)
```

Description

More observations are being deleted from the output covariance matrix than were originally entered (the corresponding row, column of the incidence matrix is less than zero).

Parameters

`j` – A `int` which specifies the row index of Variance-Covariance matrix.

`k` – A `int` which specifies the column index of Variance-Covariance matrix.

MoreObsDelThanEnteredException

```
public MoreObsDelThanEnteredException()
```

Description

More observations are being deleted from the output covariance matrix than were originally entered (the corresponding row, column of the incidence matrix is less than zero).

MoreObsDelThanEnteredException

```
public MoreObsDelThanEnteredException(string message)
```

Description

More observations are being deleted from the output covariance matrix than were originally entered (the corresponding row, column of the incidence matrix is less than zero).

Parameter

`message` – The error message that explains the reason for the exception.

MoreObsDelThanEnteredException

```
public MoreObsDelThanEnteredException(string s, System.Exception exception)
```

Description

More observations are being deleted from the output covariance matrix than were originally entered (the corresponding row, column of the incidence matrix is less than zero).

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

MoreObsDelThanEnteredException

```
MoreObsDelThanEnteredException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

More observations are being deleted from the output covariance matrix than were originally entered (the corresponding row, column of the incidence matrix is less than zero).

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NegativeFreqException Class

```
public class Imsl.Stat.NegativeFreqException :  IMSLException :  ISerializable
```

A negative frequency was encountered.

Constructors

NegativeFreqException

```
public NegativeFreqException(int rowIndex, int invocation, double val)
```

Description

A negative frequency was encountered.

Parameters

`rowIndex` – An `int` which specifies the row index of X for which the frequency is negative.

`invocation` – An `int` which specifies the invocation number at which the error occurred. A 3 would indicate that the error occurred on the third invocation.

`val` – AAn `double` which represents the value of the frequency encountered.

NegativeFreqException

```
public NegativeFreqException()
```

Description

A negative frequency was encountered.

NegativeFreqException

```
public NegativeFreqException(string message)
```

Description

A negative frequency was encountered.

Parameter

`message` – The error message that explains the reason for the exception.

NegativeFreqException

```
public NegativeFreqException(string s, System.Exception exception)
```

Description

A negative frequency was encountered.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NegativeFreqException

```
NegativeFreqException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

A negative frequency was encountered.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NegativeWeightException Class

```
public class Imsl.Stat.NegativeWeightException :  IMSLException :
ISerializable
```

A negative weight was encountered.

Constructors

NegativeWeightException

```
public NegativeWeightException(int rowIndex, int invocation, double val)
```

Description

A negative weight was encountered.

Parameters

`rowIndex` – An `int` which specifies the row index of X for which the weight is negative.

`invocation` – An `int` which specifies the invocation number at which the error occurred. A 3 would indicate that the error occurred on the third invocation.

`val` – An `double` which represents the value of the weight encountered.

NegativeWeightException

```
public NegativeWeightException()
```

Description

A negative weight was encountered.

NegativeWeightException

```
public NegativeWeightException(string message)
```

Description

A negative weight was encountered.

Parameter

`message` – The error message that explains the reason for the exception.

NegativeWeightException

```
public NegativeWeightException(string s, System.Exception exception)
```

Description

A negative weight was encountered.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NegativeWeightException

```
NegativeWeightException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

A negative weight was encountered.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NewInitialGuessException Class

```
public class Imsl.Stat.NewInitialGuessException :  IMSLException :
ISerializable
```

The iteration has not made good progress.

Constructors

NewInitialGuessException

```
public NewInitialGuessException()
```

Description

The iteration has not made good progress.

NewInitialGuessException

```
public NewInitialGuessException(string message)
```

Description

The iteration has not made good progress.

Parameter

`message` – The error message that explains the reason for the exception.

NewInitialGuessException

```
public NewInitialGuessException(string s, System.Exception exception)
```

Description

The iteration has not made good progress.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NewInitialGuessException

```
NewInitialGuessException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The iteration has not made good progress.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoAcceptableModelFoundException Class

```
public class Imsl.Stat.NoAcceptableModelFoundException :  IMSLException :
ISerializable
```

No appropriate ARIMA model could be found.

Constructors

NoAcceptableModelFoundException

```
public NoAcceptableModelFoundException()
```

Description

No appropriate ARIMA model could be found.

NoAcceptableModelFoundException

```
public NoAcceptableModelFoundException(string message)
```

Description

No appropriate ARIMA model could be found.

Parameter

`message` – The error message that explains the reason for the exception.

NoAcceptableModelFoundException

```
public NoAcceptableModelFoundException(string s, System.Exception exception)
```

Description

No appropriate ARIMA model could be found.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoAcceptableModelFoundException

```
NoAcceptableModelFoundException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

No appropriate ARIMA model could be found.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoConvergenceException Class

```
public class Imsl.Stat.NoConvergenceException :  IMSLException :  ISerializable
```

Convergence did not occur within the maximum number of iterations.

Constructors

NoConvergenceException

```
public NoConvergenceException(int maximumIterations)
```

Description

Convergence did not occur within the maximum number of iterations.

Parameter

`maximumIterations` – A `int` which specifies the maximum number of iterations allowed.

NoConvergenceException

```
public NoConvergenceException()
```

Description

Convergence did not occur within the maximum number of iterations.

NoConvergenceException

```
public NoConvergenceException(string message)
```

Description

Convergence did not occur within the maximum number of iterations.

Parameter

`message` – The error message that explains the reason for the exception.

NoConvergenceException

```
public NoConvergenceException(string s, System.Exception exception)
```

Description

Convergence did not occur within the maximum number of iterations.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoConvergenceException

```
NoConvergenceException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Convergence did not occur within the maximum number of iterations.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoDegreesOfFreedomException Class

```
public class Imsl.Stat.NoDegreesOfFreedomException :  IMSLException :
ISerializable
```

No degrees of freedom error.

Constructors

NoDegreesOfFreedomException

```
public NoDegreesOfFreedomException(int nvar, int nf)
```

Description

No degrees of freedom error.

Parameters

`nvar` – A `int` which specifies the number of variables.

`nf` – A `int` which specifies the number of factors.

NoDegreesOfFreedomException

```
public NoDegreesOfFreedomException()
```

Description

No degrees of freedom error.

NoDegreesOfFreedomException

```
public NoDegreesOfFreedomException(string message)
```

Description

No degrees of freedom error.

Parameter

`message` – The error message that explains the reason for the exception.

NoDegreesOfFreedomException

```
public NoDegreesOfFreedomException(string s, System.Exception exception)
```

Description

No degrees of freedom error.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoDegreesOfFreedomException

```
NoDegreesOfFreedomException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

No degrees of freedom error.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoProgressException Class

```
public class Imsl.Stat.NoProgressException :  IMSLException :  ISerializable
```

The algorithm is not making any progress. Try a new initial guess.

Constructors

NoProgressException

```
public NoProgressException()
```

Description

The algorithm is not making any progress. Try a new initial guess.

NoProgressException

```
public NoProgressException(string message)
```

Description

The algorithm is not making any progress. Try a new initial guess.

Parameter

`message` – The error message that explains the reason for the exception.

NoProgressException

```
public NoProgressException(string s, System.Exception exception)
```

Description

The algorithm is not making any progress. Try a new initial guess.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoProgressException

```
NoProgressException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The algorithm is not making any progress. Try a new initial guess.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoVariationInputException Class

```
public class Imsl.Stat.NoVariationInputException :  IMSLException :
ISerializable
```

There is no variation in the input data.

Constructors

NoVariationInputException

```
public NoVariationInputException()
```

Description

There is no variation in the input data.

NoVariationInputException

```
public NoVariationInputException(string message)
```

Description

There is no variation in the input data.

Parameter

`message` – The error message that explains the reason for the exception.

NoVariationInputException

```
public NoVariationInputException(string s, System.Exception exception)
```

Description

There is no variation in the input data.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoVariationInputException

```
NoVariationInputException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

There is no variation in the input data.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoVectorXException Class

```
public class Imsl.Stat.NoVectorXException :  IMSLException :  ISerializable
```

No vector X satisfies all of the constraints.

Constructors

NoVectorXException

```
public NoVectorXException()
```

Description

No vector X satisfies all of the constraints.

NoVectorXException

```
public NoVectorXException(string message)
```

Description

No vector X satisfies all of the constraints.

Parameter

`message` – The error message that explains the reason for the exception.

NoVectorXException

```
public NoVectorXException(string s, System.Exception exception)
```

Description

No vector X satisfies all of the constraints.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoVectorXException

```
NoVectorXException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

No vector X satisfies all of the constraints.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NonInvertibleException Class

```
public class Imsl.Stat.NonInvertibleException :  IMSLException :  ISerializable
```

The solution is noninvertible.

Constructors

NonInvertibleException

```
public NonInvertibleException()
```

Description

The solution is noninvertible.

NonInvertibleException

```
public NonInvertibleException(string message)
```

Description

The solution is noninvertible.

Parameter

`message` – The error message that explains the reason for the exception.

NonInvertibleException

```
public NonInvertibleException(string s, System.Exception exception)
```

Description

The solution is noninvertible.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NonInvertibleException

```
NonInvertibleException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The solution is noninvertible.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NonPosVarianceException Class

```
public class Imsl.Stat.NonPosVarianceException :  IMSLException :
ISerializable
```

The problem is ill-conditioned.

Constructors

NonPosVarianceException

```
public NonPosVarianceException(double var)
```

Description

The problem is ill-conditioned.

Parameter

`var` – A `double` which specifies the variance.

NonPosVarianceException

```
public NonPosVarianceException()
```

Description

The problem is ill-conditioned.

NonPosVarianceException

```
public NonPosVarianceException(string message)
```

Description

The problem is ill-conditioned.

Parameter

`message` – The error message that explains the reason for the exception.

NonPosVarianceException

```
public NonPosVarianceException(string s, System.Exception exception)
```

Description

The problem is ill-conditioned.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NonPosVarianceException

```
NonPosVarianceException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The problem is ill-conditioned.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NonPosVarianceXYException Class

```
public class Imsl.Stat.NonPosVarianceXYException :  IMSLException :
ISerializable
```

The problem is ill-conditioned.

Constructors

NonPosVarianceXYException

```
public NonPosVarianceXYException(string varName, double var)
```

Description

The problem is ill-conditioned.

Parameters

`varName` – A `string` which specifies either “X” or “Y”.

`var` – A `double` which specifies the variance.

NonPosVarianceXYException

```
public NonPosVarianceXYException()
```

Description

The problem is ill-conditioned.

NonPosVarianceXYException

```
public NonPosVarianceXYException(string message)
```

Description

The problem is ill-conditioned.

Parameter

`message` – The error message that explains the reason for the exception.

NonPosVarianceXYException

```
public NonPosVarianceXYException(string s, System.Exception exception)
```

Description

The problem is ill-conditioned.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NonPosVarianceXYException

```
NonPosVarianceXYException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The problem is ill-conditioned.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NonPositiveEigenvalueException Class

```
public class Imsl.Stat.NonPositiveEigenvalueException :  IMSLException :
ISerializable
```

Maximum number of iterations exceeded.

Constructors

NonPositiveEigenvalueException

```
public NonPositiveEigenvalueException(int iter, int i, double eval)
```

Description

Maximum number of iterations exceeded.

Parameters

`iter` – A `int` which specifies the iteration number.

`i` – A `int` which specifies the eigenvalue index.

`eval` – A `double` which specifies the eigenvalue.

NonPositiveEigenvalueException

```
public NonPositiveEigenvalueException()
```

Description

Maximum number of iterations exceeded.

NonPositiveEigenvalueException

```
public NonPositiveEigenvalueException(string message)
```

Description

Maximum number of iterations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

NonPositiveEigenvalueException

```
public NonPositiveEigenvalueException(string s, System.Exception exception)
```

Description

Maximum number of iterations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NonPositiveEigenvalueException

```
NonPositiveEigenvalueException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of iterations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NonStationaryException Class

```
public class Imsl.Stat.NonStationaryException :  IMSLException :  ISerializable
```

The solution is nonstationary.

Constructors

NonStationaryException

```
public NonStationaryException()
```

Description

The solution is nonstationary.

NonStationaryException

```
public NonStationaryException(string message)
```

Description

The solution is nonstationary.

Parameter

`message` – The error message that explains the reason for the exception.

NonStationaryException

```
public NonStationaryException(string s, System.Exception exception)
```

Description

The solution is nonstationary.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NonStationaryException

```
NonStationaryException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The solution is nonstationary.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoPositiveVarianceException Class

```
public class Imsl.Stat.NoPositiveVarianceException :  IMSLException :
ISerializable
```

No variable has positive variance. The Mahalanobis distances cannot be computed.

Constructors

NoPositiveVarianceException

```
public NoPositiveVarianceException()
```

Description

No variable has positive variance. The Mahalanobis distances cannot be computed.

NoPositiveVarianceException

```
public NoPositiveVarianceException(string message)
```

Description

No variable has positive variance. The Mahalanobis distances cannot be computed.

Parameter

`message` – The error message that explains the reason for the exception.

NoPositiveVarianceException

```
public NoPositiveVarianceException(string s, System.Exception exception)
```

Description

No variable has positive variance. The Mahalanobis distances cannot be computed.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoPositiveVarianceException

```
NoPositiveVarianceException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

No variable has positive variance. The Mahalanobis distances cannot be computed.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotCDFException Class

```
public class Imsl.Stat.NotCDFException :  IMSLException :  ISerializable
```

The function is not a Cumulative Distribution Function (CDF).

Constructors

NotCDFException

```
public NotCDFException(double lowerBound, double upperBound)
```

Description

The function is not a Cumulative Distribution Function (CDF).

Parameters

`lowerBound` – A `double` containing the lower bound to be displayed in message.

`upperBound` – A `double` containing the upper bound to be displayed in message.

NotCDFException

```
public NotCDFException(double range)
```

Description

The function is not a Cumulative Distribution Function (CDF).

Parameter

`range` – A `double` containing the probability of the range.

NotCDFException

```
public NotCDFException(double x1, double x2, double f1)
```

Description

The function is not a Cumulative Distribution Function (CDF).

Parameters

`x1` – is the first point

`x2` – is the second point

`f1` – is the common value for F(x1) and F(x2)

Remarks

The CDF function is not monotone, F(x1) = F(x2). No unique inverse exists.

NotCDFException

```
public NotCDFException(double lowerBound, double upperBound, double xx, int i)
```

Description

The function is not a Cumulative Distribution Function (CDF).

Parameters

`lowerBound` – A `double` containing the lower bound for the CDF value.

`upperBound` – A `double` containing the upper bound for the CDF value.

`xx` – A `double` containing the value at a cutpoint.

`i` – The index of the cutpoint that is out of range.

Remarks

The cdf function is not a cumulative distribution function because its value at a cutpoint is out of the expected range, [plower,pupper].

NotCDFException

```
public NotCDFException()
```

Description

The function is not a Cumulative Distribution Function (CDF).

NotCDFException

```
public NotCDFException(string message)
```

Description

The function is not a Cumulative Distribution Function (CDF).

Parameter

`message` – The error message that explains the reason for the exception.

NotCDFException

```
public NotCDFException(string s, System.Exception exception)
```

Description

The function is not a Cumulative Distribution Function (CDF).

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotCDFException

```
NotCDFException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The function is not a Cumulative Distribution Function (CDF).

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotPositiveDefiniteException Class

```
public class Imsl.Stat.NotPositiveDefiniteException :  IMSLException :
ISerializable
```

Covariance matrix is not positive definite.

Constructors

NotPositiveDefiniteException

```
public NotPositiveDefiniteException(int i)
```

Description

Covariance matrix is not positive definite.

Parameter

`i` – Variable i is linearly related to the other variables in the factor analysis.

NotPositiveDefiniteException

```
public NotPositiveDefiniteException()
```

Description

Covariance matrix is not positive definite.

NotPositiveDefiniteException

```
public NotPositiveDefiniteException(string message)
```

Description

Covariance matrix is not positive definite.

Parameter

`message` – The error message that explains the reason for the exception.

NotPositiveDefiniteException

```
public NotPositiveDefiniteException(string s, System.Exception exception)
```

Description

Covariance matrix is not positive definite.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotPositiveDefiniteException

```
NotPositiveDefiniteException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Covariance matrix is not positive definite.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotPositiveSemiDefiniteException Class

```
public class Imsl.Stat.NotPositiveSemiDefiniteException :  IMSLException :
ISerializable
```

Covariance matrix is not positive semi-definite.

Constructors

NotPositiveSemiDefiniteException

```
public NotPositiveSemiDefiniteException()
```

Description

Covariance matrix is not positive semi-definite.

NotPositiveSemiDefiniteException

```
public NotPositiveSemiDefiniteException(string message)
```

Description

Covariance matrix is not positive semi-definite.

Parameter

`message` – The error message that explains the reason for the exception.

NotPositiveSemiDefiniteException

```
public NotPositiveSemiDefiniteException(string s, System.Exception exception)
```

Description

Covariance matrix is not positive semi-definite.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotPositiveSemiDefiniteException

```
NotPositiveSemiDefiniteException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Covariance matrix is not positive semi-definite.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NotSemiDefiniteException Class

```
public class Imsl.Stat.NotSemiDefiniteException :  IMSLException :
ISerializable
```

Hessian matrix is not semi-definite.

Constructors

NotSemiDefiniteException

```
public NotSemiDefiniteException()
```

Description

Hessian matrix is not semi-definite.

NotSemiDefiniteException

```
public NotSemiDefiniteException(string message)
```

Description

Hessian matrix is not semi-definite.

Parameter

`message` – The error message that explains the reason for the exception.

NotSemiDefiniteException

```
public NotSemiDefiniteException(string s, System.Exception exception)
```

Description

Hessian matrix is not semi-definite.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NotSemiDefiniteException

```
NotSemiDefiniteException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Hessian matrix is not semi-definite.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoVariablesEnteredException Class

```
public class Imsl.Stat.NoVariablesEnteredException :  IMSLException :
ISerializable
```

No Variables can enter the model.

Constructors

NoVariablesEnteredException

```
public NoVariablesEnteredException()
```

Description

No Variables can enter the model.

NoVariablesEnteredException

```
public NoVariablesEnteredException(string message)
```

Description

No Variables can enter the model.

Parameter

`message` – The error message that explains the reason for the exception.

NoVariablesEnteredException

```
public NoVariablesEnteredException(string s, System.Exception exception)
```

Description

No Variables can enter the model.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoVariablesEnteredException

```
NoVariablesEnteredException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

No Variables can enter the model.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoVariablesException Class

```
public class Imsl.Stat.NoVariablesException :  IMSLException :  ISerializable
```

No variables can enter the model.

Constructors

NoVariablesException

```
public NoVariablesException()
```

Description

No variables can enter the model.

NoVariablesException

```
public NoVariablesException(string message)
```

Description

No variables can enter the model.

Parameter

`message` – The error message that explains the reason for the exception.

NoVariablesException

```
public NoVariablesException(string s, System.Exception exception)
```

Description

No variables can enter the model.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoVariablesException

```
NoVariablesException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

No variables can enter the model.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoVariationInputException Class

```
public class Imsl.Stat.NoVariationInputException :  IMSLException :
ISerializable
```

There is no variation in the input data.

Constructors

NoVariationInputException

```
public NoVariationInputException()
```

Description

There is no variation in the input data.

NoVariationInputException

```
public NoVariationInputException(string message)
```

Description

There is no variation in the input data.

Parameter

`message` – The error message that explains the reason for the exception.

NoVariationInputException

```
public NoVariationInputException(string s, System.Exception exception)
```

Description

There is no variation in the input data.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoVariationInputException

```
NoVariationInputException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

There is no variation in the input data.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

NoVectorXException Class

```
public class Imsl.Stat.NoVectorXException :  IMSLException :  ISerializable
```

No vector X satisfies all of the constraints.

Constructors

NoVectorXException

```
public NoVectorXException()
```

Description

No vector X satisfies all of the constraints.

NoVectorXException

```
public NoVectorXException(string message)
```

Description

No vector X satisfies all of the constraints.

Parameter

`message` – The error message that explains the reason for the exception.

NoVectorXException

```
public NoVectorXException(string s, System.Exception exception)
```

Description

No vector X satisfies all of the constraints.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

NoVectorXException

```
NoVectorXException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

No vector X satisfies all of the constraints.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

PooledCovarianceSingularException Class

```
public class Imsl.Stat.PooledCovarianceSingularException :  IMSLException :
ISerializable
```

The pooled variance-Covariance matrix is singular.

Constructors

PooledCovarianceSingularException

```
public PooledCovarianceSingularException()
```

Description

The pooled variance-Covariance matrix is singular.

PooledCovarianceSingularException

```
public PooledCovarianceSingularException(string message)
```

Description

The pooled variance-Covariance matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

PooledCovarianceSingularException

```
public PooledCovarianceSingularException(string s, System.Exception exception)
```

Description

The pooled variance-Covariance matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

PooledCovarianceSingularException

```
PooledCovarianceSingularException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The pooled variance-Covariance matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

RankException Class

```
public class Imsl.Stat.RankException :  IMSLException :  ISerializable
```

Rank of covariance matrix error.

Constructors

RankException

```
public RankException(int rank, int nf)
```

Description

Rank of covariance matrix error.

Parameters

`rank` – A `int` which specifies the rank of the covariance matrix.

`nf` – A `int` which specifies the number of factors.

RankException

```
public RankException()
```

Description

Rank of covariance matrix error.

RankException

```
public RankException(string message)
```

Description

Rank of covariance matrix error.

Parameter

`message` – The error message that explains the reason for the exception.

RankException

```
public RankException(string s, System.Exception exception)
```

Description

Rank of covariance matrix error.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

RankException

```
RankException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Rank of covariance matrix error.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

RankDeficientException Class

```
public class Imsl.Stat.RankDeficientException :  IMSLException :  ISerializable
```

The model has been determined to be rank deficient.

Constructors

RankDeficientException

```
public RankDeficientException(int rank)
```

Description

The model has been determined to be rank deficient.

Parameter

`rank` – An `int` which specifies the rank of the model.

RankDeficientException

```
public RankDeficientException()
```

Description

The model has been determined to be rank deficient.

RankDeficientException

```
public RankDeficientException(string message)
```

Description

The model has been determined to be rank deficient.

Parameter

`message` – The error message that explains the reason for the exception.

RankDeficientException

```
public RankDeficientException(string s, System.Exception exception)
```

Description

The model has been determined to be rank deficient.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

RankDeficientException

```
RankDeficientException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The model has been determined to be rank deficient.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ScaleFactorZeroException Class

```
public class Imsl.Stat.ScaleFactorZeroException :  IMSLException :
ISerializable
```

The computations cannot continue because a scale factor is zero.

Constructors

ScaleFactorZeroException

```
public ScaleFactorZeroException(int index)
```

Description

The computations cannot continue because a scale factor is zero.

Parameter

`index` – An `int` which specifies the index of the scale factor array at which scale factor is zero.

ScaleFactorZeroException

```
public ScaleFactorZeroException()
```

Description

The computations cannot continue because a scale factor is zero.

ScaleFactorZeroException

```
public ScaleFactorZeroException(string message)
```

Description

The computations cannot continue because a scale factor is zero.

Parameter

`message` – The error message that explains the reason for the exception.

ScaleFactorZeroException

```
public ScaleFactorZeroException(string s, System.Exception exception)
```

Description

The computations cannot continue because a scale factor is zero.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ScaleFactorZeroException

```
ScaleFactorZeroException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The computations cannot continue because a scale factor is zero.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SingularException Class

```
public class Imsl.Stat.SingularException :  IMSLException :  ISerializable
```

Covariance matrix is singular.

Constructors

SingularException

```
public SingularException(int i)
```

Description

Covariance matrix is singular.

Parameter

`i` – Variable i is linearly related to the other variables.

SingularException

```
public SingularException()
```

Description

Covariance matrix is singular.

SingularException

```
public SingularException(string message)
```

Description

Covariance matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

SingularException

```
public SingularException(string s, System.Exception exception)
```

Description

Covariance matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SingularException

```
SingularException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Covariance matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SingularTriangularMatrixException Class

```
public class Imsl.Stat.SingularTriangularMatrixException :  IMSLException :
ISerializable
```

Triangular matrix is singular.

Constructors

SingularTriangularMatrixException

```
public SingularTriangularMatrixException(int i)
```

Description

Triangular matrix is singular.

Parameter

`i` – Variable i is the index of the first zero diagonal element.

SingularTriangularMatrixException

```
public SingularTriangularMatrixException()
```

Description

Triangular matrix is singular.

SingularTriangularMatrixException

```
public SingularTriangularMatrixException(string message)
```

Description

Triangular matrix is singular.

Parameter

`message` – The error message that explains the reason for the exception.

SingularTriangularMatrixException

```
public SingularTriangularMatrixException(string s, System.Exception exception)
```

Description

Triangular matrix is singular.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SingularTriangularMatrixException

```
SingularTriangularMatrixException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Triangular matrix is singular.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

SumOfWeightsNegException Class

```
public class Imsl.Stat.SumOfWeightsNegException :  IMSLException :
ISerializable
```

The sum of the weights have become negative.

Constructors

SumOfWeightsNegException

```
public SumOfWeightsNegException(int group)
```

Description

The sum of the weights have become negative.

Parameter

`group` – An `int` which specifies the group for which the sum of the weights have become negative.

SumOfWeightsNegException

```
public SumOfWeightsNegException()
```

Description

The sum of the weights have become negative.

SumOfWeightsNegException

```
public SumOfWeightsNegException(string message)
```

Description

The sum of the weights have become negative.

Parameter

`message` – The error message that explains the reason for the exception. A `String` containing the error message.

SumOfWeightsNegException

```
public SumOfWeightsNegException(string s, System.Exception exception)
```

Description

The sum of the weights have become negative.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

SumOfWeightsNegException

```
SumOfWeightsNegException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The sum of the weights have become negative.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyCallsException Class

```
public class Imsl.Stat.TooManyCallsException :  IMSLException :  ISerializable
```

The number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters+1.

Constructors

TooManyCallsException

```
public TooManyCallsException()
```

Description

The number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters+1.

TooManyCallsException

```
public TooManyCallsException(string message)
```

Description

The number of calls to the function has exceeded the maximum number of iterations times the number of moving average (MA) parameters+1.

Parameter

`message` – The error message that explains the reason for the exception.

TooManyCallsException

```
public TooManyCallsException(string s, System.Exception exception)
```

Description

The number of calls to the function has exceeded the maximum number of iterations.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyCallsException

```
TooManyCallsException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The number of calls to the function has exceeded the maximum number of iterations.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyFunctionEvaluationsException Class

```
public class Imsl.Stat.TooManyFunctionEvaluationsException :  IMSLException :
ISerializable
```

Maximum number of function evaluations exceeded.

Constructors

TooManyFunctionEvaluationsException

```
public TooManyFunctionEvaluationsException(int maximumNumberOfEvaluations)
```

Description

Maximum number of function evaluations exceeded.

Parameter

`maximumNumberOfEvaluations` – A `int` which specifies the maximum number of function evaluations allowed.

TooManyFunctionEvaluationsException

```
public TooManyFunctionEvaluationsException()
```

Description

Maximum number of function evaluations exceeded.

TooManyFunctionEvaluationsException

```
public TooManyFunctionEvaluationsException(string message)
```

Description

Maximum number of function evaluations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

TooManyFunctionEvaluationsException

```
public TooManyFunctionEvaluationsException(string s, System.Exception
exception)
```

Description

Maximum number of function evaluations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyFunctionEvaluationsException

```
TooManyFunctionEvaluationsException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of function evaluations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyIterationsException Class

```
public class Imsl.Stat.TooManyIterationsException :  IMSLException :
ISerializable
```

Maximum number of iterations exceeded.

Constructors

TooManyIterationsException

```
public TooManyIterationsException(int maximumNumberOfIterations)
```

Description

Maximum number of iterations exceeded.

Parameter

`maximumNumberOfIterations` – A `int` which specifies the maximum number of iterations allowed.

TooManyIterationsException

```
public TooManyIterationsException()
```

Description

Maximum number of iterations exceeded.

TooManyIterationsException

```
public TooManyIterationsException(string message)
```

Description

Maximum number of iterations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

TooManyIterationsException

```
public TooManyIterationsException(string s, System.Exception exception)
```

Description

Maximum number of iterations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyIterationsException

```
TooManyIterationsException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of iterations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyIterationsReTryException Class

```
public class Imsl.Stat.TooManyIterationsReTryException :  IMSLException :
ISerializable
```

The maximum number of iterations was exceeded, increase maximum iterations or try a different parameter estimation method.

Constructors

TooManyIterationsReTryException

```
public TooManyIterationsReTryException(int maximumNumberOfIterations)
```

Description

The maximum number of iterations was exceeded, increase maximum iterations or try a different parameter estimation method.

Parameter

`maximumNumberOfIterations` – A `int` which specifies the maximum number of iterations allowed.

TooManyIterationsReTryException

```
public TooManyIterationsReTryException(string message)
```

Description

The maximum number of iterations was exceeded, increase maximum iterations or try a different parameter estimation method.

Parameter

`message` – The error message that explains the reason for the exception.

TooManyIterationsReTryException

```
public TooManyIterationsReTryException(string s, System.Exception exception)
```

Description

The maximum number of iterations was exceeded, increase maximum iterations or try a different parameter estimation method.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyIterationsReTryException

```
TooManyIterationsReTryException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

The maximum number of iterations was exceeded, increase maximum iterations or try a different parameter estimation method.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyJacobianEvalException Class

```
public class Imsl.Stat.TooManyJacobianEvalException :  IMSLException :
ISerializable
```

Maximum number of Jacobian evaluations exceeded.

Constructors

TooManyJacobianEvalException

```
public TooManyJacobianEvalException()
```

Description

Maximum number of Jacobian evaluations exceeded.

TooManyJacobianEvalException

```
public TooManyJacobianEvalException(string message)
```

Description

Maximum number of Jacobian evaluations exceeded.

Parameter

`message` – The error message that explains the reason for the exception.

TooManyJacobianEvalException

```
public TooManyJacobianEvalException(string s, System.Exception exception)
```

Description

Maximum number of Jacobian evaluations exceeded.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyJacobianEvalException

```
TooManyJacobianEvalException(System.Runtime.Serialization.SerializationInfo
info, System.Runtime.Serialization.StreamingContext context)
```

Description

Maximum number of Jacobian evaluations exceeded.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

TooManyObsDeletedException Class

```
public class Imsl.Stat.TooManyObsDeletedException :  IMSLException :
ISerializable
```

More observations have been deleted than were originally entered (the sum of frequencies has become negative).

Constructors

TooManyObsDeletedException

```
public TooManyObsDeletedException()
```

Description

More observations have been deleted than were originally entered (the sum of frequencies has become negative).

TooManyObsDeletedException

```
public TooManyObsDeletedException(string message)
```

Description

More observations have been deleted than were originally entered (the sum of frequencies has become negative).

Parameter

`message` – The error message that explains the reason for the exception.

TooManyObsDeletedException

```
public TooManyObsDeletedException(string s, System.Exception exception)
```

Description

More observations have been deleted than were originally entered (the sum of frequencies has become negative).

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

TooManyObsDeletedException

```
TooManyObsDeletedException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

More observations have been deleted than were originally entered (the sum of frequencies has become negative).

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

VarsDeterminedException Class

```
public class Imsl.Stat.VarsDeterminedException :  IMSLException :
ISerializable
```

The variables are determined by the equality constraints.

Constructors

VarsDeterminedException

```
public VarsDeterminedException()
```

Description

The variables are determined by the equality constraints.

VarsDeterminedException

```
public VarsDeterminedException(string message)
```

Description

The variables are determined by the equality constraints.

Parameter

`message` – The error message that explains the reason for the exception.

VarsDeterminedException

```
public VarsDeterminedException(string s, System.Exception exception)
```

Description

The variables are determined by the equality constraints.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

VarsDeterminedException

```
VarsDeterminedException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The variables are determined by the equality constraints.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

ZeroNormException Class

```
public class Imsl.Stat.ZeroNormException :  IMSLException :  ISerializable
```

The computations cannot continue because the Euclidean norm of the column is equal to zero.

Constructors

ZeroNormException

```
public ZeroNormException(int index)
```

Description

The computations cannot continue because the Euclidean norm of the column is equal to zero.

Parameter

`index` – An `int` which specifies the column index for which the norm has been found to be zero.

ZeroNormException

```
public ZeroNormException()
```

Description

The computations cannot continue because the Euclidean norm of the column is equal to zero.

ZeroNormException

```
public ZeroNormException(string message)
```

Description

The computations cannot continue because the Euclidean norm of the column is equal to zero.

Parameter

`message` – The error message that explains the reason for the exception.

ZeroNormException

```
public ZeroNormException(string s, System.Exception exception)
```

Description

The computations cannot continue because the Euclidean norm of the column is equal to zero.

Parameters

`s` – The error message that explains the reason for the exception.

`exception` – The exception that is the cause of the current exception. If the innerException parameter is not a null reference, the current exception is raised in a catch block that handles the inner exception.

ZeroNormException

```
ZeroNormException(System.Runtime.Serialization.SerializationInfo info,
System.Runtime.Serialization.StreamingContext context)
```

Description

The computations cannot continue because the Euclidean norm of the column is equal to zero.

Parameters

`info` – The object that holds the serialized object data.

`context` – The contextual information about the source or destination.

Chapter 30: References

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